V. Notes on the Diamond-Dybvig Model of Banking

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1 Introduction

The term *illiquid banking system* refers to a property of a consolidated balance sheet. The consolidation is over banks and may also include the central bank, the government, and even those in debt to banks. Illiquidity means that not all banking system obligations can be met if all holders of those obligations simultaneously claim what they have been promised. More generally, if the obligations are deposits that give the owners of the deposits the right to decide when to withdraw, then the banking system is illiquid if there is some possible pattern of withdrawals that cannot be accommodated. The most typical example of such a system is a fractional reserve banking system under a commodity standard such as the gold standard. Such a banking system has demand liabilities that exceed its reserves (in the form of the commodity standard).

Banking system illiquidity seems to open the door to potential difficulties. In fact, there are historical instances of what are variously called bank panics or bank runs that are generally viewed as realizations of the potential difficulties (see Friedman and Schwartz 1963). Are these realizations inevitable? Can they be avoided, and at what cost? Some economists have proposed eliminating the potential difficulties by eliminating illiquid banks. Well-known proposals include 100% reserve banking or the modern version called narrow banking. These may be good or may be silly—as silly as a proposal to reduce automobile accidents by limiting automobile speeds to zero. To appraise any such proposal, it is helpful to have a model.

Why? The alternatives are to look at history or to try an experiment using the proposal. As regards history, even if the proposal had been in effect in the past, without a model we would not know what to look for to judge its success or failure. The same difficulty arises when we consider an experiment. Moreover, experimenting on the actual economy may be very costly—particularly if the proposal is not, in fact, a good idea.

The model we use is due to Diamond and Dybvig and first appeared in print in 1983. It has two main ingredients: (i) individual uncertainty about desired time profiles of consumption, including the assumption (referred to as private information uncertainty) that realizations of this uncertainty are known to the person, but not to others; and (ii) isolation of people from each other in a way that, among other things, forces the banking system to deal with depositors on a first-come, first-served basis. The first ingredient, individual uncertainty, gives rise to a role for assets that can be cashed in at the request of the holder—something like actual demand deposit and savings deposit accounts. This ingredient is plausible in that such uncertainty has long been used to explain why people hold low-return assets such as checking accounts and traveler's checks. The second ingredient, isolation that forces first-come, first-served treatment of depositors, is also plausible—if only because something like it is necessary to account for the dominant role that first-come first-served plays in almost all retail trade.

2 The environment

There are three dates, dates $0, 1$, and 2. In addition, date 1 is split into stages, as many stages as there are people. There are N people and there is one good per date. We start with resources and technologies. The economy starts at date 0 with a resource in the form of date-0 good in amount Ny, or y per person. The economy has an inter-temporal technology which must be initiated at date 0 and which uses as input only the date 0 resource. The technology is linear and displays constant returns to scale. Its net rate-of-return from date 0 to date 1 is zero. It has a positive return over two dates, from date 0 until date 2. Let C_1 be total consumption of date-1 good and let C_2 be total consumption of date-2 good. These totals are constrained by

$$C_2 = R(Ny - C_1) \tag{1}$$

where $R > 1$. Date 1 withdrawals from the technology are perishable.

Preferences are complicated. Each person turns out to be impatient, type a, with probability π and turns out to be patient, type b, with probability $1 - \pi$. The realizations are independent across people. If the person turns out to be impatient, then the person cares only about date-1 consumption; while if the person turns out to be patient, then the person cares only about date-2 consumption. The person learns his or her type just before date 1. Prior to learning his or her type, all people have identical preferences given by expected discounted utility. If there were no separate stages at date 1, then expected discounted utility would be given by

$$\pi u(c_1^a) + (1 - \pi)\beta u(c_2^b) \tag{2}$$

where c_1^a is consumption of date-1 good *if* the person turns out to be type a and c_2^b is consumption of date-2 good *if* the person turns out to be type b. Here, the function u is strictly increasing and strictly concave and $0 < \beta < 1$. We also assume that $R < \frac{1}{\beta}$.

The date-1 stages are our way of building in the assumption that people have to be dealt with on a first-come first-served basis. We assume that the order is random and uniformly distributed. There are N date-1 stages and each person appears at one of the stages and has probability $\frac{1}{N}$ of appearing in any one of the N stages. The significance of these stages is that the person who shows up in the first stage has to be assigned some date-1 consumption at that stage. That amount cannot depend on the types who have yet to appear; and so on for the other date-1 stages. There is a single stage at date 2 at which all N people appear at the same time.

3 Two people ($N = 2$)

We discuss in detail only the case of two people. With two people, there are two date-1 stages and four possible date-1 outcomes. These are listed in the following table along with notation for consumption.

date-1 outcomes		probability	stage 1	stage 2
stage 1	stage 2	of the sequence	consumption	consumption
a	a	π^2	$c_1^a(a)$	$c_1^a(a,a)$
b	b	$(1-\pi)^2$	$c_1^b(b)$	$c_1^b(b,b)$
a	b	$\pi(1-\pi)$	$c_1^a(a)$	$c_1^b(a,b)$
b	a	$\pi(1-\pi)$	$c_1^b(b)$	$c_1^a(b,a)$

We are interested in the best allocation from the point of view of people before they realize their type and their appearance by stage. In order not to make expressions any longer than necessary, we impose some features at the outset. We do not want to give consumption to types who do not value it. Therefore, we set all type-b date-1 consumptions to zero and all type-a date-2 consumptions to zero. Then we express discounted utility as,

$$\begin{aligned}
&\pi^2\{.5u[c_1^a(a)] + .5u[c_1^a(a,a)]\} + \\
&(1-\pi)^2\beta u[c_2^b(b,b)] + \\
&\pi(1-\pi)\{.5u[c_1^a(a)] + .5\beta u[c_2^b(a,b)]\} + \\
&\pi(1-\pi)\{.5u[c_1^a(b,a)] + .5\beta u[c_2^b(b,a)]\}
\end{aligned} \tag{3}$$

Each line of this expression corresponds to the respective row of the above table. Thus, for example, consider the third line, which corresponds to the stage-1 outcome a followed by the stage-2 outcome b. The pay-off $u[c_1^a(a)]$ in the third line is to be multiplied by the probability of the following joint event: the sequence of date–1 outcomes is a followed by b *and* the person is type a. The probability of that joint event is the product of the probability of that date-1 sequence and the conditional probability that the person is type a given that date-1 sequence. And that conditional probability is simply the probability that the person appears in stage 1, which is .5. Of course, the pay-off $u[c_1^a(a)]$ also appears in the first line as part of the event that the sequence of date–1 outcomes is a followed by a. The total probability of the pay-off $u[c_1^a(a)]$, the sum of those in the first and third lines, is $.5\pi$, the product of the probabilities of the independent events: appear in stage 1 *and* be type a.

We can further simplify expression 3) using the resource constraint (1). The first line corresponds to the date-1 realization (a, a). When this occurs, then at stage 2 we want to give that person everything that is left; namely, we want to set $c_1^a(a, a) = 2y - c_1^a(a)$. The second line corresponds to the date-1 realization (b, b). Here, nothing is given out at date 1 and we simply divide up the implied total date-2 resources, $R2y$, equally between the two patient people; namely, $c_2^b(b, b) = Ry$. The third line corresponds to the date-1 realization (a, b). Here we cumulate in the technology whatever was not given to the type a person at date 1 and give the proceeds to the type b person at date 2; namely, $c_2^b(a, b) = R(2y - c_1^a(a))$. Finally, the fourth line corresponds to the date-1 realization (b, a). Here, again, we cumulate whatever was not given to the type a person at date 1 and give the proceeds to the type b person at date 2; namely, $c_2^b(b, a) = R(2y - c_1^a(b, a))$. After we substitute these expressions into (3), we are left with expected utility in terms of two unknowns, $c_1^a(a)$ and $c_1^a(b, a)$:

$$\begin{aligned}
&\pi^2\{.5u[c_1^a(a)] + .5u[2y - c_1^a(a)]\} + \\
&(1-\pi)^2\beta u(Ry) + \\
&\pi(1-\pi)\{.5u[c_1^a(a)] + .5\beta u[R(2y - c_1^a(a))]\} + \\
&\pi(1-\pi)\{.5u[c_1^a(b,a)] + .5\beta u[R(2y - c_1^a(b,a))]\}.
\end{aligned} \tag{4}$$

We can now say quite a bit about the best choices. First, because we have taken the resource constraint into account, the only remaining constraints are non negativity of consumption. Second, the expression is additively separable: one unknown, $c_1^a(a)$, appears only in the first and third lines; the other unknown, $c_1^a(b, a)$, appears only in the fourth line. It follows that we can choose $c_1^a(a)$ to maximize the sum of the first and third lines and can choose $c_1^a(b, a)$ to maximize the fourth line. We start with the choice of $c_1^a(b, a)$, which is simpler.

3.1 The choice of $c_1^a(b, a)$

Maximizing the fourth line of (4) is the same as maximizing

$$u[c_1^a(b, a)] + \beta u[R(2y - c_1^a(b, a))].$$

This is the same as problems we have seen. A person has income $2y$ in the form of the date-1 good, nothing in the form of date-2 good, and has available an inter-temporal technology such that if k is the date-1 input, then Rk is the date 2 output. To make this explicit, we can let $c_1^a(b,a) = 2y - k$. Then, in terms of k, the problem is to maximize $u(2y-k) + \beta u(Rk)$.

We can depict the best choice of k graphically. With date-1 good measured along the horizontal axis and date-2 good along the vertical axis, the pairs $(c_1^a(b,a), c_2^b(b,a))$ that can be attained by all possible choices of k between 0 and $2y$ are those on a line through the point $(2y, 0)$ with slope $-R$. The indifference curves corresponding to $u[c_1^a(b,a)] + \beta u[c_2^b(b,a)]$ are those of discounted utility. Therefore, with $R < \frac{1}{\beta}$, it follows that the best choice for $c_1^a(b,a)$, denoted $c_1^a(b,a)^*$, satisfies $c_1^a(b,a)^* > y$.

3.2 The choice of $c_1^a(a)$

We choose $c_1^a(a)$ to maximize the sum of the first and third lines of (4). We can reason as follows. If $c_1^a(a)$ were chosen to maximize only the first line, then we would choose $c_1^a(a) = y$. (Why?) If, instead, $c_1^a(a)$ were chosen to maximize only the third line, then we would choose $c_1^a(a) = c_1^a(b,a)^*$. (Why?) It makes sense and is true that when we choose $c_1^a(a)$ to maximize the sum of the first and third lines, the best choice, denoted $c_1^a(a)^*$, is a compromise between those two; it satisfies $y < c_1^a(a)^* < c_1^a(b,a)^*$.

Exercise 1 *Answer the two "why" questions posed in the last paragraph.*

3.3 Remarks about the solution

One thing to notice is that there are *gains from trade*. To see this, imagine that each of the two people is alone, starts with y units of the good at date 0, and has available the above technology. Call this the autarky outcome. The autarky outcome gives expected utility equal to $\pi u(y) + (1-\pi)\beta u(Ry)$. (Why?) Notice that (4) is equal to that expression if $c_1^a(a) = c_1^a(b,a) = y$. We know that there are gains to trade because we have just seen that a better outcome is possible. The best outcome devotes more resources on average to date-1 consumption and less to date-2 consumption than does the autarky outcome.

Exercise 2 *Answer the "why" question posed in the last paragraph.*

The best outcome is complicated and displays a feature that resembles the difficulties associated with illiquid banking system portfolios. For example, if the sequence of outcomes is a followed by a, then the stage-1 person get more than y and the stage-2 person gets less than y. Obviously, the first person is lucky and the second is unlucky. This looks a bit like a bank-run; the second person gets less for no reason other than having shown up later.

4 A Large Number of People

The above kind of problem can be formulated when N is large. Moreover, we expect that the solution will resemble the one found for $N = 2$. However, the difficulty of actually solving the problem grows with N. For that and other reasons, I suggest that we think about allocations that resemble actual banking arrangements and ask whether some such allocation might approximate the solution to the N-person problem. The allocations I have in mind are simple suspension schemes that are determined by the choice of 3 quantities: one quantity, denoted c_1, is the date-1 payment to people who are type a and that is paid until total date-1 withdrawals hit some total amount, denoted Q. After that, those who are type a get some smaller amount, denoted c_s, s for suspension. Any resources not paiid out at date 1 are cumulated until date 2 and divided equally among those who are type b.The three quantities that determine this scheme are c_1, Q, and c_s.

Here are some reasons for thinking about such a scheme. (i) As noted above, the $N-$person problem is too hard to solve. (ii) Society's banking problem is even more complicated than the above $N-$person problem. (iii) Even if that problem could be solved, it would seem difficult to convince people that they are getting the right payment. In that respect, even the simple suspension scheme should probably be thought of as involving extra costs that are like the costs borne when bankruptcy is declared. (iv) When N is large, there is reason to suspect that the above suspension scheme will approximate the solution to the $N-$person problem. (Why?)