A Geometric Approach to Expected Utility

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Abstract

We present an alternate proof and axiomatisation of the Expected Utility Theorem. The proof relies on simple geometric arguments.

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1 Introduction

The Expected Utility Theorem is the cornerstone of axiomatic choice under uncertainty. However traditional proofs rely mostly on the algebraic properties induced by the Independence and von Neumann-Morgenstern continuity axioms (see, for instance, Kreps, 1988, pp. 66). We produce here, an alternate proof based on the intuitive idea of separating hyperplanes and the notion that linear functions are translation invariant. This also results in an alternative axiomatisation.

2 The Theorem

Let Z be a compact metric space and let Δ be the space of all probability measures on Z. Let $M(Z)$ denote the (topological vector) space of countably additive measures on Z and $C(Z)$ the space of continuous functions on Z. Then, $(M(Z), C(Z))$ is a dual pair so we endow $M(Z)$ with the weak* topology and Δ with the relative topology. Let d be a metric which generates this topology. Typical elements of Δ will be denoted by

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$p, q, r, \ldots$ etc. Let $\succcurlyeq \subset \Delta \times \Delta$ be a binary relation on Δ. We will require that $\succcurlyeq$ be a preference relation, i.e. complete and transitive. The indifference class of a lottery p will be denoted by $[p]$. The ε-neighbourhood of a point q (in Δ) will be denoted by $N_\varepsilon(q)$. The following are additional axioms we will impose.

Axiom 1 (Continuity) *The sets* $\{q : q \succ p\}$ *and* $\{q : p \succ q\}$ *are open.*

Axiom 2 (Independence) $p \succ q$ *and* $\lambda \in (0,1]$ *implies* $\lambda p + (1-\lambda)r \succ \lambda q + (1-\lambda)r$.

Axiom 3 (Betweenness) (i) $p \succ q$ *and* $\lambda \in (0,1)$ *implies* $p \succ \lambda p + (1-\lambda)q \succ q$ *and* (ii) $p \sim q$ *and* $\lambda \in (0,1)$ *implies* $p \sim \lambda p + (1-\lambda)q$.

Axiom 4 (Convex Contour Sets) *The sets* $\{q : q \succ p\}$, $\{q : p \succ q\}$ *and* $[p]$ *are convex.*

Axiom 5 (Translation Invariance) $p \succcurlyeq q$ *implies* $p + c \succcurlyeq q + c$ *for all signed measures* c *such that* $c(\Delta) = 0$ *and* $p + c, q + c \in \Delta$.

Remark. The axiom *Betweenness* was first introduced by Dekel (1986) in a model of preferences under uncertainty without *Independence*. *Translation Invariance* is used by Ahn (2005) and Chatterjee and Krishna (2005) (both in different contexts) and the proof below is based on the latter's approach.

2.1 Definition. A preference relation $\succcurlyeq$ over Δ has an Expected Utility (EU) representation if there exists a continuous function $u : Z \to \mathbb{R}$ (unique up to positive affine transformations) such that $p \succcurlyeq q$ if and only if $\int u(z)\, \mathrm{d}p(z) \geqslant \int u(z)\, \mathrm{d}q(z)$.

Main Theorem. For any preference relation $\succcurlyeq$ over Δ, the following are equivalent:
(a) $\succcurlyeq$ satisfies *Continuity* and *Independence*.
(b) $\succcurlyeq$ satisfies *Continuity*, *Betweenness* and *Translation Invariance*.
(c) $\succcurlyeq$ satisfies *Continuity* and *Translation Invariance*.
(d) $\succcurlyeq$ has an EU representation.

If $p \sim q$ for all $p, q \in \Delta$, we can let u be a constant function. We shall henceforth assume that there exist $p, q \in \Delta$ so that $p \succ q$. We shall prove that (a) → (b) → (c) → (d). That (d) → (a), (b) and (c) is straightforward and omitted. Since it is immediate that *Independence* → *Betweenness* → *Convex Contour Sets*, lemma 2.2 demonstrates that (a) → (b) → (c). We give a brief sketch of the proof of (c) → (d). Lemma 2.2 shows that a consequence of *Independence* and *Continuity* is that preferences are translation invariant. A consequence of *Translation Invariance* is that for non-trivial preferences, indifference curves must necessarily be thin.[1] This is shown in lemma 2.3. That *Translation Invariance* and *Continuity* together imply *Betweenness* is shown in lemma 2.4. We then fix some $p^* \in \Delta$ that is not an extreme point of Δ. By *Convex Contour Sets*, it must be the case that there exists a hyperplane passing containing $[p^*]$ that separates the upper and lower contour sets. Since every hyperplane is the level set of some linear functional, *Translation Invariance* and the fact that indifference curves are thin implies that the indifference curve through any other point p is a translate of the indifference curve through p^*, i.e. is the level set of the same linear functional. Now a simple application of Choquet's theorem gives us the required result.

2.2 Lemma (Translation Invariance)**.** Let $\succcurlyeq$ satisfy *Independence* and *Continuity*. Then $\succcurlyeq$ is translation invariant.

Proof. Let $p \succcurlyeq q$ and c such that $c(\Delta) = 0$ and $p + c, q + c \in \Delta$. Simple geometry shows that for all $\lambda \in (0, 1)$,

$$\lambda q + (1 - \lambda)(p + c) = \lambda\{\lambda q + (1 - \lambda)(q + c)\} + (1 - \lambda)\{\lambda p + (1 - \lambda)(p + c)\}.$$

Then, since $\succcurlyeq$ is reflexive,

$$\lambda q + (1 - \lambda)(p + c) \sim \lambda\{\lambda q + (1 - \lambda)(q + c)\} + (1 - \lambda)\{\lambda p + (1 - \lambda)(p + c)\}.$$

From *Independence* we get

$$\lambda p + (1 - \lambda)(p + c) \succcurlyeq \lambda q + (1 - \lambda)(p + c).$$

Combining the relations above

$$\lambda p + (1 - \lambda)(p + c) \succcurlyeq \lambda\{\lambda q + (1 - \lambda)(q + c)\} + (1 - \lambda)\{\lambda p + (1 - \lambda)(p + c)\}.$$

From *Independence* we see that

$$\lambda p + (1 - \lambda)(p + c) \succcurlyeq \lambda q + (1 - \lambda)(q + c).$$

From *Continuity* it now follows that $p + c \succcurlyeq q + c$. □

We shall see below that (c) → (d). Let us first show that for non-trivial preferences, *Translation Invariance* implies indifference curves cannot be thick and then show that *Translation Invariance* and *Continuity* implies *Betweenness*.

[1] Recall that an indifference curve $[q]$ is *thick* if there exists $q' \in [q]$ and $\varepsilon > 0$ such that $p \in N_\varepsilon(q')$ implies $p \sim q'$. An indifference curve that is not thick will be called *thin*.

2.3 Lemma (Thin Indifference Curves)**.** Let $\succcurlyeq$ satisfy *Translation Invariance* and let $p \succ q \succ r$. Then for all $\varepsilon > 0$, there exist $p', r' \in \Delta$ such that $p' \succ q \succ r'$ and $p', q' \in N_\varepsilon(q)$.

Proof. We shall show that p' exists with the desired properties. A symmetric argument shows that r' exists. Let us fix $\varepsilon > 0$. Now $p \succ q$. Define $p_1 := (p+q)/2$ and $c := p_1 - q$. To see that $p_1 \succ q$, let us suppose the contrary, i.e. $q \succcurlyeq p_1$. Then, $q \succcurlyeq p_1 = q + c \succcurlyeq p_1 + c = p$ where the middle relation follows from *Translation Invariance* giving us the desired contradiction. Now let $p_n := (p_{n-1} + q)/2$ for all n. A simple induction shows that for all n, $p_n \succ q$. For n large enough, $d(p_n, q) < \varepsilon$. Let p' be such a p_n. □

2.4 Lemma. Let $\succcurlyeq$ satisfy *Translation Invariance* and *Continuity*. Then $\succcurlyeq$ satisfies *Betweenness*.

Proof. Let $p \succ q$ and let $\lambda \in (0,1)$. For each $n \in \mathbb{N}$, define $p_0^n := p$ and $p_{i+1}^n := p_i^n + (q-p)/2^n$ for all $i = 0, \ldots, 2^n - 1$. By *Translation Invariance*, $p \succ q$ implies $p \succ (p+q)/2 \succ q$ and repeated application gives $p = p_0^n \succ p_1^n \succ \cdots \succ p_{2^n}^n = q$. Also, there exists $j \in \{0, \ldots, 2^n - 1\}$ such that $d(p_j^n, \lambda p + (1-\lambda)q) \leqslant d(p_k^n, \lambda p + (1-\lambda)q)$ for all $k \in \{0, \ldots, 2^n - 1\}$. Define the sequence $(r_n)_0^\infty$ where $r_n := p_j^n$ so that $r_n \to \lambda p + (1-\lambda)q$. Since $p \succ r_n \succ q$ for each n and p, q are bounded away from $\lambda p + (1-\lambda)q$, from *Continuity* it follows that $p \succ \lambda p + (1-\lambda)q \succ q$. A similar argument shows that $p \sim q$ implies $p \sim \lambda p + (1-\lambda)q$ for all $\lambda \in (0,1)$. □

Let us recall some definitions of objects in linear spaces. An *affine subspace* (or linear variety) of a vector space is a translation of a subspace. A *hyperplane* is a *maximal* proper affine subspace. If H is a hyperplane in a vector space $\mathbb{V}$, then there is a linear functional f on $\mathbb{V}$ and a constant c such $H = \{x : f(x) = c\}$. Moreover, H is closed if and only if f is continuous (Luenberger, 1969, pp. 129, 130). For notational ease, we shall write H as $[f = c]$. Similarly, (two of) the negative and positive half spaces are represented as $[f \leqslant c]$ and $[f > c]$ respectively. For any subset $S \subset \mathbb{V}$, let $\operatorname{aff}(S)$ denote the affine subspace generated by S, i.e. the smallest affine subspace that contains S. Also, let $\operatorname{ext} C$ denote the extreme points of a compact convex set C in $\mathbb{V}$.

For any $p \in \Delta$, let $p_+ := \{q : q \succ p\}$, $p_- := \{q : p \succ q\}$ and recall that $[p]$ denote the indifference class of p. By *Convex Contour Sets*, p_+, p_- and $[p]$ are convex.

Let $p^* \in \Delta \setminus \operatorname{ext} \Delta$ so that $D := \Delta - p^*$ and $Y := \operatorname{span}(D)$ and also suppose p_+ and p_- are nonempty. Thus, $Y + p^*$ is $\operatorname{aff}(\Delta)$. It is clear that Y is closed and is a maximal proper subspace (ie has codimension 1) of $M(Z)$. Also, let the zero element of Y be denoted by θ. We shall also abuse notation and say that for $x, y \in D$, $x \succcurlyeq y$ if and only if $x + p^* \succcurlyeq y + p^*$.

2.5 Lemma. There exists $f : D \to \mathbb{R}$ which is continuous and linear so that $\theta_- \subset [f < f(\theta)]$, $\theta_+ \subset [f > f(\theta)]$ and $[\theta] = [f = f(\theta)]$.

Proof. From *Continuity*, θ is a boundary point of the convex set $[\theta] \cup \theta_+$. Since the convex cone generated by $[\theta] \cup \theta_+$ is not dense in Y, $[\theta] \cup \theta_+$ is supported at θ (Theorem 5.73, Aliprantis and Border, 1999). Let us denote this hyperplane by the (continuous linear functional) f so that for all $x \in \theta_+$, $f(x) > 0$. Moreover, $f([\theta]) = 0$ and for all $y \in \theta_-$, $f(y) < 0$. □

We have thus far established that for $\theta \in Y$, there exists a continuous linear functional f that represents the decision-maker's preferences at that point. This enables us to demonstrate that the same linear functional also represents the decision-maker's preferences everywhere on the domain.

2.6 Lemma. If the continuous linear functional f separates $\theta_+ \cup [\theta]$ and θ_-, then for all $x \in D$, f separates $x_+ \cup [x]$ and x_-.

Proof. By construction, there exists $b \in D$ such that $b \succ \theta$, ie $f(b) > f(\theta)$. Without loss of generality, we can take x such that $\theta \succ x$. If f does not separate $x_+ \cup [x]$ and x_-, there exists y such that $f(x) = f(y)$ and $y \nsim x$. Without loss of generality, we can take $x \succ y$. Let $E := \operatorname{conv}\{b, \theta, x, y\}$ so that $\operatorname{ri} E \neq \emptyset$ and $y_\lambda := \lambda x + (1-\lambda)y$. Then, for all $\lambda \in (0,1)$, $x \succ y_\lambda$ and $f(x) = f(y_\lambda)$. Notice that $f^{-1}(0)|_{\operatorname{span}(E)} \cap \operatorname{ri} E \neq \emptyset$. Now let $c \in E$ such that $x + c \in f^{-1}(0)|_{\operatorname{span}(E)} \cap \operatorname{ri} E$. Then, there exists $\lambda \in (0,1)$ such that $y_\lambda + c \in f^{-1}(0)|_{\operatorname{span}(E)} \cap \operatorname{ri} E$. Therefore, $f(x+c) = f(y_\lambda + c) = 0$, which implies $x + c \sim y_\lambda + c$, a contradiction. □

Thus, f separates $[x] \cup x_+$ and x_- for all $x \in D$ so that f represents $\succcurlyeq$ on D. Since, Z is a compact metric space, Choquet's theorem implies that $f(x) = \int_Z f|_{\operatorname{ext} D}(\delta_z - p^*)\, dx(\delta_z - p^*)$ (see, for instance, Lax, 2002, pp. 128). To finish the proof of the theorem, let $u(z) := f(\delta_z - p^*)$. For completeness, we now show that *Continuity* and *Translation Invariance* imply *Independence*.

2.7 Lemma. Let $\succcurlyeq$ satisfy *Continuity* and *Translation Invariance*. Then $\succcurlyeq$ satisfies *Independence*.

Proof. Let $p, q, r \in \Delta$, $p \succ q$ and $c := p - q$. By Lemma 2.4, we know that $\succcurlyeq$ satisfies *Betweenness* so that $q + \lambda c = \lambda p + (1-\lambda)q \succ q$ for all $\lambda \in (0,1)$. Now let $c' := \lambda q + (1-\lambda)r - q = q(\lambda - 1) + (1-\lambda)r$. Thus, *Translation Invariance* implies $q + \lambda c + c' \succ q + c'$. But $q + \lambda c + c' = \lambda p + (1-\lambda)q + q(\lambda-1) + (1-\lambda)r = \lambda p + (1-\lambda)r$ and $q + c' = \lambda q + (1-\lambda)r$ so that $\lambda p + (1-\lambda)r \succ \lambda q + (1-\lambda)r$ which proves that $\succcurlyeq$ satisfies *Independence*. □

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