

# Appendix to ‘Pairwise Trade and Coexistence of Money and Higher-Return Assets’

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This is the appendix to Zhu and Wallace [4]. In it, we prove the claims in section 3 of that paper that are made about the model with a non degenerate wealth distribution. We begin by formally setting out the definition of an equilibrium for that model.

Let  $\mathbf{Y} = \{y = (y_1, y_2) \in \mathbf{Z} \times \mathbf{Z} : y_1 + y_2 \leq Z\}$ . An element of  $\mathbf{Y}$  is an individual portfolio after bond purchases ( $y_1$  is the amount of money and  $y_2$  is the amount of bonds measured at maturity value) that satisfies the restriction that total nominal wealth not exceed  $Z$ . For  $y \in \mathbf{Y}$ , we again let  $y_z = y_1 + y_2$  denote the total nominal wealth implied by  $y$ . Given  $\pi_0$  (an initial distribution of money holdings over the set  $\mathbf{Z}$ ), an equilibrium is a sequence  $\{w_t, h_t, \theta_t, \pi_{t+1}\}_{t=0}^{\infty}$  that satisfies the conditions described below. The functions  $w_t$  and  $\pi_t$  pertain to the start of date  $t$ , prior to bond purchases:  $w_t : \mathbf{Z} \rightarrow \mathbb{R}$ , where  $w_t(z)$  is the expected discounted value of having wealth  $z$ , and  $\pi_t : \mathbf{Z} \rightarrow [0, 1]$ , where  $\pi_t(z)$  is the fraction of each specialization type with wealth  $z$ . The functions  $h_t$  and  $\theta_t$  pertain to the situation after bond purchases and before meetings:  $h_t : \mathbf{Y} \rightarrow \mathbb{R}$ , where  $h_t(y)$  is the expected discounted value of having the portfolio  $y$ , and  $\theta_t : \mathbf{Y} \rightarrow [0, 1]$ , where  $\theta_t(y)$  is the fraction of each specialization type with portfolio  $y$ .

We start with bond buying. We let a person with wealth  $z$  buy any lottery over portfolios in  $\mathbf{Y}$  whose expected cost does not exceed  $z$ .<sup>1</sup> Let  $\Gamma(z, p)$ , a

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<sup>1</sup>An alternative would have lotteries only over portfolios whose cost does not exceed  $z$ . All our results also hold for that version.

set of probability measures defined on  $\mathbf{Y}$ , be defined by

$$\Gamma_1(z, p) = \{\sigma : E_\sigma(y_1 + py_2) \leq z\}. \quad (1)$$

Here, for  $y \in \mathbf{Y}$ ,  $\sigma(y)$  is the probability of purchasing the portfolio  $y$  and  $E_\sigma$  is the expectation with respect to  $\sigma$ . It follows that  $w_t$  and  $h_t$  satisfy

$$w_t(z) = \max_{\sigma \in \Gamma_1(z, p)} E_\sigma h_t(y). \quad (2)$$

We next describe the law of motion of distributions induced by the choice of  $\sigma$  in (2). Let  $\Delta_1(z, h_t, p)$ , a subset of probability measures on  $\mathbf{Y}$ , be the set of maximizers in (2). If  $\delta \in \Delta_1(z, h_t, p)$ , then  $\delta$  is an optimal lottery and  $\delta(y)$  is the probability of holding portfolio  $y$ . Then, we define a set of distributions on  $\mathbf{Y}$ ,  $\Phi_\theta(h_t, \pi_t, p)$ , by

$$\Phi_\theta(h_t, \pi_t, p) = \{\theta_t : \theta_t(y) = \sum_z \pi_t(z) \delta(y) \text{ for } \delta \in \Delta_1(z, h_t, p)\}. \quad (3)$$

We next turn to trade in meetings. First, we let  $g : \mathbf{Z} \rightarrow \mathbb{R}$  denote expected discounted utility after the date- $t$  pairwise meetings but before people are taxed. (The function  $g$  is determined by  $w_{t+1}$  and  $\theta_t$  as described below.) Now consider a meeting between a buyer with portfolio  $y$  and a seller with portfolio  $y'$ . Let  $W \in \mathbb{R}$  be an upper bound on  $w_t$  (and hence on  $g$ ) defined below. (This implies a bound on output in a meeting.) We describe the asset transfers in terms of the end-of-trade wealth of the buyer, which is convenient for the updating of the wealth distribution. For step 1 of the relevant version of problem 1 (see [4]), we let  $\Gamma_{21}(y, y'; g)$ , a set of probability measures on  $[0, W] \times \{\max\{y_2, y_z - Z + y'_z\}, \dots, y_z\}$ , be defined by

$$\Gamma_{21}(y, y'; g) = \{\sigma : E_\sigma[-q + g(y'_z - z + y'_z)] \geq g(y'_z)\}. \quad (4)$$

Here,  $\sigma(q, z)$  is the probability that the step-1 trade is  $q$  amount of output and an asset transfer that leaves the buyer with  $z$  units of wealth. Notice that the cash-in-advance constraint is embedded in the definition of  $\Gamma_{21}$  through the restriction  $z \geq y_2$ . Then, the buyer's problem-1, step-1 payoff is

$$f_b(y, y'; g) = \max_{\sigma \in \Gamma_{21}(y, y'; g)} E_\sigma[u(q) + g(z)]. \quad (5)$$

For step 2, we let  $\Gamma_{22}(y, y'; g)$ , a set of probability measures on  $[0, W] \times \{\max\{0, y_z - Z + y'_z\}, \dots, y_z\}$ , be defined by

$$\Gamma_{22}(y, y'; g) = \{\sigma : E_\sigma[u(q) + g(z)] \geq f_b(y, y'; g)\}. \quad (6)$$

Then the seller's problem-1 payoff is

$$f_s(y, y'; g) = \max_{\sigma \in \Gamma_{22}(y, y'; g)} E_\sigma[-q + g(y_z - z + y'_z)]. \quad (7)$$

Because the buyer does not gain in step 2 of problem 1 (again, see[4]), it follows that the expected payoff from holding the portfolio  $y$  before random matching is

$$h_t(y) = \frac{1}{N} \sum_{y'} \theta_t(y') [f_b(y, y'; g) + f_s(y', y; g)] + (1 - \frac{2}{N})g(y_z). \quad (8)$$

Now we describe the law of motion of distributions induced by the trades in meetings. A maximizer in (7) is degenerate in  $q$  and is determined by the lottery over the buyer's end-of-trade wealth because the constraints in (4) and (6) hold with equality. Therefore, we let  $\Delta_2(y, y', g)$ , a subset of probability measures on  $\mathbf{Z}$ , be the set of maximizers in (7) described in that way; that is,  $\delta \in \Delta_2(y, y', g)$  is a lottery over the end-of-trade wealth of the buyer and  $\delta(z)$  is the probability for that maximizer that the buyer has end-of-trade wealth  $z$ . Then we define a set of post-trade and pre-tax distributions on  $\mathbf{Z}$  at date  $t$ ,  $\Omega(w_{t+1}, \theta_t)$ , by

$$\begin{aligned} \Omega(w_{t+1}, \theta_t) = \{ \omega : \omega(z) = \sum_{(y, y')} \theta_t(y) \theta_t(y') [\delta(z) + \delta(y_z - z + y'_z)] + \\ \frac{N-2}{N} \sum_y I(z; y) \theta_t(y) \text{ for } \delta \in \Delta_2(y, y'; g) \}, \end{aligned} \quad (9)$$

$\Omega(w_{t+1}, \theta_t)$  where  $\delta(y_z - z + y'_z)$ , the probability that the buyer ends up with  $y_z - z + y'_z$ , is the probability that the seller ends up with  $z$ , and where  $I(z; y) = 1$  if  $y_z = z$  and is 0 otherwise. The dependence of  $\Omega$  on  $(w_{t+1}, \theta_t)$  is through the dependence of  $g$  on  $(w_{t+1}, \theta_t)$ , which we now spell out by describing the taxing of end-of-trade wealth.

We let  $\bar{z}_{\theta_t} \equiv \sum \theta_t(y) y_z$ , the average nominal wealth implied by  $\theta_t$ . If assets were divisible, then after tax wealth,  $z$ , would be  $z'(\bar{z}/\bar{z}_{\theta_t})$ , where  $z'$  is end-of-trade wealth. To ensure that after-tax wealth is in the set  $\mathbf{Z}$ , we use a lottery and let each person choose the lottery subject to having an expected tax equal to  $z'(\bar{z}/\bar{z}_{\theta_t})$ .<sup>2</sup> Let  $\Gamma_3(z'; w_{t+1}, \theta_t)$ , a set of probability measures on  $\mathbf{Z}$ , be defined by

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<sup>2</sup>In the steady shown to exist, strict concavity of  $w$  implies that the maximizing lottery is the unique lottery over the integers closest to  $z'(\bar{z}/\bar{z}_{\theta_t})$  that satisfies the constraint. (Here and below, we apply the term *concave* to functions defined on discrete subsets of  $\mathbb{R}$ . Suppose  $X \subset \mathbb{R}$ . We say that  $f : X \rightarrow \mathbb{R}$  is (strictly) concave if there exists  $g : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f$  is the restriction of  $g$  to  $X$  and  $g$  is (strictly) concave.)

$$\Gamma_3(z'; \theta_t) = \{\sigma : E_\sigma(z) = z'(\bar{z}/\bar{z}_{\theta_t})\}. \quad (10)$$

Here, the argument of  $\sigma$  is post-tax wealth and  $E_\sigma(z)$  is the expectation of post tax-wealth implied by  $\sigma$ . Then we let

$$g(z') = \beta \max_{\sigma \in \Gamma_3(z'; \theta_t)} E_\sigma w_{t+1}(z). \quad (11)$$

To complete the description of the law of motion, let  $\Delta_3(z; w_{t+1}, \theta_t)$  be the set of maximizers in (11). Then we define a set of post-tax (and beginning-of-next-date) distributions on  $\mathbf{Z}$ ,  $\Phi_\pi(w_{t+1}, \theta_t)$ , by

$$\begin{aligned} \Phi_\pi(w_{t+1}, \theta_t) &= \{\pi_{t+1} : \pi_{t+1}(z) = \sum_{z'} \omega(z') \delta(z; z'), \\ &\text{for } \omega \in \Omega(w_{t+1}, \theta_t) \text{ and } \delta(\cdot; z') \in \Delta_3(z'; w_{t+1}, \theta_t)\}. \end{aligned} \quad (12)$$

We can now define an equilibrium and a steady state.

**Definition 1** *Given  $\pi_0$ , an initial distribution of money holdings over the set  $\mathbf{Z}$ , an equilibrium is a sequence  $\{w_t, h_t, \theta_t, \pi_{t+1}\}_{t=0}^\infty$  that satisfies (2), (8),  $\theta_t \in \Phi_\theta(h_t, \pi_t, p)$  (see (3)), and  $\pi_{t+1} \in \Phi_\pi(w_{t+1}, \theta_t)$  (see (12)). A steady state is  $(w, h, \theta, \pi)$  such that  $\{w_t, h_t, \theta_t, \pi_{t+1}\}_{t=0}^\infty = (w, h, \theta, \pi)$  is an equilibrium for  $\pi_0 = \pi$ .*

The proof of proposition 2 consists of several lemmas. We begin with some notation and assumptions. Let  $D > 0$  be the unique solution to  $u'(D) = (2/R\beta)^2$ , where  $R \equiv [N - (N-1)\beta]^{-1} < 1$ . Existence of  $D$  requires only that  $u'(0)$  is sufficiently large. Let  $\tilde{W}$  be the unique solution to  $N(1-\beta)\tilde{W} = u(\beta\tilde{W}) + N$  and let  $W = \max\{\tilde{W}, \frac{2D}{\beta}\}$ . Throughout, for  $x \in \mathbb{R}$ , we let  $x_-$  be the largest integer that does not exceed  $x$  and we let  $x_+$  be the smallest integer no less than  $x$ , so that  $x \in [x_-, x_+]$ .

Let  $\mathbf{W}$  be the set of non-decreasing and concave functions  $w : \mathbf{Z} \rightarrow [0, W]$  with  $w[(4\bar{z})_+] \geq D/\beta$ . Let  $\mathbf{K} \supset \mathbf{W}$  be the set of non-decreasing functions from  $\mathbf{Z}$  to  $[0, W]$ . Notice that the interior of  $\mathbf{W}$  (relative to  $\mathbf{K}$ ) is non-empty and that an element of the interior is strictly increasing, strictly concave, and satisfies  $w[4(\bar{z})_+] > \frac{D}{\beta}$ . Let  $\mathbf{H}$  be the set of non-decreasing functions  $h : \mathbf{Y} \rightarrow [0, W]$ . Let  $\mathbf{\Pi}$  be the set of probability measures  $\pi$  defined on  $\mathbf{Z}$  satisfying  $\sum \pi(z)z = \bar{z}$ . To save notation, we impose 0.5 as an arbitrary lower bound on  $p$ , and we let  $\mathbf{\Theta}$  be the set of probability measures  $\theta$  on  $\mathbf{Y}$

satisfying  $2\bar{z} \geq \bar{z}_\theta \geq \bar{z}$ , where  $\bar{z}_\theta \equiv \sum \theta(y)y_z$ , the average nominal wealth implied by  $\theta$ .

Now we can formally define the mapping to be studied. We let the mapping  $\Phi_h : \mathbf{W} \times \Theta \rightarrow \mathbf{H}$  be defined by

$$\Phi_h(w, \theta)(y) = \frac{1}{N} \sum_{y' \in Y} \theta(y') [f_b(y, y'; g) + f_s(y', y; g)] + (1 - \frac{2}{N})g(y_z). \quad (13)$$

where  $g$  is given by (11),  $f_b$  is given by (5), and  $f_s$  is given by (7). Let mapping  $\Phi_w : \mathbf{H} \times [0.5, 1] \rightarrow \mathbf{K}$  be defined by

$$\Phi_w(h, p)(z) = \max_{\sigma \in \Gamma(z, p)} E_\sigma h(y). \quad (14)$$

where  $\Gamma(z, p)$  is given in (4). Finally, we let  $\Phi : \mathbf{W} \times \mathbf{H} \times \Pi \times \Theta \times [0.5, 1] \rightarrow \mathbf{K} \times \mathbf{H} \times \Pi \times \Theta$  be defined by

$$\Phi(w, h, \pi, \theta, p) = (\Phi_w(h, p), \Phi_h(w, \theta), \Phi_\theta(h, \pi, p), \Phi_\pi(w, \theta)), \quad (15)$$

where  $\Phi_w(h, p)$  is given by (14),  $\Phi_h(w, \theta)$  is given by (13),  $\Phi_\theta(h, \pi, p)$  is given by (3), and  $\Phi_\pi(w, \theta)$  is given by (12). In what follows, we write  $\Phi(\cdot, \cdot, \cdot, \cdot, p) : \mathbf{W} \times \mathbf{H} \times \Pi \times \Theta \rightarrow \mathbf{K} \times \mathbf{H} \times \Pi \times \Theta$  as  $\Phi_p(\cdot, \cdot, \cdot, \cdot)$ .

**Lemma 1** *A fixed point of  $\Phi_p$  is a steady state.*

**Proof.** Obvious. ■

Our proof of proposition 2 uses a fixed-point index theorem.

**Definition 2** *Let  $\mathbf{S} \equiv \mathbf{H} \times \Pi \times \Theta$  and let  $\partial\mathbf{W}$  denote the boundary of  $\mathbf{W}$  (with respect to  $\mathbf{K}$ ). Let  $\mathcal{G}$  denote the set of upper-hemicontinuous (u.h.c.), compact valued, and convex valued mappings  $g : \mathbf{W} \times \mathbf{S} \rightarrow \mathbf{K} \times \mathbf{S}$  satisfying  $(w, s) \notin g(w, s)$  for all  $(w, s) \in \partial\mathbf{W} \times \mathbf{S}$ . Two mappings  $g_0, g_1 \in \mathcal{G}$  are said to be **homotopic on  $\partial\mathbf{W} \times \mathbf{S}$**  if there exists a u.h.c., compact valued, and convex valued mapping  $G : \mathbf{W} \times \mathbf{S} \times [0, 1] \rightarrow \mathbf{K} \times \mathbf{S}$  such that (i)  $(w, s, \alpha) \notin G(w, s, \alpha)$  for all  $(w, s, \alpha) \in \partial\mathbf{W} \times \mathbf{S} \times [0, 1]$  and (ii)  $G(w, s, \alpha) = g_\alpha(w, s)$  for all  $(w, s, \alpha) \in \partial\mathbf{W} \times \mathbf{S} \times \{0, 1\}$ .*

The version of the fixed point index theorem we need is the following (see [1, Theorem 36.1, p. 218] and [2, 13.6a, p. 604]). There exists a fixed-point index defined on  $\mathcal{G}$ , denoted *ind*, satisfying:

- (A1) If  $g$  is constant on  $\mathbf{W} \times \mathbf{S}$  with the value  $(w_0, s_0)$  where  $w_0 \in \mathbf{W} - \partial\mathbf{W}$ , then  $\text{ind}(g) = 1$ .
- (A2) If  $\text{ind}(g) \neq 0$ , then there exist some  $(w, s) \in \mathbf{W} - \partial\mathbf{W} \times \mathbf{S}$  with  $(w, s) \in g(w, s)$ .
- (A3) If  $g_0, g_1 \in \mathcal{G}$  are homotopic on  $\partial\mathbf{W} \times \mathbf{S}$ , then  $\text{ind}(g_0) = \text{ind}(g_1)$ .

The next lemma establishes properties of  $\Phi_1$ , properties of  $\Phi$  when  $p = 1$ .

**Lemma 2** (i) *There exists  $(w, h, \pi, \theta) \in \Phi_1(w, h, \pi, \theta)$  and any such  $\pi$  has full support;* (ii)  $\Phi_1 \in \mathcal{G}$ , (iii)  $\text{ind}(\Phi_1) = 1$ .

**Proof.** As noted above, if  $p = 1$ , then the model is equivalent to one in which bonds are not available. (By equivalence we mean that if  $(w, h, \pi, \theta)$  is a steady state with bonds available, then  $(w, \pi)$  is a steady state when they are not available. And vice versa in the sense that buying no bonds is an optimal portfolio when  $p = 1$ .) This is the model studied in [3] except for the lottery in meetings. For that version, parts (i) and (ii) are true (see Proposition 1 in [3]). It is straightforward to show that they also hold for a version with lotteries.

For part (iii), let  $n = (\bar{z})_+$  and let  $(w^*, s^*) \in \mathbf{W} - \partial\mathbf{W} \times \mathbf{S}$  with  $w^*(4n, 1) > \frac{D}{2n\beta}$ , where, here and below,  $w(x, y) \equiv w(x) - w(x - y)$ , the backward  $y$  increment in  $w$  at  $x$ . Then, let the mapping  $G : \mathbf{W} \times \mathbf{S} \times [0, 1] \rightarrow \mathbf{K} \times \mathbf{S}$  be defined by

$$G(w, s, \alpha) = (1 - \alpha)(w^*, s^*) + \alpha\Phi_1(w, s). \quad (16)$$

By the above fixed-point index theorem, it suffices to show that if  $(w, s, \alpha) \in \partial\mathbf{W} \times \mathbf{S} \times [0, 1]$  with  $(w, s, \alpha) \in G(w, s, \alpha)$ , then  $w$  is strictly increasing and strictly concave and satisfies  $w(4n) > \frac{D}{\beta}$ . By the definition of  $w^*$  and by parts (i) and (ii), it suffices to show this for  $\alpha \in (0, 1)$ . And because  $w^*$  is strictly increasing and strictly concave and because  $\Phi_1$  preserves monotonicity and concavity of  $w$  (see ([3])), it follows that  $w$  is strictly increasing and strictly concave. Thus, we have only to deal with the lower bound on  $w$ .

Now assume by contradiction that  $w(4n) = \frac{D}{\beta}$ . It follows that

$$w(4n, 1) < w(2n, 1) < \frac{D}{2n\beta} < w^*(4n, 1) < w^*(2n, 1). \quad (17)$$

where the first inequality follows from strict concavity of  $w$ , the second from  $2nw(2n, 1) < w(2n) < \frac{D}{\beta}$ , and the third and fourth from the assumption

of  $w^*$ . Because  $p = 1$ , we can ignore the distinction between  $\theta$  and  $\pi$  and treat the first two arguments of  $f_b$  and  $f_s$  as amounts of money. Moreover, it follows that  $f_s(m, x; w, \pi) = \beta w(x)$ . Then, because  $(w, s, \alpha)$  is a fixed point of  $G$ , it follows that

$$\begin{aligned} w(x, 1) &= (1 - \alpha)w^*(x, 1) + \alpha\left(1 - \frac{1}{N}\right)\beta w(x, 1) \\ &\quad + \alpha\frac{1}{N}\sum \pi(m)[f(x, m) - f(x - 1, m)], \end{aligned} \quad (18)$$

where we let  $f(x, m) = f_b(x, m; w, \theta)$  to simplify the notation. By (17), for  $x \in \{2n, 4n\}$ ,  $w^*(x, 1) > w(x, 1)$ . Therefore, for  $x \in \{2n, 4n\}$ , (18) implies

$$\begin{aligned} w(x, 1) &> R\sum \pi(m)[f(x, m) - f(x - 1, m)] \\ &\geq R\sum_{m=0}^{2n} \pi(m)[f(x, m) - f(x - 1, m)], \end{aligned} \quad (19)$$

where the second inequality follows from monotonicity of  $f(x, m)$  in  $x$ .

Now we apply the argument used in the proof of Lemma 3 in ([3]) to derive a contradiction. Let  $o(x, m) = \arg \max_{y \in [0, \min\{x, Z-m\}]} u[\beta\bar{w}(m + y, y)] + \beta\bar{w}(x - y)$ , where  $\bar{w}$  is the extension of  $w$  to  $[0, Z]$  defined by linear interpolation. Notice that  $o(x, m)$  is a singleton and that  $f(x, m) = u[\beta\bar{w}(m + y, y)] + \beta\bar{w}(x - y)$  with  $y = o(x, m)$ . Also,  $\pi\{m : m \leq 2n\} \geq 1/2$  because  $n \geq \bar{z}$ . Now fix  $m \leq 2n$  and let  $y = o(4n - 1, m)$ . If  $y \geq 2n$ , then, because  $y$  is a feasible offer for the buyer with  $4n$ ,

$$f(4n, m) - f(4n - 1, m) \geq \beta w(2n, 1). \quad (20)$$

If  $y < 2n$ , then, because  $y + 1$  is a feasible offer for the buyer with  $4n$ ,

$$\begin{aligned} f(4n, m) - f(4n - 1, m) &\geq u[\beta\bar{w}(m + y + 1)] - u[\beta\bar{w}(m + y)] \\ &\geq u'[\beta\bar{w}(m + y + 1, y + 1)]\beta\bar{w}(m + y + 1, 1) \\ &> \beta u'(D)w(4n, 1), \end{aligned} \quad (21)$$

where the second inequality follows from the mean value theorem and concavity, and the third from  $w(4n) = \frac{D}{\beta}$  and concavity. Either  $w(2n, 1) \geq u'(D)w(4n, 1)$  or  $w(2n, 1) < u'(D)w(4n, 1)$ . If the former, then by (19),  $w(4n, 1) > (R\beta/2)u'(D)w(4n, 1) > w(4n, 1)$ , a contradiction. (By the definition of  $D$ ,  $(R\beta/2)u'(D) > 1$ .) So the latter must hold. Then, (19) for

$x = 4n$  implies  $w(4n, 1) > (R\beta/2)w(2n, 1)$ . By exactly the reasoning used to get (21), we have  $f(2n, m) - f(2n - 1, m) > \beta u'(D)w(4n, 1)$  for  $m \leq 2n$ . But, then, by (19) for  $x = 2n$ , we have

$$w(2n, 1) > (R\beta/2)u'(D)w(4n, 1) > w(2n, 1),$$

a contradiction. ■

The proofs of the next two lemmas are standard.

**Lemma 3**  $\Phi$  is u.h.c., compact valued, and convex valued.

**Proof.** The result follows from the Theorem of Maximum and the convexification by lotteries. ■

The next lemma completes the proof of proposition 2.

**Lemma 4** There exists  $p_0 < 1$  such that if  $p \geq p_0$ , then  $\Phi_p$  has a fixed point  $(w, s) \in \mathbf{W} - \partial\mathbf{W} \times \mathbf{S}$  where  $s = (h, \pi, \theta)$  is such that  $\pi$  has full support.

**Proof.** By Lemma 3 and Lemma 2, parts (ii) and (i), there exists  $p_0 < 1$  such that if  $p \geq p_0$ , then (i)  $\Phi_p \in \mathcal{G}$  and (ii)  $(w, s) \in \Phi_p(w, s)$  implies that  $s = (h, \pi, \theta)$  is such that  $\pi$  has full support. Now we define  $\Psi : \mathbf{W} \times \mathbf{S} \times [0, 1] \rightarrow \mathbf{K} \times \mathbf{S}$  by  $\Psi_\alpha(w, s) = \Phi(w, s, (1 - \alpha)p_0 + \alpha)$ . Because  $\Phi_p \in \mathcal{G}$  for  $p \geq p_0$ , it follows that  $\Psi_\alpha \in \mathcal{G}$  all  $\alpha$ . Then by Lemma 2 (iii) and the fixed-point index theorem,  $ind(\Psi_\alpha) = 1$  for all  $\alpha$ . That is,  $ind(\Phi_p) = 1$  for  $p \geq p_0$ . And then, again, by the fixed-point index theorem,  $\Phi_p$  has a fixed point  $(w, s) \in \mathbf{W} - \partial\mathbf{W} \times \mathbf{S}$ . ■

Now we provide a proof of corollary 1.

**Proof.** We proceed by contradiction by assuming that the measure of people who leave the bond-buying stage with money is 0. For a person with wealth  $z \geq 1$ , a lower bound on leaving the bond-buying stage with 1 unit of money is  $h[1, (\frac{z-1}{p})_-]$ , while an upper bound on leaving with no money is  $h[0, (\frac{z}{p})_+]$ . In what follows, we let  $z_1 = (\frac{z-1}{p})_-$  and  $z_2 = (\frac{z}{p})_+$ . We will show that if no one has money, then  $h(1, z_1) > h(0, z_2)$  for sufficiently large  $z$ . But because the steady state distribution of wealth has full support, this implies that there exists positive measure of people leaving the bond-buying stage with money, a contradiction.

Before we proceed, recall that the value function defined on post-meeting and pre-taxing wealth is  $g$  defined in (11). Let  $\bar{g}$  and  $\bar{w}$  be the extensions of  $g$  and  $w$  to  $[0, Z]$  by linear interpolation, respectively. It is easy to see that  $\bar{g}(z) = \beta\bar{w}(\frac{\bar{z}}{\bar{z}_\theta}z)$ . Hence  $\bar{g}$  is strictly increasing and strictly concave.

A person who starts with  $z$  may end up being a buyer, a seller, or neither in a meeting. The following hold for any  $z \geq 1$ . (i) If the person is neither a buyer nor a seller, then the payoff difference between having the portfolio  $(1, z_1)$  and having  $(0, z_2)$  is

$$\bar{g}(1+z_1) - \bar{g}(z_2) > \bar{g}\left(\frac{z-2}{p}\right) - \bar{g}\left(\frac{z+1}{p}\right) = -\bar{g}\left(\frac{z+1}{p}, \frac{3}{p}\right) \geq -\frac{\bar{g}(z+1, 3)}{p}. \quad (22)$$

Here  $\bar{g}$  is the extension of  $g$  to  $[0, Z]$  by linear interpolation and  $\bar{g}(x, y) \equiv \bar{g}(x) - \bar{g}(x - y)$ . The inequalities follow from  $p \in (0, 1)$  and monotonicity and concavity of  $\bar{g}$ . (ii) If the person is a seller, then, by the contradicting assumption, with probability 1 the buyer does not hold any money. Hence, step 1 of problem 1 is null, and in step 2 a final trade that is feasible for the seller with portfolio  $(0, z_2)$  is also feasible if the seller has  $(1, z_1)$ . Because  $\bar{g}$  is concave, the payoff difference between having the portfolio  $(1, z_1)$  and having  $(0, z_2)$  is bounded below exactly as described by (22). (iii) If the person is a buyer, then the person meets either a seller with wealth (at the start of the date) that exceeds  $\bar{z}$ , or one with wealth that does not exceed  $\bar{z}$ . In the former case, a feasible choice in step-1 of problem 1 is no trade. Therefore, in the former case, the payoff difference between having the portfolio  $(1, z_1)$  and having  $(0, z_2)$  is bounded below exactly as described by (22). In the latter case, the buyer with a unit of money can offer it for at least  $\bar{g}(\frac{\bar{z}}{p} + 1, 1)$  amount of the good, where the expression for the amount of the good follows from (4) at equality and concavity of  $\bar{g}$ . Hence, in any such meeting, the payoff of the buyer with a unit of money is bounded below by  $u[\bar{g}(\frac{\bar{z}}{p} + 1, 1)] + \bar{g}(\frac{z-2}{p})$ . Now, assembling these results and recalling that a person is a buyer in a meeting with a seller with wealth that does not exceed  $\bar{z}$  with probability  $\frac{1}{2N}$ , we have

$$\begin{aligned} h(1, z_1) - h(0, z_2) &> \frac{u[\bar{g}(\frac{\bar{z}}{p} + 1, 1)]}{2N} - \frac{\bar{g}(z+1, 3)}{p} \\ &\geq \frac{u[\bar{g}(\frac{\bar{z}}{p} + 1, 1)]}{2N} - \frac{3\bar{g}(z+1, 1)}{p}, \end{aligned} \quad (23)$$

where the second inequality follows from concavity of  $\bar{g}$ . Moreover, if  $z \geq \frac{\bar{z}}{p}$ , that implies, again by concavity of  $\bar{g}$ , that

$$h(1, z_1) - h(0, z_2) > \frac{u[\bar{g}(z+1, 1)]}{2N} - \frac{3\bar{g}(z+1, 1)}{p}. \quad (24)$$

Provided  $u'(0)$  is sufficiently large, there exists a unique  $q^* > 0$  such that  $\frac{u(q^*)}{2N} = \frac{3q^*}{p}$  and  $\frac{u(q)}{2N} > \frac{3q}{p}$  for  $q \in (0, q^*)$ . Because we are free to make  $Z$  sufficiently large independent of  $\bar{z}$  and because the upper bound on  $w$  (and, hence, on  $\bar{g}$ ) is independent of  $Z$ , all sufficiently large  $z$  satisfy  $z > \frac{\bar{z}}{p}$  and  $\bar{g}(z+1, 1) \in (0, q^*)$ . Hence, for all such  $z$ , the right-hand side of (24) is positive, a contradiction. ■

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