III. Uncertainty, Expected Utility, and Competitive Equilibrium

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1 Introduction

For some of the topics we want to discuss, we need models with uncertainty. We are all familiar with the general idea of uncertainty. We are uncertain about tomorrow's weather, about whether we will wake up with a headache tomorrow morning, and about whether someone's estimate of the labor required to repair our car is correct. Considerable effort is directed toward coping with uncertainty. Some farmers have costly irrigation systems in order to make output less dependent on variations in rainfall. And many of us buy insurance of various sorts to limit our exposure to some kinds of uncertainty. Moreover, there are government programs like disaster aid and unemployment insurance that are intended to offset some of the effects of uncertainty.

In order to proceed, we need four concepts from probability theory: sets of outcomes; probability distributions; random variables; and expected values. The set of outcomes describes all possible occurrences. An example is all possible levels of rainfall in a world in which rainfall is the only source of uncertainty. We will always be assuming that the set of outcomes has a finite number of elements. A probability distribution is a function that describes the probability with which each subset of outcomes in the set of outcomes occurs. A random variable is any function whose domain is the set of outcomes and whose range is the real numbers. The expected value of a random variable is its average value computed using the probability distribution over outcomes.

2 Preferences under uncertainty

We assume that people evaluate distributions of consumption by what is called *expected utility.* Let $\{c_1, c_2, ..., c_K\}$ be a set of possible consumptions of a single underlying good for some person and let π_i denote the probability of the outcome

c_i, where $\pi_i \geq 0$ and $\sum_{i=1}^{K} \pi_i = 1$. Let u denote the utility that this person assigns to sure or certain outcomes. (An example is the kind of u function that we have used in describing discounted utility.) The *expected utility hypothesis* is that the utility that this person assigns to the above distribution is

$$\pi_1 u(c_1) + \pi_2 u(c_2) + ... + \pi_K u(c_K) = \sum_{i=1}^{K} \pi_i u(c_i). \tag{1}$$

Exercise 1 *Suppose $K = 2$ in (1). Also suppose that u is strictly increasing and strictly concave. Consider the equation $\pi_1 u(c_1) + \pi_2 u(c_2) = b$ for given π_1, π_2, and b. There are many pairs (c_1, c_2) that satisfy this equation. Sketch such pairs in a diagram.*

Exercise 2 *Let x satisfy the equation $\pi_1 u(x) + \pi_2 u(x) = b$. (i) In the diagram of the last exercise, plot all the pairs (x_1, x_2) that satisfy the equation $\pi_1 x_1 + \pi_2 x_2 = x$. (ii) If (x_1, x_2) satisfies the equation $\pi_1 x_1 + \pi_2 x_2 = x$ and $x_1 \neq x_2$, argue that $\pi_1 u(x_1) + \pi_2 u(x_2) < b$.*

3 An uncertainty interpretation of 2-good pure exchange

We saw that the static 2-good model can be interpreted to produce a model of inter-temporal trade. It can also be interpreted to deal with trade involving distributions. This was first done in the 1950's. This approach starts with some assumptions about the set of outcomes. One assumption is that the set of outcomes is not controlled by people in the model. An example might be rainfall levels in a season. Another assumption is that everyone agrees about the probability distribution over this set. This last assumption is stronger than is needed, but we will always use it. A third assumption is that people are symmetrically informed in the following sense. We must distinguish between two points in time: prior to and subsequent to the realization of the outcome. Prior to the realization of the outcome, everyone knows the set of outcomes and the probabilities of each outcome in the set. Subsequent to the realization of the outcome, everyone sees the outcome.

3.1 The environment and allocations

We start by supposing that there are only two elements in the set of possible outcomes and we label them outcome 1 and outcome 2. We assume that outcome

1 occurs with probability π_1 and outcome 2 with probability π_2, where $\pi_1 + \pi_2 = 1$.

For the inter-temporal interpretation with 2 dates, we assumed that there was one underlying good (apples) and distinguished between apples at date 1 and apples at date 2. The labeling we want now is labeling by outcome, and the results of the relabeling are outcome-contingent commodities, or, more simply, contingent commodities. We assume that there is one underlying good, apples. We distinguish between apples if the outcome is outcome 1, which we call the outcome-1 good, and apples if the outcome is outcome 2, which we call the outcome-2 good.

There is no production in this world and the total resources are total amounts of outcome-1 good, denoted W_1, and of outcome-2 good, denoted W_2. (You could think of W_1 as being the total crop of apples if outcome 1 occurs and W_2 as the total crop if outcome 2 occurs.) There are N people. If person n consumes (c_{n1}, c_{n2}), then person n gets utility $\pi_1 u^n(c_{n1}) + \pi_2 u^n(c_{n2})$, where the function u^n is strictly increasing and strictly concave. (In most applications, we assume that the everyone has the same utility function. When we do that, we drop the superscript.)

While this completes the environment, it is helpful to say something about the sequence of actions. As noted above, we have to distinguish between prior to and subsequent to the realization of the outcome. The preferences just described pertain to the situation prior to realization of the outcome. These are the preferences that are relevant to decisions about the purchase of risky assets, the purchase of insurance, and betting.

An allocation is exactly what it was before, a list of consumption pairs, one pair for each person. Also, the definition of feasible allocations is unchanged, as is the definition of Pareto efficient allocations. All of these also pertain to the situation prior to realization of the outcome.

3.2 Competitive equilibrium (CE)

As regards private ownership, we assume that each person starts out owning some of each good. We let w_{n1} denote the amount of good 1 owned by person n and let w_{n2} denote the amount of good 2 owned by person n. We will call the pair (w_{n1}, w_{n2}) person n's endowment. We also assume that everything is owned by someone. That is, $\sum_{n=1}^{N} w_{n1} = W_1$ and $\sum_{n=1}^{N} w_{n2} = W_2$.

Let p denote the price of good 2 in units of good 1 (p is measured in units of good 1 per unit of good 2). Throughout, suppose that p is positive. We say that person n can *afford* to buy the pair (c_{n1}, c_{n2}) at the price p if there exists

(q_1, q_2) such that

$$c_{n1} = w_{n1} - q_1,\ c_{n2} = w_{n2} - q_2, \text{ and } q_1 + pq_2 = 0, \tag{2}$$

where q_j is to be interpreted as sales of contingent commodity j (a negative q_j is a purchase of good j). As was demonstrated above, it is equivalent to say that person n can afford to buy the pair (c_{n1}, c_{n2}) at the price p if

$$c_{n1} + pc_{n2} = w_{n1} + pw_{n2}. \tag{3}$$

Given the above equivalence, we can again use satisfaction of (3) as our definition of affordability. The definition of CE is unchanged. That is, an allocation A and a price p is a competitive equilibrium (CE) if two conditions hold: (i) A is feasible, (ii) for each person n, the consumption pair assigned to n by A is liked by n as well as anything else that n can afford at the price p.

In the inter-temporal interpretation of two-good CE, we said that trade occurs at date 1 and involves promises. There, with one good per date, nothing happens at date 2 except that deliveries are made in accord with the promises made at date 1. Now we distinguish between *prior to* and *subsequent to* the realization of the outcome. Prior to the realization of the outcome, only promises are made. After the realization, deliveries are made and consumption occurs. In that respect, trade here is like betting with nothing changing hands at the time of the bet. After the outcome occurs, you pay if you lose and are paid if you win. Here, with two goods, trade is exactly like a bet on one outcome. If you sell outcome-1 good ($q_1 > 0$), then you are necessarily buying outcome-2 good ($q_2 < 0$). This is like a bet on outcome 2 and, therefore, against outcome 1. If outcome 1 occurs, then you lose and have to hand out q_1 units of the good. If outcome 2 occurs, then you win and collect q_2 units of the good. The price p gives the terms on which you bet.

Exercise 3 *Show the following. If $W_1 = W_2$, then there exists a CE with $p = \frac{\pi_2}{\pi_1}$. There is no other CE.*

Exercise 4 *Show the following. If $W_1 > W_2$, then any CE has $p > \frac{\pi_2}{\pi_1}$.*

4 Pure exchange CE with many outcomes

The generalization to K outcomes is straightforward. It follows exactly what we earlier said about pure exchange with K commodities. In defining CE, we now need as many prices as there are outcomes. As before, we denote them as $(p_1, p_2, ..., p_K)$ where we fix $p_1 = 1$ and where p_k is the price of outcome-k good in units of outcome-1 good.

Exercise 5 *Show the following. If* $W_1 = W_2 = ... = W_K$*, then there exists a CE with* $p_k = \frac{\pi_k}{\pi_1}$.

4.1 Competitive asset pricing*

We will use the notion of a K-outcome CE to price assets. We define assets by their payoffs. In particular, an asset is a vector of outcome-specific payoffs—say, $(a_1, a_2, ..., a_K)$ with the interpretation that the owner of this asset receives a_k units of the good if outcome k occurs. (If $a_k < 0$, then the owner must pay out a_k units of the good if outcome k occurs.) The price of the asset is defined to be the amount of outcome-1 good for which one can buy the asset prior to the realization of the outcome. Let us call this price $p(a_1, a_2, ..., a_K)$.

Proposition 1 *In a CE in which* $(p_1, p_2, ..., p_K) = (p_1^*, p_2^*, ..., p_K^*)$,

$$p(a_1, a_2, ..., a_K) = \sum_{k=1}^{K} p_k^* a_k. \tag{4}$$

Exercise 6 *Argue that proposition 1 is true. (Hint: Show that each person would profit by buying the asset if* $p(a_1, a_2, ..., a_K) < \sum_{k=1}^{K} p_k^* a_k$ *and that no-one wants to buy it and each person would profit by issuing it if* $p(a_1, a_2, ..., a_K) > \sum_{k=1}^{K} p_k^* a_k$.*)*

Exercise 7 *Consider two assets:* $(a_1, a_2, ..., a_K)$ *and* $(b_1, b_2, ..., b_K)$*, where* $b_k = za_k$ *for* $k = 1, 2, ..., K$*. Show that in a CE in which* $(q_1, q_2, ..., q_K) = (q_1^*, q_2^*, ..., q_K^*)$, $p(b_1, b_2, ..., b_K) = zp(a_1, a_2, ..., a_K)$.

4.2 The pricing of limited-liability debt and equity in CE*

We can also use proposition 1 to price the debt and equity of limited liability "corporations." We define the corporation as an asset in the sense that ownership of the entire corporation would give rise to a vector of outcome-specific payoffs—say, $(x_1, x_2, ..., x_K)$. As part of "limited liability," we assume that $x_k \geq 0$ for each k.

We next define what we mean by debt of the corporation. We consider debt to be an asset with a *promised* pay-off that is not contingent. The amount of debt will be measured by the magnitude of that *promised* pay-off. Let $P \geq 0$ be the promised pay-off. The magnitude P is to be distinguished from what the owners of the debt actually get. (After all, junk bonds are called junk because it is understood that holders of junk bonds will not always receive the promised pay-off.) Let $\min\{a, b\}$ denote the smaller of a and b. The owners of the debt of the corporation $(x_1, x_2, ..., x_K)$ with promised pay-off P receive the vector $(\min\{x_1, P\}, \min\{x_2, P\}, ..., \min\{x_K, P\})$. We let D denote the pre-outcome value of this debt in units of outcome-1 good. Using proposition 1, we have

$$D = \sum_{k=1}^{K} p_k^* \min\{x_k, P\}. \tag{5}$$

Now we define the equity of the corporation. Consider a corporation $(x_1, x_2, ..., x_K)$ with debt having the promised pay-off P. The owners of the equity of this corporation receive the vector $(\max\{x_1 - P, 0\}, \max\{x_2 - P, 0\}, ..., \max\{x_K - P, 0\})$, where $\max\{a, b\}$ means the larger of a and b. We let E denote the pre-outcome value of this equity in units of outcome-1 good. Using proposition 1, we have

$$E = \sum_{k=1}^{K} p_k^* \max\{x_k - P, 0\}. \tag{6}$$

A well-known result called the Modigliani-Miller Theorem is that in some circumstances the total value of a corporation, $D + E$, does not depend on how much debt it has. This conclusion follows from (5) and (6).

Exercise 8 *Show that* $\min(a, b) + \max(a - b, 0) = a$.

Exercise 9 *(i) Use the result of the last exercise and (5) and (6) to compute* $D+E$. *(ii) Explain why your part (i) result is a version of the Modigliani-Miller Theorem.*

Exercise 10 *We are describing a corporation by a vector of payoffs,* $(x_1, x_2, ..., x_K)$.*Suppose* $K = 2$. *(i) How would you describe the "riskiness" of a corporation? (ii) Are less risky corporations worth more than more risky corporations? Explain.*

Because D is the pre-outcome value of the debt and P is the promised pay-off, the ratio $\frac{P}{D} - 1$ is the promised yield or rate-of-return on the debt. As we next show, this yield is in general increasing in P, our measure of the amount of the debt. We draw this conclusion by studying the ratio $\frac{D}{P}$. If $b > 0$, then $\frac{\min\{a,b\}}{b} = \min\{\frac{a}{b}, 1\}$. Therefore, from (5), if $P > 0$, then

$$\frac{D}{P} = \sum_{k=1}^{K} p_k^* \min\{\frac{x_k}{P}, 1\}. \tag{7}$$

Exercise 11 *According to (7), higher debt as measured by* P *either does not affect the promised yield or increases the promised yield. Describe the circumstances in which each conclusion holds.*

Exercise 12 *At times, corporations are given loan guarantees. In a full analysis, we would have to describe how the guarantee is financed. Here we will ignore that. If the guarantee is genuine, then the owners of the debt of the corporation* $(x_1, x_2, ..., x_K)$ *with promised pay-off* P *will actually receive* P *in every outcome. Suppose that is the case. (i) Give a formula for the pre-outcome value of such debt. (ii) Give a formula for* $D + E$ *for such a corporation. (iii) Does* $D + E$ *depend on* P *and in what way? (iv) Suppose the guarantor of the debt imposes some maximum amount of debt. Given that maximum, does the value of the equity of the corporation depend on the riskiness of the corporation?*

5 Concluding remarks

We have explored in some detail what is called the contingent claims approach to uncertainty. In most of the applications we will do, we will be making assumptions inconsistent with that approach. In some, we will assume that there is asymmetric information—that not everyone sees the entire outcome. In other applications, we will say that people are strangers and, therefore, cannot credibly make promises.