The Bank and The Asset Market:

The Roles of Aggregate Uncertainty and Timing

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Abstract

This paper compares the ex ante welfare under different regimes: only the bank exists (the pure bank), only the asset market exists (the pure market) and both exist (the bank & market). People may have urgent consumption opportunities but the high-return asset is illiquid. So by satisfying the urgent opportunities without losing the investment flexibility, both the bank and the market can improve the ex ante welfare compared to autarky. The pure bank can do strictly better than the pure market. The existence of the market will affect the bank's performance negatively. But the bank & market can still do strictly better than the pure market. The last result relies crucially on two assumptions. Firstly, the aggregate liquidity need is uncertain. Secondly, timing is important. The uncertainty about the aggregate liquidity need is not solved when the depositors make their withdrawal decisions.

1 Introduction

People may have urgent consumption opportunities but the high-return asset is usually illiquid. Both the bank and the asset market can help people with taking the urgent consumption opportunities and, at the same time, enjoying high return from the illiquid asset. The bank achieves this by providing the demand deposit. The asset market achieves this by allowing people to sell their illiquid asset in the market. This paper compares the ex ante welfare under different regimes: only the bank exists (the pure bank), only the asset market exists (the pure market) and both exist (the bank & market).

The innovation I made in the paper is to introduce aggregate uncertainty and specify the time when that information is revealed. It is realistic to assume that when people decide whether to withdraw they know their own consumption need but not others' or the aggregate liquidity need. The aggregate liquidity need will affect the market price. So uncertainty on the aggregate liquidity need makes the market "risky" for people. That fact is important since it will give the bank more room to design the optimal contract when facing the competition from the market.

This paper also helps to answer a question in the literature starting from Diamond and Dybvig (1983). The bank in Diamond and Dybvig provides insurance against urgent consumption needs. But as Jacklin (1988) and others pointed out, the bank is redundant when we have the asset market. By introducing aggregate uncertainty and the specify the timing issue, I show that bank is still useful rather than redundant in face of the asset market.

2 Literature Review

Diamond and Dybvig (1983) showed how the pure bank improves the welfare when the liquidity risk is idiosyncratic. It is a three-period model. In period 0, individuals are the same and each one knows that he may be either "impatient" or "patient" in period 1. People's types will be private information. Being impatient (patient), they will only care the first (second) period consumption. The utility function is given by:

$$u_j(C_1, C_2) = \begin{cases} u(C_1) \text{ if } j = I \\ u(C_2) \text{ if } j = P \end{cases} \tag{1}$$

I (P) stands for impatient (patient). C_t is the consumption in period t ($t = 1, 2$). For every unit of the investment made in period 0, it can provide one unit of consumption goods if liquidated in period 1 or R

units of goods if held until period 2[1]. The bank will solve the following maximization problem[2]:

$$\underset{C_1,C_2}{Max} W = \alpha u(C_1) + (1-\alpha)u(C_2) \tag{2}$$

$$s.t. \begin{cases} \alpha C_1 + (1-\alpha)C_2/R = 1, & \text{(RC)} \\ C_1 \leq C_2, & \text{(IC)} \end{cases}$$

The advantage of the bank can be shown by using the following figure:

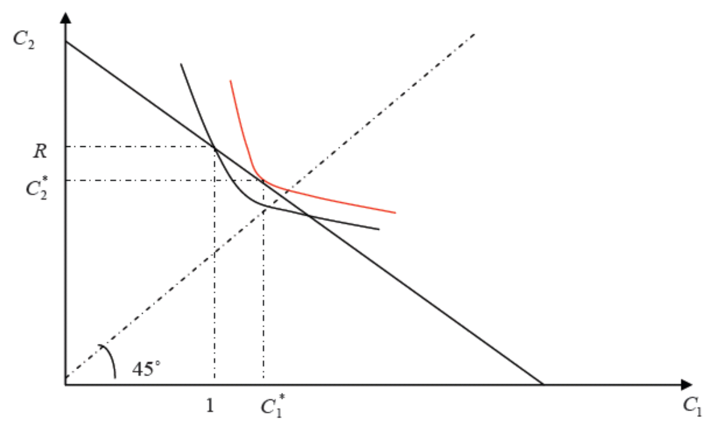

Figure 1. Pure Bank in Diamond and Dybvig

The budget line of the bank is given by the resource constraint. The incentive constraint says that the points above or on the 45-degree line is incentive compatible. So the bank can choose any point on the budget line and above the 45-degree line. But $C_1 = 1$ and $C_2 = R$ is the only consumption bundle on the budget line that is available for the people in autarky. So if people are sufficiently risk averse, bank can provide strictly higher ex ante welfare compared to autarky.[3] The demand deposit provided by the bank, like insurance, will pool the liquidity risk among people with different liquidity need.

[1]In their paper, there is another asset. Each unit of this asset always produces one unit of consumption goods no matter whether it is liquadated in period 1 or held until period 2. Since this asset is dominated by the one mentioned in the text, it will not be used.

[2]In this literature, perfect competition among the banks is assumed. So the bank will offer the contract which can maximize the depositors' welfare. I will keep the same assumption for this paper.

[3]If the utiliy function is CRRA: $u(C) = \frac{C^{1-\gamma}}{1-\gamma}$, the sufficient condition for the bank to be strictly better than the market is $\gamma > 1$

However, Jacklin (1987) showed that if the asset market exists, side trade in the market will completely restrict bank's ability to provide the insurance[4]. If W_i is the expected welfare under mechanism i, then we have:

$$W_{Pure\ Market} = W_{Bank\ \&\ Market} < W_{Pure\ Bank} \quad (3)$$

So bank is redundant if the market exists. The reason for this result is that patient people now can pretend to be impatient, get the consumption goods in the first period and using it to buy illiquid asset in the market. It is easy to see that in the market the price of the illiquid asset in terms of period 1 consumption goods is $p = 1$. So to provide enough incentives for the patient people to tell the truth, a much stronger incentive constraint must be satisfied:

$$(C_1/p) * R = C_1 * R \leq C_2 \quad (4)$$

That is to say, only the points above the blue line in Figure 2 is incentive compatible. So the bank will choose $C_1 = 1$ and $C_2 = R$, which is also the market allocation.

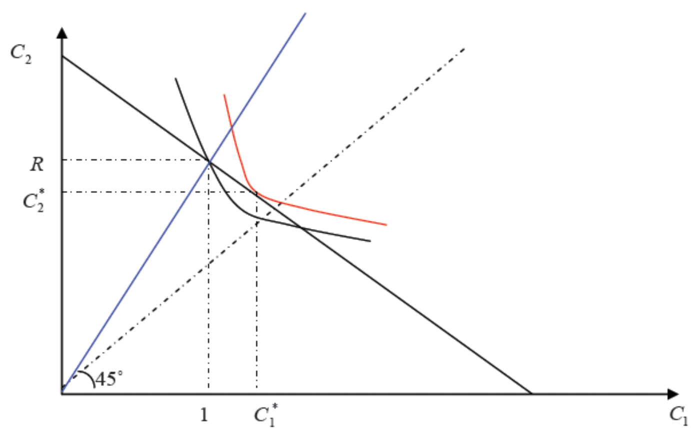

Figure 2. Jacklin's Critique

In this paper I study the question raised in Jacklin's critique again. In the original Diamond-Dybvig model, once people know their own liquidity need, there is no more uncertainty for them. The price of illiquid asset in the market is certain since the aggregate liquidity need is certain. So the patient depositors know for sure how much they can get if they withdraw in the first period and trade in the market. But in the real life, when an individual learns his own urgent consumption

[4]Several other papers also showed side trade will undermine the bank. These papers include Haubrich and King (1990), von Thadder (1997), and Hellwig (1994).

need and decides whether to withdraw, he does not know for sure how many people in the society have the same need and what the price of the illiquid asset will be. To capture this idea, I introduce the aggregate uncertainty of the liquidity need and assume that an individual just knows his own type rather than others' or the aggregate liquidity need when he makes the withdrawal decision. If we modify the model in this way, it can be shown that:

$$W_{Pure\ Market} < W_{Bank\ \&\ Market} < W_{Pure\ Bank} \tag{5}$$

As before, the bank's ability to provide insurance will be hurt by the existence of the market (the second inequality in (4))[5]. But the bank is still important , rather than redundant, even though the asset market exists (the first inequality in (4)). The intuition why the bank is not redundant is the following. The depositors do not know the state of the aggregate liquidity need. To defend the attraction from the market, the bank does not have to mimic the market allocation for each α.The bank just has to make sure that the patient depositors, given their beliefs about the aggregate liquidity need, would like to wait. This will leave room for the bank to improve the welfare compared to the pure market case. This is a big difference compared to the original Diamond and Dybvig (1983). In that model when the bank and the market coexist, as pointed out by Jacklin, the certainty of the aggregate liquidity need leaves no room for the bank but mimic the market allocation.

The importance of the bank can also be saved by limited participation of the market. Wallace (1988) argues that Diamond-Dybvig model can be interpreted as a model in which no asset market exists because people are physically separated. Diamond (1997) modified the Diamond and Dybvig model by adding a financial market with limited participation. He showed that bank is important because the bank can provide insurance among the impatient people and the patient people who can not get access to the market. Allen and Gale (2004) also studied the relation between the bank and the market. But they assume that only the bank can trade in the market. Farhi, Golosov and Tsyvinski (2009) studied the similar issue but focused on how the government regulation can make bank more efficient.

Another way to save the bank is to emphasize the role of bank to facilitate transactions like Peck and Shell (2010). They modify the D-D model to capture the roles of checking accounts and debt cards in facilitating transactions. They assume that consumption opportunity is

[5]This is not surprising. Relaxing a constraint in a second-best world will not necessarily make agents better off. See Hart (1975).

urgent and hence if checks or debit card transactions do not clear at par, the opportunity is lost. And the bank will provide the checking accounts and debt cards service.[6]

In my model, no restriction on the market participation is made. The transaction process is not part of the model either. So I show that the bank is important even though everyone can trade in the market and the transaction can be facilitated without the bank. In section 3, I will describe the setup of the model. In section 4, I study the pure market equilibrium. In section 5, the pure bank equilibrium is studied and I compare the welfare between the pure market and the pure bank. In section 6, I study the case when both the bank and the market exist. It is shown that although the bank can not do as well as before, it is still strictly better than the pure market. Bank is not redundant. In section 7, the conclusions are summarized. The proofs are in the appendix.

3 The Model

There are three dates, 0,1 and 2. There is a continuum of agents with measure 1. There is only one commodity and each agent is endowed with y units of that commodity at period 0. This commodity can be used to invest in the liquid asset or the illiquid asset in period 0. The liquid asset uses storage technology and will yield a one-period-ahead return of 1 per unit invested. The illiquid asset will yield a two-period-ahead return of $R > 1$ and nothing in one period.

	Period 0	Period 1	Period 2
Liquid Asset	-1	1 -1	 1
Illiquid Asset	-1	0	R

Table 1: Asset Return

As for people's preferences, we deviate from the original Diamond and Dybvig model by using the set up in Peck and Shell (2010)[7]. As of

[6] They also assume that there is long life ahead after the end of period 2; they capture this by positing utility of "left-over" bank balances.

[7] The reason for the deviation is to make the mechanism design question tractable. Since the bank, as well as the public, does not know the aggregate liquidity needs when people withdraw in the first peiod, the bank has to decide how much to give out without knowing the aggregate liquidity needs. The Peck and Shell set-up can simplify the answer to this. Also, in the Peck and Shell set-up, both types value the consumption in the second period. That will make the compensation to the impatient people who can not withdraw in the first period possible.

date 0, all agents are identical and are uncertain on whether he will be impatient or patient in period 1. If he is impatient, he has the urgent consumption opportunity in period 1. The consumption opportunity will cost 1 unit of the resources and increase the utility by $\overline{u}$. If an impatient individual does not have enough resources to take the urgent consumption opportunity in period 1, he has to postpone the consumption until period 2 and he can only get $\beta\overline{u}$ from it ($0 < \beta < 1$). For the patient individual, the consumption opportunity is in period 2 and he can get $\overline{u}$ from it. Both types of people also value the left-over resources after taking the consumption opportunities by the function u. u is an strictly increasing and strictly concave function. We also assume $u(0) = -\infty$[8].

If we use C_j^t to denote the the resources available for type j ($j = I, P$) in period t ($t = 1, 2$), then the utility function can be written in the following way:

$$u_I(C_I^1, C_I^2) = \begin{cases} \overline{u} + u(C_I^1 + C_I^2 - 1) & \text{if } C_I^1 \geq 1 \\ \beta\overline{u} + u(C_I^1 + C_I^2 - 1) & \text{if } C_I^1 < 1 \end{cases} \tag{6}$$

$$u_P(C_P^1, C_P^2) = \overline{u} + u(C_P^1 + C_P^2 - 1),$$

By specifying people's preferences in this way, we have made two implicit assumptions:

$$\beta\overline{u} > u'(yR - R) \tag{A1}$$

$$(1-\beta)\overline{u} > u'(yR - R)(R - 1) \tag{A2}$$

(A1) says that the utility from the inferior consumption opportunity is larger than the utility we can get by spending the same resources in the left-over. This implies that the impatient people would take the consumption opportunity if they do not have chances to do that in the first period. (A2) says that the difference between the utility from the superior and inferior consumption opportunity is larger than the extra left-over utility people can get by investing more in the illiquid asset. This implies that people will save enough resources to take the consumption opportunity if they know for sure that they are impatient.

At period 0, everyone knows that at period 1, a fraction of α people will be impatient. α can take two possible values:

$$\alpha = \begin{cases} \alpha_H, & \text{with probability } q \\ \alpha_L, & \text{with probability } (1-q) \end{cases} \tag{7}$$

[8]For example, CRRA utility funcitons with the coefficient of of relative risk aversion larger than 1 ($u(c) = \frac{c^{1-\phi}}{1-\phi}$, $\phi > 1$) satisfy all the assumptions.

And people also make their investment decisions in period 0. They will choose whether to deposit in the bank. And if they choose to invest by themselves, they need to decide their portfolios. At the beginning of period 1, everyone knows his own type and the depositors will make the withdraw decisions. But at that time no one in the society, including the bank, knows the value of α. Later, the asset market will open and people can buy and sell the illiquid asset in the market. The time line of the model can be shown by the following figure:

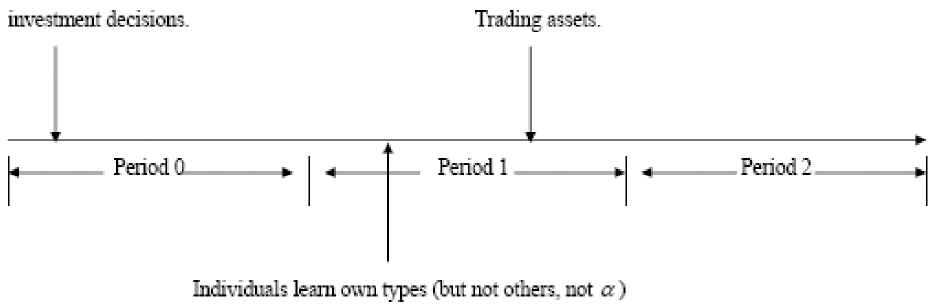

Figure 3 Time Line

As other papers in this literature, we assume perfect competition in the banking industry. So the profit of the bank is zero. The bank will just offer the contract at period 0 to maximize people's ex-ante welfare given the resource and incentive compatibility constraints.

4 The Pure Market

In this section, we will study the case when only the asset market exists. The asset market gives people a way to satisfy their urgent consumption need without keeping one unit of liquid asset beforehand. People can sell their illiquid asset in the market and get the consumption goods in period one. Of course, they may have to sell their assets at discount[9]. But they do not need to keep a lot of "idle cash" with them. So people will balance these two effects when they choose their portfolios in the initial period.

The price of the illiquid asset will depend on the aggregate liquidity need and the available resources in period one. If we use p to denote the price of the illiquid asset in terms of the first period consumption goods and γ_M to denote the fraction of resources invested in the liquid asset by the individuals in the initial period, we have the following result:

[9] We show that this is true in the model when the aggregate liquidity needs is high.

Proposition 1 *In the market, the market clearing price* p *can only take two values:* $p(\alpha) = \begin{cases} R, & \text{if } \alpha \leq \gamma_M y \\ \widehat{p}, & \text{if } \alpha > \gamma_M y \end{cases}$

$\widehat{p}$ *is given by:* $\overline{u}+u[(1-\gamma_M)yR+\frac{(\gamma_M y-1)}{\widehat{p}}R] = \beta\overline{u}+u[(1-\gamma_M)yR+\frac{\gamma_M y}{\widehat{p}}R-1]$ *and* $\widehat{p} < 1$.

$\widehat{p}$ is the price level at which the impatient people will be indifferent between selling the illiquid asset to satisfy consumption need and buying illiquid asset to increase left-over. The proof can be found in the appendix. In the initial period, people will make the investment decisions and choose the optimal $\gamma_M$to maximize the expected utility[10]. The choice of γ_M will also affect the form of the objective function through the effect on the equilibrium price.

If $\gamma_M y \geq \alpha_H$, the aggregate liquid asset in the society can always satisfy the aggregate liquidity need. So $p(\alpha_H) = p(\alpha_L) = R$. The objective function to be maximized is:

$$\overline{u} + u[(1-\gamma_M)yR + \gamma_M y - 1] \tag{8}$$

If $\alpha_L \leq \gamma_M y < \alpha_H$, the society as a whole will be short of liquid asset in the state of α_H. And $p(\alpha_H) = \widehat{p} < p(\alpha_L) = R$. The objective function is:

$$\begin{aligned} &q\{\alpha_H\beta\overline{u} + (1-\alpha_H)\overline{u} + u[(1-\gamma_M)yR + \frac{\gamma_M y}{\widehat{p}}R - 1]\} \\ &+ (1-q)\{\overline{u} + u[(1-\gamma_M)yR + \gamma_M y - 1]\} \end{aligned} \tag{9}$$

If $\gamma_M y < \alpha_L$, the society will always be short of liquid asset no matter which state of aggregate liquidity needs is realized. The objective function is:

$$\begin{aligned} &q\{\alpha_H\beta\overline{u} + (1-\alpha_H)\overline{u} + u[(1-\gamma_M)yR + \frac{\gamma_M y}{\widehat{p}}R - 1]\} \\ &+ (1-q)\{\alpha_L\beta\overline{u} + (1-\alpha_L)\overline{u} + u[(1-\gamma_M)yR + \frac{\gamma_M y}{\widehat{p}}R - 1]\} \end{aligned} \tag{10}$$

So the objective function is a continuous function of γ_M and there are two kink points at $\gamma_M y = \alpha_L$ and $\gamma_M y = \alpha_H$. Since the objective function is continuous and the choice set is compact ($0 \leq \gamma_M \leq 1$), we always have a solution γ_M^*. And it is easy to check that $\alpha_L \leq \gamma_M^* y \leq \alpha_H$.

[10]Since people are identical in the initial period, thier portfolio choices will be the same.

In the rest of the paper, we focus on the case when $\gamma_M^* y < \alpha_H$. The reason why we focus on this case is that we want the price in the market "volatile". That is the exact reason why we introduce the aggregate liquidity need and that is also more realistic.

The sufficient condition to make $\gamma_M^* y < \alpha_H$ is:

$$\begin{aligned} &qu'[(1-\gamma_M)yR + \tfrac{\gamma_M y}{\widehat{p}}R - 1]yR(\tfrac{1}{\widehat{p}} - 1 - \tfrac{\gamma_M}{(\widehat{p})^2}\tfrac{d\widehat{p}}{d\gamma_M}) \\ &+(1-q)u'[(1-\gamma_M)yR + \gamma_M y - 1]y(1-R)\Big|_{\gamma_M^* y = \alpha_H} < 0 \end{aligned} \tag{11}$$

It says that the benefit of having enough available resources in the state of α_H is lower than the cost of holding idle cash in the state of α_L. From the definition of $\widehat{p}$, we know that:

$$\begin{aligned} \frac{d\widehat{p}}{d\gamma_M} &= \frac{y[-(\widehat{p})^2+\widehat{p}]\{u'[(1-\gamma_M)yR+\frac{\gamma_M y}{\widehat{p}}R-1]-u'[(1-\gamma_M)yR+\frac{(\gamma_M y-1)}{\widehat{p}}R]\}}{\gamma_M y u'[(1-\gamma_M)yR+\frac{\gamma_M y}{\widehat{p}}R-1]+(1-\gamma_M y)u'[(1-\gamma_M)yR+\frac{(\gamma_M y-1)}{\widehat{p}}R]} \\ &< 0 \end{aligned} \tag{12}$$

5 The Pure Bank

In this section, we will study the case when only the bank exists. As other models in this literature, we assume perfect competition in the banking industry. So the bank will offer the contract that can maximize people's expected utility. Given the contract offered by the bank, people can choose whether to deposit in the bank in the initial period[11].

To find the optimal contract, it is easy to see that the bank should set the amount of resources that can be withdrew in the first period to be 1. The reason is that satisfying the urgent consumption need requires 1 unit of resources. The resources beyond 1 unit should be invested in the illiquid asset to get higher return. If the withdrawal is lower than 1 unit, then the consumption needs can not be satisfied and it is the same as no withdrawal at all[12].

So the bank just has to decide how much to invest in the liquid asset (γ_B) in the initial period and how much to give out in the second period when people withdraw. Let $c_{i,\,j}$ ($i = I, P$; $j = \alpha_H,\, \alpha_L$) be the 2nd period withdrawal by "type" i form the liquid asset when the aggregate liquidity need is j[13]. Here we abuse the notation of I and P. It is not

[11] We assume that the depositors can only choose to deposit all they have (y units of goods) or nothing.

[12] Peck and Shell (2010) first use this form of utlity function and they try to capture the importance of clearing at par in the transactions facilitated by the bank.

[13] In the second period, the aggregate liquidity needs will no longer be stochastic. So the second period withdrawal can depend on the value of α.

an individual's type but whether this individual has withdrawn in the first period. I (P) means the individual has (has not) withdrawn in the first period. If everyone tells the truth about his type and the bank has enough resources to satisfy all the urgent consumption need of the impatient people, the people who have (have not) withdrawn in the first period are exactly the people who are impatient (patient). But we can see later on that the bank will ration people when the aggregate liquidity need is high and some impatient people can not withdraw in the first period. If $\alpha_L \leq \gamma_B y < \alpha_H$, the maximization problem of the bank is[14]:

$$\begin{aligned}\underset{\gamma_B, c_{I,H}, c_{I,L}, c_{P,H}, c_{P,L}}{Max} \quad & q\{(1-\alpha_H+\gamma_B y)\overline{u}+(\alpha_H-\gamma_B y)\beta\overline{u} \\ & +\gamma_B y u[(1-\gamma_B)yR+c_{I,H}] \\ & +(1-\gamma_B y)u[(1-\gamma_B)yR+c_{P,H}-1]\} \\ & +(1-q)\{\overline{u}+\alpha_L u[(1-\gamma_B)yR+c_{I,L}] \\ & +(1-\alpha_L)u[(1-\gamma_B)yR+c_{P,L}-1]\}\end{aligned} \tag{13}$$

$$\text{s.t.} \quad c_{P,H}=\frac{-\gamma_B y c_{I,H}}{(1-\gamma_B y)} \quad c_{P,L}=\frac{(\gamma_B y-\alpha_L)-\alpha_L c_{I,L}}{1-\alpha_L} \qquad \text{(RC)} \tag{14}$$

$$\begin{aligned}& \widehat{q}\{\overline{u}+\gamma_B y u[(1-\gamma_B)yR+c_{I,H}] \\ & +(1-\gamma_B y)u[(1-\gamma_B)yR+c_{P,H}-1]\} \\ & +(1-\widehat{q})\{\overline{u}+u[(1-\gamma_B)yR+c_{I,L}]\} \\ & \leq \widehat{q}\{\overline{u}+u[(1-\gamma_B)yR+c_{P,H}-1]\} \\ & +(1-\widehat{q})\{\overline{u}+u[(1-\gamma_B)yR+c_{P,L}-1]\} \qquad \text{(IC for the patient)}\end{aligned} \tag{15}$$

$$\begin{aligned}& \widetilde{q}\{\beta\overline{u}+u[(1-\gamma_B)yR+c_{P,H}-1]\} \\ & +(1-\widetilde{q})\{\beta\overline{u}+u[(1-\gamma_B)yR+c_{P,L}-1]\} \\ \leq\ & \widetilde{q}\{\gamma_B y\overline{u}+(1-\gamma_B y)\beta\overline{u}+\gamma_B y u[(1-\gamma_B)yR+c_{I,H}] \\ & +(1-\gamma_B y)u[(1-\gamma_B)yR+c_{P,H}-1]\} \\ & +(1-\widetilde{q})\{\overline{u}+u[(1-\gamma_B)yR+c_{I,L}]\} \qquad \text{(IC for the impatient)}\end{aligned} \tag{16}$$

IC for the patient says that for the patient people, telling the truth (i.e. not withdrawing in the first period) should be at least as good as

[14] Here we just consider the "good" equilibrium: the non-run equilibrium in the model. That is the NROC in Peck and Shell (2010)

pretending to be impatient (i.e. trying to withdraw in the first period). Here $\widehat{q}$ is given by:

$$\widehat{q} = \frac{q(1-\alpha_H)}{q(1-\alpha_H)+(1-q)(1-\alpha_L)} \tag{17}$$

$\widehat{q}$ is the patient people's updated beliefs about the probability of high aggregate liquidity need. As we have discussed, at the beginning of period one, people will know their own liquidity need. Although the aggregate liquidity need is not revealed at that time, people will use the information on their own liquidity need to update their beliefs about the aggregate liquidity need by Bayes Rule.

Similarly, we have the IC for the impatient people and their updated beliefs about the probability of high aggregate liquidity need $\widetilde{q}$:

$$\widetilde{q} = \frac{q\alpha_H}{q\alpha_H+(1-q)\alpha_L} \tag{18}$$

As the unified system in Peck and Shell (2010), for a given γ_B complete consumption smoothing

$$c_{I,H} = c_{P,H} - 1 = c_{I,L} = c_{P,L} = \gamma_B y - 1 \tag{19}$$

is feasible and incentive compatible. Furthermore, it is optimal given the concavity of the function u. Similarly, we can show that for the case when $\gamma_B y < \alpha_L$ and $\gamma_B y \geq \alpha_H$, we have similar consumption smoothing. So the bank's problem will be reduced to choose the optimal γ_B. As the pure market case, the objective function will be a continuous function of γ_B with two kink points at $\gamma_B y = \alpha_L$ and $\gamma_B y = \alpha_H$.

If $\gamma_B y \geq \alpha_H$, the bank can always satisfy the liquidity need of the society. The objective function is:

$$\overline{u} + u[(1-\gamma_B)yR + \gamma_B y - 1]\} \tag{20}$$

If $\alpha_H > \gamma_B y \geq \alpha_L$, the bank will be short of resources when the aggregate liquidity need is high (α_H). The objective function is:

$$\begin{aligned} &q\{(1-\alpha_H+\gamma_B y)\overline{u} + (\alpha_H - \gamma_B y)\beta\overline{u} + u[(1-\gamma_B)yR + \gamma_B y - 1]\} \\ &+ (1-q)\{\overline{u} + u[(1-\gamma_B)yR + \gamma_B y - 1]\} \end{aligned} \tag{21}$$

If $\gamma_B y < \alpha_L$, the bank is always short of resources. The objective function is:

$$q\{(1-\alpha_H+\gamma_B y)\overline{u}+(\alpha_H-\gamma_B y)\beta\overline{u}+u[(1-\gamma_B)yR+\gamma_B y-1]\} \quad (22)$$
$$+(1-q)\{\overline{u}(1-\alpha_L+\gamma_B y)\overline{u}+(\alpha_L-\gamma_B y)\beta\overline{u}$$
$$+u[(1-\gamma_B)yR+\gamma_B y-1]\}$$

It is easy to see that the optimal γ_B^* must satisfy $\alpha_L \leq \gamma_B^* y \leq \alpha_H$. And because of the concavity of the objective function, the FOC will be sufficient to characterize γ_B^* :

If $q(1-\beta)y\overline{u}+u'[(1-\gamma_B)yR+\gamma_B y-1](1-R)y\,|_{\gamma_B y=\alpha_H} \geq 0$,
then $\gamma_B^* y=\alpha_H$.

If $q(1-\beta)y\overline{u}+u'[(1-\gamma_B)yR+\gamma_B y-1](1-R)y\,|_{\gamma_B y=\alpha_L} < 0$,
then $\gamma_B^* y=\alpha_L$.

Else, $\alpha_L < \gamma_B^* y < \alpha_H$
and $q(1-\beta)y\overline{u}+u'[(1-\gamma_B)yR+\gamma_B y-1](1-R)y\,\Big|_{\gamma_B^* y}=0$.

The next proposition compares people's welfare in the pure bank and the pure market.

Proposition 2 *The pure bank can do strictly better than the pure market.* $W_{Pure\ Market} < W_{Pure\ Bank}$

The proof can be found in the appendix. Since we have $\alpha_L \leq \gamma_M^* y < \alpha_H$, this proposition also implies that $\alpha_L \leq \gamma_B^* y < \alpha_H$[15]. In the pure market, the impatient people need to buy consumption goods from the patient people. Everyone in the market just act to maximize his own utility. There is no insurance, or consumption smoothing, between the impatient and the patient. Especially, when the liquidity need is high (α_H), a lot of impatient people are trying to sell their illiquid asset and get consumption goods. The price of the illiquid asset is driven down to $\widehat{p}$. There will be a huge volatility in the consumption between the impatient and the patient people in this case.

The bank is trying to allocate the resources through the contract it offers. The contract will smooth the consumption between the impatient

[15] From the objective function, we can know that the welfare of the pure market ($W_{Pure\ Market}$) and the welfare of the pure bank($W_{Pure\ Bank}$) will be the same if $\gamma_M^* y=\gamma_B^* y=\alpha_H$.
So if $\gamma_B^* y=\alpha_H$, we must have $\gamma_M^* y=\alpha_H$, which is not the case of our interest.

and the patient people to maximize everyone's ex ante welfare. When the aggregate liquidity need is high (α_H), the bank will use the "sequential service constraint" to allocate the consumption goods among the impatient people. Some lucky impatient can get the consumption goods they need while others have to postpone their consumption and be treated as patient. This "lottery" is desirable. It avoids the market competition which will drive down the price of the illiquid asset and make every impatient people worse off. We can also think about this in the following way. For the lucky impatient people, the bank set the shadow price for their illiquid asset as R $(> \widehat{p})$. For the unlucky impatient people, the shadow price for their illiquid asset is 0.The uncertainty in the shadow price can make the impatient people as a whole be better off.

6 The Bank & The Market

In this section, we study the case when both the bank and the asset market exist. The trading opportunity in the market will restrict the bank and the bank can not do as well as the pure bank ($W_{Bank\ \&\ Market} < W_{Pure\ Bank}$). But the optimal contract offered by the bank is still better than the pure market allocation ($W_{Pure\ Market} < W_{Bank\ \&\ Market}$).

When both the bank and the asset market exist, the objective function of the bank and the resource constraint will be the same as before. The only change is the incentive compatibility constraint. The trading opportunity in the market gives people more chances to arbitrage between the bank and the market. So making people tell the truth about their types will be even harder for the bank now. That is to say the bank must satisfy a much more restrictive incentive constraint. Before we compare the welfare between Bank & Market and the Pure Bank, let us figure out what the market clearing price is in the asset market:

Proposition 3 *When both the bank and the asset market exist, for each α_j the market clearing price of the asset can only take two possible values:*
$p(\alpha_j) = \begin{cases} R, & \text{if } \alpha_j \leq \gamma_{B\&M} y \\ \widetilde{p}_j, & \text{if } \alpha_j > \gamma_{B\&M} y \end{cases} \quad j = H, L.$
$\widetilde{p}_j$ is given by: $\overline{u} + u[(1-\gamma_{B\&M})yR + c_{P,j} - \frac{1}{\widetilde{p}}R] = \beta\overline{u} + u[(1-\gamma_{B\&M})yR + c_{P,j} - 1]$

$\widetilde{p}_j$ is the price level at which the impatient people will be indifferent between selling the illiquid asset to satisfy consumption need and buying illiquid asset to increase left-over when aggregate liquidity need is α_j. The proof is in the appendix. This is a similar result as Proposition 1. When the available resources of the bank can cover the aggregate liquidity need, the illiquid asset can be sold at a price implied by the asset return. When the bank is in short of resources, the price of the illiquid asset is lower than the return.

Given the market clearing prices, we can compare the welfare the bank can achieve when the market exists (Bank & Market) and when only the bank exists (Pure Bank).

Proposition 4 *The bank & market can not do as well as the pure bank.* $W_{Bank\ \&\ Market} < W_{Pure\ Bank}$.

The proof can be found in the appendix. Basically, we show that the pure bank allocation is not incentive compatible for bank & market. So the incentive constraints now are more restrictive compared to the pure bank case. That is why the bank can not do as well as before.

But the key question is whether the bank is redundant given the existence of the market. That is whether the bank & market can do better than the pure market. As the first step, we show that

Proposition 5 *The bank & market can do at least as well as the pure market.* $W_{Pure\ Market} \leq W_{Bank\ \&\ Market}$.

The proof can be found in the appendix. The intuition is that the bank can always mimic the market behavior. That is why the bank can not do worse than the market.

The next proposition compares the bank & market with the pure market. It shows that the bank is not redundant when the market exists.

Proposition 6 *The bank & market can do strictly better than the pure market.* $W_{Pure\ Market} < W_{Bank\ \&\ Market}$.

The proof is in the appendix. The strategy to prove this result is the following. From Proposition (5), we know that the pure market allocation is feasible for the bank & market . So it is sufficient to prove that the pure market allocation is not the optimal solution to the problem of the bank & market. The pure market allocation implies $c_{I,H} < c_{I,L} = c_{P,L} < c_{P,H}$. This can not be the optimal solution. We can find a better allocation by smoothing the consumption in state α_H ($c_{I,H} \uparrow$, $c_{P,H} \downarrow$). This may hurt the incentives for the patient. To provide enough incentives for the patient, we compensate them in state α_H ($c_{I,L} \downarrow$, $c_{P,L} \uparrow$). The above change will increase people's ex-ante welfare by making the consumption more smooth.

7 Conclusion

What is the function of bank? Diamond and Dybvig (1983) showed how bank provides insurance against liquidity shock and improves the ex-ante welfare. Jacklin (1987), among other papers, showed that bank's ability to provide insurance is completed restricted by the trade in the market. So the bank is redundant if the market is available.

In this paper, I show that if we have aggregate liquidity risk and people do not know the realized aggregate liquidity need when they make the withdrawal decisions, the bank's ability to provide insurance will only be partially restricted by the market. Compared to the pure market equilibrium allocation, the bank can still improve the ex-ante welfare. In Jacklin's critique, the aggregate liquidity need is deterministic. So depositors, especially patient depositors, know exactly what the price of the illiquid asset is. And to satisfy the IC (i.e. to prevent the patient from lying), the bank can only mimic the market allocation. But in my model the uncertainty about the aggregate liquidity need weakens the IC constraint. To make everyone tell the truth for each realized α, which is implied by the market allocation, is not required by the IC. The IC just requires the depositors to tell the truth when they do not know the value of α.Since the bank does not have to mimic the market allocation, there is room for it to improve the ex-ante welfare compared to the pure market equilibrium. And the improvement is made through increasing the risk sharing in the state of α_H at the cost of the decreasing risk sharing in the state of α_L.

Appendix

Proof of Proposition 1

Proof. For a given γ_M, in period one the excess demand for the illiquid asset by the impatient people, $D_P(p)$ (the subscript P stands for patient), will be the following correspondence:

$$D_P(p) \begin{cases} = \frac{(1-\alpha)\gamma_M y}{p}, & \text{if } p < R \\ \in [-(1-\alpha)(1-\gamma_M)y,\ (1-\alpha)\gamma_M y/R], & \text{if } p = R \\ = -(1-\alpha)(1-\gamma_M)y, & \text{if } p > R \end{cases} \tag{23}$$

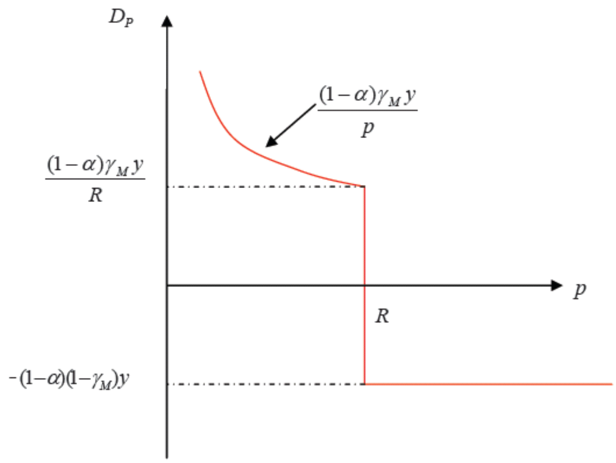

Figure 4. Excess demand for the illiquid asset by the patient

Since the patient people do not need resources to consume in period one, they will just buy or sell the illiquid asset for profit. If $p < R$, the return of the illiquid asset will be strictly higher than the cost of buying it. So a patient individual will spend all of his available resources ($\gamma_M y$) to buy the illiquid asset. And the excess demand from all patient people will be $\frac{(1-\alpha)\gamma_M y}{p}$. Similarly, if $p > R$ the asset price is too high compared to its return, so the patient people would like to sell all of their illiquid asset. The excess supply of the illiquid asset is $(1-\alpha)(1-\gamma_M)y$ (The excess demand will be $-(1-\alpha)(1-\gamma_M)y$). If $p = R$, the patient people will be indifferent between buying and selling the illiquid asset. So the excess demand can be anything is the interval $[-(1-\alpha)(1-\gamma_M)y$, $(1-\alpha)\gamma_M y/R]$.

In the same way, we can decide the excess demand for the illiquid asset by the impatient people. If $\gamma_M y \geq 1$, the impatient people have more than one unit of consumption goods in their hand. So they do not have to go to the market to get the consumption goods they need. Like the patient people, they will just buy or sell the illiquid asset for profit. Their excess demand will be given by the following correspondence (The subscript I stands for impatient):

$$D_I(p)\begin{cases} = \alpha(\gamma_M y - 1)/p, & \text{if } p < R \\ \in [-\alpha(1-\gamma_M)y,\ \alpha(\gamma_M y - 1)/R], & \text{if } p = R \\ = -\alpha(1-\gamma_M)y, & \text{if } p > R \end{cases}$$

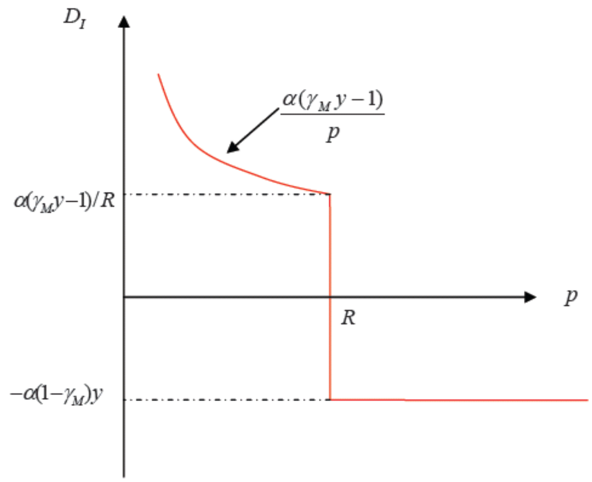

Figure 5. Excess demand for the illiquid asset by the impatient

$\gamma_M y \geq 1$

If the impatient people do not have enough resources in the first period ($\gamma_M y < 1$), they will go to the market not only for profit but also for consumption goods to satisfy their urgent consumption need. The amount of the consumption goods each impatient individual needs to buy to satisfy the consumption need is $(1-\gamma_M y)$. That means they need to sell $(1-\gamma_M y)/p$ units of illiquid asset. If the price of the illiquid asset is too low, the impatient will find it optimal to postpone the consumption until the second period and buy illiquid asset at low price. The cutting off point is $\widehat{p}$, at this price level the impatient people will be indifferent between selling the illiquid asset to satisfy consumption need and buying illiquid asset to increase left-over. So $\widehat{p}$ is defined by:

$$\overline{u}+u[(1-\gamma_M)yR+\frac{(\gamma_M y-1)}{\widehat{p}}R] = \beta\overline{u}+u[(1-\gamma_M)yR+\frac{\gamma_M y}{\widehat{p}}R-1]. \quad (24)$$

It can be proved that $\widehat{p}$ is well-defined and is unique by the following argument. It is easy to see that both the left hand side (LHS) and the right hand side (RHS) of equation (24) is a continuous function of $\widehat{p}$. LHS is increasing with $\widehat{p}$ and RHS is decreasing with $\widehat{p}$. If $\widehat{p} = 1$, LHS is larger than RHS by (A1). If $\widehat{p} \to 0$, LHS is lower than RHS since $u(0) = -\infty$. So we know that for each γ_M there is a unique $\widehat{p}$ (< 1) which satisfies the equation (24).

So when $\gamma_M y < 1$, if $p < \widehat{p}$ the impatient people will prefer to postpone the consumption opportunity and buy the illiquid asset at a low price. So the excess demand for the illiquid asset by the impatient is $\frac{\alpha\gamma_M y}{p}$. If $p = \widehat{p}$, the impatient will be indifferent between satisfying the consumption opportunity or not. So the excess demand can be anything in the interval $[\frac{\alpha(\gamma_M y-1)}{\widehat{p}}, \frac{\alpha\gamma_M y}{\widehat{p}}]$. If $\widehat{p} < p < R$, the impatient strictly prefer to satisfy their consumption opportunity. They will sell $\frac{\alpha(1-\gamma_M y)}{p}$ units of the illiquid asset (the excess demand is $\frac{\alpha(\gamma_M y-1)}{p}$) to get 1 unit of consumption goods. If $p > R$,the impatient people will sell all of their illiquid asset not only for the consumption goods but also for the profit they can get by selling asset at a high price. So their excess demand is $-\alpha(1-\gamma_M)y$. If $p = R$, the excess demand can be anything in the interval $[-\alpha(1-\gamma_M)y,\, \alpha(\gamma_M y - 1)/R]$. To summarize, the excess demand for the illiquid asset by the impatient people when $\gamma_M y < 1$ is the following correspondence and can also be shown by the following figure.

$$D_I(p) \begin{cases} = \frac{\alpha\gamma_M y}{p}, & \text{if } p < \widehat{p} \\ \in [\frac{\alpha(\gamma_M y-1)}{\widehat{p}}, \frac{\alpha\gamma_M y}{\widehat{p}}], & \text{if } p = \widehat{p} \\ = \frac{\alpha(\gamma_M y-1)}{p}, & \text{if } \widehat{p} < p < R \\ \in [-\alpha(1-\gamma_M)y,\ \alpha(\gamma_M y - 1)/R], & \text{if } p = R \\ = -\alpha(1-\gamma_M)y, & \text{if } p > R \end{cases} \tag{25}$$

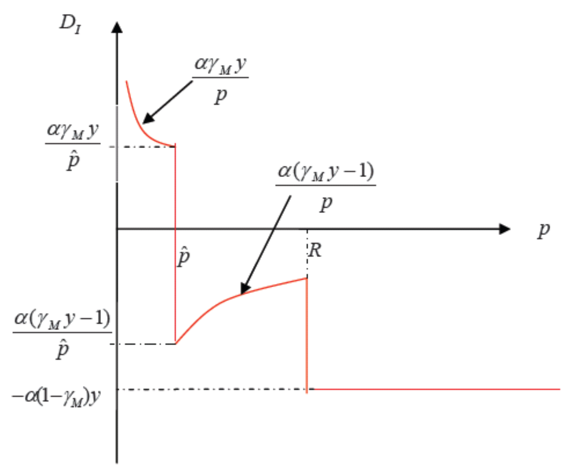

Figure 6. Excess demand for the illiquid asset by the impatient

$\gamma_M y<1$

Market clearing in the asset market requires that excess demand is zero:

$$D_P(p) + D_I(p) = 0 \tag{26}$$

So we can get that the market clearing price of the illiquid asset can only take two values:

$$p(\alpha) = \begin{cases} R, & \text{if } \alpha \leq \gamma y \\ \widehat{p}, & \text{if } \alpha > \gamma y \end{cases}. \tag{27}$$

■

Proof of Proposition 2

Proof. We can prove this proposition by showing that the objective function of the pure bank is always larger than that of the pure market.

$$\begin{aligned} W_{Pure\ Bank} =& \underset{\gamma_B}{Max} E[W_B(\gamma_B)] \\ =& \underset{\gamma_B}{Max} q\{(1-\alpha_H+\gamma_B y)\overline{u} + (\alpha_H - \gamma_B \gamma y)\beta\overline{u} \\ &+ u[(1-\gamma_B)yR + \gamma_B y - 1]\} \\ &+ (1-q)\{\overline{u} + u[(1-\gamma_B)yR + \gamma_B y - 1]\} \end{aligned} \tag{28}$$

$$\begin{aligned} W_{Pure\ Market} =& \underset{\gamma_M}{Max}\ E[W_M(\gamma_M)] \\ =& \underset{\gamma_M}{Max} q\{\overline{u} + \alpha_H u[(1-\gamma_M)yR + \frac{\gamma_M y - 1}{\widehat{p}}R] \\ &+ (1-\alpha_H)u[(1-\gamma_M)yR + \frac{\gamma_M y}{\widehat{p}}R - 1]\} \\ &+ (1-q)\{\overline{u} + u[(1-\gamma_M)yR + \gamma_M y - 1]\} \end{aligned} \tag{29}$$

We can prove that for each γ (such that $\alpha_L \leq \gamma_B = \gamma_M = \gamma < \alpha_H$), $E[W_B(\gamma)] - E[W_M(\gamma)] \geq 0$:

$$
\begin{aligned}
&E[W_B(\gamma)] - E[W_M(\gamma)] \\
&= q\{-(\alpha_H - \gamma y)(1-\beta)\overline{u} + u[(1-\gamma)yR + \gamma y - 1] \\
&- \alpha_H u[(1-\gamma)yR + \frac{\gamma y - 1}{\widehat{p}}R] \\
&- (1-\alpha_H)u[(1-\gamma)yR + \frac{\gamma y}{\widehat{p}}R - 1]\} \\
&= q\{u[(1-\gamma)yR + \gamma y - 1] - \gamma y u[(1-\gamma)yR + \frac{\gamma y - 1}{\widehat{p}}R] \\
&- (1-\gamma y)u[(1-\gamma)yR + \frac{\gamma y}{\widehat{p}}R - 1]\} \\
&> 0
\end{aligned}
$$

The last inequality is because of the strict concavity of the utility function. $E[W_M(\gamma)] < E[W_B(\gamma)] \Rightarrow W_{Pure\ Market} < W_{Pure\ Bank}$ ■

Proof of Proposition 3

Proof. Suppose that every people tells the truth when he makes the withdrawal decision[16]. That is to say, the impatient people will choose to withdraw in the first period and the patient people will choose not to withdraw in the first period. As in Proposition 1, we can have the excess demand for the illiquid asset in the following forms. For the patient people,

$$
D_P(p) \begin{cases} = 0, \quad \text{if } p < R \\ \in [-(1-\alpha_j)((1-\gamma_{B\&M})yR + c_{P,j})/R, 0], \quad \text{if } p = R \ j = H, L \\ = -(1-\alpha_j)((1-\gamma_{B\&M})yR + c_{P,j})/R, \quad \text{if } p > R \end{cases} \tag{30}
$$

[16] Telling the truth will be implied by the incentive compatibility constraints which will be discussed below.

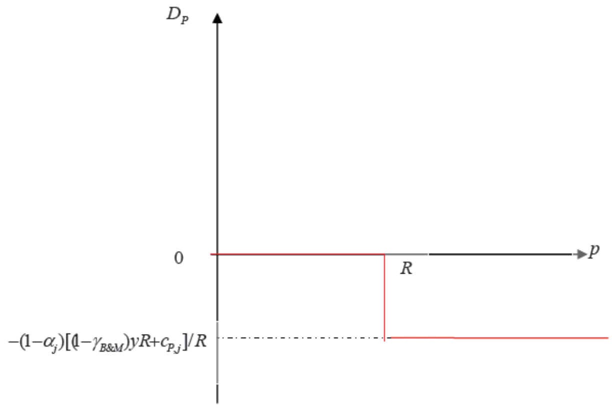

Figure 7. Excess demand for the illiquid asset by the patient

For the impatient people, when $\alpha_j \leq \gamma_{B\&M}y$:

$$D_I(p) \begin{cases} = 1/p, \quad \text{if } p < \overline{p}_{\alpha_j} \\ = 0, \quad \text{if } \overline{p}_{\alpha_j} \leq p < R \\ \in [-\alpha_j((1-\gamma_{B\&M})yR + c_{I,j})/R, 0], \quad \text{if } p = R \\ = -\alpha_j((1-\gamma_{B\&M})yR + c_{I,j})/R, \quad \text{if } p > R \end{cases} \tag{31}$$

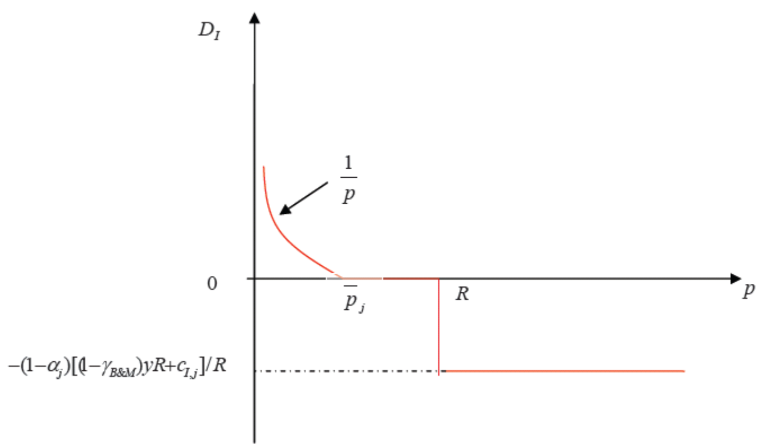

Figure 8. Excess demand for the illiquid asset by the impatient if $\alpha_j \leq \gamma_{B\&M}y$

$\overline{p}_j$ is given by:

$$\overline{U}+u[(1-\gamma_{B\&M})yR+ c_{I,j}] = \beta\overline{U}+u[(1-\gamma_{B\&M})yR+ c_{I,j}+\frac{1}{\overline{p}_j}R-1] \tag{32}$$

If we can not find a positive $\overline{p}_j$ such that equation(32) holds, let $\overline{p}_j = 0$.

When $\alpha_j > \gamma_{B\&M}y$, for the "lucky" impatient (who can withdraw in the first period) their excess demand will be similar to that of the impatient people when $\alpha_j \leq \gamma_{B\&M}y$.

$$D_I(p)\begin{cases} = 1/p, \quad \text{if } p < \overline{p}_{\alpha_j} \\ = 0, \quad \text{if } \overline{p}_{\alpha_j} \leq p < R \\ \in [-\gamma_{B\&M} y((1-\gamma_{B\&M})yR + c_{I,j})/R, 0], \quad \text{if } p = R \\ = -\gamma_{B\&M} y((1-\gamma_{B\&M})yR + c_{I,j})/R, \quad \text{if } p > R \end{cases} \tag{33}$$

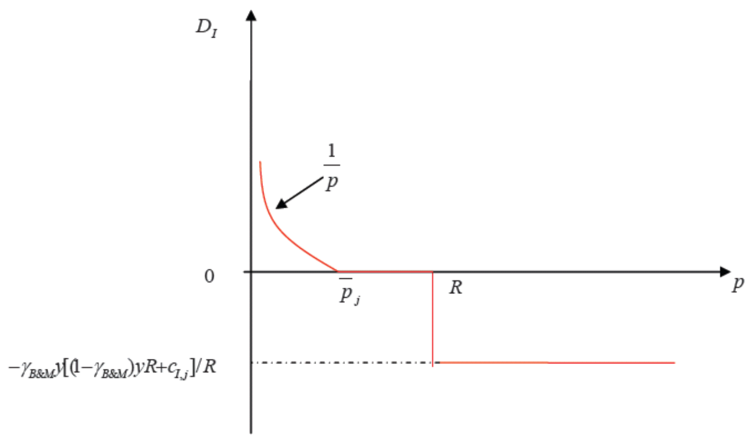

Figure 9. Excess demand for the illiquid asset by the "lucky" impatient if $\alpha_j > \gamma_{B\&M} y$

For the "unlucky" impatient (who can not withdraw in the first period):

$$D_I(p)\begin{cases} = 0, \quad \text{if } p < \widetilde{p}_j \\ \in [-1/\widetilde{p},\ 0], \quad \text{if } p = \widetilde{p}_j \\ = -1/p, \quad \text{if } \widetilde{p}_j < p < R \\ \in [(\alpha_j - \gamma_{B\&M} y)((1-\gamma_{B\&M})yR + c_{P,j})/R, 0], \ \text{if } p = R \\ = (\alpha_j - \gamma_{B\&M} y)((1-\gamma_{B\&M})yR + c_{P,j})/R, \ \text{if } p > R \end{cases} \tag{34}$$

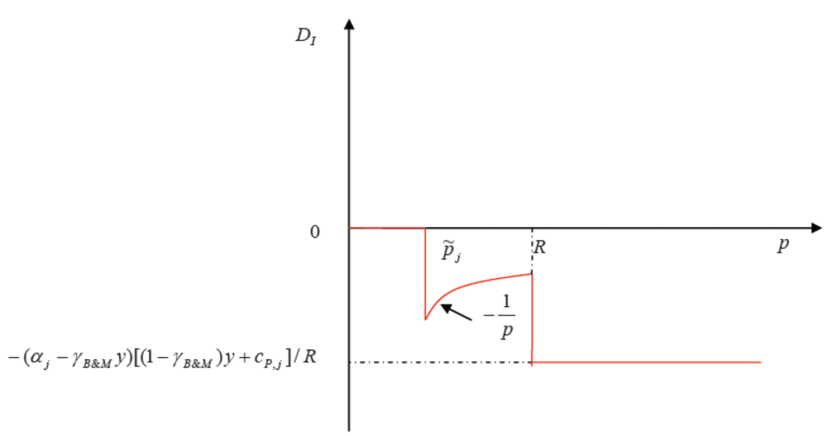

Figure 10. Excess demand for the illiquid asset by the "unlucky" impatient if $\alpha_j > \gamma_{B\&M} y$

$\widetilde{p}_j$ is given by:

$$\overline{U} + u[(1-\gamma)yR + \ c_{P,j} - \frac{1}{\widetilde{p}_j}R] = \beta\overline{U} + u[(1-\gamma)yR + \ c_{P,j} - 1] \quad (35)$$

Similar to the proof of Proposition 3, we can see that $\widetilde{p}_j$ exists and is unique.

By setting the excess demand to be 0, it is easy to see that when $\alpha_j \leq \gamma_{B\&M}y$, $p = R$ is a market clearing price[17]. And when $\alpha_j > \gamma_{B\&M}y$, $p = \widetilde{p}_j$ is a market clearing price.[18] ∎

Proof of Proposition 4

Proof. The objective function of the bank will be the same as the pure bank. And we know that $\alpha_L \leq \gamma_B^* y < \alpha_H$. So if $\gamma_{B\&M}^* y \geq \alpha_H$, we know that we must have $W_{Bank\ \&\ Market} < W_{Pure\ Bank}$. Now we just have to check the case when $\alpha_L \leq \gamma_{B\&M}^* y < \alpha_H$. From Proposition 3, we know that if $\alpha_L \leq \gamma_{B\&M}y < \alpha_H$, then the market clearing prices are $p(\alpha_L) = R$ and $p(\alpha_H) = \widetilde{p}_j$. To simplify the notation, we write $\widetilde{p}_j$ as $\widetilde{p}$. When both the bank and the asset market exist, the new incentive constraints that the bank has to satisfy are the following:

IC for the patient:

$$\begin{aligned} & \widehat{q}\{\overline{u} + \gamma_{B\&M}yu[(1-\gamma_{B\&M})yR + c_{I,H} - 1 + \frac{1}{\widetilde{p}}R] \qquad (36) \\ & + (1-\gamma_{B\&M}y)u[(1-\gamma_{B\&M})yR + c_{P,H} - 1]\} \\ & + (1-\widehat{q})\{\overline{u} + u[(1-\gamma_{B\&M})yR + c_{I,L}]\} \\ \leq\ & \widehat{q}\{\overline{u} + u[(1-\gamma_{B\&M})yR + c_{P,H} - 1]\} \\ & + (1-\widehat{q})\{\overline{u} + u[(1-\gamma_{B\&M})yR + c_{P,L} - 1]\} \end{aligned}$$

IC for the impatient:

[17] There could be other market clearing prices. We assume that people expect the highest one among all possible clearing prices.

[18] If $\widetilde{p}_j \geq \overline{p}_j$, then at $\widetilde{p}_j$ the excess demand of the patient, the "lucky" impatient and the "unlucky" impatient are all zero.

If $\widetilde{p}_j < \overline{p}_j$, then at $\widetilde{p}_j$ the excess demand of the patient is zero. The excess demand of the "lucky" patient is positive and equals $\frac{1}{\widetilde{p}_j}\gamma y$. The excess demand of the "unlucky" patient is negative and equals $-\frac{1}{\widetilde{p}_j}\gamma y$

In both cases, $\widetilde{p}_j$ is the highest price from the set of possible market clearing prices.

$$
\begin{aligned}
& \widetilde{q}\{\overline{u} + u[(1-\gamma_{B\&M})yR + c_{P,H} - \frac{1}{\widetilde{p}}R]\} \\
& + (1-\widetilde{q})\{\overline{u} + u[(1-\gamma_{B\&M})yR + c_{P,L} - 1]\} \\
\leq\ & \widetilde{q}\{\overline{u} + \gamma_{B\&M}yu[(1-\gamma_{B\&M})yR + c_{I,H}] \\
& + (1-\gamma_{B\&M}y)u[(1-\gamma_{B\&M})yR + c_{P,H} - \frac{1}{\widetilde{p}}R]\} \\
& + (1-\widetilde{q})\{\overline{u} + u[(1-\gamma_{B\&M})yR + c_{I,L}]\}
\end{aligned} \tag{37}
$$

To prove that $W_{Bank\ \&\ Market} < W_{Pure\ Bank}$, it is sufficient to show that the pure bank allocation is out of the constraint set. From the previous discussion, we know that the pure bank allocation is complete consumption smoothing: $c_{I,H} = c_{P,H} - 1 = c_{I,L} = c_{P,L} = \gamma_B y - 1$. It can be seen that this consumption smoothing is not incentive compatible. The patient people can get higher return by withdrawing in the first period and buying illiquid asset in the market. ■

Proof of Proposition 5

Proof. Let the bank chooses the pure market allocation (it is easy to see that $\widetilde{p} = \widehat{p}$ when the bank mimic the market): $\gamma_{B\&M} = \gamma_M^*$, $c_{P,H} = c_{I,H} + \frac{1}{\widehat{p}}R = \frac{\gamma_M^* y}{\widehat{p}}R$, $c_{P,L} = c_{I,L} + 1 = \gamma_M^* y$. Then it is check that both (RC) and (IC) will be satisfied. The pure market allocation is feasible and incentive compatible. So $W_{Pure\ Market} \leq W_{Bank\ \&\ Market}$. When the market exists, the bank can always mimic the pure market allocation. ■

Proof of Proposition 6

Proof. From Proposition (5), we know that the pure market allocation is feasible for the bank & market. So it is sufficient to prove that the pure market allocation is not the optimal solution to the problem of the bank & market.

Suppose that the pure market allocation is the optimal solution to the bank's problem. That is $\gamma_{B\&M}^* = \gamma_M^*$, $c_{P,H}^* = c_{I,H}^* + \frac{1}{\widehat{p}}R = \frac{\gamma_M^* y}{\widehat{p}}R$, $c_{P,L}^* = c_{I,L}^* + 1 = \gamma_M^* y$. Given $\gamma_{B\&M} = \gamma_M^*$, the necessary condition for $c_{P,H}^* = c_{I,H}^* + \frac{1}{\widehat{p}}R = \frac{\gamma_M^* y}{\widehat{p}}R$, $c_{P,L}^* = c_{I,L}^* + 1 = \gamma_M^* y$ is the FOC of the following program to be satisfied.

$$\begin{aligned}\underset{c_{P,H},\, c_{I,H},\, c_{P,L},\, c_{I,L}}{Max} \quad & q\{(1-\alpha_H+\gamma_M^* y)\overline{U}+(\alpha_H-\gamma_M^* y)\beta\overline{U} \\ & +\gamma_M^* y u[(1-\gamma_M^*)yR+c_{I,H}] \\ & +(1-\gamma_M^* y)u[(1-\gamma_M^*)yR+c_{P,H}-1]\} \\ & +(1-q)\{\overline{U}+\alpha_L u[(1-\gamma_M^*)yR+\ c_{I,L}] \\ & +(1-\alpha_L)u[(1-\gamma_M^*)yR+c_{P,L}-1]\}\end{aligned} \tag{38}$$

$$\text{s.t.} \quad c_{P,H}=\frac{-\gamma_M^* y c_{I,H}}{(1-\gamma_B y)} \quad c_{P,L}=\frac{(\gamma_M^* y-\alpha_L)-\alpha_L c_{I,L}}{1-\alpha_L} \qquad \text{(RC)} \tag{39}$$

$$\begin{aligned} & \widehat{q}\{\overline{u}+\gamma_M^* y u[(1-\gamma_M^*)yR+c_{I,H}-1+\frac{1}{\widetilde{p}}R] \\ & +(1-\gamma_M^* y)u[(1-\gamma_M^*)yR+c_{P,H}-1]\} \\ & +(1-\widehat{q})\{\overline{u}+u[(1-\gamma_M^*)yR+c_{I,L}]\} \\ \leq\ & \widehat{q}\{\overline{u}+u[(1-\gamma_M^*)yR+c_{P,H}-1]\} \\ & +(1-\widehat{q})\{\overline{u}+u[(1-\gamma_M^*)yR+c_{P,L}-1]\} \qquad \text{(IC for the patient)}\end{aligned} \tag{40}$$

$$\begin{aligned} & \widetilde{q}\{\overline{u}+u[(1-\gamma_M^*)yR+c_{P,H}-\frac{1}{\widetilde{p}}R]\} \\ & +(1-\widetilde{q})\{\overline{u}+u[(1-\gamma_M^*)yR+c_{P,L}-1]\} \\ \leq\ & \widetilde{q}\{\overline{u}+\gamma_M^* y u[(1-\gamma_M^*)yR+c_{I,H}] \\ & +(1-\gamma_M^* y)u[(1-\gamma_M^*)yR+c_{P,H}-\frac{1}{\widetilde{p}}R]\} \\ & +(1-\widetilde{q})\{\overline{u}+u[(1-\gamma_M^*)yR+c_{I,L}]\} \qquad \text{(IC for the impatient)}\end{aligned} \tag{41}$$

Plug the (RC) into the objective function to cancel $c_{P,H}$ and $c_{P,L}$. Let λ and ϕ be the Lagrangian multipliers of the two IC conditions.

We have the Lagrangian as:

$$
\begin{aligned}
L = \quad & q\{(1-\alpha_H+\gamma_M^* y)\overline{U}+(\alpha_H-\gamma_M^* y)\beta\overline{U}+\gamma_M^* y u[(1-\gamma_M^*)yR+c_{I,H}] \\
& +(1-\gamma_M^* y)u[(1-\gamma_M^*)yR+\frac{-\gamma_M^* y c_{I,H}}{(1-\gamma_M^* y)}-1]\} \\
& +(1-q)\{\overline{U}+\alpha_L u[(1-\gamma_M^*)yR+c_{I,L}]+(1-\alpha_L)u[(1-\gamma_M^*)yR \\
& +\frac{(\gamma_M^* y-\alpha_L)-\alpha_L c_{I,L}}{1-\alpha_L}-1] \\
& +\lambda\, \widehat{q}\gamma_M^* y\{u[(1-\gamma_M^*)yR+\frac{-\gamma_M^* y c_{I,H}}{(1-\gamma_M^* y)}-1]-u[(1-\gamma_M^*)yR+c_{I,H}-1+\frac{R}{\widetilde{p}}]\} \\
& +\lambda(1-\widehat{q})\{u[(1-\gamma_M^*)yR+\frac{(\gamma_M^* y-\alpha_L)-\alpha_L c_{I,L}}{1-\alpha_L}-1]-u[(1-\gamma_M^*)yR+c_{I,L}]\} \\
& +\phi\widetilde{q}\gamma_M^* y\{u[(1-\gamma_M^*)yR+c_{I,H}]-u[(1-\gamma_M^*)yR+\frac{-\gamma_M^* y c_{I,H}}{(1-\gamma_M^* y)}-\frac{R}{\widetilde{p}}]\} \\
& +\phi(1-\widetilde{q})\{u[(1-\gamma_M^*)yR+c_{I,L}]-u[(1-\gamma_M^*)yR+\frac{(\gamma_M^* y-\alpha_L)-\alpha_L c_{I,L}}{1-\alpha_L}-1]\}
\end{aligned}
$$

Take the first-order derivative with respect to $c_{I,L}$ and $c_{I,H}$. We have:

$$
\begin{aligned}
\frac{\partial L}{\partial c_{I,H}} = \; & q\{\gamma_M^* y u'[(1-\gamma_M^*)yR+c_{I,H}]-\gamma_M^* y u'[(1-\gamma_M^*)yR+\frac{-\gamma_M^* y c_{I,H}}{(1-\gamma_M^* y)}-1]\} \\
& +\lambda\, \widehat{q}\gamma_M^* y\{u'[(1-\gamma_M^*)yR+\frac{-\gamma_M^* y c_{I,H}}{(1-\gamma_M^* y)}-1](\frac{-\gamma_M^* y}{(1-\gamma_M^* y)}) \\
& -u'[(1-\gamma_M^*)yR+c_{I,H}-1+\frac{R}{\widetilde{p}}](1+\frac{-R}{(\widetilde{p})^2}\frac{d\widetilde{p}}{dc_{I,H}})\} \\
& +\phi\widetilde{q}\gamma_M^* y\{u'[(1-\gamma_M^*)yR+c_{I,H}] \\
& -u'[(1-\gamma_M^*)yR+\frac{-\gamma_M^* y c_{I,H}}{(1-\gamma_M^* y)}-\frac{R}{\widetilde{p}}](\frac{-\gamma_M^* y}{(1-\gamma_M^* y)}+\frac{R}{(\widetilde{p})^2}\frac{d\widetilde{p}}{dc_{I,H}})\}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial c_{I,L}} = \; & (1-q)\{\alpha_L u'[(1-\gamma_M^*)yR+c_{I,L}] \\
& +(1-\alpha_L)u'[(1-\gamma_M^*)yR+\frac{(\gamma_M^* y-\alpha_L)-\alpha_L c_{I,L}}{1-\alpha_L}-1](\frac{-\alpha_L}{1-\alpha_L})\} \\
& +\lambda(1-\widehat{q})\{u'[(1-\gamma_M^*)yR+\frac{(\gamma_M^* y-\alpha_L)-\alpha_L c_{I,L}}{1-\alpha_L}-1](\frac{-\alpha_L}{1-\alpha_L}) \\
& -u'[(1-\gamma_M^*)yR+c_{I,L}]\} \\
& +\phi(1-\widetilde{q})\{u'[(1-\gamma_M^*)yR+c_{I,L}] \\
& -u'[(1-\gamma_M^*)yR+\frac{(\gamma_M^* y-\alpha_L)-\alpha_L c_{I,L}}{1-\alpha_L}-1](\frac{-\alpha_L}{1-\alpha_L})\}
\end{aligned}
$$

And $\frac{d\widetilde{p}}{dc_{I,H}}$ is given by:

$$\frac{d\widetilde{p}}{dc_{I,H}}$$
$$= \frac{(\widetilde{p})^2}{R} \frac{\frac{-\gamma_M^* y}{(1-\gamma_M^* y)}\{u'[(1-\gamma_M^*)yR + \frac{-\gamma_M^* y c_{I,H}}{(1-\gamma_M^* y)} - 1] - u'[(1-\gamma_M^*)yR + \frac{-\gamma_M^* y c_{I,H}}{(1-\gamma_M^* y)} - \frac{R}{\widetilde{p}}]\}}{u'[((1-\gamma_M^*)yR + \frac{-\gamma_M^* y c_{I,H}}{(1-\gamma_M^* y)} - \frac{R}{\widetilde{p}})]}$$

It is easy to see that $\frac{d\widetilde{p}}{dc_{I,H}}\Big|_{c_{I,H}^* = \frac{\gamma_M^* y}{\widehat{p}}R - \frac{1}{\widehat{p}}R} = 0$. So we have:

$$\frac{\partial L}{\partial c_{I,H}}\Bigg|_{c_{I,H}^* = \frac{\gamma_M^* y}{\widehat{p}}R - \frac{1}{\widehat{p}}R} = q\{\gamma_M^* y u'[(1-\gamma_M^*)yR + \frac{\gamma_M^* y}{\widehat{p}}R - \frac{1}{\widehat{p}}R]$$
$$- \gamma_M^* y u'[(1-\gamma_M^*)yR + \frac{\gamma_M^* y}{\widehat{p}}R - 1]\}$$
$$- \lambda\ \widehat{q}\gamma_M^* y u'[(1-\gamma_M^*)yR + \frac{\gamma_M^* y}{\widehat{p}}R - 1](\frac{1}{(1-\gamma_M^* y)})$$
$$+ \phi\widetilde{q}\gamma_M^* y\{u'[(1-\gamma_M^*)yR + \frac{\gamma_M^* y}{\widehat{p}}R - \frac{1}{\widehat{p}}R](\frac{1}{(1-\gamma_M^* y)}$$

$$\frac{\partial L}{\partial c_{I,L}}\Bigg|_{c_{I,L} = \gamma_M^* y - 1}$$
$$= \lambda(1 - \ \widehat{q})\{u'[(1-\gamma_M^*)yR + \gamma_M^* y - 1](\frac{-1}{1-\alpha_L})\}$$
$$+ \phi(1-\widetilde{q})\{u'[(1-\gamma_M^*)yR + \gamma_M^* y - 1](\frac{1}{1-\alpha_L})\}$$

If the market allocation $c_{P,H}^* = c_{I,H}^* + \frac{1}{\widehat{p}}R = \frac{\gamma_M^* y}{\widehat{p}}R$, $c_{P,L}^* = c_{I,L}^* + 1 = \gamma_M^* y$ is optimal, the necessary condition are that

$$\frac{\partial L}{\partial c_{I,H}}\Bigg|_{c_{I,H} = \frac{\gamma_M^* y}{\widehat{p}}R - \frac{1}{\widehat{p}}R} = 0$$
$$\text{and } \frac{\partial L}{\partial c_{I,L}}\Bigg|_{c_{I,L} = \gamma_M^* y - 1} = 0.$$

$$\frac{\partial L}{\partial c_{I,L}}\Bigg|_{c_{I,L} = \gamma_M y - 1} = 0$$

$$\Rightarrow \lambda\ (1-\widehat{q}) = \phi(1-\widetilde{q})$$

$$\Rightarrow \lambda\ < \phi(since \widehat{q} < \widetilde{q}) or \lambda\ = \phi = 0 \qquad (42)$$

$$\left.\frac{\partial L}{\partial c_{I,H}}\right|_{c_{I,H}=\frac{\gamma_M y}{p}-\frac{1}{p}} = 0$$

$$\Rightarrow \lambda \; > \phi(since \widehat{q} < \widetilde{q}) \tag{43}$$

(42) and (43) give us a contradiction. So the market allocation can not be the solution to the problem of the bank & market.. ■

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