Uniquely Representing
‘A Preference for Uniformity’

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Abstract

In a model of decision making over sets of alternatives, we consider an agent who conceives of the different utilities she will receive (depending on the state of mind she is in, her *subjective state*) when she finally makes a choice from the set. We are interested in agents who have a preference for uniformity of payoffs across the various subjective states. We present a utility representation for such agents and show that our representation provides a natural measure of the agent's desire for uniformity of payoffs in the subjective states. We also show that concerns for uniformity are orthogonal to concerns about flexibility or commitment. We achieve this by relaxing the Independence axiom as applied to our environment. However, this weakening of Independence comes at the cost of our having to give up the notion of a single, long-run self and the concomitant cost interpretations of self control that have found use in the literature.

§ 1. Motivation

An agent has to choose between two sets of alternatives in the morning, knowing that she will make a choice from the delineated set of alternatives in the afternoon. What she is unsure about is how she will feel, and consequently, how she will make a final choice in the afternoon. Consider, for instance, the agent who has to choose a menu (say, from a restaurant) for lunch. There are only three menus (from three restaurants) available. Suppose the first restaurant offers only chicken (c) and the second only fish (f) and that she is indifferent between the two restaurants. Also suppose there is a third restaurant that offers either chicken or fish

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daily, but never both on the same day (so that each is offered with equal probability). We are interested in agents who strictly prefer the third restaurant to the first two.

An intuitive explanation of such preferences is the following: The agent considers possible two psychological states in the afternoon, which will govern her decision. Each psychological state corresponds to a different preference over lunch items. In the first state she strictly prefers chicken and in the second she strictly prefers fish. In the morning, when she has to make a choice of restaurant, she is unsure what state of mind she will be in at lunchtime. A desire for uniformity across psychological states means that she will hedge her bets about how she will feel at lunch time and therefore choose the third restaurant. (This example is treated more extensively in Example 1.1 below.) We shall say that the agent's preferences correspond to her having a *preference for uniformity* in terms of payoffs she receives at lunch time in the restaurant.

In this paper, we provide an axiomatic foundation and a utility representation for such preferences. Our main axiom, *Preference for Uniformity — PfU*, says that for menus A and B where $A \sim B$, it is the case that $\frac{1}{2}A + \frac{1}{2}B \succcurlyeq A$, thereby reflecting the agent's desire for uniformity in payoffs across subjective states. Her concern for uniformity in ex post payoffs is orthogonal to any concerns she might have about flexibility or commitment. For instance, suppose she is indifferent between two menus A and B. Preferences where $A \cup B \succcurlyeq A \sim B$ indicates a desire for flexibility. On the other hand, $A \sim B \succcurlyeq A \cup B$ represents a preference for commitment on her part, due to self-control issues she might have. In either case, it is not clear (and not particularly relevant) how she ranks $\frac{1}{2}A + \frac{1}{2}B$. On the other hand, if she has a preference for uniformity, then it is the case that $\frac{1}{2}A + \frac{1}{2}B \succcurlyeq A$, with her ranking of $A \cup B$ not being relevant to her desire for uniformity. Thus, a preference for flexibility/commitment and a preference for uniformity address distinct concerns of the agent.

Put differently, we claim that preferences that are linear over menus rule out many interesting phenomena. As Dekel, Lipman and Rustichini (2001) state (p 897), on models based on the subjective states of a decision maker, "… applications of the model seem to require some measure of the agent's aversion to uncertainty regarding future contingencies, which would presumably be based on the size of [the subjective states considered possible], and loosely, on the variance of [payoffs] across states." *Preference for Uniformity* captures such concerns.

Subjective state models have found great use in the literature in the modelling of temptation and related phenomenon. Typically, such models assume that preferences are linear over the space of menus. Forefront in this literature is the seminal paper of Gul and Pesendorfer (2001), who show that when preferences are linear, welfare analysis is straightforward – we only need consider the welfare of the individual, as evinced in her preferences over singletons. We discuss the model of Gul and Pesendorfer (2001) in greater detail in §1.3. and in §3.2, we show that their conclusion on welfare analysis depends heavily on the assumption that preferences over menus are linear.

There is yet another way to interpret the agent's hedging motives described above. Epstein, Marinacci and Seo (2007) interpret such hedging as evidence that the agent only has a coarse perception about how she will feel in the afternoon, ie she has a coarse perception about her future psychological states of mind. We discuss this paper more fully in §1.3. We now look at the utility representation (over menus) in a little more detail and briefly describe our main findings.

§ 1.1. *Examples and a Utility Representation*

Since we allow for choices of lotteries, we shall say that the choice in the afternoon is made according to a von Neumann-Morgentern utility function. It is useful to denote the set of all possible ex post (afternoon) vN-M utility functions by S_Z, so that a subjective, psychological state is any $s \in S_Z$. (As is standard, we identify vN-M utility functions up to positive affine transformations.)

Our representation takes the form

$$V(A) := \min\left\{ \int_{S_Z} \max_{p \in A} u(p;s)\, d\pi(s) : \pi \in \Pi \right\} \tag{$\spadesuit$}$$

where Π is a weak* compact, convex set of (finite, Borel) signed measures on S_Z. (Clearly, this means that S_Z has a topology — we shall present this in § 2.) We shall refer to this as a *multi-weight representation*. The intuition underlying the representation is simple. Faced with a menu, the agent considers some states likely to occur in the afternoon. She knows how much utility we will get in each state and the measure π weighs the various ex post utilities, aggregating them into an afternoon (ex ante) value. The representation captures the hedging motivation ascribed to the agent, since the agent evaluates according to the lower envelope of a set of linear functionals (as each π is a linear functional on the set of ex post utilities). In what follows, we shall refer to $S := \bigcup_{\pi \in \Pi} \operatorname{supp} \pi$ as the *subjective state space* (where $\operatorname{supp} \pi$ denotes the support of the signed measure π), so that in the representation above, we can take S to be the domain of integration instead of S_Z. We shall now see the representation in action in some simple examples.

Example 1.1 (Extreme Preference for Uniformity). Suppose the agent only considers the lowest utility she can get in some subset of states, $S \subset S_Z$, for each set of alternatives that she has. In other words,

$$V(A) := \min\left\{ \int_S u_A(s)\, d\delta_t(s) \right\}$$

where δ_t is the Dirac measure concentrated at $t \in S_Z$ and $u_A(s) := \max_{p \in A} u(p;s)$ for each $s \in S_Z$. Such an extreme preference for uniformity in payoffs results in extremely pessimistic behaviour.

To see that this rationalises the example given above, the subjective state space be $S := \{s_1, s_2\}$ and suppose utilities in the subjective states are as follows:

	$u(\cdot, s_1)$	$u(\cdot, s_2)$
c	4	0
f	0	4

Let $\pi_1 = (1,0)$, $\pi_2 = (0,1)$ and $\Pi := \{\pi_1, \pi_2\}$. It is easy to see that $V\left(\frac{1}{2}f + \frac{1}{2}c\right) = 2 > 0 = V(\{f\}) = V(\{c\})$, which rationalises our preferences. ◊

As a second example, consider an agent who suffers from temptation, but also has a preference for uniformity.

Example 1.2. Suppose there are four dishes: x_1 represents a healthy salad, y_1 represents the salad with a rich dressing, x_2 represents a healthy sandwich and y_2 represents the sandwich with a side order of mayonnaise and chips. Suppose x_i is tempted by y_i (ie, $\{x_i\} \succ \{x_i, y_i\} \succcurlyeq \{y_i\}$) and not y_{3-i} for $i = 1, 2$ and also suppose that the agent is indifferent between $A := \{x_1, y_1\}$ and $B := \{x_2, y_2\}$. Then, it is reasonable to expect that

$$\frac{1}{2}A + \frac{1}{2}B \succ A \sim B.$$

As above, a preference for uniformity justifies such preferences.[1] We shall demonstrate this by explicitly constructing a utility function that rationalises these preferences.

Let us suppose $S := \{s_1, s_2\}$ and suppose utilities are as given below.

	x_1	y_1	x_2	y_2
$u(\cdot, s_1)$	3	4	2	1
$u(\cdot, s_2)$	2	1	3	4

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[1] A similar example can be found in Olszewski (2008).

Now, let $\pi_1 := (2, -1)$ and $\pi_2 = (-1, 2)$ so that $\Pi := \operatorname{conv}\{\pi_1, \pi_2\} \subset \mathbb{R}^2$ (where "conv A" denotes the convex hull of the set A). Then,

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\begin{aligned}
V(A) &:= \min\left\{ \int_S u_A(s)\, d\pi(s) \right\} \\
&= \min\left\{ V_1(A), V_2(A) \right\}
\end{aligned}
$$

where $V_i(A) := \langle \pi_i, u_A \rangle$, $u_A = \big(u_A(s)\big)_{s \in S}$ and $\langle \pi, u_A \rangle$ is the standard inner product.[2] Then, $V(A) = V_2(A) = 0 = V_1(B) = V(B)$. Also, $V\big(\tfrac{1}{2}A + \tfrac{1}{2}B\big) = 3 > 0 = V(A) = V(B)$, which rationalises the preferences. It is easily seen that y_{3-i} does not tempt x_i. ◊

§ 1.2. *Nature of our Contributions*

Our first main contribution, Theorem 2.2, is to provide axioms on preferences that completely characterise multi-weight utility representations of the form (♠). At a conceptual level, the representation shows how agents' desire for uniformity in ex post payoffs can be represented and the examples above show that this desire for uniformity can result in some interesting behaviour in choice over menus. At a technical level, our representation significantly weakens the Independence axiom, while still delivering a tractable representation. The multi-weight representation (♠) is sufficiently weak that we do not require the utility functional to be linear over singletons.

The set of signed measures Π that we obtain is unique, facilitating a comparison of agents with bigger Π's (in terms of set inclusion). This leads to our second contribution, where we show how a preference for uniformity can be quantified. The set of weights (or, more generally, signed measures on S_Z) given by Π provides a natural measure of the agent's desire for uniformity in ex post payoffs, in the sense that a larger Π (in terms of set inclusion) means the agent has a greater desire for uniformity. This is the content of Theorem 2.6.

Finally, our methodological contribution, where we discuss the issue of welfare considerations in our environment of preferences over menus (ie, sets of alternatives). It is important to emphasise here that our in our model, the only observable is the choice *of* menu. The choice *from* the menu, in the afternoon, is not observed. The environment lends itself to many, internally consistent, interpretations about second period behaviour, a prime example of which is provided in Gul and Pesendorfer (2001) (and is discussed in §3.2). The interpretation of second period behaviour is facilitated by the particular utility representation and,

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[2] Notice that V_i is continuous, linear and satisfies Set Betweenness (which means that $V_i(A) \geqslant V_i(B)$ implies $V_i(A) \geqslant V_i(A \cup B) \geqslant V_i(B)$; see also §1.3) for $i = 1, 2$. Nevertheless, V as defined does not satisfy Set Betweenness. This follows from Theorems 3.4 and 3.11.

as in Gul and Pesendorfer (2001) and other papers, that interpretation itself is enabled by the Independence axiom. One of our contributions is to clarify the rôle that Independence plays in enabling these interpretations.[3]

In particular, Gul and Pesendorfer (2001) claim that in choice problems where there is no desire for flexibility, by looking at preferences over menus, we keep the link between choice and welfare intact, and that the agent's preferences over singletons tells us what she would have chosen if she did not have problems of self-control. We show that this conclusion relies heavily on the Independence axiom and therefore that the conclusion and the underlying (methodological) point of view is not robust to perturbations of the underlying axioms.

The reason for this is that our axioms are sufficiently weak that we do not require our multi-weight representation to be linear over singletons. Instead, the utility of a (singleton) lottery is given as the minimum of a collection of vN-M utility functions over the space of lotteries. Clearly, this precludes the idea of a single, coherent normative preference represented by a vN-M utility function, that represents the ideal choice of the agent in the absence of temptation.[4]

We now proceed as follows. We provide a view of the literature in §1.3 and introduce the model and the main representation theorem in §2. We discuss notions of comparative preference for uniformity in §2.4. In §3.1, we discuss how a sign can be attributed to a state and show that welfare changes resulting from local changes to menus depend on the sign of the state. We also provide a limited characterisation of the sign of a state. In §3.2, we consider some additional axioms considered in the literature on temptation, study the more refined representations that result and conclude in §4. All proofs not found in the text are in §5.

§ 1.3. *Related Literature*

That preferences over planning problems can be used to infer how an agent thinks she will behave was first pointed out by Kreps (1979). The construction of a *unique* subjective state space was carried out by Dekel, Lipman and Rustichini (2001) (henceforth DLR) by considering menus of lotteries. The subjective state space constructed by DLR is an improvement over

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[3] Our interpretation of behaviour in the afternoon continues to hold as long as preferences are continuous and the agent is indifferent between a menu and its convex hull, see Theorem 1.A in Dekel, Lipman and Rustichini (2001). For a brief description of this result, see §1.3.

[4] It may, of course, be argued that preferences over singleton menus, ie lotteries, should be linear. But this is not a methodological argument, merely a normative statement.

Kreps' for the following reason. Subjective states are important only if ascribing behavioural properties to these states translates into some behavioural properties of the preference over menus. If the state space is not unique, then different subjective states will translate into different (possibly conflicting) behavioural interpretations of preferences over menus, which diminishes the value of the theory. We now briefly describe DLR's *weak EU representation*.

DLR's weak EU representation consists of a collection of vN-M utility functions S over lotteries, and a continuous function $U : \mathbb{R}^S \to \mathbb{R}$ such that the utility of a menu A is

$$V(A) := U\left(\left(\max_{p \in A} u(p,s)\right)_{s \in S}\right).$$

The set S is referred to as the subjective state space and, as mentioned above, is shown to be unique (up to closure). The function U is called an *aggregator* since it takes a collection of ex post utilities and transforms it into an ex ante utility over a menu. Notice that there are no structural requirements on U other than continuity, so that a large class of behaviour can be captured. In particular, our multi-weight representation (♠) is also an example of a weak EU representation. Theorem 1.A of DLR shows that (only) continuous preferences over the space of menus of lotteries that have the property that a menu is indifferent to its convex hull have a weak EU representation.

The preferences considered by DLR are also not required to be monotonic (in the sense that $A \cup B \succcurlyeq A$). Preferences that are not monotonic form an important class and have been used to model, most notably, temptation by Gul and Pesendorfer (2001). However, this and subsequent papers, for instance Dekel, Lipman and Rustichini (2007) and Chatterjee and Krishna (2007), make heavy use of the Independence axiom. A study of temptation related problems while relaxing Independence is carried out by Noor (2006), whose environment is the same as ours. Olszewski (2008) considers a model of menu-dependent temptation, but looks at menus of deterministic prizes. In their (non-axiomatic) model of self control, Fudenberg and Levine (2006) argue that Independence is not an appropriate assumption for temptation and related phenomena.

From a formal viewpoint, our multi-weight representation is the subjective state equivalent of the multi-prior model of Gilboa and Schmeidler (1989). (Nevertheless, while the representations are similar, our proof is substantially different.) An excellent textbook presentation of Gilboa-Scmeidler's multi-prior model is in Ok (2007), who presents a finite dimensional version of the multi-prior model. A paper that looks at self control problems in the Gilboa-Schmeidler framework is the paper by Epstein and Kopylov (2007). They draw on the aforementioned multi-prior model (and a variant of Set Betweenness, cf §3.2). In their environment, a menu is a collection of Anscombe-Aummann acts with a finite objective state space.

The paper that is closest to ours is the the paper by Epstein, Marinacci and Seo (2007). Their motivation is the study of agents who are not sure about the contingencies that may arise. Their environment is the same as ours and their representation is also very similar to ours. In particular, they find the value of a menu to be

$$W_{\text{ems}}(A) = \min\left\{ \int_{S_Z} \max_{p\in A} u(p,s)\; d\pi(s) : \pi \in \Pi_{\text{ems}} \right\}$$

where Π_{ems} is a weak* compact, convex set of probability measures on S_Z. While this representation looks like our (♠) multi-weight representation, there are some subtle qualifications to the above representation. The first is that since they assume preferences are monotone, each π is a probability measure. We drop this assumption.

They also assume the existence of a prize z_* and define $N := \{s \in S_Z : u(z,s) \geqslant u(z_*,s) \text{ for all } z \in Z\}$, requiring that for each $\pi \in \Pi_{\text{ems}}$, $\operatorname{supp}\pi \subset N$. Put another way, the set of subjective states that are possible are restricted to those where one (predefined) prize is always the worst. In principle, there is nothing wrong with assuming that there is a prize that is always worst in every subjective state. For instance, consider a prize z_*, where the agent loses a very large sum of money. It is plausible that such a prize would always be the worst in every state of mind the agent can conceive of. Nevertheless, it is not clear why such an assumption is necessary to obtain the representation. Moreover, the set Π_{ems} is not unique, although there is a smallest such set (in terms of set inclusion) that satisfies the above.

Key to the EMS representation theorem is the assumption that the payoffs to the (worst) prize z_* and the set of lotteries Δ are certain, ie they are the same in each subjective state that the agent considers relevant (ie possible). Similarly, we assume the existence of two menus $\{q^*\}$ (where q^* is a singleton menu) and a menu D with a smooth boundary that contains q^*, that give the agent the same payoffs in every subjective state. (More precisely, we define the subjective state space so as to have this property.) This enables us to obtain a representation that is unique (up to normalisation of the subjective state space).[5]

In sum, we make three improvements on the paper of Epstein, Marinacci and Seo (2007). Firstly, we drop the assumption that preferences are monotonic. Second, we drop the require-

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[5] Actually, more is true. If we take q^* and D to be any two arbitrary menus, we can still obtain a representation of preferences of the form (♠), with the proviso that the set of (signed) measures, while weak* compact, is not unique. It is the assumption that q^* is in the relative interior of D that enables us to provide a unique representation. See also Remark 2.4.

ment that there exists a prize z_* that is the worst in every conceivable subjective state. Finally, we show that the representation obtains as long as there are two menus, the payoffs to which are certain, ie constant in all subjective states. Uniqueness of the representation obtains when the two sets satisfy additional conditions, as in our axiom D-Independence below. This is useful as it allows us to ascribe behavioural meaning to increasing sets of measures.

We end by noting (again) that the interpretation of a set of lotteries in this paper (and the ones cited above) is that a choice will be made from the set of lotteries at a later point in time. This is just an interpretation since second period choice is not observed. There is a literature that views a set of lotteries as representing ambiguity on the part of the agent, about what the "true" distribution over prizes is. Different aspects of this problem have been considered by Ahn (2007), Olszewski (2007) and Stinchcombe (2007). For more on issues regarding the interpretation of the model, see §4.

§ 2. Model and Results

§ 2.1. *Environment and Notation*

Let Z denote a finite set of prizes where the cardinality of Z is $n+1$. We shall require that $n>1$. The space of probability measures on Z is denoted by Δ and $\mathcal{F}(\Delta)$ denotes the set of closed subsets of Δ. When endowed with the Hausdorff metric, $\mathcal{F}(\Delta)$ becomes a compact metric space. A *menu* is a closed subset of Δ. Preferences over menus are complete, transitive, binary relations over $\mathcal{F}(\Delta)$, represented by $\succcurlyeq \subset \mathcal{F}(\Delta)\times\mathcal{F}(\Delta)$. We shall denote the set of all compact, convex subsets of Δ by $\mathcal{K}(\Delta)$.

Let $\{p^*\} = (1/n+1, \ldots, 1/n+1)$ denote the uniform probability measure over the prizes and $D_r := \{q \in \operatorname{aff}(\Delta) : \|q-p^*\| \leqslant r\}$ the closed disk of radius r around the uniform measure, where $\operatorname{aff}(\Delta)$ is the affine hull of Δ. Consider a convex menu $D \in \mathcal{K}(\Delta)$. We shall say that D has a *smooth boundary* if the boundary has no flats, ie each boundary point is supported by a unique hyperplane (relative to the affine hull of Δ). Notice that the set D_r has a smooth boundary according to this definition.

§ 2.2. *Axioms*

We now proceed to the axioms that lie at the heart of our model. Our theory shall require the existence of two distinguished sets; a singleton q^* with full support (so that $q^* \in \operatorname{ri}\Delta$, where "ri" refers to the relative interior), and a set $D \subset \operatorname{ri}\Delta$ where $D \supset \{q^*\}$ and D has a smooth boundary.

Axiom (Non-triviality*). There exist $D \subset \operatorname{ri}\Delta$ and $q^* \in D$ such that $D \nsim \{q^*\}$ and D has a smooth boundary.

This axiom says that preferences are non-trivial in a particular way. We note that Non-triviality* encompasses two assumptions: the first is the existence of two sets that are valued differently, and the second is that they are nested in terms of set inclusion. We can relax this second assumption, but at the cost of a weaker representation theorem, see remark 2.4. We should also mention that weakening the axiom to requiring that there exist two menus that are not indifferent but allowing $D \sim \{q^*\}$ is a case we are not able to handle. Hence this assumption has behavioural content. The next axiom is the standard continuity axiom.

Axiom (Continuity). $\succcurlyeq$ is continuous in the Hausdorff topology.

Axiom (L Continuity). There exists $N > 0$ such that for all $\varepsilon \in (0, 1/N)$, for all A, B with $d_h(A, B) \leqslant \varepsilon$,

$$(1 - N\varepsilon)A + N\varepsilon A^* \succcurlyeq (1 - N\varepsilon)B + N\varepsilon A_*$$

where $A^*, A_* \in \left\{D, \{q^*\}\right\}$ and $A^* \succ A_*$.

We need the axiom L Continuity above, because Continuity by itself is not sufficient to give us the desired representation (even in the presence of other axioms). This axiom was first introduced in Dekel et al (2007), although our version is stronger in the sense that Dekel et al (2007) allow A^* and A_* to be arbitrary compact subsets of Δ. As in that paper, assuming $D \succ \{q^*\}$, one way to think about L Continuity is as follows: For each A and B with $B \succ A$, there exists a greatest $\lambda \in (0, 1)$ such that $\lambda A + (1 - \lambda)D \succcurlyeq \lambda B + (1 - \lambda)\{q^*\}$. As $d_h(A, B) \to 0$, $\lambda \to 0$. L Continuity says that $\lambda \to 0$ smoothly. We now introduce an axiom that is a very weak version of Independence.[6]

Axiom (IR: Indifference to Randomisations[7]). $A \sim \text{conv}(A)$ for all $A \in \mathcal{F}(\Delta)$.

In line with our interpretation of a menu as representing a set of alternatives from which a choice will be made in the future, we provide a justification of this axiom. Suppose the agent knows that in the afternoon, she will make a choice according to one of a given set of vN-M utility functions (which, taken together, represent her subjective state space), then the (ex post) utility in each state remains unchanged if we replace a menu by its convex hull. Thus, the agent is indifferent between a menu and its convex hull.

We now introduce the behavioural axiom that captures a preference for uniformity. To reiterate, the intuition behind our axiom is that the agent prefers greater uniformity in the ex post payoffs he receives. In other words, he is averse to variation in payoffs in ex post payoffs.

Axiom (Preference for Uniformity — PfU). $A \sim B$ implies $(1/2)A + (1/2)B \succcurlyeq B$.

An extreme form of such behaviour is captured in example 1.1, where although the agent (weakly) prefers more alternatives to less, she always focuses on the lowest utility she might

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[6] This axiom was introduced by DLR. Dekel et al (2007) show that for a preference that satisfies vN-M continuity, Independence implies IR.

[7] We should mention that IR rules out the following example, (taken from Dekel, Lipman and Rustichini, 2007). Suppose the agent is has to make a choice between a healthy dish and an unhealthy dish. To assuage feelings of guilt, she may prefer to add a randomisation of the two dishes to the menu. This is clearly a violation of IR.

receive across all the subjective states she considers possible. It is useful to describe the converse axiom, describing the behaviour of an agent who likes extreme payoffs, perhaps because he is able to focus on the best payoff.

Axiom (Preference for Diversity — PfD). $A \sim B$ implies $B \succcurlyeq (1/2)A + (1/2)B$.

Our final axiom is a weakening of the traditional Independence axiom in that we require Independence to only hold for a singleton and one other menu.

Axiom (D-Independence). For all $A, B \in \mathcal{K}(\Delta)$ and all $\lambda \in (0,1]$, $A \succ B$ implies (i) $\lambda A + (1-\lambda)D \succ \lambda B + (1-\lambda)D$ and (ii) $\lambda A + (1-\lambda)\{q^*\} \succ \lambda B + (1-\lambda)\{q^*\}$.

(The "D" in D-Independence stands for "disk".) At a technical level, the axiom is useful since it allows for a natural parametrisation of the set of ex post utilities, S_Z, and is essential in allowing us to construct a cardinal representation of preferences.[8] More precisely, we shall construct the set of ex post utilities S_Z so that each ex post utility has the same (maximum) utility in D and $\{q^*\}$. Indeed, if the agent does have such a set of ex post utilities, then the axiom is very easy to rationalise.

A behavioural motivation for limiting Independence to D-Independence is that the agent can easily visualise such mixtures, but may not find it so easy to visualise mixtures of more general menus (which do not have uniform payoffs in every subjective state). It is important to note that even the though the set $\lambda A + (1-\lambda)D$ might be very complicated, the ex post payoffs in each subjective state are easily computed. For such an agent, if $A \succ B$, then it is reasonable, for instance, that $\lambda A + (1-\lambda)\{q^*\} \succ \lambda B + (1-\lambda)\{q^*\}$.

The natural parametrisation of ex post vN-M utility functions that gives the agent the same maximum utility with each ex post utility on both D and $\{q^*\}$ is now described. We shall let S_Z denote the set of (non-trivial) vN-M utility functions on Δ with generic element being denoted as $u(\cdot, s)$, where $s \in S_Z$. For each $s \in S_Z$, it is the case that (i) $u(q^*, s) = 0$ and (ii) $\max_{p \in D} u(p, s) = R$. The construction is made explicit in §5.1. In particular, we show that S_Z is the smooth boundary of a compact, convex subset of an n-dimensional Euclidean space.

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[8] Indeed, we shall show that a continuous preference relation that satisfies IR and D-Independence admits cardinal utility representation V that is also D-linear. For a definition of D-linearity, see the sketch of the proof of Theorem 2.2.

Notice that there are other parametrisations of the set of vN-M utility functions on Δ. But the parametrisation above has the convenient feature that if the agent satisfies D-Independence, then her behaviour can be easily rationalised. We now proceed to our representation.

§ 2.3. *Representations*

We begin with a simple representation of preferences satisfying the above axioms. First, a definition. Say that a function $V : \mathcal{F}(\Delta) \to \mathbb{R}$ is *D-linear* if, for all $\alpha \in [0,1]$ and $A \in \mathcal{F}(\Delta)$, (i) $V\big(\alpha A + (1-\alpha)D\big) = \alpha V(A) + (1-\alpha)V(D)$, (ii) $V\big(\alpha A + (1-\alpha)\{q^*\}\big) = \alpha V(A) + (1-\alpha)V\big(\{q^*\}\big)$ and (iii) $V(D) \neq V\big(\{q^*\}\big)$.

Proposition 2.1. (a) A preference $\succcurlyeq$ satisfies Non-triviality*, Continuity and D-Independence if and only if there exists a D-linear function V that represents it and that is unique up to positive affine transformation.

(b) A D-linear utility function that represents $\succcurlyeq$ is Lipschitz continuous if and only if $\succcurlyeq$ satisfies L Continuity.

(c) Moreover, a D-linear utility function V that represents $\succcurlyeq$ is concave if and only if $\succcurlyeq$ satisfies Preference for Uniformity.

Sketch of Proof. We provide a sketch of the proof here. (The proposition is proved in §5.) Throughout we assume that $D \succ \{q^*\}$. We first note that by D-Independence, the set $\Xi_D := \big\{\lambda D + (1-\lambda)\{q^*\} : \lambda \in [0,1]\big\}$ is a mixture space. Thus, preferences restricted to this domain have a continuous linear representation, V such that $V\big(\lambda D + (1-\lambda)\{q^*\}\big) = \lambda V(D) + (1-\lambda)V\big(\{q^*\}\big)$. Now, for any A such that $D \succcurlyeq A \succcurlyeq \{q^*\}$, there is a unique $\lambda_A \in (0,1)$ such that $A \sim \lambda_A D + (1-\lambda_A)\{q^*\}$, providing us with a utility representation for all such A, ie $V(A) := \lambda_A V(D) + (1-\lambda_A)V\big(\{q^*\}\big)$. If $A \succ D$, there exists a λ such that $D \succ \lambda A + (1-\lambda)\{q^*\}$, letting us attach a utility to A, such that $V\big(\lambda A + (1-\lambda)\{q^*\}\big) = \lambda V(A) + (1-\lambda)V\big(\{q^*\}\big)$. Similarly, for A such that $\{q^*\} \succ A$. We then show that such a representation is well defined and D-linear. It is straightforward to show the other parts of the proposition. $\square$

In what follows, we shall normalise V so that $V\big(\{q^*\}\big) = 0$ and $|V(D)| = R > 0$. From §2.2, we know that there exists a collection of vN-M utility functions parametrised by the set S_Z, so that $u(q^*, s) = 0$ and $\max_{p \in D} u(p, s) = R$ for each $s \in S_Z$. As we shall see below, the normalisations of V that we have chosen and those of S_Z are intimately related.

For any menu A, let $u_A(s) := \max_{p \in A} u(p,s)$. By the construction of S_Z, we see that for each A, the function $u_A : S_Z \to \mathbb{R}$ is a continuous function. Moreover, if π is a signed measure on S_Z, we define $\langle \pi, u_A \rangle := \int_{S_Z} u_A(s)\, d\pi(s)$.

A *multi-weight representation* is a function $V : \mathcal{F}(\Delta) \to \mathbb{R}$ of the form

$$V(A) := \min \left\{ \langle u_A, \pi \rangle : \pi \in \Pi \right\} \tag{$\spadesuit$}$$

where Π is a weak* compact, convex set of signed measures on S_Z such that for all $\pi, \mu \in \Pi$, $\pi(S_Z) = \mu(S_Z)$. Also, $V(D) > V(\{q^*\})$ implies $\pi(S_Z) = 1$ and $V(\{q^*\}) > V(D)$ implies $\pi(S_Z) = -1$. The subjective state is denoted by $S := \bigcup_{\pi \in \Pi} \operatorname{supp} \pi$. It is easily seen that a multi-weight representation is D-linear. We shall say that a multi-weight representation is unique (modulo the normalisations of S_Z) if the set of measures Π is unique.

It will be useful (especially in the proof of our representation theorem) to consider preferences of the form

$$V(A) := \max \left\{ \langle u_A, \pi \rangle : \pi \in \Pi \right\} \tag{$\clubsuit$}$$

We now proceed to our main theorem, the representation.

Theorem 2.2. *A. A function $V : \mathcal{F}(\Delta) \to \mathbb{R}$ is D-linear, concave and Lipschitz continuous if and only if it has a unique ($\spadesuit$) multi-weight representation.*

B. A function $V : \mathcal{F}(\Delta) \to \mathbb{R}$ is D-linear, convex and Lipschitz continuous if and only if it has a unique ($\clubsuit$) multi-weight representation.

The proof of the theorem is in §5. We present here a brief sketch of the main ideas.

Sketch of Proof. The first step in the proof is to embed the space of all convex subsets of aff Δ as a cone K^* in the function space $C(S_Z)$. (Since S_Z is the smooth boundary of a compact, convex subset of $\mathbb{R}^n$, it is homeomorphic to S^{n-1}, the $(n-1)$-dimensional sphere.) There is a standard embedding that can be found in Schneider (1993). In fact, a function $f : S_Z \to \mathbb{R}$ is in K^* if and only if $\bar{f} : \mathbb{R}^n \to \mathbb{R}$, which represents the unique extension of f to $\mathbb{R}^n$ by positive homogeneity, is sublinear. Moreover, $K^* - K^*$ is dense in $C(S_Z)$.

For the rest of the proof, we assume that $D \succ \{q^*\}$. From Proposition 2.1, we know there exists a D-linear function V that represents $\succcurlyeq$. Moreover, V is Lipschitz continuous and concave.

This shows that an equivalent way of writing preferences is via a function $\varphi : K^* \to \mathbb{R}$. We show that such a function is superlinear (assuming preferences satisfy PfU), C⁺-additive, ie $\varphi(v + \alpha\mathbf{1}) = \varphi(v) + \alpha\varphi(\mathbf{1})$, for $\alpha \geqslant 0$ and Lipschitz continuous (in the norm topology on $C(S_Z)$).

We now use the important observation that for any $f \in C^2(S_Z)$, the space of twice continuously differentiable functions on S_Z, there exists an $\alpha > 0$ such that $f + \alpha\mathbf{1} \in K_2^* := K^* \cap C^2(S_Z)$. This enables us to extend $\varphi|_{K_2^*}$ to $C^2(S_Z)$ uniquely, the extension being denoted $\psi : C^2(S_Z) \to \mathbb{R}$, such that the extension is superlinear and satisfies C⁺-additivity. Moreover, ψ is Lipschitz continuous. We now use the fact that $C^2(S_Z)$ is norm ($\| \cdot \|_\infty$) dense in $C(S_Z)$ to uniquely extend ψ to $C(S_Z)$. This extension is superlinear, satisfies C⁺-additivity and is Lipschitz continuous.

But superlinear functions that satisfy C⁺-additivity have the required characterisation, as the minimum of a family of linear functionals. (The Riesz Representation Theorem, in turn, tells us that a continuous linear functional on $C(S_Z)$ can be written as the integral with respect to a unique signed Borel measure on S_Z.) For preferences that satisfy PfD, the representation follows immediately.

Finally, for preferences such that $\{p^*\} \succ D_R$, we define the preference $\succcurlyeq^*$ as follows: $A \succcurlyeq^* B$ if and only if $B \succcurlyeq A$. We show that $\succcurlyeq^*$ satisfies Non-triviality*, D-Independence and PfU (resp PfD) if and only if $\succcurlyeq$ satisfies Non-triviality*, D-Independence and PfD (resp PfU). Then, from the result above, V^* represents $\succcurlyeq^*$ and defining $V := -V^*$, we see that V represents $\succcurlyeq$ and has the desired functional form. □

The special case where Π is a singleton is referred to as an *additive EU representation*. Such a representation is characterised by Continuity, L Continuity and Independence, where Independence says that $A \succ B$ implies $\lambda A + (1-\lambda)C \succ \lambda B + (1-\lambda)C$ for all sets C (making clear how we have weakened Independence). This brings us to the following observation.

Corollary 2.3. Let $\succcurlyeq$ be a preference relation that satisfies Non-triviality*. Then $\succcurlyeq$ also satisfies Continuity, L Continuity, D-Independence, Preference for Diversity and Preference for Uniformity if and only if $\succcurlyeq$ has an additive EU representation.

Remark 2.4. We end by considering the effect of weakening D-Independence. Instead of requiring D-Independence, suppose we require that there exist convex subsets $D_1, D_2 \in \mathcal{K}(\Delta)$ such that (i) for all $\lambda \in (0,1)$, $A \succ B$ implies $\lambda A + (1-\lambda)D_i \succ \lambda B + (1-\lambda)D_i$ for $i = 1, 2$, and (ii) $D_1 \nsim D_2$. Then, we can show that there exists a smallest set $\widetilde{\Pi}$ such that

$$\widetilde{U}(A) := \min \left\{ \langle u_A, \pi \rangle : \pi \in \widetilde{\Pi} \right\}$$

represents preferences, assuming the hypotheses of Theorem 2.2 hold. Unfortunately, since the set $\widetilde{\Pi}$ is not unique, we cannot make statements about comparative preference for uniformity, ie we can no longer state the analogue of Theorem 2.6.

§ 2.4. *Comparative Preference for Uniformity*

Our representation allows us to compare degrees of preference for uniformity. Before we proceed to the formalities, we recall an important theorem from DLR. Given preferences $\succcurlyeq_1$ and $\succcurlyeq_2$, say that agent 2 *is more uncertain than* agent 1 if $A \cup B \not\sim_1 A$ implies $A \cup B \not\sim_2 A$. Intuitively, agent 2 consider more ex post states possible than agent 1. This gives us the following characterisation.

Theorem 2.5 (Theorem 2, DLR). *If* $\succcurlyeq_2$ *is more uncertain that* $\succcurlyeq_1$*, then* $S_1 \subset S_2$*.*

As before, the menus D and $\{q^*\}$ play a special rôle in our model, ie $\{q^*\}$ represents the menu which gives the constant payoff of 0 in every subjective state and the menu D gives the constant payoff of R in every subjective state. Analogous, to the notion of the certainty equivalent of a gamble, we may define the *subjective certainty equivalent* of a menu A, and assuming $D \succ A \succ \{q^*\}$, as the (unique) menu $\lambda_A D + (1-\lambda_A)\{q^*\}$ such that $\lambda_A D + (1-\lambda_A)\{q^*\} \sim A$. Thus, the subjective certainty equivalent of a menu A is the menu $\lambda_A D_R + (1-\lambda_A)\{p^*\}$, that gives utility $\lambda_A R$ in every state and is indifferent to A.

This gives us a natural way to operationalize the notion of greater preference for uniformity. We say that $\succcurlyeq_2$ has a *greater desire for uniformity* than $\succcurlyeq_1$ if for each menu A, $\lambda D + (1-\lambda)\{q^*\} \sim_1 A$ implies $\lambda D + (1-\lambda)\{q^*\} \succcurlyeq_2 A$. Intuitively, an agent with a greater desire for uniformity has a smaller subjective certainty equivalent (in terms of set inclusion). This leads us to the following characterisation.

Theorem 2.6. *Suppose* $\succcurlyeq_1$ *and* $\succcurlyeq_2$ *both admit* ($\spadesuit$) *representations. Then,* $\succcurlyeq_2$ *has a greater desire for uniformity than* $\succcurlyeq_1$ *if and only if* $\Pi_1 \subset \Pi_2$*. Thus, if* $\succcurlyeq_2$ *desires more uniformity than* $\succcurlyeq_1$*,* $S_1 \subset S_2$*, ie* $\succcurlyeq_2$ *is more uncertain than* $\succcurlyeq_1$*.*

Proof. See Appendix. □

§ 3. Finite Representations

Our multi-weight representations allow for the subjective state space to be infinite. In this section, we look at multi-weight representations that have a finite state space. With this assumption, we characterise the notion of the sign of a state in § 3.1, which is useful since the sign of a state indicates whether local perturbations to menus leave the agent better or worse off. In §3.2, we study some additional axioms on preferences that have proved useful in the literature preferences for commitment (ie choice when in the presence of temptation). We begin with a definition.

Definition 3.1. A *finite multi-weight representation* is a multi-weight representation (♠), where the subjective state space is finite.

For concreteness, let $S := \{s_1, \dots, s_m\}$ be the finite state space. We emphasise that the cardinality of the subjective state space S is a property of the underlying preference $\succcurlyeq$, and so cannot just be assumed. In §5.9, we provide axioms that ensure that the preference indeed has a finite multi-weight representation. Let $\Delta^{|S|-1}$ be the $(|S|-1)$-dimensional, unit simplex and aff $\Delta^{|S|-1}$ be its affine hull. Then, for finite multi-weight representations, we have $\Pi \subset$ aff $\Delta^{|S|-1}$ (whenever $D_R \succ \{p^*\}$). As before, Π is compact and convex. In the rest of this section, we shall assume that preferences have a finite multi-weight representation. We now introduce the idea of the sign of a state.

§ 3.1. *Sign of the State*

In this section, we discuss how a subjective state (and the concomitant multi-weight representation) determine the impact (on the agent) of making local changes to a menu. To do this, we shall recall some ideas from DLR.

In their discussion on weak EU representations (of which the multi-weight representations are a special case), DLR show that it is possible to attach a sign to a state. A state $s \in S_Z$ is said to be *positive* if for each neighbourhod N of s, there exist menus A and B with $A \subset B$ and $B \succ A$, wherein $u_A(s') = u_B(s')$ for all $s' \in S_Z \setminus N$. The idea is that increasing the ex-post utility only in some parts of a neighbourhood state s of the menu A to give the menu B makes B more valuable than A. Similarly, a state $s \in S_Z$ is *negative* if for each neighbourhod N of s, there exist menus A and B with $A \subset B$ and $A \succ B$, wherein $u_A(s') = u_B(s')$ for all $s' \in S_Z \setminus N$. A curious property of multi-weight representations (and weak EU representations in general) is that it is possible for a state to be *both* positive *and* negative. To understand why this might be so, notice that the sign of a state is a local property, in that we are concerned with small perturbations to ex-post utility in the neighbourhood of a state and the effect that has on the menu's ex-ante value. As is commonly known, local phenomena don't usually translate into global properties.

Before proceeding to the characterisation of the sign of a state, we mention a peculiar occurrence. In what follows, we will have to restrict attention to multi-weight representations that have a finite subjective state space. The reason is that if the state space is infinite, then even under the stronger assumption of Independence (which, in our context means that Π is a singleton, see Corollary 2.3), it may be that a state is both positive and negative.[9]

We shall now demonstrate that the multi-weight representation above provides an easy characterisation of the sign of a state. Let $\Pi_i := \text{proj}_i \Pi$ for $i = 1, \ldots, m$, be the projection of Π onto the i-th axis. Say that Π_i is *negative* if $(-\infty, 0) \cap \Pi_i \neq \varnothing$ and that Π_i is *positive* if $(0, \infty) \cap \Pi_i \neq \varnothing$. In what follows, the set of extreme points of Π is denoted by $\text{ext}\,\Pi$. We begin with two important characterisations.

Proposition 3.2. Let state s_i be positive (resp negative). Then, Π_i is positive (resp negative).

Proposition 3.3. Let $\pi = (\pi^1, \ldots, \pi^i, \ldots, \pi^m) \in \text{ext}\,\Pi$ be such that $\pi^i >$ (resp $<$) 0. Then, state s_i is positive (resp negative).

For any menu A, there exists $\pi_A \in \text{ext}\,\Pi$ such that $V(A) = \langle u_A, \pi_A \rangle$. The proof of proposition 3.3 actually shows that if $\pi^i_A > 0$, then state s_i is positive *at the menu* A and if $\pi^i_A < 0$, then state s_i is negative at the menu A. Putting together the propositions above, we have the main theorem of this section.

Theorem 3.4. *State s_i is positive if and only if Π_i is positive and is negative if and only if Π_i is negative.*

It is also useful to know if a state with a particular sign exists. The following theorem, which follows immediately from Proposition 3.3, provides us with this characterisation.

Theorem 3.5. *There exists a positive state if $D \succ \{q^*\}$. There exists a negative state if $\{q^*\} \succ D$.*

We end with a remark on the appropriateness of finite state representations.

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[9] DLR claim (p 912) that if Independence holds, a state is either positive or negative, but not both. Unfortunately, this is not true in general. An example showing this is available upon request from the authors. It is, however, true if the state space is finite, and is easily seen from the definition. See also remark 3.6 below.

Remark 3.6. Our focus on finite state multi-weight representations is due to their tractability and one justification for limiting attention to such representations is that any multi-weight representation can be approximated, in utility terms, by a sequence of finite multi-weight representations. But consider an additive EU representation with an infinite state space. One can construct a sequence of finite state additive EU representations that approximate the additive EU representation in utility terms. Notice also that in a finite state additive EU representation, each state is either positive, or negative, but never both. Nevertheless, as mentioned in footnote 9, there exists an (infinite state) additive EU representation with a state that is both positive and negative. Notice also that the behavioural interpretations of the model are (to some extent) tied to the sign of each state. Thus, while we can approximate the additive EU representation in utility, the (finite) approximations may not have all the behavioural properties (such as the sign of a state) of the additive EU representations.

§ 3.2. *Special Cases*

Here we shall consider some additional axioms on preferences. These additional axioms have been useful in characterising certain kinds of behaviour. In what follows, we shall assume that preferences have a multi-weight representation with a finite state space.

Definition 3.7. A preference $\succcurlyeq$ is *upward monotonic* if $A \supset B$ implies $A \succcurlyeq B$. It is *downward monotonic* if $A \supset B$ implies $B \succcurlyeq A$.

Monotonic preferences have an easy characterisation (that we state without proof).

Proposition 3.8. Suppose $V : \mathcal{K}(\Delta) \to \mathbb{R}$ has a finite multi-weight representation (♠) where $|S| = m$. Then,
(i) V is upward monotonic if and only if $\Pi \subset \Delta^{m-1}$, and
(ii) V is downward monotonic if and only if $-\Pi \subset \Delta^{m-1}$.

Notice that if preferences are upward monotonic, then all states are positive and each measure in Π is a probability measure. If preferences are downward monotonic, then all states are negative and for each $\pi \in \Pi$, $-\pi$ is a probability measure. We now consider the axiom Set Betweenness introduced by Gul and Pesendorfer (2001) and the following weakenings of Set Betweenness, introduced by Dekel, Lipman and Rustichini (2007).

Axiom (Positive Set Betweenness). $A \succcurlyeq B$ implies $A \succcurlyeq A \cup B$.

Axiom (Negative Set Betweenness). $A \succcurlyeq B$ implies $A \cup B \succcurlyeq B$.

Axiom (Set Betweenness). $A \succcurlyeq B$ implies $A \succcurlyeq A \cup B \succcurlyeq B$.

To see the intuition behind these axioms, consider the case where $A \succcurlyeq B$. Then, Positive Set Betweenness says that the increased flexibility from considering $A \cup B$ is never sufficiently valuable for the agent, as compared to the menu A. In a similar vein, Negative Set Betweenness says that the increased flexibility from being able to choose from $A \cup B$ is at least valuable as B. The axioms are complementary in the sense that they provide upper and lower bounds on the value of $A \cup B$ in terms of A and B. Clearly, Positive and Negative Set Betweenness imply Set Betweenness. Introducing these axioms imposes a lot of structure on Π, as we shall see below.

Proposition 3.9. Suppose $V : \mathcal{K}(\Delta) \to \mathbb{R}$ has a finite multi-weight representation (♠) and satisfies Positive Set Betweenness. Then, for each $\pi = (\pi^1, \ldots, \pi^i, \ldots \pi^m) \in \Pi$, there exists exactly one i such that $\pi^i > 0$. In other words, for any menu A, we have

$$V(A) = \min\{U_\pi(A) : \pi \in \Pi\}$$

where U_π is a finite additive EU representation with exactly one positive state.

The fact that a finite additive EU representation satisfies Positive Set Betweenness if and only if it has exactly one positive state is Lemma 1 in Dekel, Lipman and Rustichini (2007). The effect of imposing Negative Set Betweenness is similar. Again, Lemma 2 from Dekel, Lipman and Rustichini (2007) is the fact that a finite additive EU representation satisfies Negative Set Betweenness if and only if it has exactly one negative state.

Proposition 3.10. Suppose $V : \mathcal{K}(\Delta) \to \mathbb{R}$ has a finite multi-weight representation (♠) and satisfies Negative Set Betweenness. Then, for each $\pi = (\pi^1, \ldots, \pi^i, \ldots \pi^m) \in \Pi$, there exists exactly one i such that $\pi^i < 0$. In other words, for any menu A, we have

$$V(A) = \min\{U_\pi(A) : \pi \in \Pi\}$$

where U_π is a finite additive EU representation with exactly one negative state.

This brings us to the consequences of assuming that $\succcurlyeq$ satisfies Set Betweenness. Assuming Set Betweenness says that there is no preference for flexibility and that there is just one source of temptation. Moreover, this allows us to completely characterise multi-weight representations that satisfy Set Betweenness.

Theorem 3.11. *Suppose $V : \mathcal{K}(\Delta) \to \mathbb{R}$ has a finite multi-weight representation ($\spadesuit$). Then, the following are equivalent.*
(i) V satisfies Set Betweenness.
(ii) The multi-weight representation has at most two subjective states, ie $|S| \leqslant 2$. If $|S| = 2$, one state is positive (but not negative) and the other state is negative (but not positive). If $|S| = 1$, then that state is either positive or negative, but not both.

It follows from Propositions 3.9 and 3.10 that if $|S| = 2$ and preferences satisfy Set Betweenness, then each $\pi \in \operatorname{ext} \Pi$ can have only two components, one of which is positive (resp negative) and one is non-positive (resp non-negative). It does not follow from this that a state cannot be both positive and negative. That this cannot be so is demonstrated in the appendix.

We now turn to the methodological implications of the above result. Notice that if we have 2 states and have $D_R \succ \{p^*\}$, then, V takes the following form:

$$V(A) := \min \left\{ \langle \pi, u_A \rangle, \langle \nu, u_A \rangle \right\}$$

where $\pi^1 + \pi^2 = \nu^1 + \nu^2 = 1$ and $\operatorname{sgn}(\pi^s) = \operatorname{sgn}(\nu^s)$. To get at an interesting interpretation of this representation, let us first assume $\pi = \nu$ (and, say, $\pi^1 = 1 - \pi^2 > 1$), so that $V(A)$ can be rewritten as

$$\begin{aligned} V(A) &= \max_{p \in A} \, v(p) - \max_{p \in A} w(p) \\ &= \max_{p \in A} \left[v(p) - c(p; A) \right] \end{aligned}$$

where $w(p) := |\pi^2| u_2(p)$, $(v + w)(p) := \pi^1 u_1(p)$ and $c(p; A) = \max_{p \in A} w(p) - w(p)$. Notice that for singletons, we now have $V(\{p\}) = v(p)$. We can interpret $c(p; A)$ as a cost function and the interpretation is that the agent believes that when she makes a choice in the second period, she will incur a self-control cost which, in turn, informs the agent's choices between menus. (This cost interpretation is central to the analysis of Gul and Pesendorfer (2001), Dekel, Lipman and Rustichini (2007) and Noor (2006). Notice that requiring a multi-weight representation to also satisfy Singleton Independence (see below for this axiom) is sufficient to ensure that $\pi = \nu$.)

Now consider the case where $\pi \neq \nu$. Then, we can write

$$V(A) = \min\left\{\max_{p\in A}\left[v_1(p) - c_1(p;A)\right], \max_{p\in A}\left[v_2(p) - c_2(p;A)\right]\right\},$$

and $c_2(p,A) = \frac{|\nu^2|}{|\pi^2|}c_1(p,A)$.

In other words, the agent can interpreted as being unsure about two things: firstly, his objective (or normative) preferences, given by his preferences over singletons, and second the cost function he will face. This brings into stark relief, the crucial rôle that Independence plays in the point of view that normative welfare judgements can be made by looking at preferences over singletons.[10]

One final observation here is that if multi-weight preferences are non-linear and satisfy Set Betweenness, then they cannot be linear over singletons. This is made precise below. We begin with an axiom.

Axiom (Singleton Independence). For all $p, q, r \in \Delta$ and $\lambda \in (0,1)$, $\{p\} \succ \{q\}$ implies $\lambda\{p\} + (1-\lambda)\{r\} \succ \lambda\{q\} + (1-\lambda)\{r\}$.

Proposition 3.12. Suppose $V : \mathcal{K}(\Delta) \to \mathbb{R}$ has a finite multi-weight representation ($\spadesuit$). Then the following are equivalent.
(i) V satisfies Set Betweenness and Singleton Independence,
(ii) V is a finite additive EU representation (ie, Π is a singleton).

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[10] We emphasise that we are not claiming that the cost interpretation is invalid. Indeed, it may well be the case that that is what is actually going on in the agent's mind, when she makes a choice. All we are pointing out is that inferring such aspects of the agent' psychological makeup seems to require information of what choices are actually made, information that is not available in the present environment.

§ 4. Discussion and Conclusion

In this paper, we introduce the notion of preference for uniformity in subjective state models. We show that a preference for uniformity in payoffs across subjective states is orthogonal to concerns for flexibility and/or commitment. We provide a simple representation theorem that captures an agent's desire for uniformity, and show how the representation can be used to compare degrees of desire for uniformity. The representation hinges on weakening the Independence axiom considerably.

We also characterise the sign of the state, ie whether a subjective state is positive or negative at a menu. This is a local characterisation in the sense that at a particular menu, a state can be positive and at another menu, the state can be negative. We end by imposing looking at some special cases of preferences that have found use in the literature. Relaxing Independence has methodological implications that we now discuss.

A particularly influential point of view is that in our environment, preferences over singletons represent the agent's objective or normative preferences. Advocates of this view are, among others, Gul and Pesendorfer (2001) and Chatterjee and Krishna (2007). Indeed, the representations obtained in these papers suggests that there is a long-run self who anticipates the possibility of a suboptimal choice being made from the menu. As we demonstrate, one consequence of weakening Independence is that it is not possible to make statements about what is normatively desirable by looking at preferences over singletons. In other words, the ability to look at the agent's preferences over singleton menus and treat them as the agent's normative preferences depends, to a large extent, on the Independence axiom.

Finally, we come to, what is in essence, a matter of interpretation. More specifically, our environment consists of looking at preferences over menus of lotteries. We interpret this as looking at the behaviour of someone who will eventually make a choice from the menu, which leads to our characterisation. An equally acceptable interpretation (of the environment) is one which views a menu of lotteries as representing *objective* ambiguity about the nature of uncertainty. Such models are discussed in Olszewski (2007), Ahn (2007) and Stinchcombe (2007). Another interpretation, subtly different from DLR's (and hence ours) is that of Gul and Pesendorfer (2001), who assume that the choice in the (unmodelled) second period is *always* made according to a utility function that is a weighted sum of two utility functions. (Thus, in our terminology, they have a subjective state space of cardinality at most 2.) In §3.2, we show that this interpretation is greatly facilitated by assuming Independence. Yet another interpretation is offered by Chatterjee and Krishna (2007), who assume that the choice in the second period is always according to one of (possibly) two utility functions.

This is because in that model, the agent does not care about the utility she will receive in the second period, but is assumed to value the choice made in the second period according to his preferences over singletons.

§ 5. Proofs

§ 5.1. *Parametrising the Set of vN-M Utility Functions*

Suppose the set of ex post utility functions can be parametrised and denoted by S_Z. Since each vN-M utility is uniquely defined up to two normalisations, the normalisations we have used (in §2.2) wherein $u(q^*, s) = 0$ and $u_D(s) = R$ for each $s \in S_Z$ give us a unique collection of ex post utilities. We shall study this set now. Recall that $p^* := \left(1/_{n+1}, \dots, 1/_{n+1}\right)$ is the uniform lottery and D_R is the disk around p^* with radius R.

It is useful to introduce the following notation. For a set A in a finite dimensional Euclidean space X, $A^{\odot}$ is the so-called *one-sided polar* of A, and is defined as

$$A^{\odot} := \{x' \in X : \langle x, x' \rangle \leqslant 1 \text{ for all } x \in A\},$$

see §5.16, Aliprantis and Border (1999). $A^{\odot}$ is nonempty, convex, closed and contains zero (Lemma 5.90, p 211 Aliprantis and Border, 1999). The Bipolar Theorem, (Theorem 5.91, p 213, Aliprantis and Border, 1999), says that if A is closed, convex and contains zero, $A^{\odot\odot} = A$.

Consider first the simplest case, where $q^* := p^*$ and $D := D_R$. Then, the set of ex post vN-M utilities is $S_Z := \left[\frac{1}{R}\left(D_R - p^*\right)\right]^{\odot} \cap \operatorname{span}\left[D_r - p^*\right]$ or more explicitly,

$$S_Z = \left\{ s \in \mathbb{R}^{|Z|} : \sum_i s_i = 0 \text{ and } \|s\| = 1 \right\}$$

where, as always, $\|s\| = \left(s_1^2 + \cdots + s_{|Z|}^2\right)^{1/2}$. (Notice that $\operatorname{span}\left[D_R - p^*\right]$ is an n-dimensional subspace of $\mathbb{R}^{n+1}$.) Clearly, for each $s \in S_Z$, $u(p, s) = \langle p, s \rangle$, so that $u(p^*, s) = 0$ and $u_{D_R}(s) = R$.

Now consider the case where $q^* - p^* = t$ and $D := D_R + t$. Then, the set of ex post utilities is still given by $S_Z = \{s \in \mathbb{R}^{|Z|} : \sum_i s_i = 0 \text{ and } \|s\| = 1\}$, so that for each $s \in S_Z$, $u(p; s) = \langle p - t, s \rangle$. Then, for each $s \in S_Z$, $u(q^*, s) = \langle p^*, s \rangle = 0$. Also, for each $s \in S_Z$, $\max_{p \in D} u(p, s) = \max_{p \in D_R} \langle p, s \rangle = R$.

Finally, consider the case where $q := p^*$ and $D \neq D_R$ (but D nevertheless has a smooth boundary). Then, define $\tilde{S}_Z := \left[\frac{1}{R}\left(D - p^*\right)\right]^{\odot} \cap \operatorname{span}\left[D - p^*\right]$. With this definition, we still have $u(p^*, \tilde{s}) = 0$ and $u_D(\tilde{s}) = R$ for all $\tilde{s} \in \tilde{S}_Z$ (from the Bipolar theorem). The case where both $q^* \neq p^*$ and $D \neq D_R$ is handled similarly.

In what follows, we shall assume, for ease of exposition, that $q^* := p^*$ and $D := D_R$, which represents a natural, symmetric normalisation. None of our results depend on this assumption although, as is obvious from the discussion above, the calculations do become slightly cleaner.

§ 5.2. *The Cone of Ex-post utilities*

Recall that S^{n-1} is the canonical $(n-1)$-dimensional sphere and let $\mathcal{K}^n$ denote the space of all compact convex subsets of $\operatorname{aff}\Delta$ (which is essentially $\mathbb{R}^n$). It is easy to show that $\mathcal{K}^n$ is a complete metric space when endowed with the Hausdorff metric, d_h. Let $S_Z := \{p \in \mathbb{R}^{|Z|} : \sum p_i = 0 \text{ and } \|p\| = 1\}$. Notice that S_Z is isometrically homeomorphic to S^{n-1}. For any $A \in \mathcal{K}^n$, its *support function* $h_A : S_Z \to \mathbb{R}$ is given by $h_A(s) := \max_{p \in A} \langle p - p^*, s \rangle$, where $p^* := ({}^{1}\!/_{n+1}, \ldots, {}^{1}\!/_{n+1})$. Notice that for each $A \in \mathcal{K}^n$, h_A is a continuous function on S_Z. This definition of a support function has some immediate consequences. Firstly, $h_{\{p^*\}} = \mathbf{0}$ and second, for $D_r := \{p \in \operatorname{aff}\Delta : \|p - p^*\| \leqslant r > 0\}$, $h_{D_r}(s) = r$ for each $s \in S$.

It should be noted that for any $\lambda \in [0,1) \cup (1,\infty)$, $\lambda A \notin \mathcal{K}^n$. So, in what follows, when we write λA where $\lambda \geqslant 0$, we shall actually mean $\lambda(A - p^*) + p^*$. Similarly, for $A, B \in \mathcal{K}^n$, $A + B$ shall mean $(A - p^*) + (B - p^*) + p^*$, which is a compact, convex subset of $\operatorname{aff}\Delta$. Notice that for any $p \in \operatorname{aff}\Delta$ and $\lambda \geqslant 0$, $\lambda(p - p^*) + p^* = \lambda p - (\lambda - 1)p^* \in \operatorname{aff}\Delta$. Thus, λA as defined above is also a subset of $\operatorname{aff}\Delta$. Moreover, for $A, B \in \mathcal{K}^n$ and $\lambda \in (0,1)$, $\lambda A + (1-\lambda)B \in \mathcal{K}^n$. We begin with some technical properties of support functions.

Proposition 5.1. The support function has the following properties:
(i) $h_{A+B} = h_A + h_B$,
(ii) $h_{\lambda A} = \lambda h_A$ for all $\lambda \geqslant 0$,
(iii) $h_A \wedge h_B = h_{A \cap B}$,
(iv) $h_A \cup h_B = h_{\operatorname{conv}(A \cup B)}$, and
(v) $\|h_A - h_B\|_\infty = d_h(A, B)$.
(vi) The (unique) extension $\bar{h}$ of a support function h to $\operatorname{span} S_Z$ by positive homogeneity is sublinear and the restriction to S_Z of a sublinear function on $\operatorname{span} S_Z$ is a support function.
(vi) Moreover, the following duality relation holds: $h_{K_h} = h$ and $K_{h_K} = K$.

Proof. See §§5.18 and 5.19 of Aliprantis and Border (1999). ☐

Thus, there is an isometry between the space of compact, convex sets of $\mathbb{R}^n$ ($\simeq \operatorname{span} S_Z$) and the space of sublinear functions on $\mathbb{R}^n$. Let $K^* \subset C(S_Z)$ denote the cone of functions on S_Z whose unique extensions to $\operatorname{span} S_Z$ by positive homogeneity are sublinear. Then, $K^* \simeq \mathcal{K}^n$.

We shall be interested in the embedding of $\operatorname{aff}\Delta$ in K^*. To this end, we define $F := \{h_{\{p\}} : p \in \operatorname{aff}\Delta\}$. Thus, F represents the space of support functions of singletons (which are also compact and convex). We show below that F is an n-dimensional subspace of K^*.

Proposition 5.2. Let $S_0 \subset S_Z$ be a linearly independent set. Then,

$$F_0 := \left\{ \left(h_{\{p\}}(s) \right)_{s \in S_0} : p \in \operatorname{aff} \Delta \right\}$$

is a (closed) $|S_0|$-dimensional subspace of K^*. Therefore, if S_0 is a maximal linearly independent set, then $F_0 = F$.

Proof. Let S_0 be a linearly independent subset of S_Z. Notice first that for any $p \in \operatorname{aff} \Delta$, there exists (a unique) $p' \in \operatorname{aff} \Delta$ such that $(p + p')/2 = p^*$. Therefore $p - p^* = -(p' - p^*)$, so that $h_{\{p\}}(s) = \langle p - p^*, s \rangle = -\langle p' - p^*, s \rangle = -h_{\{p'\}}(s)$ for all $s \in S_0$. This shows that for any $p \in \operatorname{aff} \Delta$ and any $\lambda \in \mathbb{R}$, $\lambda h_{\{p\}}(s) = h_{\lambda\{p\}}(s)$ for all $s \in S_0$ (where $\lambda\{p\}$ is actually $\lambda(p - p^*) + p^*$. Also, for any $p, q \in \operatorname{aff} \Delta$, $h_{\{p\}+\{q\}}(s) = h_{\{p\}}(s) + h_{\{q\}}(s)$ for all $s \in S_0$. Thus, F_0 is a vector subspace of K and it is easily seen to be of dimension $|S_0|$. We can similarly show that F is a vector subspace of K^* from which it follows that F_0 is a vector subspace of F.

Suppose now that S_0 spans S_Z. To show that $F = F_0$, notice that for any $s \in S_Z \setminus S_0$, there exist (unique numbers) $\lambda_1, \ldots, \lambda_k$ such that $s = \sum_{i=1}^k \lambda_i s_i$. Then, $h_{\{p\}}(s) = \langle p - p^*, s \rangle = \left\langle p - p^*, \sum_{i=1}^k \lambda_i s_i \right\rangle = \sum_{i=1}^k \lambda_i \langle p - p^*, s_i \rangle = \sum_{i=1}^k \lambda_i h_{\{p\}}(s_i)$. Thus, $F = F_0$. □

As mentioned in the text, the subjective state space can be interpreted as the set of (expected) utility functions that the agent believes can be used to make a choice from a set in the second stage. For a given menu, she will receive some utility in each state. If the subjective state space is finite, then we can say a little more about the space defined above. The following proposition is stated without proof.

Proposition 5.3. Let $S \subset S_Z$ be finite with cardinality m. Then, $K_S := \{(h_A(s))_{s \in S} : A \in \mathcal{K}^n\}$ is a closed convex cone that spans $\mathbb{R}^m$.

In general, we cannot guarantee that K is a vector space. To see this, suppose $S = \{u, -u\}$. Then, for any $x > 0$, $(-x, -x) \notin K$. We now show that K has a non-empty interior.

Proposition 5.4. K has a topological interior. Therefore, $\operatorname{span} K = \mathbb{R}^m$.

Proof. Let $r > 0$ be such that $D_r \subset \operatorname{ri} \Delta$. Since S is finite, there exists $\varepsilon > 0$ such that for any $s \in S$, there exist A^s and B^s, compact convex subsets of (the relative interior of) Δ such that (i) $h_{A^s}(s) = r - \varepsilon$ and $h_{B^s}(s) = r + \varepsilon$ and (ii) $h_{A^s}(s') = r$ and $h_{B^s}(s') = r$ for any $s' \neq s$. Thus, there is neighbourhood of $r\mathbf{1}$ in K, so K has an interior. Finally, since $0 \in K$, $\operatorname{span} K = \mathbb{R}^m$. □

The following corollary is immediate.

Corollary 5.5. $\mathbf{1} \in \operatorname{int} K$. Therefore, for each $w \in \mathbb{R}^m$, there exists $\lambda_w > 0$ such that $w + \lambda\mathbf{1} \in \operatorname{int} K$.

Proof. Since K is a convex cone, we see that if $w \in \operatorname{int} K$, $\lambda w \in \operatorname{int} K$ for all $\lambda > 0$. Therefore, $\mathbf{1} \in \operatorname{int} K$. Now take $w \in \mathbb{R}^m$. Then, there exists $\mu > 0$ such that $\mu w + (1-\mu)\mathbf{1} \in \operatorname{int} K$. This implies $\frac{1}{\mu}(\mu w + (1-\mu)\mathbf{1}) =: w + \lambda_w \mathbf{1} \in \operatorname{int} K$. □

§ 5.3. *A Utility Representation: Proof of Proposition 2.1*

We shall now construct a D-linear utility representation for our preferences. Recall that one of our main axioms is D-Independence. Now consider the following strengthening of D-Independence.

Axiom (Strong D-Independence). $A \succ B$ implies $\lambda A + (1-\lambda)D_r \succ \lambda B + (1-\lambda)D_r$ for all $\lambda \in (0,1)$ and $r > 0$ and $D_r \subset \Delta$.

It is clear that Strong D-Independence implies D-Independence. It is useful to note that the two are actually equivalent. This is shown below.

Proposition 5.6. For a preference relation $\succcurlyeq \subset \mathcal{F}(\Delta) \times \mathcal{F}(\Delta)$, strong D-Independence is equivalent to D-Independence.

Proof. As noted above, it follows from the definitions that Strong D-Independence implies D-Independence. To prove the converse, suppose $A \succ B$. We will first prove that for any $r \in (0,R)$ and any $\lambda \in (0,1]$, $\lambda A + (1-\lambda)D_r \succ \lambda B + (1-\lambda)D_r$. Fix such an r and λ and let $\mu := {}^{r}\!/\!_{R}$, so that $D_r = \mu D_R + (1-\mu)\{p^*\}$. Then, $\frac{\lambda}{\lambda+(1-\lambda)\mu}A + \frac{(1-\lambda)\mu}{\lambda+(1-\lambda)\mu}D_R \succ \frac{\lambda}{\lambda+(1-\lambda)\mu}B + \frac{(1-\lambda)\mu}{\lambda+(1-\lambda)\mu}D_R$. This implies $\gamma\left[\frac{\lambda}{\lambda+(1-\lambda)\mu}A + \frac{(1-\lambda)\mu}{\lambda+(1-\lambda)\mu}D_R\right] + (1-\gamma)\{p^*\} \succ \gamma\left[\frac{\lambda}{\lambda+(1-\lambda)\mu}B + \frac{(1-\lambda)\mu}{\lambda+(1-\lambda)\mu}D_R\right] + (1-\gamma)\{p^*\}$ where $\gamma := \lambda + (1-\lambda)\mu$. Notice that $1-\gamma = 1-\lambda-(1-\lambda)\mu = (1-\lambda)(1-\mu)$, so that we can rewrite the above relations as $\lambda A + (1-\lambda)[\mu D_R + (1-\mu)\{p^*\}] \succ \lambda B + (1-\lambda)[\mu D_R + (1-\mu)\{p^*\}]$, ie $\lambda A + (1-\lambda)D_r \succ \lambda B + (1-\lambda)D_r$ as required.

The case where $r > R$ and $D_r \subset \Delta$ is similar. Fix such an $r > R$ and notice that there exists $\nu > 0$ such that $\nu r < R$. Also fix a $\lambda \in (0,1]$. By D-Independence, $\nu A + (1-\nu)\{p^*\} \succ \nu B + (1-\nu)\{p^*\}$. Notice also that $D_{\nu r} := \nu D_r + (1-\nu)\{p^*\}$. By what we have proved in the paragraph above, we get $\lambda[\nu A + (1-\nu)\{p^*\}] + (1-\lambda)D_{\nu r} \succ \lambda[\nu B + (1-\nu)\{p^*\}] + (1-\lambda)D_{\nu r}$ which can be rewritten as $\nu\lambda A + (1-\nu)\lambda\{p^*\} + (1-\lambda)[\nu D_r + (1-\nu)\{p^*\}] \succ \nu\lambda B + (1-\nu)\lambda\{p^*\}$ which in turn can be written as $\nu[\lambda A + (1-\lambda)D_r] + (1-\nu)\{p^*\} \succ \nu[\lambda B + (1-\lambda)D_r] + (1-\nu)\{p^*\}$, which holds if and only if $\lambda A + (1-\lambda)D_r \succ \lambda B + (1-\lambda)D_r$, which is what we wanted to prove. □

By DLR's Theorem 1.A, there exists a closed subset $S \subset S_Z$ and a utility function $V : \mathcal{K}(\Delta) \to \mathbb{R}$ such that V is continuous and

$$V(A) := U(u_A).$$

This V can be chosen such that $V(p^*) = 0$ (where $p^* = (1/n+1, \dots, 1/n+1)$ is the uniform lottery). Notice that since V represents preferences, and $V(D_R)$ is positive or negative if (and only if) $D_R \succ \{p^*\}$ or $\{p^*\} \succ D_R$. Our second normalisation will set $V(D_R) \in \{-R, R\}$. Unfortunately, even with these normalisations, we do not have a utility representation with any structure.

In the rest of this subsection, we shall construct a D-linear utility function V, ie a utility function that satisfies (i) $V(\lambda A + (1-\lambda)\{p^*\}) = \lambda V(A)$ and (ii) $V(\lambda A + (1-\lambda)D_R) = \lambda V(A) + (1-\lambda)V(D_R)$. For the rest of this section (and the remainder of the proof), we shall assume that $D_R \succ \{p^*\}$, so that $V(D_R) = R$. (The other case will be derived from a simple duality result.) We shall proceed in a number of simple steps. We shall first show that there is a useful class of sets in our domain that form a mixture space.

For each $A \in \mathcal{K}(\Delta)$, let $\Xi_A := \{\lambda A + (1-\lambda)\{p^*\} : \lambda \in [0,1]\}$. Notice that each Ξ_A is a mixture space. To see this, let $\alpha, \beta, \lambda \in [0,1]$ and define $A_\alpha := \alpha A + (1-\alpha)\{p^*\}$. Then, $\lambda A_\alpha + (1-\lambda)A_\beta = \big(\lambda\alpha + (1-\lambda)\beta\big)A + \big(\lambda(1-\alpha) + (1-\lambda)(1-\beta)\big)\{p^*\} \in \Xi_A$. The following proposition is useful to record.

Proposition 5.7. If $A \succ \{p^*\}$, then for all $\mu \in (0,1)$, $A \succ \mu A + (1-\mu)\{p^*\}$. Similarly, if $\{p^*\} \succ A$, then for all $\mu \in (0,1)$, $\mu A + (1-\mu)\{p^*\} \succ A$.

Proof. We shall only prove the first part. Suppose, by way of contradiction, that $A_1 := \mu A + (1-\mu)\{p^*\} \succcurlyeq A$. Then, by D-Independence, $A_2 := \mu A_1 + (1-\mu)\{p^*\} \succcurlyeq \mu A + (1-\mu)\{p^*\} = A_1$. Defining A_n inductively, we see that $A_{n+1} \succcurlyeq A_n$. But $A_n \to \{p^*\}$, which contradicts the continuity of $\succcurlyeq$. The case where $\{p^*\} \succ A$ is proved similarly. □

The proposition above can be restated as follows.

Corollary 5.8. Let $1 \geqslant \alpha > \beta > 0$. If $A \succ \{p^*\}$, then $\alpha A + (1-\alpha)\{p^*\} \succ \beta A + (1-\beta)\{p^*\}$. If $\{p^*\} \succ A$, then $\beta A + (1-\beta)\{p^*\} \succ \alpha A + (1-\alpha)\{p^*\}$.

We shall now show that preferences restricted to each Ξ_A satisfy not only D-Independence, but also Independence.

Proposition 5.9. The naturally induced preference over the mixture space Ξ_A satisfies Independence.

Proof. Let $A_\alpha, A_\beta, A_\gamma \in \Xi_A$, where $\alpha A + (1-\alpha)\{p^*\} =: A_\alpha \succ A_\beta$ and let $\lambda \in (0,1]$. Notice that $\lambda A_\alpha + (1-\lambda)A_\gamma = \big(\lambda\alpha + (1-\lambda)\gamma\big)A + \big(\lambda(1-\alpha) + (1-\lambda)(1-\gamma)\big)\{p^*\}$ and $\lambda A_\beta + (1-\lambda)A_\gamma = \big(\lambda\beta + (1-\lambda)\gamma\big)A + \big(\lambda(1-\beta) + (1-\lambda)(1-\gamma)\big)\{p^*\}$. If $A \succ \{p^*\}$, it must be that $\alpha > \beta$, so that (by the Corollary above) $\lambda A_\alpha + (1-\lambda)\{p^*\} \succ \lambda A_\beta + (1-\lambda)\{p^*\}$ as desired. If, on the other hand, $\{p^*\} \succ A$, then $\alpha < \beta$, so that $\lambda A_\alpha + (1-\lambda)\{p^*\} \succ \lambda A_\beta + (1-\lambda)\{p^*\}$. The proof is completed with the observation that if $A \sim \{p^*\}$, then $A_\alpha \sim \{p^*\}$ for each $A_\alpha \in \Xi_A$. (This follows from D-Independence.) $\square$

We now proceed to constructing the desired utility representation. Recall that we are only considering the case where $D_R \succ \{p^*\}$. We shall proceed via a number of simple steps.

Step 1. For all $D_r \subset \Delta$, $V(D_r) = r$. This represents preferences since Ξ_{D_r} is a mixture space for each D_r and preferences on this domain satisfy Independence.

Step 2. Now suppose $A \succ \{p^*\}$. We claim that if $A \succ D_r$ (for some D_r, and since preferences are continuous, such a D_r must exist), there exists a unique $\lambda > 0$ such that $\lambda A + (1-\lambda)\{p^*\} \sim D_r$. To see that this is the case, notice that since preferences are continuous, there exists at least one λ such that $\lambda A + (1-\lambda)\{p^*\} \sim D_r$. If there are $\lambda_1 > \lambda_2 > 0$ such that $\lambda_i A + (1-\lambda_i)\{p^*\} \sim D_r$ for $i = 1, 2$, we contradict Step 1. Hence the λ must be unique. Similarly, for each A such that $D_R \succcurlyeq A \succcurlyeq \{p^*\}$, there exists a unique $\lambda \in [0,1]$ such that $A \sim \lambda D_R + (1\lambda)\{p^*\}$. For such an A, let $V(A) := \lambda$ and if $A \succ D_R$, let $V(A) := R/\lambda$. Notice that for $A_\alpha \in \Xi_A$, let $V(A_\alpha) = \alpha V(A)$.

Step 3. For $A \sim \{p^*\}$, let $V(A) = 0$ and also $V(A_\alpha) = 0$, where $A_\alpha := \lambda A + (1-\lambda)\{p^*\}$.

Step 4. Suppose $\{p^*\} \succ A$. We claim that there exists a unique $\lambda \in (0,1)$ such that $\lambda A + (1-\lambda)D_r \sim \{p^*\}$. If this is true, define $V(A)$ such that $\lambda V(A) + (1-\lambda)V(D_r) = V(\{p^*\}) = 0$, ie $V(A) = -r\frac{1-\lambda}{\lambda}$. Now extend V linearly to all Ξ_B that contain A. Clearly, this represents preferences restricted to Ξ_B.

To see that the claim is true, notice first that, since preferences are continuous, there exists a $\lambda \in (0,1)$ such that $\lambda A + (1-\lambda)D_r \sim \{p^*\}$. Suppose λ is not unique, assume there exist λ_1, λ_2 such that $\lambda_i A + (1-\lambda_i)D_r \sim \{p^*\}$, with $\lambda_1 > \lambda_2$. As before, let $\mu := \lambda_2/\lambda_1 < 1$. Then, by D-Independence, $\{p^*\} \sim \lambda_2 A + (1-\lambda_2)D_r = \mu\big(\lambda_1 A + (1-\lambda_1)D_r\big) + (1-\mu)D_r \sim \mu\{p^*\} + (1-\mu)D_R$, which contradicts Step 1.

We shall now verify that V is well defined.

Lemma 5.10. V is well defined.

Proof. Consider first the case where $A \succ \{p^*\}$. Then, for some D_r, $\lambda A + (1-\lambda)\{p^*\} \sim D_r$, giving us $V(A) = r/\lambda$. We need to show that $V(A)$ is independent of the D_r chosen. Suppose $D_s \subset D_r$, so that $D_r \succ D_s$. Letting $\eta := \nicefrac{s}{r}$, we see that $\eta\big[\lambda A + (1-\lambda)\{p^*\}\big] \sim \eta D_r + (1-\eta)\{p^*\} = D_s$. Consistency would require that $\nicefrac{s}{\eta\lambda} = \nicefrac{r}{\lambda}$ which is definitely the case. Similar arguments show this for the case where $s > r$, and also show that the value of $V(A)$ is independent of the Ξ_B it is contained in.

Now to the case where $\{p^*\} \succ A$. Recall that by finding $V(A)$, we can assign values to all A' in any Ξ_B that contains A. So suppose $\lambda A + (1-\lambda)D_r \sim \{p^*\}$ for some $\lambda \in (0,1)$, D_r, giving us $V(A) = -r(1-\lambda)/\lambda$. Suppose D_s is such that $s < r$ and let $\eta := \nicefrac{s}{r}$ so that $A' := \eta A + (1-\eta)\{p^*\}$ and $V(A') = \eta V(A)$. It will suffice to show that $\lambda A' + (1-\lambda)D_s \sim \{p^*\}$. To see that this is the case, notice that $\{p^*\} \sim \eta\big[\lambda A + (1-\lambda)D_r\big] + (1-\eta)\{p^*\} = \lambda\big[\eta A + (1-\eta)\{p^*\}\big] + (1-\lambda)\big[\eta D_r + (1-\eta)\{p^*\}\big] = \lambda A' + (1-\lambda)D_s$, as required. We have thus shown that V is well defined. $\square$

All that remains is to check that the V we have constructed is, in fact, D-linear.

Lemma 5.11. V is D-linear.

Proof. By the definition of V, all we need to verify is that $V\big(\lambda A + (1-\lambda)D_r\big) = \lambda V(A) + (1-\lambda)V(D_r)$ for all A, D_r and $\lambda \in (0,1)$. Assume first that $A \succ \{p^*\}$ and assume, without loss of generality, that $A \sim D_t$. Then, $\lambda A + (1-\lambda)D_r \sim \lambda D_t + (1-\lambda)D_r$ so that $V\big(\lambda A + (1-\lambda)D_r\big) = \lambda t + (1-\lambda)r = \lambda V(A) + (1-\lambda)V(D_r)$.

Now for the case where $\{p^*\} \succ A$. For each D_r, there exists a $\lambda \in (0,1)$ such that $\lambda A + (1-\lambda)D_r \sim \{p^*\}$. Consider first the case where $1 > \mu > \lambda$, so that $\eta := \nicefrac{\lambda}{\mu}$. Then, $\eta\big[\mu A + (1-\mu)D_r\big] + (1-\eta)D_r = \lambda A + (1-\lambda)D_r \sim \{p^*\}$ which means $V\big(\mu A + (1-\mu)D_r\big) = -r(1-\eta)/\eta = -r\frac{\mu-\lambda}{\lambda}$. But $\mu V(A) + (1-\mu)r = -\mu r\frac{1-\lambda}{\lambda} + (1-\mu)r = -r\frac{\mu-\lambda}{\lambda} = V\big(\mu A + (1-\mu)D_r\big)$, as desired. Finally consider the case where $\mu < \lambda$ and once again let $\eta := \nicefrac{\mu}{\lambda}$. Recalling that $\lambda A + (1-\lambda)D_r \sim \{p^*\}$ which means that $V(A) = -r\frac{1-\lambda}{\lambda}$, we see that $\mu A + (1-\mu)D_r = \eta\big[\lambda A + (1-\lambda)D_r\big] + (1-\eta)D_r \sim \eta\{p^*\} + (1-\eta)D_r$, which implies $V\big(\mu A + (1-\mu)D_r\big) = (1-\eta)r = r\frac{\lambda-\mu}{\lambda} = \mu V(A) + (1-\mu)V(D_r)$. $\square$

In sum, we have constructed a function $V : \mathcal{K}(\Delta) \to \mathbb{R}$ that represents $\succcurlyeq$ and is D-linear. We shall now show that if $\succcurlyeq$ also satisfies L continuity, V is actually Lipschitz continuous. It should be noted that the proof of the following lemma exactly the same as the proof of Lemma 1 in Dekel et al (2007). The key difference is that we have a stronger version of L continuity in that we require A^* and A_* to be one of D_R and $\{p^*\}$. Thus, the linearity that is so crucial in the proof of the lemma in Dekel et al (2007), is still present in our setting since V is D-linear.

Definition 5.12. A function $V : \mathcal{K}(\Delta) \to \mathbb{R}$ is Lipschitz continuous if there is an $\bar{N}$ such that $V(B) - V(A) \leqslant \bar{N} d_h(A, B)$ for all A, B.

Lemma 5.13. Let $\succcurlyeq$ have a D-linear representation V where $V(D_R) \neq V(\{p^*\})$. Then V is Lipschitz continuous if and only if $\succcurlyeq$ satisfies L continuity.

As mentioned above, the proof is essentially the same as the proof of Lemma 1 in Dekel et al (2007). Therefore, we only provide a sketch with some details missing, for which the reader is referred to the original.

Sketch of Proof. Suppose $\succcurlyeq$ satisfies L continuity. Fix an $N > 0$ and $M \in (0, 1/N)$ and fix A, B such that $\delta := d_h(A, B) \leqslant M$. Suppose $\delta > 0$ (if not, we are done) so that L continuity implies

$$(1 - N\delta)A + N\delta A^* \succcurlyeq (1 - N\delta)B + N\delta A_*.$$

Using the D-linearity of V, we can rewrite this as

$$V(B) - V(A) \leqslant \frac{N}{1 - N\delta}\big[V(A^*) - V(A_*)\big] d_h(A, B).$$

Let $\bar{N} := \frac{N}{1-NM}\big[V(A^*) - V(A_*)\big]$, so that for all A, B with $d_h(A, B) \leqslant M$, we have $V(B) - V(A) \leqslant \bar{N} d_h(A, B)$.

To show the same is true for arbitrary A and B, fix an A and B and a sequence $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_J < \lambda_{J+1} = 1$ such that $(\lambda_{j+1} - \lambda_j) d_h(A, B) \leqslant M$. Defining $B_j : \lambda_j B + (1 - \lambda_j)A$, so that $d_h(B_{j+1}, B_j) = (\lambda_{j+1} - \lambda_j) d_h(A, B)$. This implies, $V(B_{j+1}) - V(B_j) \leqslant \bar{N}(\lambda_{j+1} - \lambda_j) d_h(A, B)$ and summing over j from 0 to J gives $V(B) - V(A) \leqslant \bar{N} d_h(A, B)$, ie Lipschitz continuity of V.

To see the converse, suppose there is an $\bar{N}$ such that $V(B) - V(A) \leqslant \bar{N} d_h(A, B)$ for all A, B. Let $N := \bar{N}/|V(D_R) - V(\{p^*\})|$. Thus, for all A, B with $d_h(A, B) \leqslant 1/N$ and for every $\varepsilon \in [d_h(A, B), 1/N)$,

$$V(B) - V(A) \leqslant \frac{N\varepsilon}{1 - N\varepsilon} |V(D_R) - V(\{p^*\})|.$$

Rearranging terms gives us the L continuity of $\succcurlyeq$. $\square$

It is easy to see that extending V to $\mathcal{K}(\operatorname{aff}\Delta)$ by positive homogeneity preserves the above properties. Abusing notation, let the extension be denoted by $V : \mathcal{K}(\operatorname{aff}\Delta) \to \mathbb{R}$. (Rather than prove V is concave if preferences satisfy PfU, we shall show that the extension is concave. This is done in the next section.) We shall now use the embedding of $\operatorname{aff}\Delta$ in $C(S_Z)$ and extend V to a dense linear subspace of $C(S_Z)$. It is easy to see that the extension is also Lipschitz continuous.

§ 5.4. *Extension to a Dense Linear Subspace*

Recall that $S_Z := \{p \in \mathbb{R}^{|Z|} : \sum p_i = 0 \text{ and } \|p\| = 1\}$ is isometrically homeomorphic to S^{n-1} and that $\mathcal{K}(\operatorname{aff}\Delta)$ is isometrically embedded as $K^* \in C(S_Z)$ (which has the sup norm). For each $A \in \mathcal{K}(\operatorname{aff}\Delta)$, $h_A \in K^*$. (Also, a function $f \in K^*$ if and only if the unique extension of f to $\mathbb{R}^n$, which is isometrically homeomorphic to $\operatorname{aff}\Delta$, by positive homogeneity is superlinear. Thus, for each $f \in K^*$, there exists an $A \in \mathcal{K}(\operatorname{aff}\Delta)$ such that $f = h_A$.) Let $C^2(S_Z)$ be the space of all twice continuously differentiable functions on S_Z viewed as a subspace of $C(S_Z)$ and define $K_2^* := K \cap C(S_Z)$. We begin by defining a functional on K^*.

Define $\varphi : K^* \to \mathbb{R}$ as $\varphi\big(h_A\big) = V(A)$. Notice that since V is Lipschitz continuous, φ is too. The following property is a weakening of C-additivity introduced by Gilboa and Schmeidler (1989).

Definition 5.14. Let $K \subset C(S_Z)$ be a cone such that $\mathbf{1} \in K$. A function $\varphi : K \to \mathbb{R}$ is C^{+}-*additive* if $\varphi(f + \alpha\mathbf{1}) = \varphi(f) + \varphi(\alpha\mathbf{1})$ for all $f \in K$ and $\alpha \in \mathbb{R}_+$.

Notice that φ is, by definition, positively homogeneous. We shall now show that φ is C^{+}-additive and superlinear.[11]

Lemma 5.15. Let $\varphi : K \to \mathbb{R}$ be defined as above. Then φ is C^{+}-additive.

Proof. Recall that if $v \in K$, $v + \alpha\mathbf{1} \in K$ for all $\alpha \in \mathbb{R}_+$. Now consider $\alpha \in [0, 1)$ and let w such that $(1 - \alpha)w = v$. Then,

$$\begin{aligned}
\varphi(v + \alpha\mathbf{1}) &= \varphi\big((1-\alpha)w + \alpha\mathbf{1}\big) \\
&= V\big((1-\alpha)A_w + \alpha D\big) \\
&= V\big((1-\alpha)A_w\big) + V(\alpha D) \\
&= \varphi\big((1-\alpha)w\big) + \varphi(\alpha\mathbf{1}) \\
&= \varphi(v) + \varphi(\alpha\mathbf{1})
\end{aligned}$$

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[11] Recall that a function is superlinear if it positively homogeneous and concave.

where $A_w \in \mathcal{K}^n$ is such that $w = u_A$. To prove the general case, we shall use the fact that φ is positively homogeneous. Now let $\alpha \in [1, \infty)$. For any such α, there exists $\lambda > 0$ such that $\lambda\alpha < 1$. Then, $\varphi(v + \alpha\mathbf{1}) = (1/\lambda)\varphi(\lambda v + \lambda\alpha\mathbf{1}) = (1/\lambda)\big(\varphi(\lambda v) + \lambda\varphi(\alpha\mathbf{1})\big) = \varphi(v) + \varphi(\alpha\mathbf{1})$ which gives us the desired result. □

And now to finally show that φ is superlinear.

Lemma 5.16. The functions φ as defined above is superlinear.

Proof. We have already established that φ is positively homogeneous. It only remains to establish that φ is concave. To see this, suppose $v, w \in K$ such that $\varphi(v) = \varphi(w)$. Then, there exist $A_v, A_w \in \mathcal{K}^n$ such that $V(A_v) = V(A_w)$ and $u_{A_v} = v$ and $u_{A_w} = w$. Then, $V(A_v + A_w) = 2V\left(\frac{1}{2}(A_v + A_w)\right) \geqslant 2V(A_v)$ by PfU. Also $V(A_v) = \frac{1}{2}\left(V(A_v) + V(A_w)\right)$ so that $V(A_v + A_w) \geqslant V(A_v) + V(A_w)$. Thus, $\varphi(v + w) \geqslant \varphi(v) + \varphi(w)$.

If $v, w \in K$ are such that $\varphi(v) \neq \varphi(w)$, let us suppose $\varphi(v) > \varphi(w)$. Let $\alpha := \big(\varphi(v) - \varphi(w)\big)/\varphi(\mathbf{1})$ and define $z := w + \alpha\mathbf{1}$. Then $\varphi(z) = \varphi(w + \alpha\mathbf{1}) = \varphi(w) + \alpha\varphi(\mathbf{1}) = \varphi(w) + \varphi(v) - \varphi(w)$ by the C⁺-additivity of φ. Using the C⁺-additivity again, we see that $\varphi(v + w) + \varphi(\alpha\mathbf{1}) = \varphi(v + w + \alpha\mathbf{1}) = \varphi(v + z) \geqslant \varphi(v) + \varphi(z) = \varphi(v) + \varphi(w) + \varphi(\alpha\mathbf{1})$, which gives us $\varphi(v + w) \geqslant \varphi(v) + \varphi(w)$, the desired result. □

We now extend the function to $C^2(S_Z)$ in such a way that it preserves C⁺-additivity and superlinearity. Define $\psi : K_2^* \to \mathbb{R}$ as $\psi(f) := \varphi(f)$ for all $f \in K_2^*$. The first step we need to take is to observe that $\operatorname{span} K_2^* = C^2(S_Z)$. Let us state this formally.

Theorem 5.17. $\operatorname{span} K_2^* = C^2(S_Z)$. *In particular, for each $f \in C^2(S_Z)$, there exists a convex body K and $r > 0$ such that $f = h_K - h_{D_r}$.*

This is merely Lemma 1.7.9 from Schneider (1993). Another way of stating this is that for each $f \in C^2(S_Z)$, there exists $\alpha > 0$ such that $f + \alpha\mathbf{1} \in K_2^*$.[12] Moreover, for each $\alpha' > \alpha$, $f + \alpha'\mathbf{1} \in K_2^*$. This is because $f + \alpha'\mathbf{1} = (f + \alpha\mathbf{1}) + (\alpha' - \alpha)\mathbf{1}$, the sum of two convex sets, which is also convex.

We now show φ has a unique extension to $C^2(S_Z)$ that preserves the relevant features of φ.

Lemma 5.18. The function $\varphi : K_2^* \to \mathbb{R}$ has a unique C⁺-additive, superlinear and Lipschitz continuous extension to $C(S_Z)$, $\psi : C(S_Z) \to \mathbb{R}$.

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[12] Here, $\mathbf{1}$ is the constant function with value 1 everywhere.

Proof. We shall first prove that there exists unique C⁺-additive, superlinear and Lipschitz continuous extension of φ to $C^2(S_Z)$. We shall then extend this extension to $C(S_Z)$.

First, define $\psi(-\mathbf{1}) := -\psi(\mathbf{1})$. For $\alpha > 0$, define $\psi(-\alpha\mathbf{1}) := \alpha\psi(-\mathbf{1})$. More generally (from the theorem above), for each $f \in C^2(S_Z)$, there exists $\alpha > 0$ such that $f + \alpha\mathbf{1} \in K_2^*$. Then define,

$$\psi(f) := \psi(f + \alpha\mathbf{1}) - \alpha\psi(\mathbf{1}).$$

Now extend ψ by positive homogeneity. To see that ψ is still well defined, let $\lambda > 0$, and notice that $\lambda f + \lambda\alpha\mathbf{1} \in K_2^*$ and that $\psi\big(\lambda f + \lambda\alpha\mathbf{1}\big) = \lambda\psi(f + \alpha\mathbf{1})$.

We now show that the definition of ψ is independent of the particular α chosen for each f. Suppose $\alpha_1 > 0$ is such that $g + \alpha_1\mathbf{1} \in K_2^*$. As before, define $\psi(g) := \psi(g + \alpha_1\mathbf{1}) - \alpha_1\psi(\mathbf{1})$. Then, for $\alpha_2 > \alpha_1$, we have $\psi(g + \alpha_2\mathbf{1}) - \alpha_2\psi(\mathbf{1}) = \psi\big(g + \alpha_1\mathbf{1} + (\alpha_2 - \alpha_1)\mathbf{1}\big) - \alpha_2\psi(\mathbf{1}) = \psi(g + \alpha_1\mathbf{1}) + (\alpha_2 - \alpha_1)\psi(\mathbf{1}) - \alpha_2\psi(\mathbf{1}) = \psi(g)$ (where we have used the fact that ψ is C⁺-linear on K^*). Thus, the definition of ψ doesn't depend on the choice of $\alpha > 0$.

We also want to check that ψ is C⁺-additive everywhere on its domain. To see this, let $f \in C^2(S_Z)$ and let $\beta > 0$ be such that $f + \beta\mathbf{1} \in K_2^*$. Clearly, for all $\alpha \geqslant \beta$, $\psi(f + \alpha\mathbf{1}) = \psi(f) + \alpha\psi(\mathbf{1})$. Now suppose $\alpha \in (0, \beta)$. As before, $\psi(f) = \psi(f + \beta\mathbf{1}) - \beta\psi(\mathbf{1})$. Thus, $\psi(f + \alpha\mathbf{1}) = \psi(f + \beta\mathbf{1}) - (\beta - \alpha)\psi(\mathbf{1}) = \psi(f) + \alpha\psi(\mathbf{1})$, which shows that ψ is C⁺-additive.

Now to check that ψ is superlinear. It is clear that $\varphi : K_2^* \to \mathbb{R}$ is superlinear, hence for all $f, g \in K_2^*$, $\psi(f + g) \geqslant \psi(f) + \psi(g)$. Suppose $f, g \in C^2(S_Z)$ and $\psi(f) = \psi(g)$. Then there exists an $\alpha > 0$ such that $f + \alpha\mathbf{1}, g + \alpha\mathbf{1} \in K_2^*$. Then $\psi(f + g) = \psi(f + g + 2\alpha\mathbf{1}) - 2\alpha\psi(\mathbf{1}) \geqslant \psi(f + \alpha\mathbf{1}) - \alpha\psi(\mathbf{1}) + \psi(g + \alpha\mathbf{1}) - \alpha\psi(\mathbf{1}) = \psi(f) + \psi(g)$.

Suppose $\psi(f) > \psi(g)$. Let $\alpha\psi(\mathbf{1}) := \psi(f) - \psi(g)$ and define $\tilde{f} := g + \alpha\psi(\mathbf{1})$. Then, $\psi(\tilde{f}) = \psi(g + \alpha\mathbf{1}) = \psi(g) + \alpha\psi(\mathbf{1}) = \psi(f)$. Therefore, $\psi(f + g) + \alpha\psi(\mathbf{1}) = \psi(f + g + \alpha\mathbf{1}) = \psi(f + \tilde{f}) \geqslant \psi(f) + \psi(\tilde{f}) = \psi(f) + \psi(g) + \alpha\psi(\mathbf{1})$, ie $\psi(f + g) \geqslant \psi(f) + \psi(g)$. Thus, ψ is superlinear.

We conclude with the simple observation that since φ is Lipschitz continuous, so is ψ. Thus, there is a unique extension of ψ to $C(S_Z)$ that we shall also denote ψ. Moreover, this extension is also superlinear and C⁺-additive. □

§ 5.5. *Proof of Theorem 2.2*

Now that we constructed a D-linear utility representation for our preference and extended this utility representation to a Banach space, we are ready to proceed to the final steps of our proof.

We start with the observation that since $\psi : C(S_Z) \to \mathbb{R}$ is a superlinear function, there exists a nonempty convex set $\mathcal{L}$ of continuous linear functions on $C(S_Z)$ such that

$$\psi(f) = \min\left\{L(f) : L \in \mathcal{L}\right\} \text{ for all } f \in C(S_Z).$$

By the Riesz Representation Theorem, for each continuous linear functional L, there exists a regular, countably additive, signed measure π such that $L(f) = \langle f, \pi\rangle$. Thus, we can represent ψ as

$$\psi(f) = \min\left\{\langle f, \pi\rangle : \pi \in \Pi\right\}$$

where Π is a weak* compact, convex set in $M(S_Z)$, the space of all signed, finite Borel measures on S_Z. It is possible to show directly that Π is unique, but we shall employ the following indirect method which will be useful in the sequel.

Recall that $C(S_Z)$ is a Banach space and $M(S_Z)$ is its dual. Indeed, $\langle C(S_Z), M(S_Z)\rangle$ is a dual pair. By Theorem 6.27 of Aliprantis and Border (1999), we know that the norm topology in $C(S_Z)$ is also the Mackey topology in $C(S_Z)$ (more precisely, in $C(S_Z)$ for the dual pair $\langle C(S_Z), M(S_Z)\rangle$). Thus, ψ is Mackey continuous.

There is the obvious correspondence between sublinear and superlinear functions. Since $\psi(f) = \min\{\langle f, \pi\rangle : \pi \in \Pi\}$ is superlinear, $-\psi$ is sublinear. Moreover, $-\psi(f) = \max\{\langle f, \pi\rangle : \pi \in -\Pi\}$. Finally, by Theorem 5.102 of Aliprantis and Border (1999), we see that there is a bijection between the space of Mackey continuous sublinear functions on $C(S_Z)$ and the space of weak* compact, convex subsets of $M(S_Z)$, so that Π is unique. Indeed, $-\psi$ is the support function of $-\Pi$.

The *hypograph* of a function ψ is given by $\{(f, t) \in C(S_Z) \times \mathbb{R} : \psi(f) \geqslant t\}$. It is easy to see that since ψ is superlinear, the hypograph has a nonempty interior, so that for each $f \in C(S_Z)$, there exists $\pi_f \in \Pi$ such that $\psi(f) = \left\langle f, \pi_f\right\rangle$. Moreover, for each $\pi \in \operatorname{ext}\Pi$ (where $\operatorname{ext}\Pi$ represents the set of extreme points of Π), there exists an $f_\pi \in C(S_Z)$ such that $\psi(f_\pi) = \langle f_\pi, \pi\rangle$.

Recall that ψ is C$^+$-additive. Then, for any $\pi \in C(S_Z)$, let $f_\pi =: g + \alpha\mathbf{1}$, where $\alpha > 0$ and f_π is such that $\psi(f_\pi) = \langle f, \pi\rangle$. Then, $\psi(g + \alpha\mathbf{1}) = \psi(g) + \alpha\psi(\mathbf{1}) = \psi(f) = \langle f, \pi\rangle = \langle g, \pi\rangle + \alpha\,\langle \mathbf{1}, \pi\rangle$. Since $\psi(g) \leqslant \langle g, \pi\rangle$ and $\psi(\mathbf{1}) \leqslant \langle \mathbf{1}, \pi\rangle$, it must be the case that $\psi(\mathbf{1}) = \langle \mathbf{1}, \pi\rangle$.

Since we have assumed $D_R \succ \{p^*\}$, we can set $\varphi(\mathbf{1}) = 1$. With the observation that assuming PfD instead of PfU leads to a change in the direction of some inequalities above, we get both the (♠) and (♣) representations for this case.

We now consider the case where $\{p^*\} \succ D_R$. Define the preference relation $\succcurlyeq^* \subset \mathcal{F}(\Delta) \times \mathcal{F}(\Delta)$ as follows: $A \succcurlyeq^* B$ if and only if $B \succcurlyeq A$. Then, $D_R \succ^* \{p^*\}$ and $\succcurlyeq^*$ satisfies IR and is continuous. Now suppose $A \succ^* B$, so that $B \succ A$. Then, by D-Independence, for all $\lambda \in (0,1]$, $\lambda A + (1-\lambda)D_R \succ \lambda B + (1-\lambda)D_R$, ie, $\lambda B + (1-\lambda)D_R \succ^* \lambda A + (1-\lambda)D_R$, with a similar condition holding for the second part of D-Independence. Thus, $\succcurlyeq^*$ satisfies D-Independence. Finally, notice that $\succcurlyeq$ satisfies PfD (resp PfU) if and only if $\succcurlyeq^*$ satisfies PfU (resp PfD).

Thus, given preference relation $\succcurlyeq$ that satisfies IR, Continuity and L Continuity, D-Independence and PfD (resp PfU) and has $\{p^*\} \succ D_R$, there exists a preference relation $\succcurlyeq^*$ that satisfies IR, D-Independence and PfU (resp PfD) and has $D_R \succ \{p^*\}$ with the property that $A \succcurlyeq B$ if and only if $B \succcurlyeq^* A$. But by the proof above, we know that $\succcurlyeq^*$ has a utility representation $V^* : \mathcal{F}(\Delta) \to \mathbb{R}$ that admits a $(\spadesuit)$ representation:

$$V^*(A) := \min \left\{ \langle u_A, \pi \rangle : \pi \in \Pi \right\}.$$

Now define, $V : \mathcal{F}(\Delta) \to \mathbb{R}$ as follows: $V(A) = -V^*(A)$. Then, V represents $\succcurlyeq$, and can be written as

$$V(A) := \max \left\{ \langle u_A, \mu \rangle : \mu \in -\Pi \right\},$$

which is the $(\spadesuit)$ representation for $\succcurlyeq$. Similarly, if $\succcurlyeq$ satisfies PfU, using the arguments above we can show it has a $(\spadesuit)$ representation. This concludes the proof of Theorem 2.2. Before we proceed, we state, without proof, a simple proposition that follows from the construction of the set of support functions Π.

Proposition 5.19. For each $\pi \in \operatorname{ext} \Pi$, there exists a menu A such that $V(A) = \langle u_A, \pi \rangle$.

§ 5.6. *Proof of Theorem 2.6*

As in the text, say that $\succcurlyeq_2$ has a greater desire for uniformity than $\succcurlyeq_1$ if, for each A where $D_R \succ_1 A \succ_1 \{p^*\}$, $\lambda D_R + (1-\lambda)\{p^*\} \sim_1 A$ implies $\lambda D_R + (1-\lambda)\{p^*\} \succcurlyeq_2 A$. Let V_i represent $\succcurlyeq_i$ and be such that $V_i(\{p^*\}) = 0$ and $V(D_R) = R$. Clearly, V_i is D-linear. We shall first show that $\succcurlyeq_2$ is more pessimistic that $\succcurlyeq_1$ if and only if for each menu A, $V_1(A) \geqslant V_2(A)$.

Proposition 5.20. The preference $\succcurlyeq_2$ has a greater desire for uniformity than $\succcurlyeq_1$ if and only $V_1(A) \geqslant V_2(A)$ for each menu A.

Proof. Notice that for any A such that $D_R \succ_1 A \succ_1 \{p^*\}$, $\succcurlyeq_2$ has a greater desire for uniformity than $\succcurlyeq_1$ if and only if $V_1(A) \geqslant V_2(A)$, since $V_1(A) = V_1(\lambda_A D_R + (1-\lambda_A)\{p^*\}) = V_2(\lambda_A D_R + (1-\lambda_A)\{p^*\}) \geqslant V_2(A)$.

Now consider an A where $\{p^*\} \succcurlyeq_1 A$. For such an A, there exists a μ such that $D_R \succ_1 \mu A + (1-\mu)D_R \succ_1 \{p^*\}$ and for such a μ, $V_1(\mu A + (1-\mu)D_R) \geqslant V_2(\mu A + (1-\mu)D_R)$ which is true if and only if $V_1(A) \geqslant V_2(A)$ (using the D-linearity of V_i).

Similarly, for a menu A with $A \succcurlyeq_1 D_R$, there exists a μ such that $D_R \succ_1 \mu A + (1-\mu)\{p^*\} \succ_1 \{p^*\}$. Once again using the D-linearity of the V_i, we see that $V_1(A) \geqslant V_2(A)$ and that this is equivalent to saying that $\succcurlyeq_2$ has a greater desire for uniformity than $\succcurlyeq_1$. $\square$

Recall from the proof of theorem 2.2, the existence (and construction) of a unique function $\psi : C(S_Z) \to \mathbb{R}$ such that for any menu A, $\psi(h_A) = V(A)$. Moreover, ψ is C^{+}-additive, sublinear and continuous (and so is Lipschitz continuous). We now show that if $\succcurlyeq_2$ is more pessimistic than $\succcurlyeq_1$, it is the case that $\psi_1 \geqslant \psi_2$. To see this, let $f \in C^2(S_Z)$. Therefore, there is an $\alpha > 0$ such that $f + \alpha \mathbf{1}$ is sublinear and therefore the support function of some compact, convex set. Thus, for some $\lambda > 0$, there exists a menu A with support function $h_A = ({}^{1}\!/\!{}_{\lambda})(f + \alpha\mathbf{1})$. Using the C^{+}-additivity of the ψ_i's, we see that $\psi_1(f) \geqslant \psi_2(f)$. Using the density of $C^2(S_Z)$ and the continuity of ψ_i, we reach the desired conclusion.

As established in §5.5, ψ_i is Mackey continuous and sublinear, hence $\psi_1 \geqslant \psi_2$ implies $\Pi_1 \subset \Pi_2$ (see §5.19, Aliprantis and Border, 1999). It follows immediately that $S_1 \subset S_2$. We have thus shown that if $\succcurlyeq_2$ is more pessimistic than $\succcurlyeq_1$, then $\Pi_1 \subset \Pi_2$. The converse follows from the fact that $\Pi_1 \subset \Pi_2$ implies $\psi_1 \geqslant \psi_2$ so that $V_1(A) \geqslant V_2(A)$ for any menu A.

§ 5.7. *Proofs of Theorems 3.4 and 3.5*

Recall that a state $s_i \in S$ is *negative* if there exist menus A, B where $\max_{p \in A} u(p, s_i) < \max_{p \in B} u(p, s_i)$ and $\max_{p \in A} u(p, s_j) = \max_{p \in B} u(p, s_j)$ for all $j \neq i$ such that $V(A) > V(B)$ and *positive* if there exist menus A, B where $\max_{p \in A} u(p, s_i) > \max_{p \in B} u(p, s_i)$ and $\max_{p \in A} u(p, s_j) = \max_{p \in B} u(p, s_j)$ for all $j \neq i$ such that $V(A) > V(B)$. Also, $\Pi_i := \operatorname{proj}_i \Pi$ for $i = 1, \dots, m$, is the projection of Π onto the i-th axis. For menu A, let $u_A := \left(\max_{p \in A} u(p, s_i)\right)_{i=1}^{m} = (u_A^1, \dots, u_A^m)$ and let $\boldsymbol{e}_i := (0, \dots, 1, \dots, 0)$ with 1 being the i-th entry.

For any menu A, let $\pi_A \in \Pi$ satisfy $V(A) = \langle \pi_A, u_A \rangle$ and $\operatorname{ext} \Pi$ is the set of extreme points of Π. In proving the theorem, it is useful to recall proposition 5.19, which states that for any $\pi \in \Pi$ which is an extreme point, there exists a menu A such that $V(A) = \langle u_A, \pi \rangle$. In other words, every such π is the minimum weight for some menu. Moreover (by D-Independence), the menu can be taken to have a relative interior.

As noted in the text, it will suffice to prove Propositions 3.2 and 3.3. We shall prove these in turn.

Proof of 3.2. (i) Suppose state s_i is positive. Then, there exist menus A, B such that $u_A - u_B = (u_A^i - u_B^i)\boldsymbol{e}_i > \mathbf{0}$ (where $u_A^i - u_B^i > 0$) and $V(A) > V(B)$. Suppose Π_i is not positive, so $\Pi_i \subset (-\infty, 0]$. Then, $\langle \pi_B, u_A - u_B \rangle \leqslant 0$ since $\pi_B^i \leqslant 0$ and $u_A^i - u_B^i > 0$. Thus, $\langle \pi_B, u_A \rangle \leqslant \langle \pi_B, u_B \rangle$. Since $V(B) < V(A)$, we also have $\langle \pi_B, u_B \rangle < \langle \pi_A, u_A \rangle$. By the definition of π_A and $V(A)$, it must be that $V(A) = \langle \pi_A, u_A \rangle \leqslant \langle \pi_B, u_A \rangle$. Combining these inequalities, we get a contradiction.

Similarly, suppose state s_i is negative and by way of contradiction, Π_i is not negative, so that $\Pi_i \subset [0, \infty)$. Since s_i is negative, there exist menus A, B such that $u_A - u_B = (u_A^i - u_B^i)\boldsymbol{e}_i > \mathbf{0}$ (where $u_A^i - u_B^i > 0$) and $V(A) < V(B)$. Since $\pi_A^i \geqslant 0$, we have $\langle \pi_A, u_A - u_B \rangle \geqslant 0$, so that $\langle \pi_A, u_A \rangle \geqslant \langle \pi_A, u_B \rangle$. By the definition of π_B and $V(B)$, $\langle \pi_A, u_B \rangle \geqslant \langle \pi_B, u_B \rangle$. Finally, since $V(B) > V(A)$, we have $\langle \pi_B, u_B \rangle > \langle \pi_A, u_A \rangle$. Combining these inequalities, we get a contradiction which concludes the proof. ☐

The proof of Proposition 3.3 is similar.

Proof of Proposition 3.3. Suppose $\pi_A \in \operatorname{ext} \Pi$ is such that $\pi_A^i > 0$. By proposition 5.19, there exists a menu A that has a relative interior and has $V(A) = \langle \pi_A, u_A \rangle$. Let B be a menu such that $u_A - u_B = \varepsilon \boldsymbol{e}_i$, $\varepsilon > 0$. (Such a menu exists since A has a relative interior.) We claim that $V(A) > V(B)$. To see this, notice that $V(B) - V(A) = \langle \pi_B, u_B \rangle - \langle \pi_A, u_A \rangle \leqslant \langle \pi_A, u_B \rangle - \langle \pi_A, u_A \rangle = \langle \pi_A, u_B - u_A \rangle = \langle \pi_A, -\varepsilon \boldsymbol{e}_i \rangle < 0$. Thus, state s_i is a positive state.

Now suppose $\pi_A \in \operatorname{ext} \Pi$ is such that $\pi_A^i < 0$. Then, there exists a menu A that has a relative interior and has $V(A) = \langle \pi_A, u_A \rangle$. Let B be a menu such that $u_B - u_A = \varepsilon \boldsymbol{e}_i$. Thus, $\langle \pi_A, u_B - u_A \rangle = \langle \pi_A, \varepsilon \boldsymbol{e}_i \rangle < 0$. This means that $V(B) =: \langle \pi_B, u_B \rangle \leqslant \langle \pi_A, u_B \rangle < \langle \pi_A, u_A \rangle = V(A)$, so that s_i is a negative state. ☐

Recall that, by our normalisations, $V(D) = 1$ if $D \succ \{q^*\}$ and $V(D) = -1$ if $\{q^*\} \succ D$ and in either case, $V(\{q^*\}) = 0$. Moreover, for any $\pi \in \Pi$, $\sum_{i=1}^m \pi^i = V(D)$ (this is Theorem 2.2). The proof of theorem 3.5 now follows from Proposition 3.3 and from this normalisation of utilities.

§ 5.8. *Special Cases*

Here we shall prove Propositions 3.9 and 3.10 and Theorem 3.11.

Proof of Proposition 3.9. Suppose not, so suppose there is $\pi_A \in \operatorname{ext} \Pi$ such that $\pi_A^1, \pi_A^2 > 0$. Then, by proposition 5.19, there exists a menu A with a relative interior such that $V(A) = \langle \pi_A, u_A \rangle$. Now, for $j = 1, 2$, define B_j such that $u_{B_j}^i = u_A^i$ for $i \neq j$ and $u_{B_j}^j = u_A^j - \varepsilon > 0$ for some appropriate $\varepsilon > 0$. Then, $V(B_j) = \left\langle \pi_{B_j}, u_{B_j} \right\rangle \leqslant \left\langle \pi_A, u_{B_j} \right\rangle < \langle \pi_A, u_A \rangle = V(A)$. Thus, $V(A) = V(B_1 \cup B_2) > V(B_1), V(B_2)$, contradicting Positive Set Betweenness. $\square$

The proof of Proposition 3.10 follows from a simple transformation. Consider a preference relation $\succcurlyeq$ that admits a finite multi-weight representation, V. Now define the relation $\succcurlyeq^*$ so that for any two menus $A, B \in \mathcal{F}(\Delta)$, $A \succcurlyeq B$ if and only if $B \succcurlyeq^* A$. Then, $\succcurlyeq^*$ admits a finite multi-weight representation, namely $-V$.

Now suppose $\succcurlyeq$ satisfies Negative Set Betweenness, ie $A \succcurlyeq B$ implies $A \cup B \succcurlyeq B$. Then, $\succcurlyeq^*$ satisfies Positive Set Betweenness, ie, $B \succcurlyeq^* A$ implies $B \succcurlyeq^* B \cup A$. The proof of proposition 3.10 now follows from the observation that for any finite additive EU representation U_π that has exactly one positive state, $-U_\pi$ has exactly one negative state.

Proof of Theorem 3.11. We shall only prove that (i) implies (ii). That (ii) implies (i) is trivial.

The above proofs tell give us part of the characterisation theorem 3.11. To show that a state cannot be both positive and negative, we construct a counterexample.

Notice that if preferences satisfy Set Betweenness, it must be the case that $|S| = 2$. Now suppose a state is both positive and negative. In other words, $\Pi = \text{conv}\{\pi, \mu\}$ where $\pi^1 + \pi^2 = \mu^1 + \mu^2 \in \{-1, 1\}$. For simplicity, let us assume that $\pi^1 + \pi^2 = 1$. Suppose, $\pi = (-\pi_1, 1 + \pi_1)$ and $\mu = (\mu_1, 1 - \mu_1)$ and suppose utilities to two singletons are as given below:

	u^1	u^2
s	$\alpha + \varepsilon$	0
c	0	α

where $\alpha, \varepsilon > 0$. Notice that by propositions 3.9 and 3.10, it must be that $\pi_1, \mu_1 > 1$.

It is easily seen that for any menu A, $\langle \pi, u_A \rangle \leqslant \langle \mu, u_A \rangle$ if and only if $(\pi^1 - \mu^1)(u_A^1 - u_A^2) \leqslant 0$. Therefore, $V(\{s\}) = \left\langle \pi, u_{\{s\}} \right\rangle = -\pi_1(\alpha + \varepsilon 0$, $V(\{c\}) = \left\langle \mu, u_{\{c\}} \right\rangle = (1 - \mu_1)\alpha$ and $V(\{s, c\}) = \left\langle \pi, u_{\{s,c\}} \right\rangle = -\pi_1(\alpha + \varepsilon) + (1 + \pi_1)\alpha$.

Clearly, $V(\{s, c\}) > V(\{s\})$ if and only if $-\pi_1\varepsilon + \alpha > -\pi_1\alpha$ and $V(\{s, c\}) > V(\{c\})$ if and only if $-\pi_1\varepsilon + \alpha > (1 - \mu_1)\alpha$. The first inequality holds if and only if $\pi_1(\alpha - \varepsilon) + \alpha > 0$ and the second inequality holds if and only if $-\pi_1\varepsilon + \mu_1\alpha > 0$. It is easy to see that for any $\pi_1, \mu_1 > 1$, we can satisfy the two inequalities by taking α sufficiently large and ε sufficiently small, giving us the desired violation of Set Betweenness. $\square$

We now provide a proof of proposition 3.12.

Proof of Proposition 3.12. We shall only prove that (i) implies (ii). The other implication is trivial. Let us assume that V is a multi-weight representation and satisfies Set Betweenness and Singleton Independence. From theorem 3.11, we see that $|S| \leqslant 2$. Clearly, the only case of interest is where $|S| = 2$.

We shall first assume that for $S := \{s_1, s_2\}$, $s_1 \neq -s_2$. Let

$$K_S := \left\{u \in \mathbb{R}^2 : u = \big(u_A(s_1), u_A(s_2)\big) \text{ for some } \; A \in \mathcal{K}(\operatorname{aff} \Delta)\right\}$$

and let

$$F_S := \left\{u \in \mathbb{R}^2 : u = \big(u_{\{p\}}(s_1), u_{\{p\}}(s_2)\big) \text{ for some } \; p \in \operatorname{aff} \Delta\right\}.$$

Clearly, $F_S \subset K_S$. But since $F_S = \mathbb{R}^2$ (see proposition 5.2), it follows that $F_S = K_S$. But if V satisfies Singleton Independence, then it must be linear on F_S and consequently on K_S, which proves our claim.

We now deal with the case where $S := \{s, -s\}$. In this case, it is easy to see that $F_S := \{u \in \mathbb{R}^2 : u^1 + u^2 = 0\}$ and $K_S := \{u \in \mathbb{R}^2 : u^1 + u^2 \geqslant 0\}$. Since V is linear on F_S, it must be the case that for all $\pi, \mu \in \Pi$, $\langle u, \pi - \mu \rangle = 0$, where $u \in F_S$ that is not the zero vector. But this implies that $\pi_1 u^1 + (1 - \pi_1)(-u^1) = \mu_1 u^1 + (1 - \mu_1)(-u^1)$ which holds if and only if $\pi_1 = \mu_1$, ie if and only if $\pi = \mu$. Thus, Π must be a singleton which proves our claim. □

§ 5.9. *Finite Subjective State Space*

DLR's IR axiom for non-trivial continuous preferences ensures the presence of a unique subjective state space. However, this state space can have any cardinality between 2 and $2^{\aleph_0}$. In this section, we describe the axioms we must impose on preferences to ensure the existence of a *finite* subjective state space. The first axiom that we shall assume throughout this section is D-Independence. The second axiom, Finiteness, is introduced towards the end of the section. Recall that the space of all closed convex subsets of Δ is denoted by $\mathcal{K}(\Delta)$ and $N_\varepsilon(s)$ is the ε-neighbourhood of $s \in \mathbb{R}^n$.

Definition 5.21. A point $s \in S_Z$ is ***strongly relevant*** if for every $\varepsilon > 0$ there exist $A, A' \in \mathcal{K}(\Delta)$ such that $A \nsim A'$ and $h_A(s) = h_{A'}(s)$ for all $s \in S_Z \setminus N_\varepsilon(s)$.

Notice that we can take, without loss of generality, $A \subset A'$. This is because $\operatorname{conv}(A \cup A')$ is not indifferent to either A or A' and $A, A' \subset \operatorname{conv}(A \cup A')$. This follows from the fact that $h_{\operatorname{conv}(A \cup A')} = h_A \vee h_{A'} = \max\{h_A, h_{A'}\}$. In the presence of D-Independence, we can say even more about A and A'.

Proposition 5.22. Let $D_r(p^*) \subset \operatorname{ri}\Delta$. Then, we can take A, A' such that for all $\delta > 0$, $d_h(A, D_r) < \delta$ and $A' \subset \operatorname{ri} D_r$.

Proof. First note that, without loss of generality, we can take $A, A' \subset \operatorname{ri} D_r$. To see this, take $\varepsilon > 0$ and since s is strongly relevant, there exist A, A' such that $h_A(s) = h_{A'}(s)$ for all $s \in S_Z \smallsetminus N_\varepsilon(s)$ and $A \not\sim A'$. As noted above, we can assume $A \subset A'$. Now, by D-Independence, $\lambda A + (1-\lambda)\{p^*\} \not\sim \lambda A' + (1-\lambda)\{p^*\}$ for all $\lambda \in (0,1)$. For λ sufficiently small, $\lambda A' + (1-\lambda)\{p^*\} \subset D_r$. Moreover, since $h_{\lambda A+(1-\lambda)\{p^*\}}(s) = \lambda h_A(s) + (1-\lambda)h_{\{p^*\}}(s)$ and $h_{\{p^*\}}(s) = 0$ for all $s \in S_Z$, it follows that $h_{\lambda A+(1-\lambda)\{p^*\}}(s) = h_{\lambda A'+(1-\lambda)\{p^*\}}(s)$ for all $s \in S_Z \smallsetminus N_\varepsilon(s)$.

Since $A \subset A' \subset \operatorname{ri} D_r$, it follows that $\lambda A + (1-\lambda)D_r \subset \lambda A' + (1-\lambda)D_r \subset \operatorname{ri} D_r$ for all $\lambda \in (0,1)$. Moreover, for any $\delta > 0$, there exists $\lambda \in (0,1)$ sufficiently small such that $d_h(\lambda A + (1-\lambda)D_r) < \delta$. Finally, by D-Independence, $\lambda A + (1-\lambda)D_r \not\sim \lambda A' + (1-\lambda)D_r$ for all $\lambda \in (0,1)$. Since, $h_{D_r}(s) = r$ for all $s \in S_Z$, it follows that $h_{\lambda A+(1-\lambda)D_r}(s) = h_{\lambda A'+(1-\lambda)D_r}(s)$ for all $s \in S_Z \smallsetminus N_\varepsilon(s)$. □

Notice that for any $r > 0$ $D_r \subset \operatorname{ri}\Delta$ (where $D_r := D_r(p^*)$), D_r is isometrically homeomorphic to $r \operatorname{conv} S_Z$. Of course, in a very obvious sense, D_1 *is* $\operatorname{conv} S_Z$. Therefore, for any $s \in S_Z$, there exists a unique $p_s \in D_r$ such that $h_{D_r}(s) = \langle (p_s - p^*)/r, p_s - p^* \rangle = r$. Clearly, $p_s \in \operatorname{bd} D_r$. Conversely, for any point $p \in \operatorname{bd} D_r$, there exists a unique $s_p \in S_Z$ such that $h_{D_r}(s_p) = \langle (p-p^*)/r, p - p^* \rangle = r$. Moreover, for any $s \in S_Z$, $s_{p_s} = s$ and for any $p \in D_r$, $p_{s_p} = p$.

Recall that $S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$ and $\boldsymbol{e}_n := (0, \ldots, 1)$. If $x = (x_1, \ldots, x_{n-1}, 1-\varepsilon) \in S^{n-1}$, $\langle x, \boldsymbol{e}_n \rangle = 1 - \varepsilon$ and $x_1^2 + \cdots + x_{n-1}^2 = 1 - (1-\varepsilon)^2 = \varepsilon(2-\varepsilon)$. This implies $\|x - \boldsymbol{e}_n\|^2 = x_1^2 + \cdots + x_{n-1}^2 + (1 - \varepsilon - 1)^2 = \varepsilon(2-\varepsilon) + \varepsilon^2 = 2\varepsilon$, so that $\|x - \boldsymbol{e}_n\| = \sqrt{2\varepsilon}$.

For any point $p \in \operatorname{bd} D_r$, define the function $\psi_p : \Delta \to \mathbb{R}$ as follows:

$$\psi_p(q) := \frac{\langle p - p^*, q - p^* \rangle}{r^2}.$$

Clearly, $\psi_p(p^*) = 0$ and $\psi_p(p) = 1$. Also, the range of ψ_p for any $p \in \operatorname{bd} D_r$ is a compact set. For any $K \in \mathcal{K}(\Delta)$, we may consider $\max_{q \in K} \psi_p(q)$, the largest height in the direction of $p - p^*$ of all the lotteries in the menu K, ie the height of the menu K in the direction $p - p^*$. For instance, $h_{D_r}(s) = \max_{q \in D_r} \psi_{p_s}(q) = r$. We will be interested in the directions where increasing the height of a menu can change the value of the menu. The following definition is the central idea in this section.

Definition 5.23. A point $p \in \operatorname{bd} D_r$ is ***critical to*** $\succcurlyeq$ if for every $\varepsilon > 0$, there exist $A, A' \in \mathcal{K}(\Delta)$ with $A \nsim A'$ such that for all $\widetilde{p} \in \operatorname{bd} D_r \setminus N_\varepsilon(\widetilde{p})$, $\max_{q \in A} \psi_{\widetilde{p}}(q) = \max_{q \in A'} \psi_{\widetilde{p}}(q)$.

The intuitive idea is that p is critical if there exist menus who have equal heights in all directions $\widetilde{p} - p^*$ (where $\widetilde{p} \in D_r$) save a set of directions $p' - p^*$ where p' is an arbitrarily small neighbourhood of p. The following proposition makes this clear.

Proposition 5.24. If $s \in S^{n-1}$ is strongly relevant, p_s is critical to $\succcurlyeq$. Conversely, if p is critical to $\succcurlyeq$, s_p is strongly relevant.

Proof. First, let $s \in S^{n-1}$ be strongly relevant and fix $\varepsilon > 0$. Notice also that for any other $t \in S^{n-1}$, $\|t - s\| = \|p_t - p_s\| / r$ so that $\|s - t\| > \varepsilon / r$ if and only if $\|p_t - p_s\| > \varepsilon$. Since s is strongly relevant,, for such a $t \in S^{n-1} \setminus N_{\varepsilon/r}(s)$ there exist $A, A' \in \mathcal{K}(\Delta)$ with $A \nsim A'$ such that $h_A(t) = h_{A'}(t)$. But this is the same as saying $\max_{q \in A} \psi_{p_t}(q) = \max_{q \in A'} \psi_{p_t}(q)$ so that p_s is critical to $\succcurlyeq$. The converse is proved similarly. ☐

This brings us to the following axiom and result.

Axiom (Finiteness). The set of points critical to $\succcurlyeq$ is finite.

Theorem 5.25. *A continuous preference relation $\succcurlyeq$ satisfies IR and Finiteness if and only if it has a weak EU representation with a finite state space.*

Proof. Sufficiency is clear. To see necessity, assume that $\succcurlyeq$ has a weak EU representation V and a subjective state space S that is not finite. Then the set of points critical to $\succcurlyeq$ is also not finite, a contradiction. ☐

It should be pointed out that our axiom for ensuring the finiteness is rather stringent, especially when compared to the axiom used by Dekel, Lipman and Rustichini (2007) to ensure the finiteness of the state space. The reason for this is that in the presentation above, we are working with continuous preferences that only satisfy IR. Dekel, Lipman and Rustichini (2007) on the other hand need Independence to prove the finiteness of the state space, and noted in DLR Independence is strictly stronger than IR.

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