A Version of Lucas's Signal Extraction Model

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1 Introduction

Here I present a simplified version of the model for which R.E. Lucas, Jr. was awarded the Nobel Prize in 1995. The part we will concentrate on is the explanation provided by the model of a positive correlation between economic activity and the growth rate of the money supply, a correlation which is paradoxical. The model can account for such a correlation while implying that the correlation cannot be used to reliably predict what would happen under different growth rates for the money supply. In that respect, the model is a particular example of correlation-does-not-imply-causation.

The main idea behind the model is the following. In an economy of the sort we studied earlier where money is used, people about to acquire money do not care about what money is worth today: they care about what it will be worth in the future relative to what it is worth today. In particular, those acquiring money at date t (or holding it from date t to date $t+1$) care about the real rate-of-return on money which depends on its price at t, which they observe, and on its price at $t+1$, which may be uncertain or random. In some settings, it is sensible for people to use the price at t to draw inferences about the price at $t+1$. This will be the case if the price at t reflects several underlying random processes which are not directly observed and which also influence the price at $t+1$. Such forecasting may give rise to correlations between actions that people take and some of those processes which may seem paradoxical. Moreover, the replacement of one underlying random process by another, which is one way to think of a change in policy, may change the way the current price is best used for forecasting and, may, thereby, give rise to different correlations. A consequence is that one such correlation does not predict what happens if one policy is replaced by another. This will be deomonstrated via exercises which describe how to compute the consequences of a policy.

2 The environment.

The original contribution used an overlapping generations (OLG) model of two-date lived people. I will exposit it against the background of the model used in section II—a model of infinitely-lived people with preferences given by discounted-utility, with one good per date, and with the special alternating pattern of endowments. Because we will here be concerned with the descriptive implications of the model, the distinction between that version and the OLG version is not important. In other respects, I will present the simplest possible version of the model.

The model has two underlying sources of uncertainty, two kinds of shocks. One shock will be labeled an *aggregate supply* (of goods) shock. The other is labeled an *aggregate demand* (for goods) shock.

The aggregate-supply uncertainty.

In the background model of section II, there are $2N$ people, each of whom has an endowment stream of goods that alternates between y and 0. Half are endowed with y at odd dates and half at even dates. Now we assume that the actual number of people with an endowment of y is a random subset of those who have an endowment in the model of section II. The number of them is denoted N_t at date t. We assume that

$$N_t = \begin{cases} N^{(1)} \text{ with probability } 1/2 \\ N^{(2)} \text{ with probability } 1/2 \end{cases} \tag{1}$$

where $N^{(1)} \leq N^{(2)} \leq N$. For reasons to be explained later, we assume that $N^{(1)}$ and $N^{(2)}$ are not too different. In particular, we assume that $\beta \frac{N^{(2)}}{N^{(1)}} < 1$, where β, as above, is the discount factor in preferences.

What happens to the $N - N_t$ who do not have an endowment at t? (This question does not arise in the OLG version of the model. There, N_t can be taken to be the size of generation t.) We assume that these $N - N_t$ people are *drop-outs* for two dates: a drop-out at date t does not get an endowment at t and does not value consumption at either t or $t + 1$. In other words, a drop-out gets sleeping sickness for two dates. It is okay to suppose that each person looks ahead and supposes that at every date at which they would ordinarily have an endowment of y, there is a probability that they become a drop-out for two dates. Because they know they will have an endowment when they next become a non drop-out, they do not have to make special plans for drop-out dates. (If they dropped out for one date, then they would be concerned about how they would get consumption during the date subsequent to the drop-out date.)

The aggregate-demand uncertainty.

The aggregate demand shock is the percentage change in the amount of money. We assume that

$$M_t = (1 + \theta_t) M_{t-1}, \tag{2}$$

where M_t is the total amount of money that people carry over from t to $t+1$ and where

$$\theta_t = \begin{cases} \theta^{(1)} \text{ with probability } 1/2 \\ \theta^{(2)} \text{ with probability } 1/2 \end{cases}, \tag{3}$$

where $\theta^{(2)} \geq \theta^{(1)} \geq 0$. We assume that newly created money is used to finance government expenditures.

As suggested by (1) and (3), the realizations for N_t and θ_t are independent of each other and over time.

Exercise 1 *What is the probability of the following event: $\theta_t = \theta^{(2)}$ and $N_t = N^{(1)}$? What is the probability of the following event: $N_t = N^{(2)}$ and $N_{t+1} = N^{(2)}$?*

An information Lag.

A crucial assumption is that there is incomplete information about realizations of the shocks. Although the date t realizations of N_t and θ_t occur before people trade at date t, the assumption is that people at t do not see those realizations. They do see all earlier realizations and the starting money supply M_0. It is as if there is a one-period lag in gathering data. People do know the random processes as described by (1) and (3).

Gross-substitutes preferences.

We also make an additional assumption about the utility function u. The function u is such that a two-date lived person with utility function $u(c_1)+\beta u(c_2)$ and who could borrow or lend at the real rate r would choose choose a smaller c_1 the higher is r. (That is, *the substitution effect* of a higher r dominates *the income effect*.) One class of examples satisfying this assumption is $u(x) = x^\alpha$ for any α strictly between 0 and 1. This gross-substitutes assumption in our context is equivalent to assuming that the function $xu'(x)$ is strictly increasing. (Recall that u' denotes the derivative of the function u.)

Exercise 2 *Let $f(x) = xu'(x)$ and let $u(x) = \sqrt{x}$. (i)Give an explicit expression for $f(x)$. (ii) Is $f(x)$ strictly increasing? (iii) Use excel to plot $f(x)$ for $0 \leq x \leq 2$.*

Measured GDP.

Finally, we want to reinterpret the endowment y as production and want to call the person with such an endowment at t (a non drop-out) a producer at t.

However, we do not want to count all of y as part of measured GDP. Descriptions of how GDP is measured often note that if you and your neighbor mow your own lawns, then that activity does not show up in GDP, but that if you and your neighbor hire each other to mow each other's lawn, then the mowing becomes part of GDP. We will make use of this feature of measured economic activity. We will say that only the part of y that is sold (for money) counts toward measured GDP. There is an equivalent alternative interpretation in terms of leisure, which only relies on leisure not being counted toward measured economic activity. The alternative is that y is an endowment of effort which is split between leisure and productive effort (which gives rise to production of the date t consumption good), and that people who are endowed at t care only about leisure at t and consumption at $t+1$.

Before turning to an analysis of the model, it will be helpful to have first studied another version of a Robinson-Crusoe economy.

3 Crusoe with a random technology

We here consider another one-person world. Time is discrete and is indexed by $t = 1, 2, ...$ This person maximizes expected discounted utility over an infinite horizon. There is one good per date and the person's endowment alternates between odd and even dates: $w_t = y > 0$ if t is odd and $w_t = 0$ if t is even. (Because there is only one person, I am dropping the subscript that identifies the person.) The person also has access to the following inter-temporal technology: if k_t denotes date t input into the technology, then the date $t+1$ output is $R_{t+1}k_t$, where $R_{t+1} = R^{(i)}$ with probability π_i. We assume that there are I possible returns, $R^{(1)}, R^{(2)}, ..., R^{(I)}$, and that for each i, $0 < R^{(i)} < \frac{1}{\beta}$. Note that when Crusoe chooses k_t, he faces uncertainty. He knows the distribution of R_{t+1}, but does not know the realization. He will learn it at date $t+1$ before he makes a decision about his consumption at $t+1$ and k_{t+1}.

We will construct a guess about the solution to Crusoe's problem. The guess is based on the conjecture that Crusoe will place into storage the same amount at each date when $w_t = y$ and will not store anything at dates when $w_t = 0$. Therefore, let's consider the following problem.

Problem 3 *Choose $k \geq 0$ to maximize*

$$u(y-k)+\beta[\pi_1 u(R^{(1)}k)+\pi_2 u(R^{(2)}k)+...+\pi_I u(R^{(I)}k)] \quad = \quad u(y-k)+\beta\sum_{i=1}^{I}\pi_i u(R^{(i)}k) \quad \equiv \quad g(k). \tag{4}$$

The function g is strictly concave. Therefore, there is a unique best choice for k. Moreover, if $u'(0)$ is sufficiently large, then the best choice is the solution to the equation $g'(k) = 0$. Using simple rules for differentiation,

$$g'(k) = -u'(y-k)+\beta\sum_{i=1}^{I}\pi_i R^{(i)}u'(R^{(i)}k). \tag{5}$$

Therefore, the equation $g'(k) = 0$ is equivalent to

$$\frac{u'(y-k)}{\beta} = \sum_{i=1}^{I}\pi_i R^{(i)}u'(R^{(i)}k). \tag{6}$$

Multiplying both side of (6) by k, we can rewrite it as

$$\frac{u'(y-k)k}{\beta} = \sum_{i=1}^{I}\pi_i f(R^{(i)}k) \tag{7}$$

where

$$f(x) \equiv xu'(x).$$

Exercise 4 *Let $u(x) = \sqrt{x}, y = 1, \beta = .5,$ $I = 2$ and $\pi_i = \frac{1}{2}$. (i) Use Excel to plot $g(k)$ for $0 \leq k \leq 1$ if $R^{(1)} = 1.0$ and $R^{(2)} = 1.5$. (ii) Use Excel to plot $g(k)$ for $0 \leq k \leq 1$ if $R^{(1)} = .5$ and $R^{(2)} = 1.0$.*

Exercise 5 *Let $u(x) = \sqrt{x}$. For this case, solve (7) for k.*

In general, let k^* denote the solution to (7). We want to argue that $k_t = k^*$ when t is odd and $k_t = 0$ when t is even is the solution to Crusoe's problem. Although the randomness makes Crusoe's objective complicated, it continues to be true that the objective can be expressed in terms of the sequence of storage decisions and that the objective is strictly concave in those decisions. Therefore, it is valid to consider small deviations from our candidate solution one variable at a time. Moreover, it continues to be true that the storage decision for date t appears in only two terms of the objective, that which pertains to date t and that which pertains to date $t+1$. Given our candidate, there are only two such decisions that have to be checked, one which applies when t is odd and another which applies when t is even.

When t is odd, we take as given that the storage decisions at $t-1$ and $t+1$ are those given by the candidate—namely, 0—and we ask whether the sum of the two terms of the objective in which k_t appears can be made larger by departing locally from $k_t = k^*$. Because the sum of those two terms is nothing but $\beta^{t-1}g(k)$, no such departure makes that sum larger.

When t is even, we take as given that the storage decisions at $t-1$ and $t+1$ are those given by the candidate—namely, k^*—and we ask whether the sum of the two terms of the objective in which k_t appears can be made larger by departing locally from $k_t = 0$. The sum of those two terms, expressed in terms of k_t, is proportional to

$$u(R^{(j)}k^* - k_t) + \beta \sum_{i=1}^{I} \pi_i u(y - k^* + R^{(i)}k_t) \equiv h(k_t). \tag{8}$$

Because it is enough to check for local departures, it is enough to show that the function h is declining at $k_t = 0$. That is, it is enough to compute the derivative of the function h, evaluate it at $k_t = 0$, and show that it is negative. The derivative of the function h, denoted h', is

$$h'(k_t) = -u'(R^{(j)}k^* - k_t) + \beta \sum_{i=1}^{I} \pi_i R^{(i)} u'(y - k^* + R^{(i)}k_t). \tag{9}$$

Exercise 6 *Let $u(x) = \sqrt{x}$. (i) For this case, give an exression for $h'(0)$. (ii) Substitute into your part (i) expression the formula for k^* that you found in the previous exercise. (iii) Is $h'(0) < 0$? Explain.*

Exercise 7 *Show that $h'(0) < 0$. (To do this, evaluate $h'(k_t)$ at $k_t = 0$ while making use of the fact that k^* satisfies (7). To reach the conclusion, you must also use the gross-substitutes assumption, the assumption that $f(x) \equiv xu'(x)$ is strictly increasing.)*

This completes what we want to say about this version of a Robinson-Crusoe problem. What we have done will turn out to be useful later. In what follows, those who are producers at a date will face a version of Crusoe's problem, and their choice of an amount of production to devote to purchasing money will play the role of k in Crusoe's problem. Therefore, that amount will satisfy a version of (7). However, in what follows the $R^{(i)}$'s are not exogenous. Instead, as is usual in a model of many interacting people, the $R^{(i)}$'s depend on the actions that people take. Even so, equation (7) holds because each person makes a choice taking the return distribution as a given—the usual price-taking assumption of competitive behavior.

4 Supply shocks only

Here we look at the special case in which $\theta^{(1)} = \theta^{(2)} = \theta$. We also assume, as we will throughout, that $N^{(1)} < N^{(2)}$. We again proceed using a guess and verify procedure.

Consider the N_t producers at date t. We conjecture that each does the same thing. In particular, let x_t denote the amount of production that each of them sells for money at date t. The simplest conjecture that we could make about x_t is that it is constant. That, however, is too simple. Our conjecture is that there are two possible magnitudes of x_t, one of which occurs when $N_t = N^{(1)}$ and the other when $N_t = N^{(2)}$. And, as is usual, it is helpful to adopt symbols for our unknowns. That is, we conjecture that

$$x_t = \begin{cases} x^{(1)} \text{ whenever } N_t = N^{(1)} \\ x^{(2)} \text{ whenever } N_t = N^{(2)} \end{cases}. \tag{10}$$

Of course, this does not tell us the magnitudes of $x^{(1)}$ and $x^{(2)}$. At this point, you may also wonder how a producer at date t could produce different amounts depending on the magnitude of N_t given that N_t is not observed then. We will answer that later when we will see that the price of money at t, again denoted v_t, reveals N_t.

We also conjecture, as we have been doing for related models, that the producers at t do not start with any money and that they end up buying all of it. If so, then

$$N_t x_t = v_t M_t. \tag{11}$$

We treat (10) and (11) as conjectured features of an equilibrium. Our next step is to complete the conjecture.

We first describe the return distributions implied by (10) and (11). Using the expression for v_{t+1} implied by (11), we have

$$\frac{v_{t+1}}{v_t} = \frac{N_{t+1}x_{t+1}/M_{t+1}}{v_t} \tag{12}$$

If (10) and (11) are correct, then a producer at date t who sees the price v_t faces a rate-of-return distribution on money that has two possible outcomes, the two possible outcomes for the numerator in (12). There are only two because M_{t+1} is known—$M_{t+1} = (1+\theta)^{t+1}M_0$—and because (10) implies that there are only two possible outcomes for the product $N_{t+1}x_{t+1}$. That is, $N_{t+1}x_{t+1} = N^{(1)}x^{(1)}$ or $N_{t+1}x_{t+1} = N^{(2)}x^{(2)}$.

Equation (12) takes us part of the way toward expressing the rate of return distribution that each person faces in terms of the choices that people make. To complete doing that, we use (11) again to replace v_t. We then get

$$\frac{v_{t+1}}{v_t} = \frac{N_{t+1}x_{t+1}}{N_t x_t(1+\theta)}. \tag{13}$$

This and (10) imply that there are four different possible rates of return, two possibilities when $N_t = N^{(1)}$ and a (possibly) different two when $N_t = N^{(2)}$.

In particular, when $N_t = N^{(1)}$, a producer faces a return distribution with two equally likely outcomes. They are

$$\frac{N^{(1)}x^{(1)}}{N^{(1)}x^{(1)}(1+\theta)} \text{ and } \frac{N^{(2)}x^{(2)}}{N^{(1)}x^{(1)}(1+\theta)}. \tag{14}$$

Our conjecture is that $x^{(1)}$ is the best choice for spending on money when the return distribution has two equally likely outcomes given by (14). Using what we learned from our Robinson Crusoe analysis, our conjecture is that $x^{(1)}$ satisfies (7) with the following substitutions: replace k by $x^{(1)}$, set $I = 2$, $\pi_1 = \pi_2 = \frac{1}{2}$ (two equally likely outcomes), and replace $R^{(1)}$ by the first expression in (14) and $R^{(2)}$ by the second expression in (14). The result can be written

$$\frac{2u'(y - x^{(1)})x^{(1)}}{\beta} = f(\frac{N^{(1)}x^{(1)}}{N^{(1)}(1+\theta)}) + f(\frac{N^{(2)}x^{(2)}}{N^{(1)}(1+\theta)}). \tag{15}$$

This is one equation in two unknowns, $x^{(1)}$ and $x^{(2)}$. We get a second equation in those same two unknowns by considering what our conjectures imply when $N_t = N^{(2)}$.

When $N_t = N^{(2)}$, a producer also faces a return distribution with two equally likely outcomes. They are

$$\frac{N^{(1)}x^{(1)}}{N^{(2)}x^{(2)}(1+\theta)} \text{ and } \frac{N^{(2)}x^{(2)}}{N^{(2)}x^{(2)}(1+\theta)}. \tag{16}$$

Our conjecture is that $x^{(2)}$ is the best choice for spending on money when the return distribution has two equally likely outcomes given by (16). That is, our conjecture is that $x^{(2)}$ satisfies (7) with the following substitutions: replace k by $x^{(2)}$, set $I = 2$, $\pi_1 = \pi_2 = \frac{1}{2}$ (two equally likely outcomes), and replace $R^{(1)}$ by the first expression in (16) and $R^{(2)}$ by the second expression in (16). The result can be written

$$\frac{2u'(y - x^{(2)})x^{(2)}}{\beta} = f(\frac{N^{(1)}x^{(1)}}{N^{(2)}(1+\theta)}) + f(\frac{N^{(2)}x^{(2)}}{N^{(2)}(1+\theta)}). \tag{17}$$

Equations (15) and (17) are two simultaneous equations in two unknowns, $x^{(1)}$ and $x^{(2)}$. The following is true about positive solutions to these equations.

Proposition 8 *Equations (15) and (17) have a unique positive solution that satisfies (i)* $x^{(1)} > x^{(2)}$ *and (ii)* $N^{(1)}x^{(1)} < N^{(2)}x^{(2)}$. *Moreover, the solution is such that (iii)* $N^{(i)}x^{(i)}$ *is decreasing in* θ.

Proving existence of a positive solution to equations (15) and (17) is straightforward, but requires some advanced mathematics. Uniqueness and the other properties are established using the approach taken to prove analogous properties for the version below with both shocks.

Exercise 9 *Use inequality (ii) in proposition 8 and (14) and (16) to show that a more favorable rate of return distribution occurs when* $N_t = N^{(1)}$ *than when* $N_t = N^{(2)}$.

Exercise 10 *Use inequality (ii) in proposition 8 and (11) to show that a lower* v_t *occurs when* $N_t = N^{(1)}$ *than when* $N_t = N^{(2)}$.

The conclusion in the last exercise confirms that people can deduce N_t from their observation of v_t in an equilibrium that satisfies the conclusions of proposition 8.

There is an additional step to confirming that the solution to equations (15) and (17) is an equilibrium. That additional step involves showing that those who have acquired money at t, as described by the solution to equations (15) and (17), want to spend it all at $t+1$. This is implied if each rate of return is less than $\frac{1}{\beta}$.

It is enough to consider the more favorable rate-of-return distribution, the one that occurs when $N_t = N^{(1)}$. There are two possible outcomes, depending on the outcome for N_{t+1}. If $N_{t+1} = N^{(1)}$, then the rate-of-return is $\frac{1}{1+\theta} \leq 1$. If $N_{t+1} = N^{(2)}$, then the rate of return is

$$\frac{N^{(2)}x^{(2)}}{(1+\theta)N^{(1)}x^{(1)}} \leq \frac{N^{(2)}}{(1+\theta)N^{(1)}} \leq \frac{1}{\beta}, \tag{18}$$

where the first inequality follows from $x^{(1)} \geq x^{(2)}$ and the second from our assumptions that $\beta\frac{N^{(2)}}{N^{(1)}} < 1$ and $\theta \geq 0$. By our discussion of the Robinson Crusoe economy, these conclusions imply that each person who acquires money at date t wants to spend all of it at date $t+1$.

Exercise 11 *Offer an explanation of conclusion (iii) in Proposition 8.*

We now use Excel to compute some solutions for specific examples.

In the examples, we let $u(x) = x^{1/2}$. It follows that the derivative function is $u'(x) = \frac{1}{2}x^{-1/2}$. That, in turn, implies that $f(x) \equiv xu'(x) = \frac{1}{2}x^{1/2}$. Therefore, if $u(x) = x^{1/2}$, then (15) takes the form

$$\frac{2x^{(1)}}{\beta(y - x^{(1)})^{\frac{1}{2}}} = [\frac{N^{(1)}x^{(1)}}{N^{(1)}(1+\theta)}]^{\frac{1}{2}} + [\frac{N^{(2)}x^{(2)}}{N^{(1)}(1+\theta)}]^{\frac{1}{2}}, \tag{19}$$

while (17) takes the form

$$\frac{2x^{(2)}}{\beta(y - x^{(2)})^{\frac{1}{2}}} = [\frac{N^{(1)}x^{(1)}}{N^{(2)}(1+\theta)}]^{\frac{1}{2}} + [\frac{N^{(2)}x^{(2)}}{N^{(2)}(1+\theta)}]^{\frac{1}{2}}. \tag{20}$$

Equations (19) and (20) are two non-linear simultaneous equations in two unknowns, $x^{(1)}$ and $x^{(2)}$. Although I will not prove this, these equations are such that a simple iterative scheme can be used to obtain a solution. Below, I describe how to apply this scheme using Excel. Before doing that, I describe the nature of the iterative scheme. To do this, it is helpful to represent equations (19) and (20) using a more compact notation. Thus, we represent equations (19) and (20) as

$$h(x^{(1)}) = g_1(x^{(1)}, x^{(2)}) \text{ and } h(x^{(2)}) = g_2(x^{(1)}, x^{(2)}), \tag{21}$$

respectively. Because of the appearance of the square root on the left-hand sides of equations (19) and (20), it is convenient to square both sides of the equations and write them as

$$[h(x^{(1)})]^2 = [g_1(x^{(1)}, x^{(2)})]^2 \text{ and } [h(x^{(2)})]^2 = [g_2(x^{(1)}, x^{(2)})]^2 \tag{22}$$

The iterative scheme starts with a not-too-crazy guess for $(x^{(1)}, x^{(2)})$. For example, $(x^{(1)}, x^{(2)}) = (\frac{y}{2}, \frac{y}{2})$ is such a guess, but any values strictly between 0 and y would be okay. Step 1: Evaluate the right-hand sides of the equations in (22) at the initial guess. That is, compute $[g_1(\frac{y}{2}, \frac{y}{2})]^2$ and $[g_1(\frac{y}{2}, \frac{y}{2})]^2$ and solve the two equations, $[h(x^{(1)})]^2 = [g_1(\frac{y}{2}, \frac{y}{2})]^2$ and $[h(x^{(2)})]^2 = [g_2(\frac{y}{2}, \frac{y}{2})]^2$. Each equation has one unknown. In fact, each is a quadratic equation, and, therefore, is easy to solve. Were it to turn out that the solution to these equations is the same as our initial guess, then we would have found a solution to (19) and (20). If not, then we take the solution as our new initial guess and repeat the process. We continue until the new initial guess and the solution coincide. When that happens, we have a solution to (19) and (20).

Here is a detailed description of one way to use Excel to find solutions to equations (19) and (20) for given numerical values of $y, \beta, N^{(1)}, N^{(2)}$, and θ.

Each row will have 8 columns. (In Excel, columns are designated by capital letters and rows by numbers so that, for example, $B1$ stands for the entry in the second column of row 1.) I will describe the entries one row at a time.

Row 1.

A1 $= \frac{y}{2}$; B1 $= \frac{y}{2}$;

(Comment: These are the initial guesses for $x^{(1)}$ and $x^{(2)}$.)

C1 = the right-hand side of (19) when $x^{(1)} =$ A1 and $x^{(2)} =$ B1;

D1 = the right-hand side of (20) when $x^{(1)} =$ A1 and $x^{(2)} =$ B1;

(Comment: columns 3 and 4 evaluate the right-hand sides of (19) and (20), respectively, at the initial guess.)

E1 = (C1)^2; F1 = (D1)^2;

(Comment: These columns compute the squares of the right-hand sides of (19) and (20), respectively, at the initial guesses.)

$\text{G1} = \frac{-\text{E1} + [(\text{E1})^2 + (16)\text{yE1}/\beta^2]^{\frac{1}{2}}}{8/\beta^2}$

$\text{H1} = \frac{-\text{F1} + [(\text{F1})^2 + (16)\text{yF1}/\beta^2]^{\frac{1}{2}}}{8/\beta^2}$.

(Comment: G1 is the solution to the quadratic equation that results from squaring the left-hand side of (19) and equating it the square of the right-hand side of (19), when the latter is evaluated at the initial guess. H1 does the same for (20).)

Row 2.

A2 = G1; B2 = H1;

(Comment: These replace the initial guesses by the solution to the quadratics.)

C2=C1, D2=D1,...,H2=H1

(Comment: These repeat the row 1 calculations except for new initial guesses.)

Row 3 = Row 2, ..., Row k = Row 2.

(Comment: These again repeat the row 1 calculations except for new initial guesses.)

Stop when two rows are identical. Then the entries in the first two columns are a solution. This should happen once you have about 10 rows.

Exercise 12 *(i) Let* $y = 1, \beta = .95, N^{(1)} = 10^6, N^{(2)} = 1.05N^{(1)}$, *and* $\theta = 0$. *Use the above scheme to find a positive solution to (19) and (20). (ii) Let* $y = 1, \beta = .95, N^{(1)} = 10^6, N^{(2)} = 1.05N^{(1)}$, *and* $\theta = .05$. *Use the above scheme to find a positive solution to (19) and (20). (iii) Verify that your solutions satisfy conditions (i)-(iii) of proposition 8.*

Exercise 13 *Consider two distinct economies, economy 1 and economy 2, where the former has the parameters of part (i) of the last exercise and the latter has* *those of part (ii). In accord with our discussion above, $N_t x_t$ is our measure of GDP. What is average GDP in economy 1? What is it in economy 2? Is the higher average GDP for economy 1 a consequence of proposition 8? Explain.*

We will use the numerical solutions to these exercises later.

5 Both shocks

We now return to the complete model with both kinds of shocks present. We do, however, make a special assumption about the possible outcomes of the two kinds of shocks, namely

$$\frac{N^{(2)}}{N^{(1)}} = \frac{1+\theta^{(2)}}{1+\theta^{(1)}}. \tag{23}$$

Although this assumption seems very special, it is needed only because we are working with discrete distributions. Therefore, you should not be too concerned about it. Its role is to insure that an observation of the price v_t does not necessarily reveal N_t and θ_t. (If N_t and θ_t could each take on any values in an interval, the case studied by Lucas in his original contribution, then an observation of the price v_t would quite generally not reveal N_t and θ_t.)

It is convenient to introduce a symbol for the ratio, $\frac{1+\theta_t}{N_t}$. We call it z_t. Notice that (23) implies that there are only 3 possible outcomes for z_t. We denote them as follows,

$$z^{(1)} = \frac{1+\theta^{(1)}}{N^{(2)}},\ z^{(2)} = \frac{1+\theta^{(1)}}{N^{(1)}} = \frac{1+\theta^{(2)}}{N^{(2)}},\ \text{and } z^{(3)} = \frac{1+\theta^{(2)}}{N^{(1)}}. \tag{24}$$

Exercise 14 *(i) What is the probability that $z_t = z^{(1)}$? What is the probability that $z_t = z^{(2)}$? (ii) Suppose someone told you that $z_t = z^{(3)}$. What could you say about N_t and θ_t? Suppose someone told you that $z_t = z^{(2)}$. What could you say about N_t and θ_t?*

We again proceed using a guess and verify procedure. Consider the N_t producers at date t. We again conjecture that each does the same thing and let x_t denote the amount of production that each of them sells for money at date t. And, we again conjecture that the producers at t do not start with any money and that they end up buying all of it. If so, then, as above,

$$N_t x_t = v_t M_t. \tag{25}$$

Because we assume that M_{t-1} is known at t, it is helpful to express M_t in terms of M_{t-1} and θ_t. Then (25) can be written as

$$\frac{x_t}{z_t} = v_t M_{t-1}. \tag{26}$$

Our information assumption is that people at t know M_{t-1} and observe v_t. *If* x_t depends only on z_t, then (26) implies that conditional on M_{t-1}, v_t depends only on z_t. We pursue the following two-part conjecture: (i) x_t depends only on z_t and (ii) the dependence of x_t on z_t is such that knowledge of the right-hand side of (26) implies knowledge of z_t. We express the conjectured dependence of x_t on z_t using the following notation,

$$x_t = x^{(i)} \text{ whenever } z_t = z^{(i)} \text{ for } i = 1, 2, 3. \tag{27}$$

Next, we examine the rate-of-return distributions on money implied by the conjecture. Using the expression for v_{t+1} implied by (26), we have

$$\frac{v_{t+1}}{v_t} = \frac{x_{t+1}/z_{t+1}}{v_t M_t} \tag{28}$$

or

$$\frac{v_{t+1}}{v_t} = \frac{x_{t+1}/z_{t+1}}{v_t(1+\theta_t)M_{t-1}} \tag{29}$$

If the conjecture is correct, then there are three distinct distributions of $\frac{v_{t+1}}{v_t}$. We describe those distributions using our conjecture that observing v_t allows people to deduce z_t.

Suppose first that $z_t = z^{(1)}$. Then there is only one combination of N_t and θ_t consistent with $z_t = z^{(1)}$: in particular, it must be that $\theta_t = \theta^{(1)}$. And because M_{t-1} is known by assumption, it follows that the denominator on the right-hand side of (29) is known. Therefore, randomness in this case comes only from the numerator on the right-hand side of (29). According to the conjecture, see (27), the numerator takes on 3 possible values according to what z_{t+1} turns out to be.

Next, suppose that $z_t = z^{(3)}$. Again, there is only one combination of N_t and θ_t consistent with $z_t = z^{(3)}$: in particular, it must be that $\theta_t = \theta^{(2)}$. This also implies that the denominator on the right-hand side of (29) is known. Therefore, randomness in this case also comes only from the numerator on the right-hand side of (29), which can take on 3 possible values according to what z_{t+1} turns out to be.

What if $z_t = z^{(2)}$? Now the denominator on the right-hand side of (29) is not known. In particular, given our assumptions, the observation that $z_t = z^{(2)}$ is not informative about the distribution of θ_t : it remains the case that that distribution is described by (3). Moreover, because θ_t and z_{t+1} are independent of each other, there are 6 possible outcomes for $\frac{v_{t+1}}{v_t}$ when $z_t = z^{(2)}$.

To further describe these distributions, we substitute into the denominator on the right-hand side of (29) the expression for $v_t M_{t-1}$ from (26). The result is

$$\frac{v_{t+1}}{v_t} = \frac{x_{t+1}/z_{t+1}}{(1+\theta_t)(x_t/z_t)} = \frac{x_{t+1}/z_{t+1}}{N_t x_t} \tag{30}$$

Using the last expression and (27), we can describe the three rate-of-return distributions in terms of the unknowns $x^{(1)}, x^{(2)}$, and $x^{(3)}$.

$$\text{If } z_t = z^{(1)}, \text{ then } \frac{v_{t+1}}{v_t} = \begin{cases} \frac{x^{(1)}/z^{(1)}}{N^{(2)}x^{(1)}} \text{ with prob } 1/4 \\ \frac{x^{(2)}/z^{(2)}}{N^{(2)}x^{(1)}} \text{ with prob } 1/2 \\ \frac{x^{(3)}/z^{(3)}}{N^{(2)}x^{(1)}} \text{ with prob } 1/4 \end{cases}. \tag{31}$$

$$\text{If } z_t = z^{(2)}, \text{ then } \frac{v_{t+1}}{v_t} = \begin{cases} \frac{x^{(1)}/z^{(1)}}{N^{(2)}x^{(2)}} \text{ with prob } 1/8 \\ \frac{x^{(2)}/z^{(2)}}{N^{(2)}x^{(2)}} \text{ with prob } 1/4 \\ \frac{x^{(3)}/z^{(3)}}{N^{(2)}x^{(2)}} \text{ with prob } 1/8 \\ \frac{x^{(1)}/z^{(1)}}{N^{(1)}x^{(2)}} \text{ with prob } 1/8 \\ \frac{x^{(2)}/z^{(2)}}{N^{(1)}x^{(2)}} \text{ with prob } 1/4 \\ \frac{x^{(3)}/z^{(3)}}{N^{(1)}x^{(2)}} \text{ with prob } 1/8 \end{cases}. \tag{32}$$

And

$$\text{If } z_t = z^{(3)}, \text{ then } \frac{v_{t+1}}{v_t} = \begin{cases} \frac{x^{(1)}/z^{(1)}}{N^{(1)}x^{(3)}} \text{ with prob } 1/4 \\ \frac{x^{(2)}/z^{(2)}}{N^{(1)}x^{(3)}} \text{ with prob } 1/2 \\ \frac{x^{(3)}/z^{(3)}}{N^{(1)}x^{(3)}} \text{ with prob } 1/4 \end{cases}. \tag{33}$$

Although not a necessary step at this stage, it may be helpful to express some further conjectures about the 3 return distributions in (31)-(33). When there are only supply shocks, the return distribution when $N_t = N^{(1)}$ is more favorable than when $N_t = N^{(2)}$. Even when both shocks are present, the only aspect of the current realization of the shocks that matters for the return distribution is N_t. (That is because θ_{t+1} is independent of θ_t.) Therefore, comparing the first and third distributions, we expect that the distribution when $z_t = z^{(3)}$ will be more favorable than that when $z_t = z^{(1)}$. And we expect that the distribution when $z_t = z^{(2)}$ will be between the other two. Thus, we should not be surprised to find that $x^{(3)} > x^{(2)} > x^{(1)}$.

Our next task is to express the condition that $x^{(i)}$ is the producer's best choice for x_t when $z_t = z^{(i)}$. As in the case of supply shocks only, this condition is expressed by making the appropriate substitutions in equation (7).

Our conjecture is that $x^{(1)}$ is the best choice for spending on money when the return distribution is given by (31). That is, our conjecture is that $x^{(1)}$ satisfies (7) when we replace k by $x^{(1)}$ and use the distribution in (31) for the π_i and $R^{(i)}$. The result is,

$$\frac{u'(y-x^{(1)})x^{(1)}}{\beta} = \frac{1}{4}f(\frac{x^{(1)}/z^{(1)}}{N^{(2)}}) + \frac{1}{2}f(\frac{x^{(2)}/z^{(2)}}{N^{(2)}}) + \frac{1}{4}f(\frac{x^{(3)}/z^{(3)}}{N^{(2)}}) \quad (34)$$

Our conjecture is that $x^{(2)}$ is the best choice for spending on money when the return distribution is given by (32). That is, our conjecture is that $x^{(2)}$ satisfies (7) when we replace k by $x^{(2)}$ and use the distribution in (32) for the π_i and $R^{(i)}$. The result is,

$$\begin{aligned}\frac{u'(y-x^{(2)})x^{(2)}}{\beta} &= \frac{1}{8}f(\frac{x^{(1)}/z^{(1)}}{N^{(2)}}) + \frac{1}{4}f(\frac{x^{(2)}/z^{(2)}}{N^{(2)}}) + \frac{1}{8}f(\frac{x^{(3)}/z^{(3)}}{N^{(2)}}) + \\ &\quad \frac{1}{8}f(\frac{x^{(1)}/z^{(1)}}{N^{(1)}}) + \frac{1}{4}f(\frac{x^{(2)}/z^{(2)}}{N^{(1)}}) + \frac{1}{8}f(\frac{x^{(3)}/z^{(3)}}{N^{(1)}}) \quad (35)\end{aligned}$$

Our conjecture is that $x^{(3)}$ is the best choice for spending on money when the return distribution is given by (33). That is, our conjecture is that $x^{(3)}$ satisfies (7) when we replace k by $x^{(3)}$ and use the distribution in (33) for the π_i and $R^{(i)}$. The result is,

$$\frac{u'(y-x^{(3)})x^{(3)}}{\beta} = \frac{1}{4}f(\frac{x^{(1)}/z^{(1)}}{N^{(1)}}) + \frac{1}{2}f(\frac{x^{(2)}/z^{(2)}}{N^{(1)}}) + \frac{1}{4}f(\frac{x^{(3)}/z^{(3)}}{N^{(1)}}) \quad (36)$$

Equations (34)-(36) are 3 simultaneous equations in 3 unknowns: $x^{(1)}$, $x^{(2)}$, and $x^{(3)}$. The following is true about solutions to those equations.

Proposition 15 *Equations (34)-(36) have one and only one positive solution. That solution satisfies (i)* $x^{(3)} > x^{(2)} > x^{(1)}$ *and (ii)* $x^{(3)}/z^{(3)} < x^{(2)}/z^{(2)} < x^{(1)}/z^{(1)}$.

Proving existence of a positive solution requires some advanced mathematics and, therefore, will not be proved here. Nor will we prove uniqueness of a positive solution, although that is not hard. In the appendix, we prove that such a solution satisfies (i) and (ii). Inequality (ii) in proposition 15 and (26) imply that for a given M_{t-1}, which is known at date t, there are 3 possible values v_t and that the lowest one occurs when $z_t = z^{(3)}$, the next lowest when $z_t = z^{(2)}$, and the highest when $z_t = z^{(1)}$. This confirms our conjecture that observing M_{t-1} and v_t allows people to determine which of the 3 possible values of z_t has occurred.

Next, we want to indicate why people who acquire money at t want to spend it all at $t+1$. As in the last subsection, we do this by showing that each possible rate of return is less than $\frac{1}{\beta}$.

Exercise 16 *Consider the* 3 *return distributions in (31)-(33). For each distribution, proposition 15 implies that there is a highest realization. What is it? Show that proposition 15 implies that the highest realization is less than* $\frac{1}{\beta}$.

Next, we use the inequalities in 15 to describe the model's implication for the correlation between the total output and the growth rate of the quantity of money. In particular, we will look at the distribution of the pairs $(1+\theta_t, N_t x_t)$ in a time series implied by this equilibrium. Only 4 distinct pairs would occur.

Exercise 17 *They are* $(1+\theta^{(1)}, N^{(2)}x^{(1)})$, $(1+\theta^{(1)}, N^{(1)}x^{(2)})$, $(1+\theta^{(2)}, -)$, $(1+\theta^{(2)}, -)$. *Fill in the blanks. What is the frequency with which each of these pairs occurs?*

By the definition of the $z^{(i)}$,

$$x^{(2)}/z^{(2)} = \frac{N^{(1)}x^{(2)}}{1+\theta^{(1)}} \text{ and } x^{(1)}/z^{(1)} = \frac{N^{(2)}x^{(1)}}{1+\theta^{(1)}}. \tag{37}$$

Exercise 18 *Use inequality (ii) in 15 and (37) to show that* $N^{(1)}x^{(2)} < N^{(2)}x^{(1)}$.

Exercise 19 *(a) With* $1+\theta_t$ *on the horizontal axis and* $N_t x_t$ *on the vertical axis, sketch, using the appropriate labels, the scatter diagram of pairs* $(1+\theta_t, N_t x_t)$ *that is implied by the equilibrium. (Make use of the inequality obtained in the last exercise and inequality (i) in 15.) (b) Consider a least square regression of* $N_t x_t$ *on* $1+\theta_t$ *for data implied the equilibrium. What would the sign of the regression coefficient on* $1+\theta_t$*? Explain.*

Our last task is to use Excel to compute some solutions for specific examples. We again let $u(x) = x^{1/2}$. In this case, (34) takes the form

$$\frac{x^{(1)}}{\beta(y-x^{(1)})^{\frac{1}{2}}} = \frac{1}{4}[\frac{x^{(1)}/z^{(1)}}{N^{(2)}}]^{\frac{1}{2}} + \frac{1}{2}[\frac{x^{(2)}/z^{(2)}}{N^{(2)}}]^{\frac{1}{2}} + \frac{1}{4}[\frac{x^{(3)}/z^{(3)}}{N^{(2)}}]^{\frac{1}{2}}, \tag{38}$$

(35) takes the form

$$\frac{x^{(2)}}{\beta(y-x^{(2)})^{\frac{1}{2}}} = \frac{1}{8}[\frac{x^{(1)}/z^{(1)}}{N^{(2)}}]^{\frac{1}{2}} + \frac{1}{4}[\frac{x^{(2)}/z^{(2)}}{N^{(2)}}]^{\frac{1}{2}} + \frac{1}{8}[\frac{x^{(3)}/z^{(3)}}{N^{(2)}}]^{\frac{1}{2}} + \frac{1}{8}[\frac{x^{(1)}/z^{(1)}}{N^{(1)}}]^{\frac{1}{2}} + \frac{1}{4}[\frac{x^{(2)}/z^{(2)}}{N^{(1)}}]^{\frac{1}{2}} + \frac{1}{8}[\frac{x^{(3)}/z^{(3)}}{N^{(1)}}]^{\frac{1}{2}}, \quad (39)$$

and (36) takes the form

$$\frac{x^{(3)}}{\beta(y-x^{(3)})^{\frac{1}{2}}} = \frac{1}{4}[\frac{x^{(1)}/z^{(1)}}{N^{(1)}}]^{\frac{1}{2}} + \frac{1}{2}[\frac{x^{(2)}/z^{(2)}}{N^{(1)}}]^{\frac{1}{2}} + \frac{1}{4}[\frac{x^{(3)}/z^{(3)}}{N^{(1)}}]^{\frac{1}{2}}. \quad (40)$$

We will use the same kind of iterative scheme we used in the last section. The main difference is that we now have 3 equations and 3 unknowns. Here is a way to use Excel to find solutions to equations (38)-(40) for given numerical values of $y, \beta, N^{(1)}, N^{(2)}, \theta^{(1)}$,and $\theta^{(2)}$.

Each row will have 15 columns.

Row 1.

A1= an initial guess for $x^{(1)}$, B1= an initial guess for $x^{(2)}$, C1= an initial guess for $x^{(3)}$.

D1=A1/$10^6 z^{(1)}$, E1=B1/$10^6 z^{(2)}$, F1=C1/$10^6 z^{(3)}$; (Comment: these evaluate $x^{(i)}/z^{(i)}$ for $i = 1, 2, 3$ for the initial guesses with some scaling to keep the numbers in a reasonable range.)

G1 = [.25(D1)$^{1/2}$+ .5(E1)$^{1/2}$+ .25(F1)$^{1/2}$]/$(N^{(2)}/10^6)^{1/2}$; (Comment: This evaluates the right-hand side of (38).)

H1 = [.25(D1)$^{1/2}$+ .5(E1)$^{1/2}$+ .25(F1)$^{1/2}$]/$(N^{(1)}/10^6)^{1/2}$; (Comment: This evaluates the right-hand side of (40).)

I1 = .5G1+.5H1 (Comment: This evaluates the right-hand side of (39) by taking advantage of the fact that the right-hand side of (39) is the average of the right-hand sides of (38) and (40).)

J1 = (G1)^2, K1 = (I1)^2, L1 = (H1)^2 (Comment: This squares the right-hand sides of (38)-(40), respectively.)

M1 = $\frac{-\text{J1} + [(\text{J1})^2 + (4y\text{J1}/\beta^2)]^{\frac{1}{2}}}{2/\beta^2}$

N1 = $\frac{-\text{K1} + [(\text{K1})^2 + (4y\text{K1}/\beta^2)]^{\frac{1}{2}}}{2/\beta^2}$.

O1 = $\frac{-\text{L1} + [(\text{L1})^2 + (4y\text{L1}/\beta^2)]^{\frac{1}{2}}}{2/\beta^2}$

(Comment: These are the solutions to the quadratic equation which results from squaring both side of each of equations (38)-(40) while evaluating the right-hand sides at the initial guess.)

Row 2.

A2 = M1, B2 = N1, C2 = O1; (Comment: These set new initial values at the previous solution.)

The other entries in row 2 are copied from row 1.

Row 3 = Row 2, ..., *Row k = Row 2*. (Comment: The other rows are copies of row 2.)

Stop when two rows are identical. Then the entries in the first three columns are a solution. This should happen once you have about 10 rows.

Exercise 20 *(i) Let* $y = 1, \beta = .95, N^{(1)} = 10^6, N^{(2)} = 1.05N^{(1)}$, $\theta^{(1)} = 0$; *and* $\theta^{(2)} = 0.05$. *Use the above scheme to find a positive solution to (38)-(40). (ii) Let* $y = 1, \beta = .95, N^{(1)} = 10^6, N^{(2)} = 1.05N^{(1)}$, $\theta^{(1)} = 0.05$, *and* $(1 + \theta^{(2)}) = 1.05(1+\theta^{(1)})$. *Use the above scheme to find a positive solution to (38)-(40). (iii) Verify that your solutions satisfy conditions (i)-(ii) of proposition 15.*

Exercise 21 *Call the economy described in part (i) of exercise 20 economy 3. Suppose that time series data on two variables are collected for this economy: one variable is* $\frac{M_t}{M_{t-1}} = (1+\theta_t)$; *the other variable is* $N_t x_t$, *our measure of GDP. (i) With* $\frac{M_t}{M_{t-1}}$ *measured on the horizontal axis and* $\frac{N_t x_t}{10^6}$ *measured on the vertical axis, graph carefully all the different* $(\frac{M_t}{M_{t-1}}, \frac{N_t x_t}{10^6})$ *pairs that you would observe. (ii) Label each pair with the fraction of the observations equal to that pair that you would expect to see in a very long time series. (iii) Corresponding to each observed value of* $\frac{M_t}{M_{t-1}}$ *is an average value of* $\frac{N_t x_t}{10^6}$, *denoted* $X(\frac{M_t}{M_{t-1}})$. *Plot the pairs* $(\frac{M_t}{M_{t-1}}, X(\frac{M_t}{M_{t-1}}))$. *There should be two such pairs. Connect them by a line. This line is the least squares regression line in a least squares regression of* $\frac{N_t x_t}{10^6}$ *on* $\frac{M_t}{M_{t-1}}$. *I will call this line regression-line* 3 *because it comes from economy* 3.

Exercise 22 *Call the economy described in part (ii) of exercise 20 economy 4. Repeat for economy 4 everything you did for economy 3 in exercise 21. (It is best to put the results for both economies on the same graph. However, you should use different symbols for the two economies—for example, x's for one economy and dots for the other.)*

Exercise 23 *Still using the same graph, plot the* $1 + \theta$ *and average GDP pairs for economies 1 and 2 of exercise 13. (In doing this, divide the average GDP by* 10^6.*)*

Exercise 24 *(i) How would you explain the relative qualitative positions of regression lines 3 and 4? (ii) Can you explain the relative qualitative positions of the pairs for economies 1 and 2 of exercise 13?*

These examples illustrate the nature of Lucas's contribution. Although the model is very special, some of its implications are quite general. The exercises show that the relationship between GDP and the growth rate of the money that turns up for economy 3 cannot be extrapolated from to accurately predict what happens in economies 1, 2, and 4. That is an instance of correlation-does-not-imply-causation.

Usually, examples of correlation-does-not-imply-causation are explained by missing underlying causes. For example, suppose it was found that grades are positively correlated with smoking across college students. Would we expect that if we somehow encouraged more smoking, then that would be accompanied by higher grades? Not likely. The reason is that both smoking and grades are caused by other things. In Lucas's model, the time series for the economy depends on the *distribution* of the money-supply process. That is, both the realizations of the money supply for different dates and those for measured GDP are caused by the *distribution* of the money supply. Put somewhat differently, according to the model, the outcomes for different dates for a single economy are not separate experiments. That feature of the model is almost perfectly general. If a model is inter-temporal in the sense that actions at one date have future consequences, then outcomes for different dates implied by such a model should not be viewed as separate experiments.

6 Appendix

Here we prove a part of proposition 2; namely, that a positive solution to equations (34)-(36) satisfies (i) $x^{(3)} > x^{(2)} > x^{(1)}$ and (ii) $x^{(3)}/z^{(3)} < x^{(2)}/z^{(2)} < x^{(1)}/z^{(1)}$.

Proof. Let $g(x) \equiv u'(y-x)x$ and let

$$H(N) \equiv \frac{1}{4}f(\frac{x^{(1)}/z^{(1)}}{N}) + \frac{1}{2}f(\frac{x^{(2)}/z^{(2)}}{N}) + \frac{1}{4}f(\frac{x^{(3)}/z^{(3)}}{N}), \tag{41}$$

where, as above $f(x) \equiv xu'(x)$ and where, by assumption, $f' > 0$. Then, (34)-(36) can be written, respectively, as

$$g(x^{(1)})/\beta = H(N^{(2)}), \tag{42}$$

$$g(x^{(2)})/\beta = (H(N^{(1)}) + H(N^{(2)}))/2, \tag{43}$$

and

$$g(x^{(3)})/\beta = H(N^{(1)}). \tag{44}$$

Because the function f is strictly increasing and $N^{(2)} > N^{(1)}$, it follows that $H(N^{(1)}) > H(N^{(2)})$. And because the function g is strictly increasing, it follows that any positive solution satisfies (i).

Inequality (ii) requires more work. First, (42) and (43) imply

$$1 - \frac{g(x^{(1)})}{g(x^{(2)})} = \frac{H(N^{(1)}) - H(N^{(2)})}{H(N^{(1)}) + H(N^{(2)})} < 1 - \frac{H(N^{(2)})}{H(N^{(1)})}, \tag{45}$$

while (43) and (44) imply

$$1 - \frac{g(x^{(2)})}{g(x^{(3)})} = \frac{H(N^{(1)}) - H(N^{(2)})}{2H(N^{(1)})} < 1 - \frac{H(N^{(2)})}{H(N^{(1)})}. \tag{46}$$

Now, because $d\ln g(x)/d\ln x > 1$, $\ln[g(x^{(i+1)})/g(x^{(i)})] > \ln[x^{(i+1)}/x^{(i)}]$. Therefore,

$$1 - \frac{x^{(i)}}{x^{(i+1)}} < 1 - \frac{g(x^{(i)})}{g(x^{(i+1)})}. \tag{47}$$

Also,

$$-\frac{d\ln H(N)}{d\ln N} = \frac{1}{H}[\frac{1}{4}h(\frac{x^{(1)}/z^{(1)}}{N}) + \frac{1}{2}h(\frac{x^{(2)}/z^{(2)}}{N}) + \frac{1}{4}h(\frac{x^{(3)}/z^{(3)}}{N})$$

where $h(x) \equiv xf'(x) = x^2u''(x) + xu'(x) < xu'(x) = f(x)$. Therefore, $-d\ln H(N)/d\ln N \in (0,1)$. It follows that $-\ln[H(N^{(2)})/H(N^{(1)})] < \ln[N^{(2)}/N^{(1)}] = \ln[z^{(i+1)}/z^{(i)}]$. Hence,

$$1 - \frac{H(N^{(2)})}{H(N^{(1)})} < 1 - \frac{z^{(i)}}{z^{(i+1)}}. \tag{48}$$

Then, by (45)-(48), we have

$$1 - \frac{x^{(i)}}{x^{(i+1)}} < 1 - \frac{z^{(i)}}{z^{(i+1)}}, \tag{49}$$

which is equivalent to conclusion (ii). ■