

Appendix to ‘Pairwise Trade and Coexistence of Money and Higher-Return Assets’

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This is the appendix to Zhu and Wallace [4]. In it, we prove the claims in section 3 of that paper that are made about the model with a non degenerate wealth distribution. We begin by formally setting out the definition of an equilibrium for that model.

Let $\mathbf{Y} = \{y = (y_1, y_2) \in \mathbf{Z} \times \mathbf{Z} : y_1 + y_2 \leq Z\}$. An element of \mathbf{Y} is an individual portfolio after bond purchases (y_1 is the amount of money and y_2 is the amount of bonds measured at maturity value) that satisfies the restriction that total nominal wealth not exceed Z . For $y \in \mathbf{Y}$, we again let $y_z = y_1 + y_2$ denote the total nominal wealth implied by y . Given π_0 (an initial distribution of money holdings over the set \mathbf{Z}), an equilibrium is a sequence $\{w_t, h_t, \theta_t, \pi_{t+1}\}_{t=0}^{\infty}$ that satisfies the conditions described below. The functions w_t and π_t pertain to the start of date t , prior to bond purchases: $w_t : \mathbf{Z} \rightarrow \mathbb{R}$, where $w_t(z)$ is the expected discounted value of having wealth z , and $\pi_t : \mathbf{Z} \rightarrow [0, 1]$, where $\pi_t(z)$ is the fraction of each specialization type with wealth z . The functions h_t and θ_t pertain to the situation after bond purchases and before meetings: $h_t : \mathbf{Y} \rightarrow \mathbb{R}$, where $h_t(y)$ is the expected discounted value of having the portfolio y , and $\theta_t : \mathbf{Y} \rightarrow [0, 1]$, where $\theta_t(y)$ is the fraction of each specialization type with portfolio y .

We start with bond buying. We let a person with wealth z buy any lottery over portfolios in \mathbf{Y} whose expected cost does not exceed z .¹ Let $\Gamma(z, p)$, a

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¹An alternative would have lotteries only over portfolios whose cost does not exceed z . All our results also hold for that version.

set of probability measures defined on \mathbf{Y} , be defined by

$$\Gamma_1(z, p) = \{\sigma : E_\sigma(y_1 + py_2) \leq z\}. \quad (1)$$

Here, for $y \in \mathbf{Y}$, $\sigma(y)$ is the probability of purchasing the portfolio y and E_σ is the expectation with respect to σ . It follows that w_t and h_t satisfy

$$w_t(z) = \max_{\sigma \in \Gamma_1(z, p)} E_\sigma h_t(y). \quad (2)$$

We next describe the law of motion of distributions induced by the choice of σ in (2). Let $\Delta_1(z, h_t, p)$, a subset of probability measures on \mathbf{Y} , be the set of maximizers in (2). If $\delta \in \Delta_1(z, h_t, p)$, then δ is an optimal lottery and $\delta(y)$ is the probability of holding portfolio y . Then, we define a set of distributions on \mathbf{Y} , $\Phi_\theta(h_t, \pi_t, p)$, by

$$\Phi_\theta(h_t, \pi_t, p) = \{\theta_t : \theta_t(y) = \sum_z \pi_t(z) \delta(y) \text{ for } \delta \in \Delta_1(z, h_t, p)\}. \quad (3)$$

We next turn to trade in meetings. First, we let $g : \mathbf{Z} \rightarrow \mathbb{R}$ denote expected discounted utility after the date- t pairwise meetings but before people are taxed. (The function g is determined by w_{t+1} and θ_t as described below.) Now consider a meeting between a buyer with portfolio y and a seller with portfolio y' . Let $W \in \mathbb{R}$ be an upper bound on w_t (and hence on g) defined below. (This implies a bound on output in a meeting.) We describe the asset transfers in terms of the end-of-trade wealth of the buyer, which is convenient for the updating of the wealth distribution. For step 1 of the relevant version of problem 1 (see [4]), we let $\Gamma_{21}(y, y'; g)$, a set of probability measures on $[0, W] \times \{\max\{y_2, y_z - Z + y'_z\}, \dots, y_z\}$, be defined by

$$\Gamma_{21}(y, y'; g) = \{\sigma : E_\sigma[-q + g(y'_z - z + y'_z)] \geq g(y'_z)\}. \quad (4)$$

Here, $\sigma(q, z)$ is the probability that the step-1 trade is q amount of output and an asset transfer that leaves the buyer with z units of wealth. Notice that the cash-in-advance constraint is embedded in the definition of Γ_{21} through the restriction $z \geq y_2$. Then, the buyer's problem-1, step-1 payoff is

$$f_b(y, y'; g) = \max_{\sigma \in \Gamma_{21}(y, y'; g)} E_\sigma[u(q) + g(z)]. \quad (5)$$

For step 2, we let $\Gamma_{22}(y, y'; g)$, a set of probability measures on $[0, W] \times \{\max\{0, y_z - Z + y'_z\}, \dots, y_z\}$, be defined by

$$\Gamma_{22}(y, y'; g) = \{\sigma : E_\sigma[u(q) + g(z)] \geq f_b(y, y'; g)\}. \quad (6)$$

Then the seller's problem-1 payoff is

$$f_s(y, y'; g) = \max_{\sigma \in \Gamma_{22}(y, y'; g)} E_\sigma[-q + g(y_z - z + y'_z)]. \quad (7)$$

Because the buyer does not gain in step 2 of problem 1 (again, see[4]), it follows that the expected payoff from holding the portfolio y before random matching is

$$h_t(y) = \frac{1}{N} \sum_{y'} \theta_t(y') [f_b(y, y'; g) + f_s(y', y; g)] + (1 - \frac{2}{N})g(y_z). \quad (8)$$

Now we describe the law of motion of distributions induced by the trades in meetings. A maximizer in (7) is degenerate in q and is determined by the lottery over the buyer's end-of-trade wealth because the constraints in (4) and (6) hold with equality. Therefore, we let $\Delta_2(y, y', g)$, a subset of probability measures on \mathbf{Z} , be the set of maximizers in (7) described in that way; that is, $\delta \in \Delta_2(y, y', g)$ is a lottery over the end-of-trade wealth of the buyer and $\delta(z)$ is the probability for that maximizer that the buyer has end-of-trade wealth z . Then we define a set of post-trade and pre-tax distributions on \mathbf{Z} at date t , $\Omega(w_{t+1}, \theta_t)$, by

$$\begin{aligned} \Omega(w_{t+1}, \theta_t) = \{ \omega : \omega(z) = \sum_{(y, y')} \theta_t(y) \theta_t(y') [\delta(z) + \delta(y_z - z + y'_z)] + \\ \frac{N-2}{N} \sum_y I(z; y) \theta_t(y) \text{ for } \delta \in \Delta_2(y, y'; g) \}, \end{aligned} \quad (9)$$

$\Omega(w_{t+1}, \theta_t)$ where $\delta(y_z - z + y'_z)$, the probability that the buyer ends up with $y_z - z + y'_z$, is the probability that the seller ends up with z , and where $I(z; y) = 1$ if $y_z = z$ and is 0 otherwise. The dependence of Ω on (w_{t+1}, θ_t) is through the dependence of g on (w_{t+1}, θ_t) , which we now spell out by describing the taxing of end-of-trade wealth.

We let $\bar{z}_{\theta_t} \equiv \sum \theta_t(y) y_z$, the average nominal wealth implied by θ_t . If assets were divisible, then after tax wealth, z , would be $z'(\bar{z}/\bar{z}_{\theta_t})$, where z' is end-of-trade wealth. To ensure that after-tax wealth is in the set \mathbf{Z} , we use a lottery and let each person choose the lottery subject to having an expected tax equal to $z'(\bar{z}/\bar{z}_{\theta_t})$.² Let $\Gamma_3(z'; w_{t+1}, \theta_t)$, a set of probability measures on \mathbf{Z} , be defined by

²In the steady shown to exist, strict concavity of w implies that the maximizing lottery is the unique lottery over the integers closest to $z'(\bar{z}/\bar{z}_{\theta_t})$ that satisfies the constraint. (Here and below, we apply the term *concave* to functions defined on discrete subsets of \mathbb{R} . Suppose $X \subset \mathbb{R}$. We say that $f : X \rightarrow \mathbb{R}$ is (strictly) concave if there exists $g : \mathbb{R} \rightarrow \mathbb{R}$ such that f is the restriction of g to X and g is (strictly) concave.)

$$\Gamma_3(z'; \theta_t) = \{\sigma : E_\sigma(z) = z'(\bar{z}/\bar{z}_{\theta_t})\}. \quad (10)$$

Here, the argument of σ is post-tax wealth and $E_\sigma(z)$ is the expectation of post tax-wealth implied by σ . Then we let

$$g(z') = \beta \max_{\sigma \in \Gamma_3(z'; \theta_t)} E_\sigma w_{t+1}(z). \quad (11)$$

To complete the description of the law of motion, let $\Delta_3(z; w_{t+1}, \theta_t)$ be the set of maximizers in (11). Then we define a set of post-tax (and beginning-of-next-date) distributions on \mathbf{Z} , $\Phi_\pi(w_{t+1}, \theta_t)$, by

$$\begin{aligned} \Phi_\pi(w_{t+1}, \theta_t) &= \{\pi_{t+1} : \pi_{t+1}(z) = \sum_{z'} \omega(z') \delta(z; z'), \\ &\text{for } \omega \in \Omega(w_{t+1}, \theta_t) \text{ and } \delta(\cdot; z') \in \Delta_3(z'; w_{t+1}, \theta_t)\}. \end{aligned} \quad (12)$$

We can now define an equilibrium and a steady state.

Definition 1 *Given π_0 , an initial distribution of money holdings over the set \mathbf{Z} , an equilibrium is a sequence $\{w_t, h_t, \theta_t, \pi_{t+1}\}_{t=0}^\infty$ that satisfies (2), (8), $\theta_t \in \Phi_\theta(h_t, \pi_t, p)$ (see (3)), and $\pi_{t+1} \in \Phi_\pi(w_{t+1}, \theta_t)$ (see (12)). A steady state is (w, h, θ, π) such that $\{w_t, h_t, \theta_t, \pi_{t+1}\}_{t=0}^\infty = (w, h, \theta, \pi)$ is an equilibrium for $\pi_0 = \pi$.*

The proof of proposition 2 consists of several lemmas. We begin with some notation and assumptions. Let $D > 0$ be the unique solution to $u'(D) = (2/R\beta)^2$, where $R \equiv [N - (N-1)\beta]^{-1} < 1$. Existence of D requires only that $u'(0)$ is sufficiently large. Let \tilde{W} be the unique solution to $N(1-\beta)\tilde{W} = u(\beta\tilde{W}) + N$ and let $W = \max\{\tilde{W}, \frac{2D}{\beta}\}$. Throughout, for $x \in \mathbb{R}$, we let x_- be the largest integer that does not exceed x and we let x_+ be the smallest integer no less than x , so that $x \in [x_-, x_+]$.

Let \mathbf{W} be the set of non-decreasing and concave functions $w : \mathbf{Z} \rightarrow [0, W]$ with $w[(4\bar{z})_+] \geq D/\beta$. Let $\mathbf{K} \supset \mathbf{W}$ be the set of non-decreasing functions from \mathbf{Z} to $[0, W]$. Notice that the interior of \mathbf{W} (relative to \mathbf{K}) is non-empty and that an element of the interior is strictly increasing, strictly concave, and satisfies $w[4(\bar{z})_+] > \frac{D}{\beta}$. Let \mathbf{H} be the set of non-decreasing functions $h : \mathbf{Y} \rightarrow [0, W]$. Let $\mathbf{\Pi}$ be the set of probability measures π defined on \mathbf{Z} satisfying $\sum \pi(z)z = \bar{z}$. To save notation, we impose 0.5 as an arbitrary lower bound on p , and we let $\mathbf{\Theta}$ be the set of probability measures θ on \mathbf{Y}

satisfying $2\bar{z} \geq \bar{z}_\theta \geq \bar{z}$, where $\bar{z}_\theta \equiv \sum \theta(y)y_z$, the average nominal wealth implied by θ .

Now we can formally define the mapping to be studied. We let the mapping $\Phi_h : \mathbf{W} \times \Theta \rightarrow \mathbf{H}$ be defined by

$$\Phi_h(w, \theta)(y) = \frac{1}{N} \sum_{y' \in Y} \theta(y') [f_b(y, y'; g) + f_s(y', y; g)] + (1 - \frac{2}{N})g(y_z). \quad (13)$$

where g is given by (11), f_b is given by (5), and f_s is given by (7). Let mapping $\Phi_w : \mathbf{H} \times [0.5, 1] \rightarrow \mathbf{K}$ be defined by

$$\Phi_w(h, p)(z) = \max_{\sigma \in \Gamma(z, p)} E_\sigma h(y). \quad (14)$$

where $\Gamma(z, p)$ is given in (4). Finally, we let $\Phi : \mathbf{W} \times \mathbf{H} \times \Pi \times \Theta \times [0.5, 1] \rightarrow \mathbf{K} \times \mathbf{H} \times \Pi \times \Theta$ be defined by

$$\Phi(w, h, \pi, \theta, p) = (\Phi_w(h, p), \Phi_h(w, \theta), \Phi_\theta(h, \pi, p), \Phi_\pi(w, \theta)), \quad (15)$$

where $\Phi_w(h, p)$ is given by (14), $\Phi_h(w, \theta)$ is given by (13), $\Phi_\theta(h, \pi, p)$ is given by (3), and $\Phi_\pi(w, \theta)$ is given by (12). In what follows, we write $\Phi(\cdot, \cdot, \cdot, \cdot, p) : \mathbf{W} \times \mathbf{H} \times \Pi \times \Theta \rightarrow \mathbf{K} \times \mathbf{H} \times \Pi \times \Theta$ as $\Phi_p(\cdot, \cdot, \cdot, \cdot)$.

Lemma 1 *A fixed point of Φ_p is a steady state.*

Proof. Obvious. ■

Our proof of proposition 2 uses a fixed-point index theorem.

Definition 2 *Let $\mathbf{S} \equiv \mathbf{H} \times \Pi \times \Theta$ and let $\partial\mathbf{W}$ denote the boundary of \mathbf{W} (with respect to \mathbf{K}). Let \mathcal{G} denote the set of upper-hemicontinuous (u.h.c.), compact valued, and convex valued mappings $g : \mathbf{W} \times \mathbf{S} \rightarrow \mathbf{K} \times \mathbf{S}$ satisfying $(w, s) \notin g(w, s)$ for all $(w, s) \in \partial\mathbf{W} \times \mathbf{S}$. Two mappings $g_0, g_1 \in \mathcal{G}$ are said to be **homotopic on $\partial\mathbf{W} \times \mathbf{S}$** if there exists a u.h.c., compact valued, and convex valued mapping $G : \mathbf{W} \times \mathbf{S} \times [0, 1] \rightarrow \mathbf{K} \times \mathbf{S}$ such that (i) $(w, s, \alpha) \notin G(w, s, \alpha)$ for all $(w, s, \alpha) \in \partial\mathbf{W} \times \mathbf{S} \times [0, 1]$ and (ii) $G(w, s, \alpha) = g_\alpha(w, s)$ for all $(w, s, \alpha) \in \partial\mathbf{W} \times \mathbf{S} \times \{0, 1\}$.*

The version of the fixed point index theorem we need is the following (see [1, Theorem 36.1, p. 218] and [2, 13.6a, p. 604]). There exists a fixed-point index defined on \mathcal{G} , denoted *ind*, satisfying:

- (A1) If g is constant on $\mathbf{W} \times \mathbf{S}$ with the value (w_0, s_0) where $w_0 \in \mathbf{W} - \partial\mathbf{W}$, then $\text{ind}(g) = 1$.
- (A2) If $\text{ind}(g) \neq 0$, then there exist some $(w, s) \in \mathbf{W} - \partial\mathbf{W} \times \mathbf{S}$ with $(w, s) \in g(w, s)$.
- (A3) If $g_0, g_1 \in \mathcal{G}$ are homotopic on $\partial\mathbf{W} \times \mathbf{S}$, then $\text{ind}(g_0) = \text{ind}(g_1)$.

The next lemma establishes properties of Φ_1 , properties of Φ when $p = 1$.

Lemma 2 (i) *There exists $(w, h, \pi, \theta) \in \Phi_1(w, h, \pi, \theta)$ and any such π has full support;* (ii) $\Phi_1 \in \mathcal{G}$, (iii) $\text{ind}(\Phi_1) = 1$.

Proof. As noted above, if $p = 1$, then the model is equivalent to one in which bonds are not available. (By equivalence we mean that if (w, h, π, θ) is a steady state with bonds available, then (w, π) is a steady state when they are not available. And vice versa in the sense that buying no bonds is an optimal portfolio when $p = 1$.) This is the model studied in [3] except for the lottery in meetings. For that version, parts (i) and (ii) are true (see Proposition 1 in [3]). It is straightforward to show that they also hold for a version with lotteries.

For part (iii), let $n = (\bar{z})_+$ and let $(w^*, s^*) \in \mathbf{W} - \partial\mathbf{W} \times \mathbf{S}$ with $w^*(4n, 1) > \frac{D}{2n\beta}$, where, here and below, $w(x, y) \equiv w(x) - w(x - y)$, the backward y increment in w at x . Then, let the mapping $G : \mathbf{W} \times \mathbf{S} \times [0, 1] \rightarrow \mathbf{K} \times \mathbf{S}$ be defined by

$$G(w, s, \alpha) = (1 - \alpha)(w^*, s^*) + \alpha\Phi_1(w, s). \quad (16)$$

By the above fixed-point index theorem, it suffices to show that if $(w, s, \alpha) \in \partial\mathbf{W} \times \mathbf{S} \times [0, 1]$ with $(w, s, \alpha) \in G(w, s, \alpha)$, then w is strictly increasing and strictly concave and satisfies $w(4n) > \frac{D}{\beta}$. By the definition of w^* and by parts (i) and (ii), it suffices to show this for $\alpha \in (0, 1)$. And because w^* is strictly increasing and strictly concave and because Φ_1 preserves monotonicity and concavity of w (see ([3])), it follows that w is strictly increasing and strictly concave. Thus, we have only to deal with the lower bound on w .

Now assume by contradiction that $w(4n) = \frac{D}{\beta}$. It follows that

$$w(4n, 1) < w(2n, 1) < \frac{D}{2n\beta} < w^*(4n, 1) < w^*(2n, 1). \quad (17)$$

where the first inequality follows from strict concavity of w , the second from $2nw(2n, 1) < w(2n) < \frac{D}{\beta}$, and the third and fourth from the assumption

of w^* . Because $p = 1$, we can ignore the distinction between θ and π and treat the first two arguments of f_b and f_s as amounts of money. Moreover, it follows that $f_s(m, x; w, \pi) = \beta w(x)$. Then, because (w, s, α) is a fixed point of G , it follows that

$$\begin{aligned} w(x, 1) &= (1 - \alpha)w^*(x, 1) + \alpha\left(1 - \frac{1}{N}\right)\beta w(x, 1) \\ &\quad + \alpha\frac{1}{N}\sum \pi(m)[f(x, m) - f(x - 1, m)], \end{aligned} \quad (18)$$

where we let $f(x, m) = f_b(x, m; w, \theta)$ to simplify the notation. By (17), for $x \in \{2n, 4n\}$, $w^*(x, 1) > w(x, 1)$. Therefore, for $x \in \{2n, 4n\}$, (18) implies

$$\begin{aligned} w(x, 1) &> R\sum \pi(m)[f(x, m) - f(x - 1, m)] \\ &\geq R\sum_{m=0}^{2n} \pi(m)[f(x, m) - f(x - 1, m)], \end{aligned} \quad (19)$$

where the second inequality follows from monotonicity of $f(x, m)$ in x .

Now we apply the argument used in the proof of Lemma 3 in ([3]) to derive a contradiction. Let $o(x, m) = \arg \max_{y \in [0, \min\{x, Z-m\}]} u[\beta\bar{w}(m + y, y)] + \beta\bar{w}(x - y)$, where \bar{w} is the extension of w to $[0, Z]$ defined by linear interpolation. Notice that $o(x, m)$ is a singleton and that $f(x, m) = u[\beta\bar{w}(m + y, y)] + \beta\bar{w}(x - y)$ with $y = o(x, m)$. Also, $\pi\{m : m \leq 2n\} \geq 1/2$ because $n \geq \bar{z}$. Now fix $m \leq 2n$ and let $y = o(4n - 1, m)$. If $y \geq 2n$, then, because y is a feasible offer for the buyer with $4n$,

$$f(4n, m) - f(4n - 1, m) \geq \beta w(2n, 1). \quad (20)$$

If $y < 2n$, then, because $y + 1$ is a feasible offer for the buyer with $4n$,

$$\begin{aligned} f(4n, m) - f(4n - 1, m) &\geq u[\beta\bar{w}(m + y + 1)] - u[\beta\bar{w}(m + y)] \\ &\geq u'[\beta\bar{w}(m + y + 1, y + 1)]\beta\bar{w}(m + y + 1, 1) \\ &> \beta u'(D)w(4n, 1), \end{aligned} \quad (21)$$

where the second inequality follows from the mean value theorem and concavity, and the third from $w(4n) = \frac{D}{\beta}$ and concavity. Either $w(2n, 1) \geq u'(D)w(4n, 1)$ or $w(2n, 1) < u'(D)w(4n, 1)$. If the former, then by (19), $w(4n, 1) > (R\beta/2)u'(D)w(4n, 1) > w(4n, 1)$, a contradiction. (By the definition of D , $(R\beta/2)u'(D) > 1$.) So the latter must hold. Then, (19) for

$x = 4n$ implies $w(4n, 1) > (R\beta/2)w(2n, 1)$. By exactly the reasoning used to get (21), we have $f(2n, m) - f(2n - 1, m) > \beta u'(D)w(4n, 1)$ for $m \leq 2n$. But, then, by (19) for $x = 2n$, we have

$$w(2n, 1) > (R\beta/2)u'(D)w(4n, 1) > w(2n, 1),$$

a contradiction. ■

The proofs of the next two lemmas are standard.

Lemma 3 Φ is u.h.c., compact valued, and convex valued.

Proof. The result follows from the Theorem of Maximum and the convexification by lotteries. ■

The next lemma completes the proof of proposition 2.

Lemma 4 There exists $p_0 < 1$ such that if $p \geq p_0$, then Φ_p has a fixed point $(w, s) \in \mathbf{W} - \partial\mathbf{W} \times \mathbf{S}$ where $s = (h, \pi, \theta)$ is such that π has full support.

Proof. By Lemma 3 and Lemma 2, parts (ii) and (i), there exists $p_0 < 1$ such that if $p \geq p_0$, then (i) $\Phi_p \in \mathcal{G}$ and (ii) $(w, s) \in \Phi_p(w, s)$ implies that $s = (h, \pi, \theta)$ is such that π has full support. Now we define $\Psi : \mathbf{W} \times \mathbf{S} \times [0, 1] \rightarrow \mathbf{K} \times \mathbf{S}$ by $\Psi_\alpha(w, s) = \Phi(w, s, (1 - \alpha)p_0 + \alpha)$. Because $\Phi_p \in \mathcal{G}$ for $p \geq p_0$, it follows that $\Psi_\alpha \in \mathcal{G}$ all α . Then by Lemma 2 (iii) and the fixed-point index theorem, $ind(\Psi_\alpha) = 1$ for all α . That is, $ind(\Phi_p) = 1$ for $p \geq p_0$. And then, again, by the fixed-point index theorem, Φ_p has a fixed point $(w, s) \in \mathbf{W} - \partial\mathbf{W} \times \mathbf{S}$. ■

Now we provide a proof of corollary 1.

Proof. We proceed by contradiction by assuming that the measure of people who leave the bond-buying stage with money is 0. For a person with wealth $z \geq 1$, a lower bound on leaving the bond-buying stage with 1 unit of money is $h[1, (\frac{z-1}{p})_-]$, while an upper bound on leaving with no money is $h[0, (\frac{z}{p})_+]$. In what follows, we let $z_1 = (\frac{z-1}{p})_-$ and $z_2 = (\frac{z}{p})_+$. We will show that if no one has money, then $h(1, z_1) > h(0, z_2)$ for sufficiently large z . But because the steady state distribution of wealth has full support, this implies that there exists positive measure of people leaving the bond-buying stage with money, a contradiction.

Before we proceed, recall that the value function defined on post-meeting and pre-taxing wealth is g defined in (11). Let \bar{g} and \bar{w} be the extensions of g and w to $[0, Z]$ by linear interpolation, respectively. It is easy to see that $\bar{g}(z) = \beta\bar{w}(\frac{\bar{z}}{\bar{z}_\theta}z)$. Hence \bar{g} is strictly increasing and strictly concave.

A person who starts with z may end up being a buyer, a seller, or neither in a meeting. The following hold for any $z \geq 1$. (i) If the person is neither a buyer nor a seller, then the payoff difference between having the portfolio $(1, z_1)$ and having $(0, z_2)$ is

$$\bar{g}(1+z_1) - \bar{g}(z_2) > \bar{g}\left(\frac{z-2}{p}\right) - \bar{g}\left(\frac{z+1}{p}\right) = -\bar{g}\left(\frac{z+1}{p}, \frac{3}{p}\right) \geq -\frac{\bar{g}(z+1, 3)}{p}. \quad (22)$$

Here \bar{g} is the extension of g to $[0, Z]$ by linear interpolation and $\bar{g}(x, y) \equiv \bar{g}(x) - \bar{g}(x - y)$. The inequalities follow from $p \in (0, 1)$ and monotonicity and concavity of \bar{g} . (ii) If the person is a seller, then, by the contradicting assumption, with probability 1 the buyer does not hold any money. Hence, step 1 of problem 1 is null, and in step 2 a final trade that is feasible for the seller with portfolio $(0, z_2)$ is also feasible if the seller has $(1, z_1)$. Because \bar{g} is concave, the payoff difference between having the portfolio $(1, z_1)$ and having $(0, z_2)$ is bounded below exactly as described by (22). (iii) If the person is a buyer, then the person meets either a seller with wealth (at the start of the date) that exceeds \bar{z} , or one with wealth that does not exceed \bar{z} . In the former case, a feasible choice in step-1 of problem 1 is no trade. Therefore, in the former case, the payoff difference between having the portfolio $(1, z_1)$ and having $(0, z_2)$ is bounded below exactly as described by (22). In the latter case, the buyer with a unit of money can offer it for at least $\bar{g}(\frac{\bar{z}}{p} + 1, 1)$ amount of the good, where the expression for the amount of the good follows from (4) at equality and concavity of \bar{g} . Hence, in any such meeting, the payoff of the buyer with a unit of money is bounded below by $u[\bar{g}(\frac{\bar{z}}{p} + 1, 1)] + \bar{g}(\frac{z-2}{p})$. Now, assembling these results and recalling that a person is a buyer in a meeting with a seller with wealth that does not exceed \bar{z} with probability $\frac{1}{2N}$, we have

$$\begin{aligned} h(1, z_1) - h(0, z_2) &> \frac{u[\bar{g}(\frac{\bar{z}}{p} + 1, 1)]}{2N} - \frac{\bar{g}(z+1, 3)}{p} \\ &\geq \frac{u[\bar{g}(\frac{\bar{z}}{p} + 1, 1)]}{2N} - \frac{3\bar{g}(z+1, 1)}{p}, \end{aligned} \quad (23)$$

where the second inequality follows from concavity of \bar{g} . Moreover, if $z \geq \frac{\bar{z}}{p}$, that implies, again by concavity of \bar{g} , that

$$h(1, z_1) - h(0, z_2) > \frac{u[\bar{g}(z+1, 1)]}{2N} - \frac{3\bar{g}(z+1, 1)}{p}. \quad (24)$$

Provided $u'(0)$ is sufficiently large, there exists a unique $q^* > 0$ such that $\frac{u(q^*)}{2N} = \frac{3q^*}{p}$ and $\frac{u(q)}{2N} > \frac{3q}{p}$ for $q \in (0, q^*)$. Because we are free to make Z sufficiently large independent of \bar{z} and because the upper bound on w (and, hence, on \bar{g}) is independent of Z , all sufficiently large z satisfy $z > \frac{\bar{z}}{p}$ and $\bar{g}(z+1, 1) \in (0, q^*)$. Hence, for all such z , the right-hand side of (24) is positive, a contradiction. ■

References

- [1] M. A. Krasnoel'skiĭ and P. P. Zabreĭko, *Geometrical Methods of Nonlinear Analysis*, Springer-Verlag, New York, 1984.
- [2] E. Zeidler, *Nonlinear Functional Analysis and its Applications I: Fixed-Point Theorems*, Springer-Verlag, New York, 1985.
- [3] T. Zhu, Existence of a monetary steady state in a matching model: indivisible money, *J. of Econ. Theory* **112** (2003), 307-324.
- [4] T. Zhu and N. Wallace, Pairwise trade and coexistence of money and higher-return assets. Manuscript, 2005.