II. Competitive Trade Using Money*

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1 Introduction

Here we introduce our first serious model of money. We now assume that there is no record keeping. As discussed earler, the role of this assumption is to preclude borrowing. And we add money to the model. Money is a perfectly durable and divisible object which, however, is not ever consumed because it is stuff that no one wants to consume. (It is paper, sand, or stones.) This model is extreme in a number of respects. First, all borrowing is precluded. Second, money is the only asset.

2 The environment.

Time is discrete and indexed by $t = 1, 2, ...$. There are $2N$ people, where N is a large integer. There is one good per date. All $2N$ people have the same preferences which satisfy the discounted utility specification. There is no production and society's total endowment of each good is the same and denoted W. (Note that if borrowing and lending were possible, then proposition 4 of the previous section of notes would apply.)

We begin by supposing that there is a fixed stock of money and that the amount of it never changes. We denote the total amount of money by M. We do not concern ourselves with where this money came from. It is part of resources, although a part which is not productive in any ordinary sense.

3 Allocations and feasible allocations

Now that we have added money to the model, we will say that an allocation describes for each person both how much they consume at each date and how much money they *end* up with at each date.

*Sections marked with an asterisk (*) can be skipped without affecting the understanding of later results.

Definition 1 *An allocation is* $(c_{n1}, c_{n2}, ..., c_{nt}, ...)$ *and* $(m_{n1}, m_{n2}, ..., m_{nt}, ...)$ *for each person* n*, where* m_{nt} *is the amount of money that person* n *ends up with at date* t*.*

An allocation is feasible if $\sum_{n=1}^{2N} c_{nt} \leq W$ and $\sum_{n=1}^{2N} m_{nt} \leq M$.

4 Equilibrium

We start out by describing what people own. We assume that each person has an income stream in the form of some of the good at each date and that each person starts out owning some money. Person n's endowment of goods is $(w_{n1}, w_{n2}, ..., w_{nt}, ...)$ and person n's initial ownership of money is $m_{n0} \geq 0$. And, again, everything is owned by someone. That is, $\sum_{n=1}^{2N} w_{nt} = W$ and $\sum_{n=1}^{2N} m_{n0} = M$.

Although people cannot borrow, we assume that they can trade money for goods at each date. We also assume that this trade is competitive in the sense that each person acts as if he or she can trade any amounts at the prices that prevail. We let v_t denote the price per unit of money in units of the date t good. (This symbol has nothing to with the velocity symbol that appears in the quantity equation of the classical dichotomy model.) That is, v_t is the value of a unit of money at date t in terms of the date t good. In terms of v_t, the date-t price level is $\frac{1}{v_t}$.

The next step is to describe what a person can afford.

Definition 2 *We say that person* n *can afford* $(c_{n1}, c_{n2}, ..., c_{nt}, ...)$ *and* $(m_{n1}, m_{n2}, ..., m_{nt}, ...)$ *at the prices (at the price vector or price sequence)* $(v_1, v_2, ..., v_t, ...)$ *if* $m_{nt} \geq 0$ *and*

$$c_{nt} + v_t m_{nt} = w_{nt} + v_t m_{n,t-1} \tag{1}$$

for $t = 1, 2, ...$ *.*

Now we can define an equilibrium.

Definition 3 *An allocation* A *and the price sequence* $(v_1, v_2, ..., v_t, ...)$ *will be said to be an equilibrium if two conditions hold: (i)* A *is feasible, (ii) for each person* n*, the sequences* $(c_{n1}, c_{n2}, ..., c_{nt}, ...)$ *and* $(m_{n1}, m_{n2}, ..., m_{nt}, ...)$ *are liked by person* n *as well as any other sequences that* n *can afford at the price sequence* $(v_1, v_2, ..., v_t, ...)$*.*

Exercise 1 *Suppose the above economy has a last date instead of going on forever. Could there be an equilibrium in which* $v_t > 0$ *for some date* t*? Explain.*

Exercise 2 *Decsribe equation (1) in words.*

5 A special case

As you might guess, describing equilibrium in the above model for a general pattern of endowments is not easy. Moreover, not every pattern of endowments is consistent with money being valuable and used. For both reasons, we now assume very special income streams and a very special initial ownership pattern of money.

Let $y = \frac{W}{N}$. Each person labeled $1, 2, 3, ..., N$ has income y at each odd date and income 0 at each even date. Each person numbered $N+1, N+2, ..., 2N$ has income 0 at each odd date and income y at each even date. That is, if a person is a "low numbered" person ($n \leq N$), then the person's endowment of goods is $(w_{n1}, w_{n2}, ..., w_{nt}, ...) = (y, 0, y, 0, y, ...)$, while if a person is a "high numbered" person ($n > N$), then the person's endowment of goods is $(w_{n1}, w_{n2}, ..., w_{nt}, ...)$ $= (0, y, 0, y, 0, ...)$. As regards the initial ownership of money, let $m = \frac{M}{N}$. We assume that each high-numbered person enters date 1 with an amount of money m, and that each low-numbered person enters date 1 with no money.

The above income-stream assumption sets up a simple absence-of-coincidence in goods. Roughly speaking, each person wants to consume at every date. However, low-numbered people only have income in the form of goods at odd dates. Therefore, they cannot support their consumption at even dates with their income in that period. They have to get that consumption from high-numbered people, but they have no good to offer. The only thing they might offer is money. To do that, they must have acquired money earlier. And, of course, the situation is similar for high-numbered people with the role of odd and even dates reversed.

We pursue a guess and verify procedure to find an equilibrium for the above special case. Before doing that, though, we study a particular one-person economy. What happens in that one-person economy will turn out to be closely related to what happens in our many-person monetary economy.

6 Crusoe with a linear storage technology

Consider a one-person world, sometimes called a Robinson-Crusoe economy. Time is discrete and is indexed by $t = 1, 2, ..., T$. This person maximizes discounted utility. There is one good per date and the person's endowment alternates between odd and even dates: $w_t = y > 0$ if t is odd and $w_t = 0$ if t is even. (Because there is only one person, the subscript that identifies the person is dropped.) The person also has access to the following inter-temporal technology: if k_t denotes date t input into the technology, then the date $t+1$ output is Rk_t, where R is given and satisfies $0 < R \leq \frac{1}{\beta}$. We want to describe what is best for this person.

6.1 $T = 2$ (A two-date lived person).

The problem is to choose (c_1, c_2) and k_1 to maximize $u(c_1) + \beta u(c_2)$ subject to $c_1 + k_1 = y$ and $c_2 = Rk_1$. As indicated in the following exercise, this problem can be solved using a diagram.

Exercise 3 *In a diagram with date-*1 *good on the horizontal axis and date-*2 *good on the vertical axis, depict the pairs* (c_1, c_2) *that satisfy the constraints. Using what you know about the indifference curves corresponding to* $u(c_1) + \beta u(c_2)$*, depict the best choice of* (c_1, c_2)*.*

I will do it using calculus. Using the constraints, the objective can be written as a function of a single unknown, k_1 :

$$u(y - k_1) + \beta u(Rk_1) \equiv f(k_1), \tag{2}$$

where the only constraint on k_1 is $0 \leq k_1 \leq y$.

The function f defined in (2) is strictly concave and differentiable. Its derivative is

$$f'(k_1) = -u'(y - k_1) + R\beta u'(Rk_1). \tag{3}$$

Because f is strictly concave, it has a unique maximum which occurs either at an endpoint of the domain or where $f'(k_1) = 0$. The domain for k_1 is the interval $[0, y]$. To see whether the maximum occurs at an endpoint of this domain, let's evaluate $f'(0)$ and $f'(y)$. We have from (3),

$$f'(0) = -u'(y) + R\beta u'(0)$$

and

$$f'(y) = -u'(0) + R\beta u'(Ry).$$

If $u'(0)$ is large enough, then $f'(0) > 0$ and $f'(y) < 0$. These inequalities imply that the maximum of f does not occur at an endpoint of the domain. Therefore, it occurs where $f'(k_1) = 0$. That is, by (3), the maximum occurs at the value of k_1 which satisfies

$$-u'(y - k_1) + R\beta u'(Rk_1) = 0, \tag{4}$$

or, equivalently,

$$u'(y - k_1) = R\beta u'(Rk_1). \tag{5}$$

Let k^* denote the solution for k_1 to this equation. Because $R\beta \leq 1$, (5) implies that k^* is such that $y - k^* \geq Rk^* > 0$. This is equivalent to $c_1 \geq c_2 > 0$, the conclusion obtained from the diagram.

Exercise 4 *Suppose* $u(x) = x^{1/2}$*. Then* $u'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}}$*. (i) Using Excel, plot the function* f *in (2) for* $y = 2$*,* $R = 1$*, and* $\beta = .9$*. (ii) Solve (5) for* k^* *in terms of* $y, R,$ *and* β*.*

6.2 $T = 3$ (A three-date lived person).

Now the problem is to choose (c_1, c_2, c_3) and (k_1, k_2) to maximize $u(c_1)+\beta u(c_2)+\beta^2 u(c_3)$ subject to $c_1+k_1 = y$, $c_2+k_2 = Rk_1$, and $c_3 = y+Rk_2$. If we substitute for the consumption variables using the constraints, our problem is to choose (k_1, k_2) to maximize

$$u(y - k_1) + \beta u(Rk_1 - k_2) + \beta^2 u(y + Rk_2) \equiv g(k_1, k_2), \tag{6}$$

subject to $y - k_1 \geq 0$ and $Rk_1 - k_2 \geq 0$,which reflect the constraints that consumption cannot be negative.

Exercise 5 *In a diagram with k_1 on the horzontal axis and k_2 on the vertical axis, sketch the pairs (k_1, k_2) that satisfy $y - k_1 \geq 0$ and $Rk_1 - k_2 \geq 0$.*

We take the result of this exercise to be the domain for the function g defined in (6). So our problem is find the pair (k_1, k_2) in this domain that maximizes g. We proceed by guessing at the best pair and then verifying that it is best. There is an obvious guess to make. It is $(k_1, k_2) = (k^*, 0)$, where k^* is the solution to (5). Our way of confirming this guess appeals to the fact that g is strictly concave and differentiable. We will do this confirmation twice, using diagrams and using calculus. The strict concavity of g implies that it is enough to check departures from $(k_1, k_2) = (k^*, 0)$ one variable at a time and to check small departures.

Diagrams. Checking departures one variable at a time means setting the other variable at its conjectured best value and checking on departures for the other one.

Let's start by checking departures from $k_1 = k^*$, while holding $k_2 = 0$. This means examining how the objective varies with k_1, while holding $k_2 = 0$. Because k_1 does not appear in the third term of the left-hand side of (6) and because when we set $k_2 = 0$, the first two terms become the objective in our two-date problem. Therefore, this is nothing but the two-date problem for which we know that $k_1 = k^*$ is best.

Next we want to check departures from $k_2 = 0$, while holding $k_1 = k^*$. Because k_2 does not appear in the first term of the left-hand side of (6), we can proceed by studying only how variations in k_2 affect c_2 and c_3. I outline how to proceed in two exercises.

Exercise 6 *With date-2 good on the horizontal axis and date-3 good on the vertical axis, depict the pairs (c_2, c_3) that satisfy the constraints while holding $k_1 = k^*$.*

Exercise 7 *Given what you know about the indifferences corresponding to $\beta u(c_2)+\beta^2 u(c_3)$, argue, using your diagram, that $k_2 = 0$ is best.*

Calculus. We let $g_1(k_1, k_2)$ denote the partial derivative of g with respect to its first argument and let $g_2(k_1, k_2)$ denote the partial derivative of g with respect to its second argument. It follows that

$$g_1(k_1, k_2) = -u'(y - k_1) + R\beta u'(Rk_1 - k_2) \tag{7}$$

and

$$g_2(k_1, k_2) = -\beta u'(Rk_1 - k_2) + R\beta^2 u'(y + Rk_2) \tag{8}$$

Exercise 8 *(i) Write an expression for* $g_1(k^*, 0)$ *and one for* $g_2(k^*, 0)$. *In other words, evaluate the expressions in (7) and (8) at* $(k_1, k_2) = (k^*, 0)$. *(ii) Argue that* $g_1(k^*, 0) = 0$. *(iii) Argue that* $g_2(k^*, 0) \leq 0$.

Because g is strictly concave, the conclusions in the last exercise imply that $(k_1, k_2) = (k^*, 0)$ is the solution to the problem.

6.3 $T = \infty$ (an infinitely-lived person)

Now the problem is to choose $(c_1, c_2, c_3, ...)$ and $(k_1, k_2, ...)$ to maximize $u(c_1) + \beta u(c_2) + \beta^2 u(c_3) + ... + \beta^{t-1} u(c_t) + ...$ subject to

$$c_t + k_t = y + Rk_{t-1} \text{ when } t \text{ is odd} \tag{9}$$

and

$$c_t + k_t = Rk_{t-1} \text{ when } t \text{ is even} \tag{10}$$

and $k_0 = 0$.

Notice that the only difference between (9) and (10) is the presence or absence of y. It is convenient to replace these by a single constraint,

$$c_t + k_t = w_t + Rk_{t-1}, \tag{11}$$

where

$$w_t = \begin{cases} y \text{ if } t \text{ is odd} \\ 0 \text{ if } t \text{ is even} \end{cases}. \tag{12}$$

When we substitute (11) into the objective, we can write our problem in the compact form: choose $k_1, k_2, ..., k_t, ...$to maximize

$$\sum_{t=1}^{\infty} \beta^{t-1} u(w_t + Rk_{t-1} - k_t) \equiv h(k_1, k_2, ..., k_t, ...), \tag{13}$$

where w_t satisfies (12) and where $k_0 = 0$. A significant feature of (13) is that each particular term in the sequence $k_1, k_2, ..., k_t, ..$ appears in exactly two terms of the infinite sum. In particular, k_τ appears in the terms

$$\beta^{\tau-1} u(w_\tau + Rk_{\tau-1} - k_\tau) + \beta^\tau u(w_{\tau+1} + Rk_\tau - k_{\tau+1}). \tag{14}$$

By now, it should be no surprise that we guess that a solution to the above problem is

$$k_t = \begin{cases} k^* \text{ if } t \text{ is odd} \\ 0 \text{ if } t \text{ is even} \end{cases} \tag{15}$$

Although I will not prove this for you, the function h is strictly concave and differentiable. Therefore, it is enough to check departures from (15) one variable at a time and locally. And this is easy to do. There are only two things to check.

Exercise 9 *Let τ be odd. (i) Give the expression for (14) that is relevant for checking the one-variable-at-a-time departure for k_τ from the guess given by (15). (ii) Explain using previous results why no departure from $k_\tau = k^*$ is best.*

Exercise 10 *Let τ be even. (ii) Give the expression for (14) that is relevant for checking the one-variable-at-a-time departure for k_τ from the guess given by (15). (ii) Explain using previous results why no departure from $k_\tau = 0$ is best.*

Several remarks are in order about the above description. First, there is a technical matter that arises in the infinite horizon case. There is another condition to check, a condition called the tranversality condition. We do not have to worry about it; all the solutions that we propose satisfy it.[1] Second, you may wonder why the solution that applies when Crusoe lives only two dates also holds when he lives for any number of dates. This happens for two reasons. His preferences (discounted utility), his income stream (y at odd dates 0 at even dates), and his technology (R is constant) imply that *if* he enters each odd date with the same pay-off from previous investment, then his opportunities and preferences for the current and all future dates look the same. This *if* clause holds because R is so low that Crusoe wants to enter each odd date with a zero payoff from previous investment.

All these features will turn out to be features of the equilibrium in the special case of our monetary economy. There, acquiring money will replace storage. However, in other respects, matters will be similar.

7 Equilibrium in the Special Case

The simplest equilibrium with valued and traded money that could exist is one in which the price of a unit of money is constant. Let's try out this feature as step (i) of the guess and verify procedure. The feature is that there is an equilibrium with $(v_1, v_2, ..., v_t, ...) = (v, v, ..., v, ...)$ with $v > 0$, or, more succintly, an equilibrium with $v_t = v > 0$ for all t. Notice that the guess does not pick out a particular magnitude for v. Doing that is part of step (ii) of the guess and verify procedure. (There is no single right way of doing steps (i) and (ii) of a guess and verify procedure. In this case, as a way of explaining how the guess is constructed, I am trying to put very little into step (i).) I now describe a way to complete the guess, a way to carry out step (ii).

Consider a particular low-numbered person and the first 2 dates in isolation. With date-1 good on the horizontal axis and date-2 good on the vertical axis, let's sketch the opportunities for date-1 and date-2 consumption implied by being able to buy money at price v at date 1, storing the purchased money until date 2, and then selling at date 2 at the price v all the money acquired at date 1.

The $t = 1$ version of (1) for a low numbered person under the conjectured price sequence is

$$c_{n1} + vm_{n1} = y \tag{16}$$

[1] See the appendix for a demonstration.

while that for $t = 2$ assuming all the money is sold is

$$c_{n2} = vm_{n1} \tag{17}$$

If both hold, then the sum holds. The sum of (16) and (17) is

$$c_{n1} + c_{n2} = y. \tag{18}$$

Exercise 11 *In your diagram, sketch all the pairs* (c_{n1}, c_{n2}) *that satisfy (18).*

Exercise 12 *If* (c_{n1}, c_{n2}) *satisfies (18), then there exists* $m_{n1} \geq 0$ *such that it and* (c_{n1}, c_{n2}) *satisfy (16) and (17). True or false? Explain.*

Exercise 13 *Explain in words why* v *does not appear in (18).*

Exercise 14 *Show that the consumption opportunities depicted by (18) are identical to those for the two-date lived Crusoe of the last section when* $R = 1$. *Explain why that is not a surprise.*

Our guess will come from depicting what is best for this person from among the opportunities given by (18).

Exercise 15 *Sketch some indifference curves over pairs* (c_{n1}, c_{n2}) *for this low-numbered person and depict what is best for the person from among the opportunities given by (18).*

Let's denote the best consumption pair as depicted in this exercise, (c_y, c_0). (Think of c_y as consumption when the income is y and c_0 as consumption when the income is 0.)

Exercise 16 *Is* $c_y > \frac{y}{2}$*? Is* $c_y < y$*? Explain.*

Notice that the pair (c_y, c_0) depends only on y and on the indifference curves implied by our assumed discounted utility specification. Now we complete the guess in terms of (c_y, c_0) and state the guess as part of a proposition.

Proposition 1 *Let* (c_y, c_0) *be the best consumption pair as depicted in the last exercise. The special case has the following as an equilibrium. The allocation is as follows. If* $n \leq N$, *then*

$$(c_{n1}, c_{n2}, ..., c_{nt}, ...) = (c_y, c_0, c_y, c_0, ...) \tag{19}$$

and

$$(m_{n1}, m_{n2}, ..., m_{nt}, ...) = (m, 0, m, 0, ...). \tag{20}$$

If $n > N$, *then*

$$(c_{n1}, c_{n2}, ..., c_{nt}, ...) = (c_0, c_y, c_0, c_y, ...) \tag{21}$$

and

$$(m_{n1}, m_{n2}, ..., m_{nt}, ...) = (0, m, 0, m, ...). \tag{22}$$

The price sequence is

$$v_t = \frac{c_0}{m} \text{ for all } t \geq 1. \tag{23}$$

Exercise 17 *Suppose* $u(x) = x^{1/2}$. *Solve for* c_0 *in terms of* y *and* β. *Solve for* v_t *in terms of* y, β, *and* m.

We will not do a formal proof of proposition 1, but will go through most of the steps, some in exercises. A proof involves the verification part of the guess and verify procedure. Before we do that, some remarks are in order. You cannot understand the statement of 1 unless you understand exactly how the pair (c_y, c_0) was constructed in the last exercise. Indeed, that description is part of the statement of the proposition. The last claim in the proposition, equation (23), picks out a particular magnitude for the price of money. Because the right-hand side of that equation is a constant, the proposal is consistent with the initial feature of our guess—namely, that v_t is constant. The magnitude of v_t is chosen so that total spending on money at each date, Nc_0, is equal to the value of the total money supply, $v_t M$.

Now we sketch the steps in a proof of proposition 1.

Exercise 18 *Show that the allocation given by (19)-(22) is feasible—that is, that it satisfies condition (i) in the definition of equilibrium.*

Now we turn to condition (ii) in the definition of equilibrium. This is the part that says that each person's part of the allocation is best from among those that the person can afford. A necessary condition is that each person's part of the allocation is affordable. (If it is not affordable, then it cannot be best from among those that are affordable.)

Exercise 19 *Show that if* $n \leq N$, *then (19)-(20) is affordable for* n.

Exercise 20 *Show that if* $n > N$, *then (21)-(22) is affordable for* n.

Now for the hard part of condition (ii)—showing that each person's part of the allocation is best from among those that the person can afford. We start by rewriting (1) slightly. Let $s_{nt} = v_t m_{nt}$. In words, s_{nt} is the value in terms of date t good of the money with which person n leaves date t. Or you can call it saving in terms of the date-t good. Then, we can write (1) as

$$c_{nt} + s_{nt} = w_{nt} + \frac{v_t}{v_{t-1}} s_{n,t-1} \tag{24}$$

or, using (23),

$$c_{nt} + s_{nt} = w_{nt} + s_{n,t-1} \tag{25}$$

Now let's express discounted utility in terms of the sequence $s_{n0}, s_{n1}, \ldots$, where all but the first term are choice variables for person n. That is, we can solve (25) for c_{nt} and write

$$\sum_{t=1}^{\infty} \beta^{t-1} u(c_{nt}) = \sum_{t=1}^{\infty} \beta^{t-1} u(w_{nt} + s_{n,t-1} - s_{nt}) \tag{26}$$

The right-hand side is helpful because we can think directly about the consequences for discounted utility of varying the s_{nt} terms. The only constraints we have to worry about are $c_{nt} \geq 0$ and $s_{nt} \geq 0$.

Now suppose that someone presents us with a particular sequence $s_{n1}, s_{n2}, ...$ and asserts that it maximizes the right-hand side of (26). Because the right-hand side of (26) is strictly concave, we can check departures one variable at a time and locally, just we did for Crusoe in the last section.

Exercise 21 *Show that our assumptions about initial holdings of money and our guess in (19)-(23) imply the following candidates for the sequence* $s_{n0}, s_{n1}, ...$:

$$(s_{n0}, s_{n1}, ...) = (0, c_0, 0, c_0, ...) \text{ if } n \leq N \tag{27}$$

and

$$(s_{n0}, s_{n1}, ...) = (c_0, 0, c_0, 0, ...) \text{ if } n > N. \tag{28}$$

A feature of discounted utility is that the term s_{nt}, for any $t \geq 1$, appears in exactly two terms in the infinite summation in the right-hand side of (26). Those two terms are

$$\beta^{t-1} u(w_{nt} + s_{n,t-1} - s_{nt}) + \beta^t u(w_{n,t+1} + s_{n,t} - s_{n,t+1}). \tag{29}$$

Then the check of deparures one variable at a time takes the following form: $s_{n,t}$ as given by the candidate in (27) and (28) maximizes the expression in (29) when $s_{n,t-1}$ and $s_{n,t+1}$ are given by the candidate. Moreover, there are only two different versions of (29). One version applies to every low-numbered person at every odd date and to every high-numbered person at every even date; that is, whenever a person has a date t income of y (and, therefore, a date $t+1$ income of 0):

$$\beta^{t-1} u(y + 0 - s_{nt}) + \beta^t u(0 + s_{n,t} - 0) = \beta^{t-1}[u(y - s_{nt}) + \beta u(s_{n,t})] \tag{30}$$

The other applies to every low-numbered person at every even date and to every high-numbered person at every odd date; that is, whenever a person has a date t income of 0 (and, therefore, a date $t+1$ income of y):

$$\beta^{t-1} u(0+c_0-s_{nt})+\beta^t u(y+s_{n,t}-c_0) = \beta^{t-1}[u(c_0-s_{nt})+\beta u(y+s_{n,t}-c_0)]. \tag{31}$$

I will present the analysis of (31) and leave for an exercise the analysis of (30).

We are trying to show that $s_{n,t} = 0$ maximizes the expression in square brackets on the right-hand side of (31). As a first step, we sketch all pairs $(c_{n,t}, c_{n,t+1})$ implied by all possible choices for $s_{n,t}$. So let's consider a diagram with date t good on the horizontal axis and date $t+1$ good on the vertical axis. Any magnitude of $s_{n,t}$ between 0 and c_0 is a possible choice. If $s_{n,t} = 0$, then $(c_{n,t}, c_{n,t+1}) = (c_0, y - c_0)$. Sketch this point in the diagram. If $s_{n,t} = c_0$, then $(c_{n,t}, c_{n,t+1}) = (0, y)$. Sketch this point in the diagram. What about the other possible choices of $s_{n,t}$? Just connect the dots; that is, connect the points $(c_0, y - c_0)$ and $(0, y)$. (If you have done this correctly, then you should have a line segment whose slope is -1 and that lies entirely above the line through the origin with slope 1.

The next step is to represent the indifference curves that correspond to what we are trying to maximize. We have picked out two terms of the infinite sum on the left-hand side of (26); namely

$$\beta^{t-1}u(c_{nt}) + \beta^t u(c_{n,t+1}) = \beta^{t-1}[u(c_{nt}) + \beta u(c_{n,t+1})] \tag{32}$$

The constant β^{t-1} on the right-hand side of (32) plays no role in determining the shapes of the indifference curves. Therefore, we want to sketch the indifference curves implied by $u(c_{nt}) + \beta u(c_{n,t+1})$. We have done this before. These are nothing but the indifference curves for two goods implied by discounted utility. From the above facts about the line segment that depict opportunities, it follows that the indifference through any point on the line segment is steeper at that point than the line segment. That, in turn, implies that the highest indifference curve is reached at the point $(c_{n,t}, c_{n,t+1}) = (c_0, y - c_0)$, which is the point that corresponds to $s_{n,t} = 0$. This is what we set out to prove.

Exercise 22 *Argue, using a diagram, that $s_{nt} = c_0$ maximizes the right-hand side of (30).*

We have now completed the verification part. Therefore, subject to your acceptance of my claims about strict concavity and its consequences, we have proved proposition 1.

Exercise 23 *Consider two economies that are identical except for the amounts of money. According to proposition 1, how do the equilibria for the two economies differ?*

8 Other income streams*

The main features of the equilibrium just described hold for income streams that do not vary over time as much as those of the special case. Suppose the low-numbered people have income $y - \varepsilon$ when t is odd and have income ε when t is even and vice versa for the high-numbered people, where ε is between 0 and $\frac{y}{2}$. An equilibrium with a constant price of money that is positive exists if the following condition holds. Consider the consumption pair $(c_{nt}, c_{n,t+1}) = (y - \varepsilon, \varepsilon)$ and consider the indifference curves over pairs $(c_{nt}, c_{n,t+1})$ implied by discounted utility. If the indifference curve through the point $(c_{nt}, c_{n,t+1}) = (y - \varepsilon, \varepsilon)$ has slope greater than -1 at that point (is flatter than a line with slope -1), then a monetary equilibrium can be constructed exactly as we did above except that wherever we used the point $(y, 0)$ in that argument, we have to replace it with $(y - \varepsilon, \varepsilon)$. That slope condition will not hold if ε is sufficiently close to $\frac{y}{2}$.

Exercise 24 *State a proposition about an equilibrium for the economy of this section that is analogous in form to 1.*

Exercise 25 *Suppose* $u(x) = \sqrt{x}$*,* $\beta = .9$*,* $y = 1$ *and* $m = 1$*. (i) If* $\varepsilon = .25$*, then describe numerically an equilibrium with a positive and constant value of money. (ii) For what magnitudes of* ε *is there an equilibrium with a positive and constant value of money?*

9 Other initial distributions of money*

Part of the special case is a very particular distribution of initial holdings of money. Each low numbered person has no money initially and each high numbered person has the same amount of money. In the equilibrium displayed above, this distribution of money persists through all time. This is not an accident. I chose that initial distribution because I knew that there is an equilibrium in which it persists. Such an initial condition is called a steady state.

Let's briefly consider other initial distributions such that all low numbered people start with the same amount of money and all high numbered people start with the same amount of money. In particular, let each low numbered person start with αm amount of money and let each high numbered person start with $(1 - \alpha)m$ amount of money, where α is between 0 and 1. Here is a guess and verify argument for this model.

Consider the high numbered people at the initial date. Even if they have all the money, we found an equilibrium in which they spend it all at date 1. It seems plausible that if they have less of it, then there would also be an equilibrium in which they spend all of it. If they do spend all of it, then the low numbered people at date 2 have it all. But, then date 2 looks exactly like date 1 looked when $\alpha = 0$. Therefore, a plausible surmise for this model is that only date 1 quantities and prices depend on α.

To pursue this surmise and construct the complete date 1 guess, we use the following technique. We first ask the following question. Given that each low numbered person will consume c_0 at date 2 and that v_2 is given by (23), what does v_1 have to be in order that that a low numbered person be willing to acquire $(1 - \alpha)m$ amount of money? After answering this question, we must confirm that each high numbered person is willing to spend that amount of money.

Let's start with a diagram with date 1 good on the horizontal axis and date 2 good on the vertical axis. On this diagram, let's start by plotting two points: (c_y, c_0) and (y, c_0). The first of these is the equilibrium (c_{n1}, c_{n2}) for a low numbered person when $\alpha = 0$. The second is the equilibrium (c_{n1}, c_{n2}) for such a person when $\alpha = 1$. Next, let's think about the indifference curves over date 1 and date 2 consumption pairs implied by the discounted utility assumption. By construction of (c_y, c_0), we know that the indifference through that point has slope -1 at that point. Now consider the points on the line segment that connects the two points and the slopes of the indifferences curves through those points. As we move from (c_y, c_0) to (y, c_0), the indifference curves get flatter and flatter. Now pick out any point on that line segment and call it (x, c_0). Consider the slope of the indifference curve through that point. In order that the low numbered person be willing to consume that point, it must be that the ratio $\frac{v_2}{v_1}$is equal to minus the slope of the indifference curve at the point (x, c_0). That condition tells us what v_1 must be because we are surmising that v_2 is given by (23). Moreover, because we know that the indifference curves get flatter the larger is x, we know how v_1 varies with x. Finally, what α goes along the point (x, c_0)? That can be answered by using

$$x = y - v_1(1-\alpha)m. \tag{33}$$

Having picked out an x and the v_1 associated with it, this equation now has one unknown, α, which can be solved for uniquely from this equation. Moreover, although it takes a little bit of mathematical reasoning, as we vary x from c_y to y, the set of solutions for α to (33) covers all the possible α's.

This may seem like a backward procedure because it does not start from an α. It is equivalent to the following "forward" procedure which is sketched out as part of the next exercise.

Exercise 26 *Fix α at some value between 0 and 1. Now consider a diagram with date 1 good on the horizontal axis and the price of money in terms of date 1 good on the vertical axis. (i) For x between c_y and y, plot the pairs (x, v_1) that satisfy (33). (ii) For x between c_y and y, plot the pairs (x, v_1) that satisfy the following condition: the slope of the indifference through the point (x, c_0) is equal to $-\frac{v_2}{v_1}$, where v_2 is given by (23). (iii) Argue that there is exactly one pair (x, v_1) that satisfies the condition in (i) and the condition in (ii) This pair is the candidate for the equilibrium c_{n1} for a low numbered person and for v_1. (iv) How does that candidate vary with α?*

Exercise 27 *If $\alpha = 1$, then the equilibrium c_{n1} for an odd person is y. Why must that be the case?*

We now argue that faced with the v_1 as constructed in part (iii) of the next to last exercise and v_2 as given by (23), a high numbered person wants to spend all her or his money at date 1. The argument is similar to our analysis above which we did for $\alpha = 0$. Now we do it for date 1 for an arbitrary α between 0 and 1. Our goal is to show that $s_{n1} = 0$ is best for a high numbered person when the person takes as given $(s_{n2}, s_{n3}, ..) = (c_0, 0, c_0, 0, ...)$.

We start by writing (24) for a high numbered person for date 1 and for date 2. For date 1, it is

$$c_{n1} + s_{n1} = 0 + v_1(1-\alpha)m; \tag{34}$$

For date 2, it is

$$c_{n2} + c_0 = y + \frac{v_2}{v_1} s_{n,1}, \tag{35}$$

where we have inserted the equilibrium value for s_{n2} in accord with studying the necessary condition for a maximizing choice of s_{n1}. It is helpful to replace $v_1(1-\alpha)m$ in (34) using (33) to get

$$c_{n1} + s_{n1} = y - x \tag{36}$$

where $y - x$ is between 0 and c_0. Now let's plot the pairs (c_{n1}, c_{n2}) implied by all possible choices for s_{n1}, the possible choices being anything between 0 and $y - x$. The points are a line segment that connects the pairs implied by $s_{n1} = 0$ and $s_{n1} = y - x$. The choice $s_{n1} = 0$ implies $(c_{n1}, c_{n2}) = (y - x, y - c_0)$. The choice $s_{n1} = y - x$ implies $(c_{n1}, c_{n2}) = (0, y - c_0 + \frac{v_2}{v_1}(y - x))$. Because $\frac{v_2}{v_1} \leq 1$, it follows that this line segment has slope between -1 and 0. Also, because $y - x \leq c_0 \leq y - c_0$, all points on it satisfy $c_{n1} < c_{n2}$.

Now let's consider the indifference curves over the pairs (c_{n1}, c_{n2}) implied by discounted utility. From the above facts about the line segment that depict opportunities, it follows that the indifference through any point on the line segment is steeper at that point than the line segment. That, in turn, implies that the highest indifference curve is reached at the point corresponding to the choice $s_{n1} = 0$, which is what we set out to show.

So we have shown how to construct an equilibrium for any initial condition such that all people of the same type in terms of income streams have the same money holdings.

10 Welfare

We have described a monetary equilibrium in great detail. The equilibrium we have described is not Pareto Efficient. We list this as a proposition.

Proposition 2 *The equilibrium we have described is not Pareto Efficient.*

Exercise 28 *Prove this proposition.*

The assumptions we made to rule out borrowing have welfare consequences. Under those assumptions, a fixed stock of money can help, but cannot help enough to produce an allocation that is Pareto Efficient.

Faced with this conclusion in this and closely related models, some economists have suggested that the problem would be fixed if it could be arranged to have money bear interest at the real rate $\frac{1}{\beta} - 1$. In the equilibrium we have found the real interest rate on money is either 0 or negative (at date 1 if the low numbered people start with some money). We will put off an analysis of paying interest on money until we study money creation and inflation. There, our conclusion will be that it does not make sense in the kind of world we have been studying, a world of strangers, to assume that it is easy to finance the payment of interest on money.

11 Relationship to an OLG model*

The model of infinitely lived people set out above and a simple OLG model are similar in some respects. Here I want to point out the similarities and the differences. Finally, I will say why I favor the model of infinitely lived people.

Suppose the OLG model has two-date lived people who have the same preferences and that these are described by discounted utility. Also, suppose that each generation has the same number, N, of members, that there is one good per date, no production, and that society's endowment of date t good is W and not dependent on t. We call generation t the generation who are young at t and old at $t+1$.

We assign the indexes $1, 2, ..., N$ to the young people at date t and the indexes $N+1, N+2, ..., 2N$ to the old people at date t, but with a particular understanding. If $n \leq N$, then the index n at t and the index $N+n$ at $t+1$ refer to the same person. Then we use the same notation for allocations that we used earlier, but with that understanding. That is, if $n \leq N$, then the pair $(c_{nt}, c_{N+n,t+1})$ is the lifetime consumption stream of person n who is young at t and old at $t+1$, and $(m_{nt}, m_{N+n,t+1})$ is the lifetime stream of end-of-date money holdings. With that understanding, an allocation can be denoted by two sequences $(c_{n1}, c_{n2}, ..., c_{nt}, ...)$ and $(m_{n1}, m_{n2}, ..., m_{nt}, ...)$ for each n from $1, 2, ..., 2N$, just as we did in the model of infinitely lived people. And the definition of feasibility is the same.

Exercise 29 *Suppose $N = 1,000$. Using the applicable subscripts, denote the lifetime consumption of person 50 who is young at $t = 3$?*

As regards individual endowments, let each young person at t have the income stream in goods consisting of $\frac{W}{N} = y$ units of the date t good and 0 of the date $t+1$ good. Finally, assume that the economy has a total of M amount of money and that each person who is old at the first date starts out with $\frac{M}{N} = m$ amount of money.

We next give a definition of equilibrium. We define prices exactly as we did above. We let v_t denote the date t price of money in units of date t good. We start with a definition of what a person can afford over the person's lifetime.

Definition 4 *If $n \leq N$, then the pair $(c_{nt}, c_{N+n,t+1})$ and the pair $(m_{nt}, m_{N+n,t+1})$ is affordable for person n of generation t if*

$$c_{nt} = y - v_t m_{nt} \text{ and } c_{N+n,t+1} + v_{t+1} m_{N+n,t+1} = v_{t+1} m_{n,t} \tag{37}$$

If $n > N$, then c_{n1} and m_{n1} is affordable for person n of generation 0 if

$$c_{n,1} + v_1 m_{n,1} = v_1 m.$$

Definition 5 *An allocation A and the price sequence $(v_1, v_2, ..., v_t, ...)$ will be said to be an equilibrium if two conditions hold: (i) A is feasible, (ii) for each person $n \leq N$ in every generation $t \geq 1$, the part of A assigned to that person is liked by the person as well as anything else that the person can afford at the price sequence $(v_1, v_2, ..., v_t, ...)$ and similarly for the members of generation 0 who are old at $t = 1$.*

Proposition 3 *The allocation and price sequence given by (19)-(23) is an equilibrium of the OLG model.*

The proof, which consists again of verifying that the candidate equilibrium satisfies the equilibrium conditions, is a simplified version of the argument we presented for the special-case economy with infinitely lived people. There we had to argue at some length that people who had no income wanted to spend all their money. Now such people are about to die; obviously, they want to spend all their money. Second, as regards people when young, because they live for two dates their entire problem can immediately be represented by thinking about their choices in the two-dimensional diagram we have been using. That simplicity is a virtue of the OLG model.

Indeed, given that simplicity, you may wonder why we did not reverse the order of propositions 1 and 2. We could have. I chose not to because I want to do some subsequent analysis directly in the model of infinitely-lived agents.

Although there is an association between equilibrium in the the model of infinitely-lived agents and equilibrium in the OLG model, the welfare interpretation of the equilibrium is different in the two models. We have seen that the equilibrium in the model of infinitely-lived agents is not PE. The source of the inefficiency is that a person giving up money to acquire some of good at a date and a person giving up some of the good to acquire money at the same date have different marginal rates of substitution between the good at that date and the good at the next date. That source of inefficiency is not present in the OLG model because the people giving up money to acquire some of the good at a date are old people who will be dead at the next date.

Because one of the main reasons we build models is to derive conclusions about welfare, this difference between the two models is important. Which model should we like better? I think we should like the model of infinitely lived people better because it is more realistic in the following sense. Most people in the actual economy who give up money to acquire goods will be around at the next date. That suggests that the source of inefficiency in the model of infinitely-lived agents is present in the actual economy. Therefore, we should favor a model that allows for such inefficiency rather than one that rules it out by its special demographic structure. Of course, a more complicated version of the OLG model with longer-lived people amd, therefore, more overlap between generations, would also display ineffciency. However, such version would not be simpler than our model of infinitely lived people.

12 Money Creation and Inflation

Here we study money creation and inflation using the model of infinitely lived people. We will consider different cases regarding how the inflation comes about and regarding the environment within which it happens. In some, the money creation will be unambiguously a bad thing. In others, the conclusions are ambiguous.

12.1 Transfers

Consider the economy that we are calling the special case. Low numbered people have income in the form of goods of y in odd periods and 0 in even periods and vice versa for high numbered people. We now suppose that the money supply can be varied and that in each period some additional money is created and handed out to the people who have no income in the form of goods at that date. We label those handouts of money tranfers and we think of them as being a government policy.

We begin by describing the amounts transferred. We want to choose those amounts so that the resulting equilibrium is simple. We expect that increases in the amount of money will produce inflation, a falling value of money. Such an equilibrium will be relatively simple if it produces a constant inflation *rate*. To have that happen, we choose the tranfers so that the total amount of money increases at a constant *rate*.

With the total amount of money changing, we have to be careful about notation. At any date, there is money carried over from the last date, some newly created money that is tranferred to people, and the total amount carried forward. We let M_t denote the total amount carried forward from date t, or the post-transfer amount of money at date t. We assume that

$$M_{t+1} = (1+\theta)M_t \tag{38}$$

where θ is a positive number. This implies that the amount of new money created at date $t+1$, $M_{t+1} - M_t$, is equal to θM_t. Now that the amount of money is changing, the condition for feasibility of money holdings becomes $\sum_{n=1}^{2N} m_{nt} \leq M_t$.

At each date, there are N people who have no income in the form of goods at that date. We assume that the new money that is created goes to them. Thus, at date $t+1$, each of the people who have no income in goods at that date get $\frac{\theta M_t}{N}$ units of money.

Although we will study only this special transfer scheme, it is helpful to have a general notation for transfers in the form of money. Thus, we let T_{nt} denote the transfer at date t to person n. Given these transfers, we must amend the definition of affordability.

Definition 6 *We say that person n can afford* $(c_{n1}, c_{n2}, ..., c_{nt}, ...)$ *and* $(m_{n1}, m_{n2}, ..., m_{nt}, ...)$ *at the prices (at the price vector or price sequence)* $(v_1, v_2, ..., v_t, ...)$ *if* $m_{nt} \geq 0$ *and*

$$c_{nt} + v_t m_{nt} = w_{nt} + v_t m_{n,t-1} + v_t T_{nt} \tag{39}$$

for $t = 1, 2, ...$ *.*

There is no reason to amend the definition of equilibrium. We can summarize our assumptions about both income streams in goods and transfers of money and inital (pre-transfer) holdings of money as follows:

$$w_{nt} = \begin{cases} y \text{ if } n \leq N \text{ and } t \text{ is odd, or if } n > N \text{ and } t \text{ is even} \\ 0 \text{ if } n \leq N \text{ and } t \text{ is even, or if } n > N \text{ and } t \text{ is odd} \end{cases}, \tag{40}$$

$$T_{nt} = \begin{cases} 0 \text{ if } n \leq N \text{ and } t \text{ is odd, or if } n > N \text{ and } t \text{ is even} \\ \frac{\theta M_{t-1}}{N} \text{ if } n \leq N \text{ and } t \text{ is even, or if } n > N \text{ and } t \text{ is odd} \end{cases}, \tag{41}$$

and

$$m_{n0} = \begin{cases} 0 \text{ if } n \leq N \\ \frac{M_0}{N} \text{ if } n \leq N \end{cases}. \tag{42}$$

As we have been doing, we will pursue a guess and verify procedure to find an equilibrium. We begin by conjecturing that there is an equilibrium in which

$$\frac{v_{t+1}}{v_t} = \frac{1}{1+\theta}. \tag{43}$$

This is the conjecture that the value of money falls at a rate equal to that at which the money supply increases. Based on this conjectured feature of an equilibrium, we form a complete candidate for equilibrium as follows.

We surmise that whenever a person has the same income in the form of goods, the person will consume the same amount. That is, let c_y denote consumption when a person has income y and let c_0 denote consumption when a person has income 0. If this happens, then by feasibility, it must be that $c_y + c_0 \leq y$. Moreover, we do not expect goods to be wasted or thrown away in an equilibrium. Therefore, we construct a candidate that satisfies

$$c_y + c_0 = y. \tag{44}$$

Exercise 30 *If c_y is consumption of every person who has income y and c_0 is consumption of every person who has income 0, then feasibility implies $c_y + c_0 \leq y$. True or false. Explain.*

With c_y on the horizontal axis and c_0 on the vertical axis, let's sketch the pairs that satisfy (44). These fall on a line with slope -1 and intercept y. The next step is to make a guess about a point on this line. This guess is based on the following reasoning. We expect people whose income is y to be acquiring money and to be planning to use it to acquire consumption at the next date. If so, then the pair (c_y, c_0) must be a best choice for $(c_{nt}, c_{n,t+1})$ when current income is y and income at the next date is 0. So let's consider the indifference curves over pairs $(c_{nt}, c_{n,t+1})$ implied by discounted utility. Imagine that these indifference curves are on our diagram. Now according to (43), the opportunities faced by trading off current consumption against consumption at the next date for those with current income y imply that one unit of current consumption can be traded against $\frac{1}{1+\theta}$ units of consumption at the next date. Therefore, our guess for the pair (c_y, c_0) is that it is the point satisfying (44) at which an indifference curve has slope $-\frac{1}{1+\theta}$. Let us denote this point (c_y^*, c_0^*).

Exercise 31 *If $u(x) = \sqrt{x}$, express c_y^* and c_0^* in terms of β, y, and θ.*

Exercise 32 *How does the point (c_y^*, c_0^*) vary with θ? Explain.*

Given (43) and (c_y^*, c_0^*), the rest of our guess comes from conjecturing that at each date, all the money held by people with income 0 (in goods) is used to purchase the good at that date. In particular, this gives us a candidate for v_1; namely, the solution for v_1 to

$$c_0^* = v_1(1+\theta)\frac{M_0}{N}. \tag{45}$$

We assemble our conjecture in the form of a proposition.

Proposition 4 *Under the money transfer scheme and endowments as given by (40)-(42), there is an equilibrium with a price sequence given by (45) and (43) and with*

$$c_{nt} = \begin{cases} c_y^* \text{ if } n \leq N \text{ and } t \text{ is odd, or if } n > N \text{ and } t \text{ is even} \\ c_0^* \text{ if } n \leq N \text{ and } t \text{ is even, or if } n > N \text{ and } t \text{ is odd} \end{cases} \tag{46}$$

and

$$m_{nt} = \begin{cases} \frac{M_t}{N} \text{ if } n \leq N \text{ and } t \text{ is odd, or if } n > N \text{ and } t \text{ is even} \\ 0 \text{ if } n \leq N \text{ and } t \text{ is even, or if } n > N \text{ and } t \text{ is odd} \end{cases}. \tag{47}$$

The proof of this proposition consists of the verification part of the guess and verify procedure. We will do this in exercises which to a large extent mimic what we did for $\theta = 0$. In doing these exercises, you should make use of the following expressions for M_t and v_t which are implied by (38) and (43),

$$M_t = (1+\theta)^t M_0 \text{ and } v_t = \frac{1}{(1+\theta)^{t-1}} v_1. \tag{48}$$

Exercise 33 *Verify that the allocation given in (46) and (47) is affordable at the price sequence given by (45) and (43). (Hint: Show that (39) holds. There are two versions that have to verified. One version is for a person with income y and the other version is for a person with income 0.)*

The next two exercises confirm that the part of the allocation that pertains to each person is best for that person from among those that the person can afford.

Exercise 34 *Consider a person n with income y at some date t. Suppose this person enters date t with no money and will leave date $t+1$ with no money. (i) With date-t good on the horizontal axis and date-$t+1$ good on the vertical axis, sketch the opportunities for date-t and date-$t+1$ consumption implied by buying money at price v_t at date t, storing the purchased money until date $t+1$, and then selling at date $t+1$ at the price v_{t+1} all the money acquired at date t, when v_t and v_{t+1} satisfy (43). (ii) Explain why $(c_{nt}, c_{n,t+1}) = (c_y^*, c_0^*)$ is this person's preferred choice from among those opportunities.*

Exercise 35 *Consider a person n with income 0 (in goods) at some date t. Suppose this person enters date t with an amount of money equal to $\frac{M_{t-1}}{N}$ and will leave date $t+1$ with an amount of money equal to $\frac{M_{t+1}}{N}$. (i) With date-t good on the horizontal axis and date-$t+1$ good on the vertical axis, sketch the opportunities for date-t and date-$t+1$ consumption implied by selling money at price v_t at date t and storing any unsold money until date $t+1$ when its price is v_{t+1}, when v_t and v_{t+1} satisfy (43). (ii) Explain why $(c_{nt}, c_{n,t+1}) = (c_0^*, c_y^*)$ is this person's preferred choice from among those opportunities.*

These last three exercises accomplish the verification, subject, as above, to your acceptance of the claims about strict concavity and its consequences.

This model gives rise to a strong welfare conclusion about the transfer scheme.

Proposition 5 *Consider the equilibrium of proposition 4. In that equilibrium, the discounted utility of each person is decreasing in θ.*

Exercise 36 *Argue that Proposition 5 is true.*

12.2 Government expenditures

Governments sometimes use money creation as a way to finance expenditures. We now analyze such a situation against the background of the same model we used to study transfers. We assume that the goods the government acquires are used to provide a public good that is valued by people, but in a way that does not affect their preferences over private consumption as represented by the consumption streams we have so far studied. We do not attempt to determine the best amount of the public good. Instead, we assume a constant rate of money creation and describe a resulting equilibrium, including the amount of the good at each date that the government acquires.

We should make one change now that we are introducing the government as a user of goods. We should redefine allocations and feasible allocations to include the amount of the date-t good used by the government. We let G_t denote that amount.

Definition 7 *An allocation is $(c_{n1}, c_{n2}, ..., c_{nt}, ...)$ and $(m_{n1}, m_{n2}, ..., m_{nt}, ...)$ for each person n, where m_{nt} is the amount of money that person n ends up with at date t, and $(G_1, G_2, ..., G_t, ...)$.*

An allocation is feasible if $G_t + \sum_{n=1}^{2N} c_{nt} \leq W$ and $\sum_{n=1}^{2N} m_{nt} \leq M_t$.

We continue to assume that the path of the money supply satisfies (38). The amount of newly created money at date t is again $M_t - M_{t-1} = \theta M_{t-1}$. This newly created money is used to buy the date t good. In particular, the condition that government can afford $(G_1, G_2, ..., G_t, ...)$ is

$$G_t = v_t \theta M_{t-1} \tag{49}$$

for all $t \geq 1$.

We again find an equilibrium using a guess and verify procedure. We start with the conjecture that the equilibrium price sequence satisfies (43). And we also surmise that whenever a person has the same income in the form of goods, the person consumes the same amount. That is, we let c_y denote consumption when a person has income y and let c_0 denote consumption when a person has income 0. Now, however, we do not expect that (44) holds.

Exercise 37 *Explain why we do not expect that (44) holds.*

Instead, we do the following. We consider what a person with income y at t and income 0 at $t+1$ would do if faced with the opportunity to purchase money at t at the price v_t and to sell it at $t+1$ at the price v_{t+1}, where v_t and v_{t+1} satisfy (43). In a diagram with date t good on the horizontal axis and date $t+1$ good on the vertical axis, these opportunities lie on a line with slope $-\frac{1}{1+\theta}$ that passes through the point $(y, 0)$. (These opportunities are exactly those of the two-date lived Crusoe who faces $R = \frac{1}{1+\theta}$.) Denote by (c_y', c_0') the pair on the line which gets to the highest indifference curve.

Exercise 38 *If $u(x) = \sqrt{x}$, express c_y' and c_0' in terms of β, y, and θ.*

The rest of our guess comes from conjecturing that at each date, all the money held by people with income 0 (in goods) is used to purchase the good at that date. In particular, this gives us a candidate for v_1; namely, the solution for v_1 from

$$c_0' = v_1 \frac{M_0}{N} \tag{50}$$

With v_1 given by (50), (43) gives us the entire price sequence.

We now assemble our guess in the form of a proposition:

Proposition 6 *Under the endowments as given by (40)-(42), with money creation as given by (38), and with newly created money used by the government to purchase the good, there is an equilibrium with a price sequence given by (50) and (43), with govervnment expenditures given by (49) and with*

$$c_{nt} = \begin{cases} c_y' \text{ if } n \leq N \text{ and } t \text{ is odd, or if } n > N \text{ and } t \text{ is even} \\ c_0' \text{ if } n \leq N \text{ and } t \text{ is even, or if } n > N \text{ and } t \text{ is odd} \end{cases}, \tag{51}$$

and

$$m_{nt} = \begin{cases} \frac{M_t}{N} \text{ if } n \leq N \text{ and } t \text{ is odd, or if } n > N \text{ and } t \text{ is even} \\ 0 \text{ if } n \leq N \text{ and } t \text{ is even, or if } n > N \text{ and } t \text{ is odd} \end{cases}. \tag{52}$$

We again leave the verification for exercises.

Exercise 39 *Show that $G_t + Nc_y' + Nc_0' = W$.*

Exercise 40 *Verify that the consumption and money holdings in (51) and (52) are affordable at the price sequence given by (50) and (43). (Hint: Show that (39) holds when transfers are zero for everyone. There are two versions that have to verified. One version is for a person with income* y *and the other version is for a person with income* 0*.)*

Exercise 41 *Consider a person* n *with income* y *at some date* t*. Suppose this person enters date* t *with no money and will leave date* $t+1$ *with no money. (i) With date-*t *good on the horizontal axis and date-*$t+1$ *good on the vertical axis, sketch the opportunities for date-*t *and date-*$t+1$ *consumption implied by buying money at price* v_t *at date* t*, storing the purchased money until date* $t+1$*, and then selling at date* $t+1$ *at the price* v_{t+1} *all the money acquired at date* t*, when* v_t *and* v_{t+1} *satisfy (43). (ii) Explain why* $(c_{nt}, c_{n,t+1}) = (c'_y, c'_0)$ *is this person's preferred choice from among those opportunities.*

Exercise 42 *Consider a person* n *with income* 0 *(in goods) at some date* t*. Suppose this person enters date* t *with an amount of money equal to* $\frac{M_{t-1}}{N}$ *and will leave date* $t+1$ *with an amount of money equal to* $\frac{M_{t+1}}{N}$*. (i) With date-*t *good on the horizontal axis and date-*$t+1$ *good on the vertical axis, sketch the opportunities for date-*t *and date-*$t+1$ *consumption implied by being able to sell money at price* v_t *at date* t*, and storing any unsold money until date* $t+1$ *when its price is* v_{t+1}*, when* v_t *and* v_{t+1} *satisfy (43). (ii) Explain why* $(c_{nt}, c_{n,t+1}) = (c'_0, c'_y)$ *is this person's preferred choice from among those opportunities.*

These steps complete the verification. Now we turn to welfare.

It may seem surprising, but we cannot indict money creation here in the same way that we could in the last subsection. The government here is getting control of some goods at each date. Because we cannot say much about how worthwhile are the uses of those goods, the standard approach is to compare money creation as a way to finance the acquisition of those goods with other ways of financing that acquisition—the other ways being other forms of taxation. But even that does not lead to an unambiguous answer. If we ascribe to the government a great deal of power to tax—for example, via lump-sum taxes—then it would be better to use that taxation power to finance its expenditures. However, such an assumption seems to conflict with our assumption that the use of money is important for accomplishing trade. If the government can tax easily, then it can accomplish what monetary trade can accomplish simply by mandating those trades.

12.3 Paying interest on money

We noticed above that an equilibrium with a fixed stock of money was not Pareto Efficient. The inefficiency would be corrected if somehow money could be made to bear interest at the real rate $\frac{1}{\beta} - 1$. But how can that be done?

One conceivable way is to create additional money at each date using it to pay interest on the money that people carried over from the previous date. It is well known that that will not work. What happens is that the value of money falls in a way to exactly offset the interest. We (you) will demonstrate that in the context of the special model we have been using.

Suppose the money supply follows (38) and that this money is used to pay interest in the following sense.

Definition 8 *We say that person n can afford $(c_{n1}, c_{n2}, ..., c_{nt}, ...)$ and $(m_{n1}, m_{n2}, ..., m_{nt}, ...)$ at the prices (at the price vector or price sequence) $(v_1, v_2, ..., v_t, ...)$ if $m_{nt} \geq 0$ and*

$$c_{nt} + v_t m_{nt} = w_{nt} + v_t(1+\theta)m_{n,t-1} \tag{53}$$

for $t = 1, 2, ...$.

Notice the appearance of θ. It represents the payment of interest on money. I state an assertion in the form of a proposition.

Proposition 7 *If interest is paid on money at the rate θ and financed by creating money at that rate, according to (38), then there is an equilibrium in which each person's consumption and, therefore, welfare does not depend on θ.*

Exercise 43 *Sketch a proof of proposition 7.*

If payment of interest on money could be financed without creating money, then the amount of interest paid would matter. The other ways involve taxes. If it is easy to tax, then payment of interest on money would seem to be a good thing. However, as noted above, if it is easy to tax, then the allocations that trade accomplishes could also be achieved by taxing.

13 Appendix: Infinite-horizon Crusoe problems*

We study three versions of the following saving-consumption problem. The objective is

$$\sum_{t=0}^{\infty} \beta^t u(\omega_t + Rx_t - x_{t+1}).$$

The constraints are $\omega_t + Rx_t - x_{t+1} \geq 0$, $x_t \geq 0$ (no borrowing), and a given x_0. The assumptions are $\beta \in (0,1)$, $R \in [0, 1/\beta]$, $\omega_t \geq 0$, and u strictly increasing, strictly concave, and continuously differentiable with $u'(0) = \infty$. Each version will have a simple pattern for the endowment sequence, the ω_t sequence: a constant sequence and two periodic sequences. Each will also have a special initial condition. In each case, I propose an optimum and show that it is an optimum. Uniqueness of the optimum is obvious because the constraint set is convex and the objective is strictly concave.

After presenting the claims and the proofs, I comment on the close relationship between the arguments and standard proofs of the sufficiency of the Euler equation and the transversality condition.

Problem 1: $\omega_t = \omega > 0$ and $x_0 = 0$.

Claim 1: The optimum is $x_t = 0$. (This implies $u(\omega)/(1-\beta)$ for the value of the objective.)

Proof. Let $l(c) = u(\omega) + u'(\omega)(c - \omega)$. Because l, the tangent line to the function u at $(\omega, u(\omega))$, is a subgradient of u at ω, $l(c) \geq u(c)$. Also $u'(\omega) > 0$. Therefore, for any feasible path,

$$\begin{aligned}
\sum_{t=0}^{n} \beta^t u(\omega + Rx_t - x_{t+1}) &\leq \sum_{t=0}^{n} \beta^t l(\omega + Rx_t - x_{t+1}) \\
&= u(\omega) \sum_{t=0}^{n} \beta^t + u'(\omega) \sum_{t=0}^{n} \beta^t (Rx_t - x_{t+1}) \\
&= u(\omega) \sum_{t=0}^{n} \beta^t + u'(\omega)[\sum_{t=1}^{n} \beta^{t-1}(\beta R - 1)x_t - \beta^n x_{n+1}] \\
&\leq u(\omega) \sum_{t=0}^{n} \beta^t
\end{aligned}$$

where the last equality uses $x_0 = 0$ and the last inequality uses $x_t \geq 0$ and $\beta R \leq 1$. Therefore,

$$\lim\sup_{n\to\infty} \sum_{t=0}^{n} \beta^t u(\omega + Rx_t - x_{t+1}) \leq u(\omega) \lim\sup_{n\to\infty} \sum_{t=0}^{n} \beta^t = \frac{u(\omega)}{1-\beta},$$

which establishes the claim. □

Problem 2:

$$\omega_t = \begin{cases} 0 \text{ if } t \text{ is odd} \\ \omega > 0 \text{ if } t \text{ is even} \end{cases}$$

and $x_0 = 0$.

Claim 2: The unique optimum is

$$x_{t+1} = \begin{cases} 0 \text{ if } t \text{ is odd} \\ k^* \text{ if } t \text{ is even} \end{cases},$$

where k^* satisfies

$$u'(\omega - k^*) = \beta R u'(Rk^*).$$

(This implies $[u(\omega - k^*) + \beta u(Rk^*)]/(1-\beta^2)$ for the value of the objective.)

Proof. By the intermediate value theorem, k^* exists, is unique, and satisfies $k^* \in (0, \omega)$. We proceed by grouping two adjoining dates. For $(x, y) \in [-\omega/R, \infty) \times (-\infty, R(\omega + Rx)]$, let

$$F(x, y) = \max_{k\in[\max\{0,y/R\},\omega+Rx]} [u(\omega + Rx - k) + \beta u(Rk - y)].$$

By the Wierstrass theorem, F exists. Moreover, because the objective is strictly concave in k and the constraint set is convex, the maximum is unique, and, therefore, can be denoted $h(x,y)$. Also, F is easily shown to be strictly concave.

For (x,y) in a neighborhood of $(0,0)$, let $G(x,y) = u(\omega + Rx - k^*) + \beta u(Rk^* - y)$. Then, because $h(0,0) = k^*$, $F(x,y) \geq G(x,y)$ and $F(0,0) = G(0,0)$. Moreover, G is differentiable at $(0,0)$. Now let $\bar{F}(x,y)$ be any subgradient of F at $(0,0)$. It follows that $\bar{F}(x,y)$ is a subgradient of G at $(0,0)$. Therefore,

$$\bar{F}(x,y) = F(0,0) + Ru'(\omega - k^*)x - \beta u'(Rk^*)y.$$

Notice that the objective at the candidate optimum is $F(0,0)/(1-\gamma)$, where $\gamma = \beta^2$.

For any feasible path of x_t, we have

$$\begin{aligned}
\sum_{t=0}^{2n} \beta^t u(\omega_t + Rx_t - x_{t+1}) &\leq \sum_{t=0}^{n} \gamma^t F(x_{2t}, x_{2t+2}) \leq \sum_{t=0}^{n} \gamma^t \bar{F}(x_{2t}, x_{2t+2}) \\
&= \sum_{t=0}^{n} \gamma^t [F(0,0) + Ru'(\omega - k^*)x_{2t} - \beta u'(Rk^*)x_{2t+2}] \\
&= F(0,0)\sum_{t=0}^{n} \gamma^t + \sum_{t=1}^{n} \gamma^{t-1}\beta[\beta Ru'(\omega - k^*) - u'(Rk^*)]x_{2t} - \gamma^n \beta u'(Rk^*)x_{2n+2} \\
&= F(0,0)\sum_{t=0}^{n} \gamma^t + u'(\omega - k^*)\sum_{t=1}^{n} \gamma^{t-1}[(\beta R)^2 - 1)x_{2t} - \beta^{2n+1}u'(Rk^*)x_{2n+2} \\
&\leq F(0,0)\sum_{t=0}^{n} \gamma^t
\end{aligned}$$

where the last equality follows from the definition of k^*. Therefore,

$$\limsup_{n\to\infty} \sum_{t=0}^{2n} \beta^t u(\omega + Rx_t - x_{t+1}) \leq F(0,0) \limsup_{n\to\infty} \sum_{t=0}^{n} \gamma^t = \frac{F(0,0)}{1-\gamma},$$

which establishes the claim. □

Problem 3:

$$\omega_t = \begin{cases} \omega > 0 \text{ if } t \text{ is odd} \\ 0 \text{ if } t \text{ is even} \end{cases},$$

and $x_0 = k^*$, where k^* is as defined in claim 2.

Claim 3: The unique optimum is

$$x_{t+1} = \begin{cases} k^* \text{ if } t \text{ is odd} \\ 0 \text{ if } t \text{ is even} \end{cases}.$$

(This implies $[u(Rk^*) + \beta u(\omega - k^*)]/(1-\beta^2)$ for the value of the objective.)

Proof. Again, I group two adjoining dates. For $(x, y) \in [0, \infty) \times (-\infty, \omega + R^2 x)]$, let

$$F(x, y) = \max_{k \in [\max\{0, (y-\omega)/R\}, Rx]} [u(Rx - k) + \beta u(\omega + Rk - y)]. \tag{54}$$

By the Wierstrass theorem, F exists. Moreover, because the objective is strictly concave in k and the constraint set is convex, the maximum is unique, and, therefore, can be denoted $h(x, y)$. Again, F is easily shown to be strictly concave.

It is easy to show that $h(k^*, k^*) = 0$. Then, for (x, y) in a neighborhood of (k^*, k^*), let $G(x, y) = u(Rx) + \beta u(\omega - y)$. It follows that $F(x, y) \geq G(x, y)$ and that $F(k^*, k^*) = G(k^*, k^*)$. Moreover, G is differentiable at (k^*, k^*). Now let $\bar{F}(x, y)$ be any subgradient of F at (k^*, k^*). It follows that $\bar{F}(x, y)$ is a subgradient of G at (k^*, k^*). Therefore,

$$\bar{F}(x, y) = F(k^*, k^*) + Ru'(Rk^*)(x - k^*) - \beta u'(\omega - k^*)(y - k^*).$$

Notice, also, that the objective at the candidate optimum is $F(k^*, k^*)/(1 - \gamma)$, where $\gamma = \beta^2$.

For any feasible path, we have

$$\begin{aligned}
\sum_{t=0}^{2n} \beta^t u(\omega_t + Rx_t - x_{t+1}) &\leq \sum_{t=0}^{n} \gamma^t F(x_{2t}, x_{2t+2}) \leq \sum_{t=0}^{n} \gamma^t \bar{F}(x_{2t}, x_{2t+2}) \\
&= \sum_{t=0}^{n} \gamma^t [F(k^*, k^*) + Ru'(Rk^*)(x_{2t} - k^*) - \beta u'(\omega - k^*)(x_{2t+2} - k^*)] \\
&= F(k^*, k^*) \sum_{t=0}^{n} \gamma^t + \sum_{t=1}^{n} \gamma^{t-1} \beta [\beta R u'(Rk^*) - u'(\omega - k^*)](x_{2t} - k^*) \\
&\quad - \gamma^n \beta u'(\omega - k^*)(x_{2n+2} - k^*) \\
&= F(k^*, k^*) \sum_{t=0}^{n} \gamma^t - \gamma^n \beta u'(\omega - k^*)(x_{2n+2} - k^*) \\
&\leq F(k^*, k^*) \sum_{t=0}^{n} \gamma^t + \gamma^n \beta u'(\omega - k^*) k^*
\end{aligned}$$

where the last equality follows from the definition of k^* and the last inequality follows from the constraint, $x_t \geq 0$. Therefore,

$$\begin{aligned}
\limsup_{n \to \infty} \sum_{t=0}^{2n} \beta^t u(\omega + Rx_t - x_{t+1}) &\leq F(k^*, k^*) \limsup_{n \to \infty} \sum_{t=0}^{n} \gamma^t + \beta u'(\omega - k^*) k^* \limsup_{n \to \infty} \gamma^n \\
&= \frac{F(k^*, k^*)}{1 - \gamma},
\end{aligned}$$

which establishes the claim. □

Remark. All three proofs are versions of the standard proof of sufficiency of the Euler equation and the transversality condition (see, for example, the proof of Theorem 4.15 in Stokey, Lucas, with Prescott 1989, page 98).

The proof of claim 3 is exactly that proof applied to the return function F as defined in (54). Notice that the standard proof requires differentiability of the return function only at the candidate optimum. That feature is used in the proof of claim 3.

The proof of claim 1 differs a bit from the standard proof because the candidate optimum is not internal in the state variable. However, it is internal in consumption and that suffices. (No transversality term appears because the candidate for the optimum has $x_t = 0$.) Problem 1 is a special case of that described in Stokey, Lucas, with Prescott 1989, section 5.17, special in that I assume $x_t = 0$, while they assume $x_0 \geq 0$. The method described above can, I think, be adapted to an arbitrary initial condition.

The proof of claim 2 is a variant of that of claim 1.