

Concepts of Strategic Complexity in Extensive-Form Unanimity Bargaining*

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Abstract

In an earlier paper, we showed that a new complexity measure on strategies, response complexity, implied that strategies chosen in equilibrium were stationary or minimally complex, in the unanimity bargaining game. In this paper, we explore alternative specifications of machines and the standard complexity measure, that of counting the number of states, and show that, except for two-player games, state complexity is not strong enough to reflect the intuition that costs of complexity lead to stationary strategies being played.

*This paper is part of a larger project on strategic complexity that we began in 1994. One part of this has appeared in Chatterjee and Sabourian (2000). We thank Anil Dasgupta, Vijay Krishna and Ariel Rubinstein for comments on drafts of work related to this project over the years. This current draft is still not expected to be the final version.

[†]This paper was written for the conference in honour of Ken Binmore's 65th birthday. Binmore has been a pioneer in the understanding and use of complexity in game theory.

1 Introduction

In an earlier paper (Chatterjee and Sabourian (2000)), we addressed the question of the multiplicity of subgame perfect equilibria in the unanimity bargaining game and showed that, if players' preferences lexicographically accounted for costs of implementing complex strategies, a rationale could be found for the equilibrium in stationary strategies. The concept of strategic complexity used in that paper, *response complexity*, is somewhat stronger than the counting states measure of complexity used in the repeated Prisoners' Dilemma paper of Abreu and Rubinstein (1988) and in subsequent papers on two-player repeated games and complexity (see Piccione and Rubinstein (1992) and the list of references at the end of the current paper). The unanimity bargaining game, to be described below, is a single extensive-form game, not a repeated game, and some actions, such as accepting an offer end the game. Additionally, the multiplicity of subgame perfect equilibria arises only for three or more players, while the literature has concerned itself mainly with two-player games.¹

The aim of this paper is to explore various specifications of automata and concepts of strategic complexity in the setting of the unanimity bargaining game. We find that, though the response complexity measure used in our earlier paper is plausible, there are other plausible measures that do not give us the result on stationarity. There are also different ways of defining machines that give rise to different results. While the conclusions of this paper are not a clear-cut as that of our earlier paper, it still serves to illustrate how complexity and the extensive form interact and might therefore be useful.

We now turn to the description of the unanimity bargaining game (n -player, one-cake problem) of Binmore (1985), Herrero (1985) and Shaked (1986). The game is as follows: There are n players $(1, \dots, n)$ who have joint access to a 'cake' of size unity, if they can agree on how to share it amongst themselves. Player 1 makes the first offer at time $t = 1$, offering $x = (x_1, \dots, x_n)$, where x_i is player i 's proposed share and $\sum_i x_i = 1$. Player 2 to player n then respond sequentially, each saying either A(cept) or R(eject) to the proposal. If all responders accept then the game ends with player i obtaining a payoff of x_i . A rejection takes the game to the next period, where

¹Ideas similar to response complexity are used in more general market settings to refine the set of subgame perfect equilibria by Gale and Sabourian (2005).

player 2 now makes an offer and Players $(3, \dots, n, 1)$ sequentially respond. If one of the responders rejects at time $t = 2$, then the game goes to the next stage with player 3 making the offer and so on.² If the offer x is accepted by all responders in period t , the payoff is $\delta^{t-1}x_i$ to player i , where $\delta < 1$ is the common discount factor. There is no exogenously imposed limit on the duration of the game. Absence of agreement, that is bargaining forever, leads to a payoff of 0 for all players.

It is easy to see that any partition of the cake at any time t can be sustained as a Nash equilibrium of this game. This is also true of the two-player version. However, Rubinstein (1982) demonstrated that using the refinement of subgame perfectness was sufficient to pick out a unique equilibrium in this game. If three or more players are involved, though, the nature of the game changes as Shaked showed (see Osborne and Rubinstein (1990) for an exposition). In fact, for the 3-player game every allocation of the cake, including the extreme points, can be sustained as a subgame perfect equilibrium (SGPE) if $\delta > \frac{1}{2}$. Outcomes with delay can also be sustained as SGPE for 3-player games. The multiplicity result extends for high enough δ to general characteristic function games of coalition formation (see Chatterjee et al. (1993) for the formal statement). Jun (1987) and Krishna and Serrano (1996), among others, have considered alternative extensive forms that give a unique SGPE with the limiting equal split. Their models have the feature that a player is able to leave with his share before the entire bargaining process is completed, in contrast with the previously-cited work of Binmore-Shaked-Herrero, where the entire allocation has to be unanimously approved before anyone can leave.

If we could restrict consideration to equilibria in stationary strategies, a unique, no-delay subgame perfect equilibrium can be obtained (Herrero, (1985)). The allocation corresponding to the unique stationary SGPE is $(\frac{1}{1+\delta+\dots+\delta^{n-1}}, \frac{\delta}{1+\delta+\dots+\delta^{n-1}}, \dots, \frac{\delta^{n-1}}{1+\delta+\dots+\delta^{n-1}})$, which also tends to the equal split as $\delta \rightarrow 1$, an outcome consistent with the two-player result of Rubinstein (1982).

Stationary strategies (to be defined more precisely later) are thought to be simpler to implement than strategies requiring complicated forms of history dependence. We start with one way of discussing simplicity or complexity of strategies, by representing a strategy as a finite-state machine. Restrictions

²Thus player k makes an offer at $t = k + \tau n$ for every non-negative integer τ .

on strategies are introduced either by some exogenous bounds on the complexity of the machines (usually given by counting the number of states in a machine) or by making complexity a decision variable - complex strategies can be used but the complexity comes at a cost compared to simpler strategies. In this paper, we adopt the latter approach, as we did in our earlier work as well.

As mentioned before, most of the literature in this area has dealt with two-player repeated, normal-form games. (See Kalai (1990) for a survey.) Unanimity bargaining is, however, a single, extensive-form game, possibly with more than two players.

Focussing on complexity of implementation rather than of computation has its critics, for example, Papadimitriou (1992). We will not address this issue here, or more philosophical aspects of representing players as automata.

An alternative, evolutionary, approach to *two-person bargaining* was taken in Binmore et al. (1998). Their primary aim is to derive the Rubinstein result based, not on subgame perfectness, but on the interaction between complexity and evolutionary stability. We do not consider evolutionary ideas or learning in this paper.³

The rest of the paper is organised as follows. In the next section, we adopt a particular definition of an automaton and discuss the resulting interaction between complexity measures and the machine specification adopted. It is important to note that, while the different specifications of machines are equivalent, in the sense of being able to implement the same set of strategies, the refining⁴ effect of different complexity criteria could depend on the particular specification.⁵

In Section 3, we consider the “counting states” measure of strategic complexity. This turns out to be quite powerful in two-player games but does not refine equilibria in games of more than two players. There are two counter-

³Maenner (2004) has an interesting model of learning, in repeated Prisoners’ Dilemma games, involving minimal inferences where strategies to be inferred are finite automata. One of the inference rules he considers gives rise to stationarity. It is not immediately clear whether his work has more general implications in extensive-form games

⁴We mean refining the set of equilibria.

⁵In this paper, we do not need to assume finiteness of the machines: the number of states, the set of possible inputs and the set of possible outputs of a machine could be infinite. The structure is therefore only a small departure from that of the standard bargaining game.

examples; we discuss one of these in some detail. Section 4 contains a different specification of a player—we divide a player into “agents” corresponding to the role a given player is fulfilling in a particular period and endow each agent with one sub-machine. We discuss how far we can go with this specification, where once again counting states is used to measure complexity. Section 5 contains a brief discussion of the main result in Chatterjee and Sabourian (2000) along with a measure stronger than counting states. Section 6 concludes.⁶

2 Nash equilibria with complexity

The basic definition of Nash equilibrium with complexity used in our paper is a modification of the one introduced by Abreu and Rubinstein (1988) to take into account the nature of the game we study. Consider a n – *player* game with strategy sets $\Lambda_i, i = 1, 2, 3..n$. Here Λ_i represents the set of “machines” available to Player i , where the definition of machine is postponed until later. Let $|M_j|$ denote the “complexity” of machine $M_j \in \Lambda_i$, where complexity is also defined in several different ways later in this paper. One can think of $|M_j|$ as the “cost” of implementing the strategy given by the machine.

As usual, the payoffs to players from a strategy profile are given by some real-valued function of the strategies, so that the payoff to Player i is $\pi_i(M_i, M_{-i})$ where $M_i \in \Lambda_i$ and $M_{-i} \in \times \Lambda_k, k \neq i, k = 1, 2, 3..n$. Then the two conditions, from Abreu and Rubinstein, are as follows:

(M_i, M_{-i}) constitute a NEC (Nash equilibrium with complexity) if :

(i) $\pi_i(M_i, M_{-i}) \geq \pi_i(M'_i, M_{-i})$ for all $M'_i \in \Lambda_i$,

(ii) If $\pi_i(M_i, M_{-i}) = \pi_i(M'_i, M_{-i})$, then $|M_i| \leq |M'_i|$.

The modifications we will make will have to do with the definition of machine and the associated definition of complexity. But before this, we need to discuss the specific features of the bargaining game in more detail.

In this game, each player has to play different roles, as proposer or k^{th} responder, $k \geq 1$, and thus must have different action sets corresponding to these roles. An *offer* consists of a non-negative vector $x = (x_1, \dots, x_n)$ such that $\sum_i x_i = 1$. The actions available to each responder in the sequence must come from $\{A, R\}$ (accept or reject). If an offer is accepted by all responders,

⁶For a fuller account of the related literature, we refer to our working paper “N-Person Bargaining and Strategic Complexity”, posted on our respective web sites.

the game ends or enters a terminal state T with the payoffs as described in the introduction.

A ‘period’ consists of an offer and the sequence of responses to it, so it could consist of at most n actions of which $n - 1$ are responses and it could be fewer if one of the responses is a rejection (in which case the game moves to the next period). A ‘stage’ of the bargaining game consists of n consecutive periods. A stage could be terminated before n periods has elapsed if the bargaining is terminated by an agreement before the n -th period. In any given *period*, a player is in *one* of the following n roles: the proposer or the k -th responder for some $k = 1, \dots, (n - 1)$, while in any *stage* the player must be prepared to play *all* n roles. We shall denote a history of outcomes in a stage by e . Also we shall denote the set of all such possible histories of a stage by E . Thus a particular history of a stage, $e = \{(x^1, A, R), (x^2, A, A, A, A, R), \dots, (x^n, R)\} \in E$, could, for example, consist of an offer x^1 by player 1, an acceptance by the first and rejection by the second responder, followed by an offer x^2 , an acceptance by the first four responders and a rejection by the fifth responder in the second period of the stage, .., and finally an offer x^n rejected by the first responder in the n -th period of the stage. A stage can therefore be described by at most $n \times n$ choices (offers and responses) by the players.

We also need notation for partial descriptions of choices (partial history) within a stage. We shall denote such a partial history by s and the set of such partial histories by S . Thus, for example, the second responder in period 2 in a given stage moves after a history in that stage of a first-period offer, its acceptance by some responders and rejection by one and a second-period offer and its acceptance by the first responder. The proposer in the first period of a stage faces a null history \emptyset . Thus $S = \emptyset \cup \{s = (c^1, \dots, c^r) \in C^r \mid (s, d^1, \dots, d^{r'}) \in E \text{ for some sequence of choices } (d^1, \dots, d^{r'}) \in C^{r'}\}$, where for any τ , C^τ is the τ -fold Cartesian product of C .

Also, we shall denote the information sets (the sets of partial histories) for player i in any stage by S_i . Thus $S_i \equiv \{s \in S \mid \text{it is } i\text{'s turn to play after } s\}$.

Finally, denote the set of choices available to a player i , given a partial description $s \in S_i$, by $C_i(s)$. Thus

$$C_i(s) = \begin{cases} \Delta^n & \text{if } i \text{ is the proposer (if } i = 1 \text{ and } s = \emptyset \text{ or if } i > 1 \text{ and } s \text{ is a complete} \\ & \text{history of the first } i - 1 \text{ periods of the stage)} \\ \{A, R\} & \text{if } s \text{ is such that } i \text{ is the responder to some offer } x. \end{cases}$$

Similarly, we can describe partial histories within a period, for example an offer and two acceptances constitute a partial history.

We are now ready to consider specifications of machines or automata.⁷ The sequential offers bargaining game has each player playing different roles in different periods, so we can choose to specify a machine to implement a particular strategy in several different ways.

Our first definition, equivalent to the one used in Chatterjee-Sabourian (2000) specifies that the states of a player's machine do not change during each *stage* of the bargaining game and transitions from a state to another state in the same player's machine take place at the *end* of a stage. An action would be specified for each role of a player given the state her machine is in and the partial history s prior to the action being taken. A referee (labelled "Master of the Game" by Piccione and Rubinstein, (1992)) would activate each player's machine when needed. We now set down the formal definition.

Definition 1 (D1) *A machine M_i is a five-tuple $(Q_i, q_i^1, T, \lambda_i, \mu_i)$, where*
 Q_i is a set of states;
 q_i^1 is a distinguished initial state belonging to Q_i ;
 T is a distinguished absorbing or terminating state (T for "Termination");
 $\lambda_i : Q_i \times S_i \rightarrow C$, describes the output function of the machine given the state of the machine and given the partial history that has occurred during the current stage of the bargaining before i 's move such that $\lambda_i(q_i, s) \in C_i(s)$, $\forall q_i \in Q_i$ and $\forall s \in S_i$;
 $\mu_i : Q_i \times E \rightarrow Q_i \cup T$ is the transition function, specifying the state of the machine in the next stage of the bargaining as a function of the current state and the realised history of the stage.⁸

Remark 1 *If we denote the set of strategies for a player i in any stage of the bargaining game by $\mathcal{F}_i \equiv \{f : S_i \rightarrow C \mid f(s) \in C_i(s) \forall s \in S_i\}$, then the output*

⁷Specifying a machine means, as usual, specifying a set of states, a set of inputs and outputs (not necessarily finite), a function prescribing an output deterministically given the state and the input, and a transition function prescribing the next state as a function of the current state and the history up to the point of transition.

⁸Henceforth, we shall not always explicitly refer to the terminal state T . We are assuming that if an offer is accepted by all responders, M_i enters state T and shuts off. Thus $\mu_i(q_i, x, A, \dots, A) = T$ for any state q_i and any $x \in \Delta^n$. Also, unless stated otherwise, we shall drop the term 'non-terminal' from the 'non-terminal states' of the machine and simply refer to them as the states of the machine.

function λ_i in Definition 1 can be thought of as a mapping $\tilde{\lambda}_i : Q_i \rightarrow \mathcal{F}_i$ where $\tilde{\lambda}_i(q_i)(s) = \lambda_i(q_i, s)$.

The remaining definitions highlight the differences with D_1 and are not stated formally to avoid repetition.

Definition 2 (D_2) Divide a player i into n “agents”, i_0, i_1, \dots, i_{n-1} , where i_0 is the proposer and $i_k, k = 1, 2, \dots, n-1$ is the k th responder role. Let $M_i = (M_{i_0}, \dots, M_{i_k}, M_{i_{n-1}})$. Here $C_{i_k}(s) = \{A, R\}$ for $k > 0$ and $C_{i_k}(\emptyset) = \Delta^n$. Transitions take place from a state in each sub-machine M_{i_k} to a state in the same sub-machine one stage (n periods) after the last action prescribed by the sub-machine, for all sub-machines (roles), $k = 0, 1, 2, \dots, n-1$.⁹

Definition 3 (D_3) This is the same as D_1 , except that the transitions now take place at the end of each **period**, rather than at the end of each stage. Naturally, here again the action chosen by the machine in any state also depends on the partial history \bar{s} within the period.

Definition 4 (D_4) As in D_2 , each machine is partitioned into n parts; the transitions now take place at the end of each period, rather than at the end of each stage. Since the transitions can be from a state in one sub-machine to a state in a different sub-machine for each player, the change of roles from period to period can be automated. Actions can depend upon partial histories within the period.

The definition D_1 has the advantage that the game “looks” the same at the beginning of each stage, so that the output and transition maps remain the same in each stage. However, this is not necessarily a compelling justification for using this definition.

We now turn our attention to defining the complexity of a machine. The most popular measure of complexity in this literature is the number of states in Q . In repeated strategic form games, where both of the two players move simultaneously, the set of states induces a partition on the set of histories and a finer partition can be considered more complex. However, when a player

⁹It is clear therefore that the transitions do not depend just on e but on the n periods separating the successive activations of the sub-machine. We have not introduced additional notation for this.

moves during a stage and not simultaneously with the other players, her response to the choices of earlier players within a stage also become relevant. This suggests that we need a strengthening of the complexity measure to take into account complexity of *responses*.

We shall denote the number of states of a machine M_i by $|M_i|$. This refers to the number of states in the set Q_i (we shall ignore the terminal state).

Definition 5 (*State complexity, denoted by ‘s-complexity’*) A machine M'_i is more s-complex than another machine M_i , denoted by $M'_i \succ^s M_i$, if

$$|M'_i| > |M_i|.$$

We shall use $M'_i \succeq^s M_i$ to denote “ M'_i is at least as s-complex as M_i ”.

The strengthening of this measure to take into account the nature of the game is given below.

Definition 6 (*Response-State complexity, denoted ‘r-complexity’*) A machine $M'_i = \{Q'_i, q_i^1, T, \lambda'_i, \mu'_i\}$ is more r-complex than another machine $M_i = \{Q_i, q_i^1, T, \lambda_i, \mu_i\}$, denoted by $M'_i \succ^r M_i$, if

either (i) $M'_i \succ^s M_i$

or (ii) the machines M_i and M'_i are otherwise identical except that given some non-null partial history $s' \in S_i$, M_i always responds the same way to s' irrespective of the state of the machine and M'_i responds differently to s' depending on the state of the machine. Formally, $Q_i = Q'_i, q_i^1 = q_i^1, \mu_i = \mu'_i$ and there exists a non-empty partial history $s' \in S_i$ such that

$$\lambda'_i(q_i, s) = \lambda_i(q_i, s) \quad \forall s \neq s' \text{ and } \forall q_i \in Q_i = Q'_i$$

$$\lambda_i(q_i, s') = \lambda_i(q'_i, s') \quad \forall q_i, q'_i \in Q_i,$$

$$\lambda'_i(q_i, s') \neq \lambda'_i(q'_i, s') \quad \text{for some } q_i, q'_i \in Q'_i.$$

As before, we shall use $M'_i \succeq^r M_i$ to refer to M'_i is at least as r-complex as M_i .

Part (ii) of the above definition distinguishes between two otherwise identical machines that are such that one chooses a constant action given a partial history s' and the other does not. This definition is the one used in our earlier paper and we refer to that for a discussion of its properties.

Stationarity. A machine that is minimally complex but can be used to play against any strategy is called stationary. For example, under definition D_1 , a minimally complex machine will have one state (in addition to the

termination state). A minimally complex machine under definition D_2 must have one state for each sub-machine; responder sub-machines also need the termination state T .

3 State complexity

We now examine what the use of state complexity can give us in terms of characterising the set of NEC. We start with two-player games. Before this, we state a lemma that says that a zero payoff in equilibrium can be implemented with a minimally complex machine. We shall use definition D_1 and mention D_3 when needed.

Lemma 1 *For any NEC profile $M = (M_1, \dots, M_n)$, if $\pi_i(M) = 0$ for some i then M_i is minimal.*

The proof is obvious.

3.1 Stationarity in two-player games

We first show that using the weaker complexity criterion \succ^s , with some special arguments that do not extend to the n-player ($n > 2$) case, suffices to obtain stationarity. Thus for the 2-player case the result is stated for NEC (both NECs and NECr).

Proposition 1 *Consider the alternating offers, two-player Rubinstein bargaining game. Let (z, t) be an agreement reached at a time period $t < \infty$. Suppose that (z, t) is induced by a NEC. Then $t \leq 2$.*

Proof. Suppose not, then there exists a machine profile $M = (M_1, M_2)$, where $M_i = \{Q_i, q_i^1, T, \lambda_i, \mu_i\}$ for $i = 1, 2$, that constitute a NEC and results in an agreement (z, t) with $t > 2$. Let q_i^τ be the state of player i at period $\tau \leq t$ on the equilibrium path. Suppose that player i is the responder and $j \neq i$ is the proposer at t . The rest of the proof is in several steps.

Step 1: The state of the responder at t , q_i^t , could not have occurred on the equilibrium path before the last stage.

Suppose not; then q_i^t occurs on the equilibrium path at some time period $\tau = t - 2l$ for some $l > 1$. Now, consider two cases.

Case A: $z_j = 0$. Then, by Lemma 1, M_j must have just one state q_j^t . Since, M_i is in the same state in the last stage of the game as in stage $(\frac{\tau}{2})^*$, where for any real number r , r^* stands for the smallest integer greater than or equal to r , it follows that the outcome of the game is the same in the last stage of the game as in stage $(\frac{\tau}{2})^*$. Therefore, the game would end at $\tau < t$. But this is a contradiction.

Case B: $z_j > 0$. Then, player j (the proposer at t) could change his machine by making a transition to q_j^t at stage $(\frac{\tau}{2})^*$. Since i is in state q_i^t at stage $(\frac{\tau}{2})^*$, this change by j results in the same outcome in stage $(\frac{\tau}{2})^*$ as in the last stage of the original game. Thus j could obtain the payoff $z_j > 0$ sooner at $\tau < t$ as a result of this change. But this is a contradiction.

Step 2: z could not have occurred on the equilibrium path before t when i is the responder.

Suppose not; then z has been offered by $j \neq i$ on the equilibrium path at some period $\tau < t$ and the responder i must have rejected it. Now consider two cases.

Case A: $z_i = 0$. In this case, it follows from Lemma 1 that M_i must have one state. Therefore player i 's behaviour must be the same in all stages. But this contradicts i rejecting z at τ and accepting it at t .¹⁰

Case B: $z_i > 0$. Suppose that i , the responder at τ , modifies its machine by changing its output function to accepting offer z in all states. Since this will yield the same agreement z before t , it must be better for the responder i , given discounting. Therefore, the original path could not have been a Nash equilibrium.

Step 3: A contradiction .

Let s^t (it could be null) denote the partial history within the last stage prior to z being offered at t . Either (1) s^t occurs at some previous period $\tau < t$ on the equilibrium path (for example this would be the case if s^t is null), or (2) s^t does not occur elsewhere on the equilibrium path. Suppose (1) holds. Let player i deviate to a machine where q_i^t is eliminated in favor of q_i^τ , with the one change that $\lambda_i(q_i^\tau, s^t z) = A$. From the previous Steps, q_i^t and z do not appear on the equilibrium path before t when i is the responder. Therefore the new machine for i implements the same outcome path as the equilibrium

¹⁰If i is the responder in the second period of a stage, having one state implies that i always makes the same offer and this offer must be rejected on the equilibrium path before period t . So the partial histories prior to periods τ and t must be the same.

machine and has one fewer state; therefore the original path could not have been an equilibrium. Suppose (2). Now s^t has not occurred before the final stage (thus, $s^t \neq \emptyset$). Player j must therefore be in the same state $q_j^t = q_j^\tau$ for some period $\tau = t - 2l$, $l > 1$; otherwise he could deviate to a machine without q_j^t and save a state. (Actions can be conditioned on the history s^t , which does not occur anywhere else.). Again, consider two cases.

Case A: $z_i = 0$. Then, by Lemma 1 i would have one state and thus, given that $q_j^t = q_j^\tau$, the game would end at $\tau < t$; a contradiction.

Case B: $z_i > 0$. Let Player i now deviate and make $q_i^\tau = q_i^t$. Then the outcome of the game in the stage containing τ would be the same as the outcome in the last stage. This would end the game at $\tau < t$, for the same agreement z , and is therefore a profitable deviation. But this is a contradiction. ■

3.2 Counter-example for three players.

Unfortunately, we cannot go much further with s -complexity as the following examples indicate.

3.2.1 Stage specification, definition D_1 .

This is a 30-period counter-example—we were unable to find anything simpler—and is discussed in an Appendix. The counter-example shows that the NEC with state complexity is not sufficient to generate minimally complex machines in equilibrium

3.2.2 Period specification

A simpler example is available with specification D_3 . We now discuss this.

We note that the states of each machine are allowed to change every period and the output function $\lambda_i(\cdot)$ is a function of the state of the machine and the partial history within a period.

Each machine has two states 1 and 2 and an agreement z^2 is reached in period 6. The actions and transitions on the equilibrium path are as shown above, with the states for each player as shown in brackets in each entry. The offers, apart from z^2 , are such that the second responder prefers waiting until

period 6 to obtain the equilibrium payoff rather than accepting the current offer and ending the game. All the offers are distinct. (We assume $z^2 \gg 0$.)

	1	2	3	4	5	6
Pl.1	$x^1(1)$	$R(1)$	$A(1)$	$x^2(2)$	$R(2)$	$A(2)$
2	$A(1)$	$y^1(1)$	$R(1)$	$A(2)$	$y^2(2)$	$A(2)$
3	$R(1)$	$A(1)$	$z^1(1)$	$R(2)$	$A(2)$	$z^2(2)$

Out of equilibrium, the actions and transitions are as follows.

For any offer x , we define the output function λ_i out of equilibrium as:

$$\lambda_i(q_i^k, x) = R, \text{ if } (q_i^k, x) \text{ does not occur on the equilibrium path}$$

Also

$$\lambda_i(q_i^k, x, A) = R, \text{ if } (q_i^k, x, A) \text{ does not occur on the equilibrium path}$$

For any state-history pair (q_i, e) that does not occur on the equilibrium path in a given period, the transition functions are constructed as follows

$$\mu_i(q_i, e) = \begin{cases} q_i' & \text{if } i \text{ is the proposer next period} \\ q_i & \text{if } i \text{ is the first responder next period} \\ q_i' & \text{if } i \text{ is the second responder next period} \end{cases}$$

where $q_i' \neq q_i$ (the other state being used on the path).

In this example the two states of each machine are essential because each is required to make a separate offer (e.g. x^1, x^2). Thus any attempt to save states results in an out-of-equilibrium outcome. Given the above transition and response functions any deviation resulting in an out-of-equilibrium outcome makes the player worse off by delaying the agreement.

Consider, as an example, a deviation by player 1, who drops state 1 from his machine, and thus begins the game with x^2 . This is rejected by player 2, who moves to state 2, while 3 stays in state 1. In the next period, the outcome is (y^2, R) , and in period 3, player 3 offers z^2 , which is accepted by player 1 and rejected by player 2 (who has moved back to state 1). The outcome (z^2, A, R) is not on the equilibrium path, and player 3 moves back to state 1, while player 2 stays in state 1, so the game moves back to the first period making the initial deviation unprofitable for player 1.

4 Sub-machines for each role, three players and more

We turn to definition D_2 , more formally stated here. It turns out that with s-complexity and three players, we can guarantee that a NEC implies minimally complex machines (i.e. stationarity). However, unfortunately, the situation is different with more than three players as a four-player counter-example illustrates.

We give the formal definition of D_2 below. (An informal definition was given in Section 2.)

Definition 7 (D_2) *A machine M_i consists of n -submachines, one for each role, $M_{ik} = (Q_{ik}, q_{ik}^1, T, \lambda_{ik}, \mu_{ik})$ for $k = 0, 1, \dots, n-1$ ¹¹, where Q_{ik} is a finite set of non-terminal states used by the machine M_i in the k -th role,*

$q_{ik}^1 \in Q_{ik}$ is a distinguished initial state in the k -th role,

T is a distinguished absorbing or terminating state,

$\lambda_{ik} : Q_{ik} \times \overline{S}_{ik} \rightarrow C$, describes the output function of the machine in the k -th role given the state of the machine in that role and given a partial history that has occurred during the current stage of the bargaining, where $\overline{S}_{ik} \subset \overline{S}_i$ is the set of partial histories in a period at which i is in the k -th role

$\mu_{ik} : Q_{ik} \times E \rightarrow Q_{ik} \cup T$, is the transition function specifying the state of the k -th sub-machine one stage (n periods) after the current move of the sub-machine as a function of the current state and the realized endpoint of the stage.

Note the following three points concerning the above definition. Firstly, the output function λ_{ik} can be thought of as a mapping $\tilde{\lambda}_{ik} : Q_i \rightarrow \mathcal{F}_i^k$ where $\mathcal{F}_{ik} = \{a : \overline{S}_{ik} \rightarrow C \mid a(\overline{s}) \in C_i(\overline{s}) \forall \overline{s} \in \overline{S}_{ik}\}$ and $\tilde{\lambda}_{ik}(q_i)(\overline{s}) = \lambda_{ik}(q_i, \overline{s})$ for all $q_i \in Q_{ik}$ and for all $\overline{s} \in \overline{S}_{ik}$. Thus for each $q_i \in Q_{ik}$, $\tilde{\lambda}_{ik}(q_i) \in \mathcal{F}_{ik}$ specifies a mapping from histories within a period in the k -th role, \overline{S}_{ik} , to set of actions C . Secondly, the number of states $|M_i|$ in the above definition refers to the sum of all states in the different roles $\sum_k |Q_i^k|$. And thirdly, note that a machine in the above specification has to be able to play in each role, so it needs a minimal number of n states.

¹¹Note that $k = 0$ refers to the proposal role and $k > 0$ refers to the k -th responder role.

We demonstrate that the results of the paper on stationarity of NEC hold for 3-player games when counting number of states criterion \succ^s is used if machines are specified as in D_2 . Henceforth, in this appendix we assume there are 3 players and machines and equilibrium in machines refers to the D_2 specification.

As we mentioned before, a minimal machine for the D_2 specification in 3 player games has 3 (non-terminal) states - one for each role.

Proposition 2 *For any Nash equilibrium in the machine game $M = (M_1, M_2, M_3)$, if $\pi_i(M) = 0$ for some i then M_i is minimal (has one non-terminating state for each role), irrespective of the complexity criterion used.*

The proof of the above is also obvious.

Proposition 3 *Let (z, t) be an agreement $z \in \Delta^2$ at period $t < \infty$ for a 3-player game. Suppose that (z, t) is induced by a Nash equilibrium of the machine game $M = (M_1, M_2, M_3)$ with the D_2 specification. Then $t \leq 3$.*

Proof. Suppose not. Then there exists some $t > 3$, for which this is not true. W.l.o.g let Player 3 be the second responder in period t . Also for each player i , let q_i^τ be the state of i at period τ . The rest of the proof is in several steps.

Step 1: M_3 has one state as the second responder: $|Q_{32}| = 1$.

If $z_3 = 0$ then this step follows from the previous Proposition. So let us consider the case in which $z_3 > 0$.

We note that the partial history (z, A) cannot have appeared on the equilibrium path in any period $t - 3\tau, \tau \geq 1$. If (z, A) had occurred previous to t , Player 3 could have chosen a machine M'_3 such that $\lambda_3(q_3, z, A) = A$, for all $q_3 \in Q_{32}$, and would therefore have obtained a payoff δ^{-3} times greater than the equilibrium payoff, without adding states, thus contradicting the definition of a *NEC*.

For any other partial history $s^\tau \in S_{32}$ such that it occurs on the equilibrium path at period τ , it must be the case that $\lambda_3(q_3^\tau, s^\tau) = R$, where $q_3^\tau \in Q_{32}$ is the state of the machine as a second responder at τ . This is because if Player 3's move as a second responder appears on the equilibrium path in these periods, the action chosen must be R . (An A ends the game at a period before t , contrary to hypothesis.) Therefore, if Q_{32} contains two or

more states, Player 3 can replace the candidate equilibrium machine M_3 by M'_3 , where M'_3 differs from M_3 only in that Q'_{32} has one state (in addition to the terminal state T). Let this one state be q' ; let

$$\begin{aligned}\lambda'_3(q', s) &= A \text{ if } s = (z, A) \\ \lambda'_3(q', s) &= R \text{ if } s \neq (z, A) \\ \mu'_3(q', e) &= q', \text{ for all } e \neq (z, A, A).\end{aligned}\tag{1}$$

Clearly, M'_3 is less complex and reproduces the same equilibrium path as before given no other changes in the other players' machines. This contradicts M_3 being part of a NEC. Therefore Q_{32} must have one state.

Step 2: z is not offered by player 1 before t .

Consider two cases. If $z_2 = 0$, then by the previous Proposition and player 2 accepting z at t , 2 will always accept z . From the previous step, Player 3 would choose A in response to (z, A) . Therefore, if z happens before t an agreement would have taken place before t . But this is a contradiction. Now consider the case in which $z_2 > 0$. If the offer z occurs on the equilibrium path at $t - 3\tau, \tau \geq 1$, for the same reason as before Player 2 must reject it in equilibrium (because from the previous step, Player 3 would choose A in response to (z, A)). Player 2 should therefore deviate and accept z , thus ending the game at least three periods earlier and obtaining a payoff greater by a factor of δ^{-3} . Therefore, z must not have appeared in equilibrium in periods $t - 3l, l \geq 1$.

Step 3: $q_2^t = q_2^\tau$ for some $\tau < t$.

Suppose that this step is false. Then, consider a machine M'_2 where M'_2 is the same machine as M_2 without the state q_2^t and the output function is changed to

$$\lambda'_2(q_2, s) = \begin{cases} A & \text{if } s = z \\ \lambda_2(q_2, s) & \text{if } s \neq z \end{cases}$$

Since z does not occur on the path before t and $q_2^t \neq q_2^\tau$ for all $\tau < t$, this construction does not affect the equilibrium path. Since, M'_2 has one less state than M_2 the above construction contradicts the hypothesis of the machines M being a *NEC*.

Step 4: z will be agreed to at some period $\tau < t$ of the game, resulting in a contradiction.

We have already seen that (z, A) will be followed by an acceptance by Player 3. Also, $z_1 > 0$ because if $z_1 = 0$ then M_1 would be minimal and would always offer z as a proposer; but this contradicts the second step above.

Now by the previous step $q_2^t = q_2^\tau$ for some $\tau < t$. Therefore, if z appears in period τ , it will be accepted and the game will end.

Consider finally Player 1, the proposer in period $t-3k, k \geq 0$. Player 1 will deviate in period τ and offer z , which will be accepted by both responders, thus ending the game before t . Therefore (z, t) cannot be a *NEC* outcome for $\infty > t > 3$, for $z \gg 0$.

■

This result also gives us stationarity. However, stationarity does not extend beyond three players as the counter-example in Appendix B illustrates.

The above specifications of machines are similar to that of Piccione and Rubinstein (1992). In these specifications, changes in the state of the machine are allowed after each stage (or after each period) of the game and the output of the machine in each state depends on the partial history within the stage (or period). Yet another different specification (similar to Binmore et al. (1998)) of machines would be to allow changes in the states to occur also *within* each stage of the bargaining, after every piece of new information. Thus in this specification the output function is only a function of the state and not of any partial history, and transitions take place just before a player is required to move. For this last specification, we can show (see Appendices D and E) that the result on stationarity (and uniqueness with ‘noise’) for two-player games continues to hold with s-complexity, but s-complexity is insufficient for three or more players, just as in the definitions used in this paper.¹²

5 Response complexity and stationarity

5.1 Response complexity

As we have seen, state complexity is not strong enough in general to obtain a result that NEC implies minimally complex machines. However, a strength-

¹²Since output, in this last specification, is a function only of the state, r-complexity and s-complexity are equivalent; strengthening the s-complexity measure to obtain our results for the case of $n > 2$ with this specification requires considering complexity of transitions.

ening of the complexity measure to include complexity of responses across states, as defined in Section 2 as r – *complexity* turns out to be sufficient to obtain this in all the specifications we have discussed in this paper. The proof given in our earlier paper (Chatterjee-Sabourian (2000)) deals with the most plausible specification, where states change at the end of each stage. We refer to that paper for the proof.

Of course, stationarity by itself is not able to resolve the multiplicity problem pointed out for the unanimity game by Shaked, though it does reduce the set of outcomes to agreement in n periods or indefinite delay. For uniqueness, either we have to restrict ourselves to the least complex machines that implement *subgame perfect equilibrium payoffs*, as we did in our earlier paper, or we can introduce machines that make errors in outputs and assume that the order of the expected payoff change due to the mistake is small compared to the fixed cost of an additional state. Defining “noisy NEC” or NNEC this way, with response complexity, also enables us to obtain uniqueness in the unanimity game. We do not give the details here; they are available from the authors.

In this section we have spoken of strategies implementable by NEC machines being made credible by introducing arbitrarily ‘small’ noise in the output of the machines. The noise does not affect transitions. The reason for not introducing errors in transitions is the following: If the equilibrium machines have a finite number of states then, at the end of any stage, there are only a finite number of states to make transitions to, and even a completely mixed transition would generate at most a finite number of offers. This is not sufficient to induce credibility in almost all subgames. In particular, since Nash equilibrium in machines are each one-state, noise in transitions would not refine NEC¹³

5.2 Alternative refinements of NEC

An alternative definition of equilibrium with machines is contained in Rubinstein (1986). A machine profile is a semi-perfect equilibrium (SPE) if the two NEC conditions, in the definition at the beginning of Section 2, *hold after*

¹³Also, transitions to different states are made in order to play different actions in different stages, after identical partial stage histories. Therefore, noise in transitions has the effect of introducing a limited degree of noise in the output; noise directly placed on output would include the possible effects of noisy transitions.

every history induced by the machine profile. This means that a player's machine has to be the least complex one implementing the equilibrium payoff after every history reached *on the equilibrium path*. Thus states that are not used in the future can be dropped and other states combined, provided the equilibrium path is unchanged. A refinement of SPE would be to require subgame perfectness in the choice of machines. This would require that the NEC conditions are satisfied after every history, not just those induced on the candidate equilibrium path. (Neme and Quintas (1995) explore this refinement of Rubinstein's solution concept.) The results for n -player games on no delay in agreement beyond period n and stationary behaviour can be obtained with state complexity if we consider SPE rather than NEC. The relevant proposition is stated and proved in Appendix C. Therefore, a Noisy SPE will also give the unique stationary subgame perfect equilibrium.

6 Conclusions

This paper has attempted to explore the consequences on multiperson bargaining of incorporating costs of complexity, with different specifications of automata and two different notions of complexity. The choice of the complexity of equilibrium machines, modelled along the lines proposed in the papers of Abreu and Rubinstein and Piccione and Rubinstein (but not identical to these), yield the result that the machines will be stationary ones for special cases with state complexity and in all the specifications considered with response complexity. This is not sufficient to give the stationary, subgame perfect equilibrium allocation as the sole equilibrium; this requires an additional property of robustness against errors. It is interesting to note that, except for two-player games, not all definitions of complexity and specification of machines give us stationarity; the intuition behind stationarity relies on a notion of flatness of responses and not just on the number of states of the machines.

It would be of interest to explore the issue of stationarity in other contexts where multiplicity of equilibria is a theoretical problem. This might be true especially in other bargaining games, which share the special feature that no payoffs are obtained unless there is an agreement.

7 Appendix A: Counter example for three players with state-complexity

As mentioned earlier in the paper, the state complexity measure, while sufficient to obtain stationarity in 2-player games, is not strong enough to rule out non-stationary equilibria and delays beyond period n , when $n > 2$. We sketch below a counter-example that constructs a NECs in a three-player game in which each player has four states, denoted by a, b, c, d , and an agreement $z'' \gg 0$ is reached in 30 periods. The construction is itself of some independent interest, in bringing out some of the considerations that make n -player ($n > 2$) bargaining games played by machines different from the 2-player case.

The equilibrium consists of 29 periods of offers followed by rejection by the first responders and an agreement z'' in the 30th period (the last period of the 10-th stage). Table 1 below describes *the equilibrium path* of the offers and the responses for the 30 periods and the states (and thus the transitions among the states) for different players in the 10 different stages, *along the equilibrium path*. In this table, (x^a, x^b, x^c, x^d) , $(y^a, y^b, y^c, y^d, y', y'')$ and $(z^a, z^b, z^c, z^d, z', z'')$ represent different offers made by players 1, 2 and 3 respectively; ‘A’ and ‘R’ represent acceptance and rejection respectively.

In order to ensure the path described in Table 1 can be induced as a NECs, we have to specify other actions as follows.

We require that any responder in any state to reject all offers unless the partial history and the state of the responder are those that occur on the equilibrium path in period 30 when the agreement is reached. Thus, player 3 rejects all offers; player 1 rejects all offers unless, as the first responder, he is in state d and the partial history is that on the equilibrium path in period 30 before 1's move - namely (x^d, R, y'', R, z'') ; and player 2 rejects all offers unless, he, as the second responder, is in state c and the partial history is that on the equilibrium path in period 30 before 2's last move - namely (x^d, R, y'', R, z'', A) . If a partial history is observed in any stage that is not on the equilibrium path for a proposer, given his state, he chooses an offer that is not on the equilibrium path at any stage, and thus creates an off-equilibrium outcome e .

The above ensures at any period $t \leq 30$ no player can improve his payoff by deviating unless the deviation can induce a sequence of actions that results

in an agreement z'' being concluded earlier than in period 30. This is the phenomenon of “speedup”, which is not important in repeated games, but *is* important in the context of a single extensive-form bargaining game.

Speedup can happen directly, if any two of the three players have the same states in any two stages, one earlier than the other. (From Table 1, we see this is not the case.) The third player can then deviate from his state in the earlier stage to his state in the later stage, and thus cause intermediate stages to be skipped, hence “speeding up”. There could also be an indirect way of speeding up; a deviation is designed to cause the play of the game to go off the equilibrium path (by which we mean both the outcome path and the machine states that gave rise to the outcome path) and to rejoin it at a later stage, so as to reduce the total number of stages before the final agreement is reached. In order to make such deviations unprofitable, out-of-equilibrium transitions are constructed (see below) to have the property that a subgame following a deviation can rejoin the equilibrium path only in the first stage.

Table 1

period	stage	1's state	1's action	2's state	2's action	3's state	3's action
1	1	a	x^a	a	R	a	-
2	1	a	-	a	y^a	a	R
3	1	a	R	a	-	a	z'
4	2	a	x^a	b	R	b	-
5	2	a	-	b	y^b	b	R
6	2	a	R	b	-	b	z'
7	3	a	x^a	c	R	c	-
8	3	a	-	c	y^c	c	R
9	3	a	R	c	-	c	z'
10	4	a	x^a	d	R	d	-
11	4	a	-	d	y^d	d	R
12	4	a	R	d	-	d	z'
13	5	b	x^b	b	R	a	-
14	5	b	-	b	y'	a	R
15	5	b	R	b	-	a	z^a
16	6	b	x^b	c	R	b	-
17	6	b	-	c	y'	b	R
18	6	b	R	c	-	b	z^b
19	7	b	x^b	d	R	c	-
20	7	b	-	d	y'	c	R
21	7	b	R	d	-	c	z^c
22	8	b	x^b	a	R	d	-
23	8	b	-	a	y'	d	R
24	8	b	R	a	-	d	z^d
25	9	c	x^c	c	R	a	-
26	9	c	-	c	y''	a	R
27	9	c	R	c	-	a	z''
28	10	d	x^d	c	R	d	-
29	10	d	-	c	y''	d	R
30	10	d	A	c	A	d	z''

Consider any out-of-equilibrium history of actions in a stage e' that results from a deviation by say player 1, and the transitions for Player 2 and 3. For any $i = 2, 3$ and for any state q_i , if (q_i, e') does not occur on the equilibrium path for both $i = 2$ and 3 then go to the beginning of the game: $\mu_i(q_i, e') = a$.

If (q_i, e') does not occur on the equilibrium path for some $i \neq 1$ and does occur for the other player on the equilibrium path then $\mu_i(q_i, e')$ is constructed so that either both 2 and 3 are in states a in the next stage or the final outcome of the next stage, e'' , and the states of i in the next stage, do not occur on the equilibrium path. Thus the players who see histories that are not on the equilibrium path for them generate transitions that ultimately lead back to the beginning of the game. Such a construction implies that any agreement reached as a result of a deviation ultimately does not succeed in accelerating the agreement. We omit details and simply give an example.

Suppose player 1 considers a deviation by offering x^b instead of x^a in stage 2 (period 4) at which the states of the players are (a, b, b) . Now (x^b, R) occurs on the equilibrium path for player 2 in his state b , at period 13 (in stage 5). Therefore player 2 must play the same way in period 5, after observing (x^b, R) , as in period 14 on the equilibrium path and thus he plays y' and player 3 rejects. Player 3 then faces the history (x^b, R, y', R) and is in state b . This partial history and state b occur for player 3 on the equilibrium path in stage 6 (period 17), and he responds by a play of z^b , as the equilibrium path action in period 18. The history $e' \equiv (x^b, R, y', R, z^b, R)$ does not appear anywhere else on the equilibrium path for player 2 in state b . Then suppose $\mu_2(b, e') = b$. Since e' does occur on the equilibrium path for player 3 in state b , player 3 makes an equilibrium path transition to state c (as if going to stage 7). Thus in stage 3 players 2 and 3 are in states b and c respectively. Notice that this pair of states for players 2 and 3, never appears on the equilibrium path. Also notice that the only offers by 1 that appear on the equilibrium path with either state b for player 2 or with state c for player 3 are x^a or x^b . Thus any other offer in stage 3 by player 1 will take the game back to the beginning of the game by the construction in the previous paragraph.. Clearly, player 1 does not want to offer x^a (no speeding up will take place). Player 1 could repeat x^b in stage 3, inducing an offer z^c from player 3, again a surprise (i.e. not on the equilibrium path) for player 2 in state b . In the next stage, suppose player 2 will be in state b again; also the equilibrium transitions takes player 3 to state d . Given that in this stage (stage 4) players 2 and 3 are in states (b, d) respectively, for any offer of player 1, we can describe transitions (we omit details) in the same way, consistent with the equilibrium path, that take players 2 and 3 to states (a, a) respectively in the next stage (stage 5). Thus player 2's transitions are designed, using player 3's different offers, to delay agreement

sufficiently to deter this possible deviation from player 1. The checking that other deviations are unprofitable could be done in the same way, *essentially* because the example has been constructed with distinct outcomes for distinct stages.

Finally we have to check that no player can save a state. Note that each state is used to make a specific offer by each player. Player 1 uses states 1,2,3,4 to make distinct offers in stages 1-4, 5-8, 9 and 10 respectively. Player 2 makes distinct offers in stages 1,2,3 and 4 following identical histories (an offer x^a followed by rejection) in every stage. Distinct states are needed for player 2, because the stage histories preceding his distinct offers are identical. Similarly player 3 needs distinct states to make distinct offers in stages 5, 6, 7 and 8. Note that in these stages, player 1 uses the same state to make the same offer and player 2 uses different states and conditioning on player 1's offer of x^b to make the same offer, so that the partial histories preceding player 3's distinct offers are the same. So every state of every player is essential - saving a state results in an 'off-the-equilibrium' outcome.

8 Appendix B: Counter-example to stationarity with state complexity with four players for the sub-machine specification (D2).

This is a counter-example for four players, showing that for the definition of a machine as the combination of four sub-machines, (one for each role), state complexity is not sufficient to lead to stationarity. The equilibrium path has been constructed so as to make all states essential (no saving of a state is possible without inducing an 'out-of-equilibrium outcome'). The Table below describes the equilibrium path of actions and states. The states for each player in each stage are listed in brackets in each entry in the table; note there are four sub-machines per player, since each player must play four roles. Note also that the sub-machine that implements each player's behaviour as the third responder contains one state, whereas each of the other three sub-machines, for each player, contains two states (denoted by 1 & 2). All offers are distinct, unless explicitly noted. Transitions take place from sub-machine to sub-machine just before a player has to move in the role corresponding to the sub-machine. Such *transitions* could, of course,

depend on the entire history of the game since the last transition for that sub-machine. The *actions* depend on partial history within a period.

Period	1	2	3	4	5	6	7	8
Player 1	$x^1(2)$	(1)	R(1)	A(2)	$x^1(2)$	R(1)	A(2)	A(1)
2	A(2)	$x^2(2)$	(1)	R(1)	A(1)	$x^2(2)$	R(1)	A(2)
3	R(1)	A(2)	$x^3(2)$	(1)	A(2)	A(1)	$x^3(2)$	R(1)
4	(1)	R(1)	A(2)	$x^4(2)$	R(1)	A(2)	A(1)	$x^4(2)$
9	10	11	12	13	14	15	16	
$z^1(1)$	(1)	R(2)	R(2)	$z^1(1)$	(1)	(1)	A(1)	
A(2)	$z^2(1)$	(1)	(2)	R(1)	$z^2(1)$	(1)	A(1)	
R(2)	A(2)	$z^3(1)$	(1)	(1)	R(1)	$z^3(1)$	A(1)	
(1)	R(2)	A(2)	$z^4(1)$	(1)	(1)	R(1)	$z^4(1)$	

States are made essential for each player because each sub-machine, on the equilibrium path, is required to play different actions, after the same history, in different periods where a player has the same role.

As in the paper, deviations from the above equilibrium path are deterred by

(i) constructing responses that reject all offers not on the equilibrium path

(ii) constructing transition functions that go to the beginning of the game if any out-of-equilibrium behaviour is observed. For example, if out-of-equilibrium behaviour is observed by player 1, his sub-machines in the roles of the proposer, first responder and second responder will move to state 2, 1 and 1 respectively (the states of the sub-machines in period 1,2 and 3 respectively).

An example is given below of how such a construction ensures that no player would deviate from the equilibrium path.

If player 1, for example, attempts to speed up the game by offering z^1 in period 5, then player 2, in state 1, rejects this offer (because player 2 in this state rejects z^1 on the equilibrium path at period 13). Thus the history of actions in the last 4 periods, before period 6, will be $e = (x^2, A, R; x^3, A, R; x^4, A, R; z^1, R)$. Since this history does not occur on the equilibrium path, by construction the other players move back to the states in their appropriate roles at the beginning of the game. Thus in period 6, the players 2,3 and 4 move to the states (2,2,1) respectively (to the states on the equilibrium path at period 2). This makes the deviation by 1 in period

5 unprofitable. The unprofitability of other deviations can be checked in the same way.

Note that every sequence of four periods has a path distinct from that of any other corresponding four periods (each sequence beginning with periods where players have the same role). This means any attempt to substitute an action by one occurring at a later period will be out-of-equilibrium and be detected; the transitions for out-of-equilibrium histories then ensure that such deviations do not increase a player's payoff.

9 Appendix C: The results for SPE with state complexity

Proposition 4 *Let (z, t) be an agreement reached in a SPE¹⁴. Then $t \leq n$, or $t = \infty$.*

Proof. We demonstrate this result by showing that no equilibrium machine can have more than one non-terminal state. Suppose $t > n$, and some machine M_i has more than one state in equilibrium. Consider the last stage of the game, the stage that contains period t . At the beginning of this stage, suppose M_i is in state q_i^t . Define an alternative machine M'_i , with one non-terminal state q_i and output and transition functions determined as follows: (we use primes for machine M'_i).

$\lambda'_i(q_i, s) = \lambda_i(q_i^t, s)$, for all s on the equilibrium path and $\lambda'_i(q_i, s) = R$, for all s off the equilibrium path for which Player i is a responder, and $\mu'_i(q_i, e) = q_i, \forall e \in E$.

The machine M_i replicates equilibrium play for the last stage for Player i , and does so with one state. Therefore, any M_i with two or more non-terminal states can not be a NECs at the beginning of the last stage. Therefore, such a machine M_i can not be part of a SPE, which requires the NEC conditions hold at every period.

Since the minimal machine must have one state, it follows that a SPE machine is minimal. Therefore, either $t \leq n$, or $t = \infty$. ■

Using this result and the reasoning of Section 3, it is clear that a Noisy SPE will also give the unique stationary subgame perfect equilibrium.

¹⁴We recall that a machine profile is a semi-perfect equilibrium if the NEM conditions are satisfied at every point along the outcome path induced by the machine profile.

10 Appendix D: Stationarity result for two-player game with another alternative specification¹⁵.

Here we consider a specification, which we shall not write out formally, where each state corresponds to the output of an offer, except for a termination state T (equivalent to accepting an existing offer), and transitions can take place after every piece of new information is available. We obtain the result that any Nash equilibrium machine must be stationary (one-state) (and any agreement must occur by period 2). State-complexity is sufficient for this.

Here is an outline of the argument.

Clearly, if there is an agreement in the first two periods or if there is no agreement the equilibrium machine must have one state (exactly analogous to the treatment in the paper -Lemma 2).

Suppose now that there is an agreement $(z, t), t > 2, z \gg 0$.

1. All states for both players must be used on the equilibrium path. If not, a player could save a state by dropping one that is unused, and this contradicts the definition of equilibrium.

2. No state for any player i can occur twice on the equilibrium path. Suppose otherwise, so that $q_i^\nu = q_i^\tau$ (the states at ν and τ are the same), where $\tau < \nu \leq t$. Then the other player can deviate, by playing the game from period τ as if he/she is at ν , on the equilibrium path, thus eliminating the states between period τ and period ν . This will speed up the game by $\nu - \tau$ periods, and make players better off (because of discounting and $z_i > 0$).

3. Player 1 must have one state, other than T . Suppose not; then if Player 1 has two states, he can drop q_1^1 (that occurs only in period 1 by the previous step) and replace it with the other non-terminal state. Player 2 cannot punish, because the worst punishment (using one of the distinct states used on the equilibrium path) would be to make a transition to period 2, and this is what would have happened on the original equilibrium path.

4. Player 2 must have one state, other than T . Suppose not; then if player 2 has two states, he can drop q_2^2 (that occurs only in period 2) and replace it with the other non-terminal state. Player 1 cannot punish, because he has only one state.

¹⁵This specification, which is similar to Binmore, Piccione and Samuelson (1998).

The case of $z_i = 0$, for some i , is exactly analogous to the treatment in the paper.

11 Appendix E: Counter-example to stationarity with state complexity, with three players, for the machine specification of Appendix D.

This three-player counter-example involves an agreement on a partition x^1 at period 4. (Here too, all offers are distinct).

M_1 has two states, q_1^P and q_1^R ; M_2 has three, q_2^P, q_2^R, q_2^A ; and M_3 has three states q_3^P, q_3^R, q_3^A .

The output maps are defined as follows:

$$\begin{aligned}\lambda_i(q_i^P) &= x^i. \\ \lambda_i(q_i^R) &= R, \\ \lambda_i(q_i^A) &= A.\end{aligned}$$

The transitions are as follows. For any state q_1 for player 1,

$$\mu_1(q_1, e) = \begin{cases} q_1^R & \text{if } e \text{ is such that player 1 is a responder} \\ q_1^P & \text{otherwise} \end{cases}$$

where e is the new information preceding a transition and contains enough information to indicate the role that each player is going to play in his or her next move. Also,

$$\mu_2(q_2^P, e) = q_2^R$$

$$\mu_2(q_2^R, e) = \begin{cases} q_2^P & \text{if } e \text{ is such that player 2 is the proposer} \\ q_2^A & \text{if } e \text{ is such that } x^1 \text{ has been offered by player 1} \\ q_2^R & \text{otherwise} \end{cases}$$

$$\mu_3(q_3^P, e) = \begin{cases} q_3^P & \text{if } e \text{ is such that player 1 has just made an offer} \\ q_3^R & \text{otherwise} \end{cases}$$

$$\mu_3(q_3^R, e) = \begin{cases} q_3^P & \text{if } e \text{ is s.t. player 3 is the proposer} \\ q_3^A & \text{if } e \text{ is s.t. player 3 is the second responder} \\ & \text{and an offer has been accepted by the first responder} \\ q_3^R & \text{otherwise} \end{cases}$$

The initial states are (q_1^P, q_2^P, q_3^P) .
 The outcome path in this example will be

Period			
1	2	3	4
x^1	x^2	x^3	x^1
R	R	R	A
			A

The states for player 1 on the equilibrium path in the first four periods will be $q_1^P, q_1^R, q_1^R, q_1^P$. The transitions of player 2 on the equilibrium path will be: start initially in state q_2^P , change to q_2^R after receiving the offer x^1 in period 1, switch back to q_2^P after rejecting the offer in the first period, change to q_2^R after making the offer x^2 in period 2, stay in q_2^R until period 4, then switch to q_2^A in period 4 after receiving the offer x^1 . The states of player 3 on the equilibrium path are initially q_3^P , switching to q_3^R after player 2's response in period 1 and staying in this state until period 3, switching to q_3^P at the beginning of period 3, switching back to q_3^R after making his offer x^3 in this period, and finally switching to q_3^A in period 4 after player 2 accepts the offer x^1 .

Note that no saving of states is possible, since all states are used on the equilibrium path to take different actions. The offer x^1 appears at $t = 1$, but is rejected by player 2, because an acceptance by player 2 would lead to a rejection by player 3 (here player 3 is in state q_3^R).

Clearly, no deviation can make any player better off.

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