I. The Classical Dichotomy

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1 Introduction

Roughly a century ago, the standard model of money was called the classical dichotomy. This model continues to play a role in the thinking of many economists about money. That is one reason we start by reviewing it. The other reason we do so is that its well-known defects provide hints toward better ways to model money. The model has two parts: one part is called the general competitive equilibrium model; the other part is called the quantity theory of money.

2 2-good pure exchange competitive equilibrium

The model we start with is one of the simplest which economists use. It and variants of it will serve two purposes for us. First, it is one version of the general competitive equilibrium model which is part of the classical dichotomy model. Second, we will build on it to construct other monetary models.

There is a more-or-less standard way to describe an economic model. The description has two groups of assumptions. One group describes the setting or background world, which we will call the environment. The other group describes how people in that setting interact, which tends to be called the equilibrium concept.

2.1 Environment

There are three things to describe: resources, technologies, and people and their preferences. This economy has no technologies and, hence, no production. There are two distinct goods, which we label good 1 and good 2. The resources consist of the total amount of each good available. Let W_1 denote the total amount of good 1 available and W_2 that of good 2. There are N people labeled by the integers $1, 2, ..., N$.

Each person cares only about his or her own consumption of the two goods. For two goods, we will often represent preferences by an indifference curve map where each indifference curve has the usual shape. We also at times represent preferences by a utility function. (I assume that students have some familiarity with both.)

Exercise 1 *Let c_1 denote consumption of good 1 and let c_2 denote consumption of good 2. Suppose that a person's utility function is $\sqrt{c_1} + \sqrt{c_2}$, where here and and in what follows $\sqrt{x}$ means the positive square root of x. An indifference curve corresponding to this utility function consists of all ordered pairs (c_1, c_2) that satisfy the equation $\sqrt{c_1} + \sqrt{c_2} = b$, for some $b \geq 0$. (i) Solve the equation $\sqrt{c_1} + \sqrt{c_2} = b$ for c_2 as a function of (in terms of) c_1. (ii) Use your part (i) answer and the spread-sheet program Excel to plot the indifference curve corresponding to $b = 2$ for values of c_1 between 0 and 4. (This is the first of several uses we will make of Excel. Although I will keep referring to Excel, students can use other software if they prefer.) (iii) Repeat part (ii) but for $b = 3$ and c_1 between 0 and 9.(iv) Use Excel to plot the indifference curve through the point $(c_1, c_2) = (2, 1)$.*

2.2 Allocations and Feasible Allocations

In general economies, an allocation describes who does what and who gets what—sometimes described as answers to "who?, what?, and for whom?". Because there is no production, an allocation here describes only "for whom." An allocation is a list of consumptions pairs, one pair for each person. Let c_{n1} denote the consumption of person n (the first subscript) of good 1 (the second subscript) and let c_{n2} denote the consumption of person n of good 2. An allocation is (c_{n1}, c_{n2}) for $n = 1, 2, ..., N$.

An allocation is said to be feasible if it is consistent with the resources and the technologies. Here feasibility is simple. An allocation is feasible if $\sum_{n=1}^{N} c_{n1} \leq W_1$ and $\sum_{n=1}^{N} c_{n2} \leq W_2$.

2.3 Pareto Efficient (PE) Allocations

Let A and B be two allocations. A is said to be Pareto-Superior to B if no one prefers B to A and at least one person strictly prefers A to B.

Let A be an allocation. A is said to be Pareto Efficient (PE)—some prefer the term Pareto Optimal—if A is feasible and if there is no feasible allocation that is Pareto-Superior to A.

Exercise 2 *Let A be an allocation that assigns positive amounts of each good to each person. Two simple conditions on A are both necessary and sufficient for A to be PE. What are the conditions?*

2.4 Competitive Equilibrium (CE)

CE is the most commonly used equilibrium concept in economics. It is intended to describe what happens if there is private ownership of all resources and if people trade taking prices as given.

As regards private ownership, we assume that each person starts out owning some of each good. We let w_{n1} denote the amount of good 1 owned by person n and let w_{n2} denote the amount of good 2 owned by person n. We will call the pair (w_{n1}, w_{n2}) person n's endowment. We also assume that everything is owned by someone. That is, $\sum_{n=1}^{N} w_{n1} = W_1$ and $\sum_{n=1}^{N} w_{n2} = W_2$.

Let p denote the price of good 2 in units of good 1 (p is measured in units of good 1 per unit of good 2). Throughout, suppose that p is positive. We say that person n can *afford* to buy the pair (c_{n1}, c_{n2}) at the price p if there exists (q_1, q_2) such that

$$c_{n1} = w_{n1} - q_1,\ c_{n2} = w_{n2} - qs_2, \text{ and } q_1 + pq_2 = 0, \tag{1}$$

where q_j is to be interpreted as sales of good j (a negative q_j is a purchase of good j). As is to be demonstrated in the next exercise, it is equivalent to say that person n can afford to buy the pair (c_{n1}, c_{n2}) at the price p if

$$c_{n1} + pc_{n2} = w_{n1} + pw_{n2}. \tag{2}$$

Exercise 3 *Show the following. (i) If person n can afford to buy the pair (c_{n1}, c_{n2}) at the price p, then (c_{n1}, c_{n2}) satisfies (2). (ii) If (c_{n1}, c_{n2}) satisfies (2), then person n can afford to buy the pair (c_{n1}, c_{n2}) at the price p.*

Exercise 4 *With good 1 on the horizontal axis and good 2 on the vertical axis, sketch the pairs (c_{n1}, c_{n2}) that satisfy (2). (The symbols p, w_{n1} and w_{n2} should appear as labels on your sketch.)*

Given the above equivalence, we can use satisfaction of (2) as our definition of affordability.

We say that an allocation A and a price p is a competitive equilibrium (CE) if two conditions hold: (i) A is feasible, (ii) for each person n, the consumption pair assigned to n by A is liked by n as well as anything else that n can afford at the price p.

We say that an allocation A is a competitive allocation if it is the allocation part of a CE; that is, A is a competitive allocation if there exists some price p such that A and p is a CE. Also, we say that p is a CE price if there exists some allocation A such that A and q is a CE.

Exercise 5 *Suppose that people are identical in preferences and endowments. Show that the endowment allocation is a competitive allocation.*

Let $(d_{n1}(p), d_{n2}(p))$ denote person n's demand at the price p (preferred pair from among those affordable at the price p). Let $D_1(p) = \sum_{n=1}^{N} d_{n1}(p)$ and let $D_2(p) = \sum_{n=1}^{N} d_{n2}(p)$.

Exercise 6 *With good* 1 *on the horizontal axis and good* 2 *on the vertical axis, sketch the pair* $(d_{n1}(2), d_{n2}(2))$*. (Person* n'*s endowment and some of person* n'*s indifference curves should appear in the sketch.)*

Exercise 7 *Show that* $D_1(p) + pD_2(p) = W_1 + pW_2$*. (This is called Walras' Law.)*

Exercise 8 *Suppose that* p^* *satisfies* $D_1(p^*) = W_1$*. Show that* p^* *is a CE price.*

Proposition 1 *A competitive allocation is PE.*

Exercise 9 *Prove proposition 1.*

3 Many-good pure exchange CE

The assumption that there are two goods is convenient mainly because preferences and affordable sets can then be depicted in two dimensions. However, we need to deal with more than two goods. Therefore, I want to briefly set out how competitive equilibrium is defined for an economy with K goods labeled $1, 2, ..., K$.

As above, this economy has no technologies and, hence, no production. The resources consist of the total amount of each good available. Let W_k denote the total amount of good k available. Again, there are N people labeled by integers $1, 2, ..., N$. Each person cares only about his or her own consumption. But now person n's consumption is a commodity vector denoted $(c_{n1}, c_{n2}, ..., c_{nK})$. And although we cannot draw indifference curve surfaces in K dimensions unless K is two or three, preferences can still be represented conceptually in that way.

An allocation is a commodity vector for each person. An allocation is feasible if $\sum_{n=1}^{N} c_{nk} \leq W_k$ for $k = 1, 2, ..., K$. The definition of Pareto-Efficiency is unaffected.

To define CE, we again need to describe what each person starts out owning. We let $(w_{n1}, w_{n2}, ..., w_{nK})$ denote the K-component commodity vector owned by person n and call it n's endowment. We also assume that everything is owned by someone. That is, $\sum_{n=1}^{N} w_{nk} = W_k$ for $k = 1, 2, ..., K$.

With K goods, we need at least $K - 1$ prices. Let p_k denote the price of good k in units of good 1 (p_k is measured in units of good 1 per unit of good k). Although it is convenient now to carry along the symbol p_1, it should be understood as being fixed at unity. Throughout, we suppose that p_k is positive for each k. We say that person n can *afford* to buy the vector $(c_{n1}, c_{n2}, ..., c_{nK})$ at the price vector $(p_1, p_2, ..., p_K)$ if there exists $(q_1, q_2, ..., q_K)$ such that

$$c_{nk} = w_{nk} - q_k \text{ for } k = 1, 2, ..., K \text{ and } \sum_{k=1}^{K} p_k q_k = 0. \tag{3}$$

As above, that notion of what person n can afford is equivalent to

$$\sum_{k=1}^{K} p_k c_{nk} = \sum_{k=1}^{K} p_k w_{nk} \tag{4}$$

The definition of CE is the same as for two goods. However, just to emphasize that there are many prices, I will write it out.

We say that an allocation A and a price vector $(p_1, p_2, ..., p_K)$ is a competitive equilibrium (CE) if two conditions hold: (i) A is feasible, (ii) for each person n, the consumption vector assigned to n by A is liked by n as well as anything else that n can afford at the price vector $(p_1, p_2, ..., p_K)$.

Proposition 2 *A competitive allocation is PE.*

Exercise 10 *Prove proposition 2.*

4 The Static Classical Dichotomy Model

The theory set out above is a theory of relative prices and allocations. The classical dichotomy was an attempt to reconcile some version of that model with the observation that most goods trade for something called money and not directly for other goods. Put another way, in an economy with money, there are prices that are not accounted for by the CE model—namely, the prices in terms of money. These are sometimes called absolute or nominal prices or , to distinguish them from the relative prices of the CE model. The classical dichotomy is the name given to a theory that says that real things—allocations and relative prices—are determined separately from nominal things—the quantity of money, nominal prices, nominal interest rates, and exchange rates.

4.1 Money

The word money is used in a variety of ways. Sometimes it is used to mean wealth. Economists do not use it that way. Instead, they tend to use it to refer to those components of wealth that serve as media of exchange. We will use it that way. First, we will consider a world that consists of a single country. One feature of the thing labeled money in the classical-dichotomy model and in many other models is that it is not a good in the ordinary sense. People acquire it not for using it as in input into a production process or for consuming it. Instead, it is acquired in order to subsequently trade it. It is best, therefore, to think of it as a durable physical object that is intrinsically useless—pieces of paper, stones, or sea shells. When the object serving as a medium of exchange is a good in the ordinary sense, then it is called commodity money. Unless other wise noted, when we refer to money here, we not mean a commodity money.

4.2 The Quantity Theory

The quantity theory of money is best regarded as a theory of the "demand" for money. The simplest version of the theory is that the "demand" for money at a date is proportional to the nominal value of total output for that date.

What do we mean by the nominal value of total output? Users of the quantity theory were not and did not have to be explicit about the model of the real part of the economy. It was in the background. Here, we want to be explicit about the real part of the economy. Therefore, we exposit the quantity theory against the background of the pure exchange model set out above (where we think of the different goods as distinct goods at the same date).

If there are 2 goods, then we let $Y = W_1 + p^* W_2$, where p^* is the CE price. If there are K goods, then $Y = \sum_{k=1}^{K} p_k^* W_k$ where the vector $(p_1^*, p_2^*, ..., p_k^*)$ is the CE price vector. In either case, Y is to be interpreted as total real income for the economy measured in units of good 1. Let P denote the price of good 1 in terms of money. Then PY, the product of P and Y, is total nominal income. The quantity theory posits that the "demand" for money is proportional to PY. We can denote it by $\frac{PY}{V}$, where V is a given positive constant which is often called the income velocity of money. Next suppose that the quantity of money is given and denoted by M. The quantity theory in this form says that the price level P is determined by equating the supply of money M to the "demand" for money $\frac{PY}{V}$. That is

$$M = \frac{PY}{V} \tag{5}$$

The classical-dichotomy theory in its static form consists of the CE model with the goods interpreted as distinct goods at the same date and equation (5).

Exercise 11 *A famous statement about money is: "money is a veil." In a few sentences, describe the sense in which the above classical-dichotomy model is consistent with money being a veil.*

5 A 2-date model

When first formulated, the goods in the above CE model were interpreted as different goods at the same date. In the late 19th century, a famous American economist, Irving Fisher, showed that they could also be interpreted as goods at different dates. His contribution, among other things, opened the way to applying the CE model to borrowing and lending. Here we briefly review what Fisher did in the context of the 2-good pure exchange model set out above. We have many reasons for wanting to do that. First, money is a durable object and, hence, lasts for more than one period. To study it seriously, we need to have a multi-period model. Second, as we will emphasize throughout, money and credit are rival ways of accomplishing trade. Hence, we want to think about credit, which also necessitates having a multi-period model.

Fisher changed the interpretation of the goods. Rather than thinking of the two goods in the above 2-good model as different objects at the same date, apples and oranges, he said that we can think of them as the same physical object at different dates—for example, date-1 apples and date-2 apples.

While we could simply end there, we want to point out the association between objects in the model and some of the terminology usually used to describe multi-period trade. In addition, we want to introduce a special form of preferences and some special endowments that we will use repeatedly. By the way, the assumption that there are no technologies is carried over directly to the 2-date interpretation of the model. In particular, date-1 good, good 1, cannot be converted into date-2 good, good 2. In other words, it cannot be stored from one date to the next; it is perishable. Also, society's endowment of date-2 good, and, hence, the individual endowments of date-2 good do not physically exist at date 1.

5.1 Terminology.

Let's interpret the objects in the model using the terms borrowing, lending, interest rate, and present value. All appear in the two equivalent definitions of affordability. First, person n's endowment, the pair (w_{n1}, w_{n2}), can be thought of as n's *income stream* with w_{n1} being income at date 1 and w_{n2} being income at date 2. If we do that, then the right-hand side of (2) is the present value, the date-1 value, of n's income stream. And using this terminology, equation (2) says that the present value of consumption is equal to the present value of income. We could also call the right-hand side of (2) person n's wealth. Present values are usually written using interest rates. To do that, we can simply replace p by $\frac{1}{1+r}$ where r is the interest rate at date 1 on one-period loans in the form of goods. (Because it applies to loans and repayments in the form of goods, r is called a real interest rate.) In accord with that terminology, we should think of q_1 in (1) as lending of person n in the form of date-1 good (borrowing by n in that form if q_1 is negative) and of q_2 as repayment of what was borrowed with interest (repayment of what was lent with interest if q_2 is negative). The equality $q_1 + pq_2 = 0$ with p replaced by $\frac{1}{1+r}$ says that loans are repaid with interest.

Exercise 12 *Describe an environment and individual endowments such that there is a CE with a positive interest rate. Also describe one with a negative interest rate. (Hint: See exercise 5.)*

5.2 Some special preferences and endowments

In multi-period models, it is common to use a special form of preferences called *discounted utility*. We will do that and will, in addition, assume that everyone has the same preferences. In the 2-date model with one good per date, the assumption is described as follows: the utility that person n gets from consuming the pair (c_{n1}, c_{n2}) is $u(c_{n1}) + \beta u(c_{n2})$, where β is a number strictly between 0 and 1 and where u is a function that is strictly increasing, is smooth (has a derivative), and displays diminishing marginal utility (is strictly concave). (The function $u(x) = \sqrt{x}$ is one possible u function.) In this specification, u is called the *period utility function* and β is called the *discount factor*. We denote the derivative of the function u by the symbol u'. For example, if $u(x) = \sqrt{x}$,then $u'(x) = \frac{1}{2\sqrt{x}}$.

Exercise 13 *Let the utility function be $u(c_{n1}) + \beta u(c_{n2})$ with $u(x) = \sqrt{x}$. (i) Solve the equation $\sqrt{c_1} + \beta\sqrt{c_2} = b$ for c_2 as a function of (in terms of) c_1. (ii) Let $\beta = .5$. Use your part (i) answer and the spread-sheet program Excel to plot the indifference curve corresponding to $b = 2$ for values of c_1 between 0 and 4.*

Exercise 14 *Let $u(x) = \sqrt{x}$. Use Excel and plot on the same chart $u(x)$ and $u'(x)$ for $0 \leq x \leq 4$.*

One fact about the indifference curve map implied by discounted utility that we will use is the following:

Claim 1 *Let preferences be given by* $u(c_{n1}) + \beta u(c_{n2})$ *and consider the implied indifference curve map with good* 1 *on the horizontal axis and good* 2 *on the vertical axis. The slope of the indifference curve through the point* (c_{n1}, c_{n2}) *at the point* (c_{n1}, c_{n2}) *is* $-\frac{u'(c_{n1})}{\beta u'(c_{n2})}$.

If you know enough calculus, then you can derive this fact. Whether you can derive it or not, use it from now on.

Exercise 15 *Let preferences be given by* $u(c_{n1})+\beta u(c_{n2})$ *and let* $u(x) = \sqrt{x}$. *If* $\beta = .5$, *what is the slope of the indifference curve through the point* $(c_{n1}, c_{n2}) = (2, 1)$ *at the point* $(c_{n1}, c_{n2}) = (2, 1)$?

Exercise 16 *Show the following. If preferences are given by* $u(c_{n1}) + \beta u(c_{n2})$, *then the indifference curve through the point* $(c_{n1}, c_{n2}) = (x, x)$ *has slope* $-\frac{1}{\beta}$ *at the point* $(c_{n1}, c_{n2}) = (x, x)$. *(We will use this result repeatedly.)*

In what follows, use is often made of a way of finding an equilibrium called *guess and verify.* In fact, this procedure was already used in several exercises. There are three steps to the procedure: (i) make a guess about some feature of an equilibrium; (ii) construct an entire guess based on that feature; and (iii) verify that the step (ii) construction is an equilibrium. In exercise 5, the step (i) feature is an allocation; in exercise 8, it is a price. In the following proposition, it is also a price.

Proposition 3 *Consider a two-good pure exchange environment in which everyone has the same preferences which satisfy the discounted utility description and in which* $W_1 = W_2$. *(i) There is a CE with* $p = \beta$. *(ii) There is no other CE.*

Exercise 17 *Prove conclusion (i).*

Exercise 18 *Prove conclusion (ii).*

6 More than 2 dates

Just as we could interpret the 2-good model as an inter-temporal model with 2 dates and one good per date, we can also interpret the K-good model as a K-date model with one good per date. In particular, we assume that all K goods are the same physical object but at different dates with good k being date-k good.

Again, we can interpret (4) as saying that the present value of consumption is equal to the present value of income, where we label w_{nk} as person n's date-k

income. Suppose we want to express these present values in terms of interest rates. Let's do it in terms of one-period interest rates at each of the dates $1, 2, ..., K-1$.

To see how to do this, suppose for a moment that the only q's that are not zero are q_k and q_{k+1}. Then, by the third equation in (3), q_k and q_{k+1} must satisfy

$$q_{k+1} = -\frac{p_k}{p_{k+1}} q_k.$$

But such a transaction can be interpreted as a loan made at date k that is repaid with interest at date $k+1$. Therefore, if r_k denotes the interest rate at date k on one-period loans made in the from of goods, it must be that

$$1 + r_k = \frac{p_k}{p_{k+1}}. \tag{6}$$

Then $p_1 = 1$ and a little algebra gives

$$p_2 = \frac{1}{1+r_1},$$

$$p_3 = \frac{p_2}{1+r_2} = \frac{1}{(1+r_1)(1+r_2)},$$

and, in general,

$$p_k = \frac{1}{(1+r_1)(1+r_2)...(1+r_{k-1})}.$$

Notice that if $r_1 = r_2 = ... = r_{K-1} = r$, then $p_k = \frac{1}{(1+r)^{k-1}}$ and (4) takes the familiar form

$$\sum_{k=1}^{K} \frac{c_{nk}}{(1+r)^{k-1}} = \sum_{k=1}^{K} \frac{w_{nk}}{(1+r)^{k-1}}.$$

This case of a constant interest rate is relevant in the special environments we now describe. The special form of preferences called *discounted utility* can also be defined for a K-date model. In the K-date model with one good per date, the assumption is described as follows: the utility that person n gets from consuming the vector $(c_{n1}, c_{n2}, ..., c_{nK})$ is

$$u(c_{n1}) + \beta u(c_{n2}) + \beta^2 u(c_{n3}) + ... + \beta^{K-1} u(c_{nK}) = \sum_{k=1}^{K} \beta^{k-1} u(c_{nk}), \tag{7}$$

where β and u satisfy the assumptions made above. And there is an analogue of proposition 3.

Proposition 4 *Consider a K-good pure exchange environment in which everyone has the same preferences which satisfy the discounted utility description, (7) and in which $W_1 = W_2 = ... = W_K$. (i) There is CE with $p_k = \beta^{k-1}$. (ii) There is no other CE.*

Notice from (6), that $p_k = \beta^{k-1}$ implies $1 + r_k = \frac{1}{\beta}$.

Here is a sketch of a proof of this proposition. It is only a sketch because one step, which I will point out, is left out. The proposition proposes a price sequence. I will propose an allocation and then verify that the two conditions for a CE hold for the complete proposal.

Let

$$x_n = \frac{\sum_{k=1}^{K} \beta^{k-1} w_{nk}}{\sum_{k=1}^{K} \beta^{k-1}}. \tag{8}$$

The proposed allocation is $c_{nk} = x_n$ for $k = 1, 2, ..., K$ and each n.

Exercise 19 *Describe in words the propsoed allocation.*

Now we turn to verification.

Feasibility of the allocation. We have

$$\sum_{n=1}^{N} x_n = \sum_{n=1}^{N} \left\{ \frac{\sum_{k=1}^{K} \beta^{k-1} w_{nk}}{\sum_{k=1}^{K} \beta^{k-1}} \right\} = \frac{\sum_{n=1}^{N} \left\{ \sum_{k=1}^{K} \beta^{k-1} w_{nk} \right\}}{\sum_{k=1}^{K} \beta^{k-1}} =$$

$$\frac{\sum_{k=1}^{K} \left\{ \sum_{n=1}^{N} \beta^{k-1} w_{nk} \right\}}{\sum_{k=1}^{K} \beta^{k-1}} = \frac{\sum_{k=1}^{K} \beta^{k-1} \left\{ \sum_{n=1}^{N} w_{nk} \right\}}{\sum_{k=1}^{K} \beta^{k-1}} = \frac{\sum_{k=1}^{K} \beta^{k-1} W}{\sum_{k=1}^{K} \beta^{k-1}} = W.$$

where the third equality involves a change in the order of summation.

Individual optimization. At the proposed prices, person n chooses $(c_{n1}, c_{n2}, ..., c_{nK})$ to maximize discounted utility subject to

$$\sum_{k=1}^{K} \beta^{k-1} c_{nk} = \sum_{k=1}^{K} \beta^{k-1} w_{nk}.$$

The part that I cannot prove for you is that a maximum exists. That is true, but cannot be established without some advanced math. Just accept it. Now, suppose by contradiction that the maximum is not achieved by the proposed path of consumption for some person n, but by something else. I will denote the alternative consumption vector for person n by attaching $*$'s to the consumption symbols. Is it another constant sequence of consumptions? No. (Why?) Then it must be achieved by a non constant sequence. If so, then for person n the best affordable sequence has two adjacent terms, call them c^*_{nk}, c^*_{nk+1}, where $c^*_{nk} \neq c^*_{nk+1}$. We now prove that this leads to a contradiction.

Let $z = \beta^{k-1} c^*_{nk} + \beta^k c^*_{nk+1}$, where z is total spending on the pair c^*_{nk}, c^*_{nk+1} at the proposed prices. Now consider the following problem.

Problem A. Choose (c_{nk}, c_{nk+1}) to maximize

$$\beta^{k-1} u(c_{nk}) + \beta^k u(c_{nk+1}) \tag{9}$$

subject to

$$\beta^{k-1}c_{nk} + \beta^k c_{nk+1} = z. \tag{10}$$

I will sketch two ways to describe the solution to problem A—one using a diagram, the other using calculus.

A Diagram. The key is to sketch the indifference curves implied by (9) and the opportunities implied by (10), with c_{nk} measured along the horizontal axis and c_{nk+1} measured along the vertical axis. Recall that a single indifference for (9) consists of all pairs c_{nk}, c_{nk+1} that leave the expression in (9) constant. But, if so, then $u(c_{nk}) + \beta u(c_{nk+1})$ is also constant. But, by claim 1, we know that the indifference curves for $u(c_{nk}) + \beta u(c_{nk+1})$ have slope $-\frac{1}{\beta}$ at any point at which $c_{nk} = c_{nk+1}$. Therefore, the same is true for the indifference curves for (9). Next, sketch the pairs c_{nk}, c_{nk+1} that satisfy (10). These are on a line with slope $-\frac{1}{\beta}$ and a positive intercept. It follows that the highest indifference curve is reached by choosing $c_{nk} = c_{nk+1}$. This conclusion contradicts the assumption that $c^*_{nk} \neq c^*_{nk+1}$ is best for person n from among what is affordable.

Calculus. In problem A, (9) is called the objective and (10) is called the constraint. One way to solve problem A is to use the constraint to eliminate one of the choice variables. So, for example, we could solve (10) for c_{nk} and substitute the result into (9). When we do that, we are left with the problem of choosing c_{nk+1} to maximize

$$u(\frac{z}{\beta^{k-1}} - \beta c_{nk+1}) + \beta u(c_{nk+1}). \tag{11}$$

The objective in (11) is a function of a single variable, c_{nk+1}. The maximum occurs where the derivative of this function is zero. As above, we let u' denote the function that is the derivative of the function u. Then, using what is called the chain rule for differentiation, the maximum occurs where

$$-\beta u'(\frac{z}{\beta^{k-1}} - \beta c_{nk+1}) + \beta u'(c_{nk+1}) = 0. \tag{12}$$

Because the function u' is decreasing, (12) holds if and only if

$$\frac{z}{\beta^{k-1}} - \beta c_{nk+1} = c_{nk+1}. \tag{13}$$

Because the left-hand side of (13) is c_{nk}, (13) implies $c_{nk} = c_{nk+1}$. This contradicts the assumption that c^*_{nk}, c^*_{nk+1} where $c^*_{nk} \neq c^*_{nk+1}$ is best for person n from among what is affordable. This completes the proof.

Exercise 20 *Describe problem A in words.*

Exercise 21 *In problem A, why is $z > 0$?*

Because environments which have last dates, finite horizons, are awkward for many purposes, it is usual to let K be infinite. Although an infinite K raises some mathematical difficulties, we can for the most part ignore those. Provided that we take satisfaction of (4) to be our definition of affordability, the last proposition also holds if K is infinite.

7 A More General Classical Dichotomy Model

We now have an explicit theory of the real one-period interest rate at each date, exactly the kind of theory that Irving Fisher used in his expanded version of the classical dichotomy model. Fisher's version of the classical dichotomy model has a theory of nominal rates, interest rates on loans in the form of money. We now describe that version against the background of a K-date pure exchange model with one good per date, not necessarily the model with the special assumptions of proposition 4. However, now it is convenient to replace the index k by t for date (time).

With one good per date, total real income at date t is simply the total amount of date t good, W_t. Let P_t denote the price of date t good in units of money, the date t price level. Then the nominal value of total output at date t is the product $P_t W_t$.

Let's start with the simple version of the quantity theory according to which the aggregate "demand" for money is proportional to $P_t W_t$—$\frac{P_t W_t}{V}$ for some positive and known constant V. If we equate the amount of money to this "demand", then we have

$$M_t = \frac{P_t W_t}{V}. \tag{14}$$

And if we regard the total amount of money as given, then equation (14) determines P_t, the price level.

7.1 The nominal interest rate

Now we use the quantity theory and an arbitrage argument to get a theory of the nominal interest rate. The arbitrage argument implies a relationship between interest rates on money loans, nominal interest rates, and loans made in the form of goods, the real interest rates in the inter-temporal version of the pure exchange model.

Let i_t denote the interest rate on one-period money loans at date t. We suppose that the goal is to surrender one unit of date t good to get some amount of date $t+1$ good. One way to do this is to (i) sell the date t good for money obtaining P_t units of money, (ii) lend the money at i_t thereby acquiring $(1+i_t)P_t$ units of money at date $t+1$, and (iii) use that money to buy date $t+1$ good in the amount $\frac{(1+i_t)P_t}{P_{t+1}}$. An alternative is to lend the date t good at the real rate r_t. This gives $1+r_t$ units of the date $t+1$ good. Equating the two amounts of date $t+1$ good, we get

$$1+i_t = (1+r_t)\frac{P_{t+1}}{P_t} \tag{15}$$

This is often called the Fisher equation.

Exercise 22 *Suppose that* $W_{t+1} = W_t$, *that* $r_t = .02$, *and that* $M_{t+1}/M_t = 1.10$. *According to the quantity theory and the Fisher equation, equations (14) and (15), what is the one-period nominal interest rate at* t*?*

Exercise 23 *Suppose equation (15) does not hold and that, instead, the left-hand side of equation (15) exceeds the right-hand side. Describe borrowing and lending in goods and in money that is profitable.*

Exercise 24 *Consider an environment that satisfies the hypotheses of proposition 4 for* $K = \infty$. *Suppose that* $M_{t+1} = zM_t$, *where* $z > 0$ *for all dates* t. *Use Proposition 4 and equations (14) and (15) to express* i_t *in terms of* β *and* z.

7.2 Many countries

The quantity theory can be extended to many countries and many monies. To show how that is done, it is sufficient to suppose that there are two countries, country A and country B. We suppose that each country consists of people who live for T dates and that there is one good per date in the entire world. We suppose that people cannot move between countries, but that people in one country can trade with those in the other country and that the single good can be moved between the two countries at each date without any transport costs being incurred. In particular, we suppose there is a single world-wide market in loans made in the form of goods.

Exercise 25 *Does the definition of competitive equilibrium need to amended to deal with this world of two countries? Explain.*

Exercise 26 *In accord with standard usage, we define the date* t *trade deficit of a country as the value of its imports at date* t *minus the value of its exports at date* t. *(i) In a competitive equilibrium for the above two-country world, would you expect each country to have a zero trade deficit at each date? Explain. (ii)*

In a 2-date version of the model, could a country have a positive trade deficit at both dates? Explain.

We now suppose that each country has its own money supply and that the quantity theory in the form of equation (14) holds for each country. That is, corresponding to equation (14), we have

$$M_t^A = \frac{P_t^A W_t^A}{V} \text{ and } M_t^B = \frac{P_t^B W_t^B}{V}. \tag{16}$$

We also suppose that the Fisher equation, (15), holds for each country:

$$1 + i_t^A = (1 + r_t)\frac{P_{t+1}^A}{P_t^A} \text{ and } 1 + i_t^B = (1 + r_t)\frac{P_{t+1}^B}{P_t^B}. \tag{17}$$

Exchange rates describe prices of one country's money in terms of other countries' monies. With two countries, let E_t denote the price of country A's money in terms of country B's money at date t. The requirement that there be no profit from trading one money for the other and importing or exporting the good at date t gives rise to the following relationship among the exchange rate and the price levels in the two countries,

$$E_t = \frac{P_t^B}{P_t^A}. \tag{18}$$

Equation (18), another no-arbitrage condition, is sometimes called the purchasing power parity theory of exchange rates.

Exercise 27 *Suppose equation (18) does not hold and that, instead, the left-hand side of equation (18) exceeds the right-hand side. Describe transactions in the date t good and in the two monies that are profitable.*

Exercise 28 *The assumption that there is a single world-wide market in loans made in terms of goods implies that there is a common real interest rate for the world consisting of countries A and B. This is built into (17). Does the model above imply that the nominal interest in terms of country A money is equal to the nominal interest in terms of country B money? Explain.*

7.3 Dependence of money "demand" on the interest rate

The classical-dichotomy model can be extended to permit the "demand" for money to depend on the nominal interest rate. The idea behind such dependence is that people are sacrificing interest at the nominal rate by holding non-interest-bearing money and, therefore, that the higher the nominal interest rate, the smaller the "demand" for money. In the one-country version, this can be incorporated by making V in (14) an increasing function of the nominal interest rate. If this is done, then the dichotomy between real and nominal objects is preserved. However, the determination of the price level is more complicated. Instead of being able to solve (14) for the price level at date t without regard to what it is at other dates, the new version of (14) involves the price level at different dates. It is obtained by making V in (14) a function of the right-hand side of (15). Because this more complicated version of the classical dichotomy model suffers from all the defects of the simpler version, we will not spend time studying it.

8 The Classical Dichotomy Model: Defects and Remedies

There are at least two serious defects of the classical dichotomy model; it is incoherent and it fails to describe the benefits of monetary exchange.

There are various ways to describe the incoherence. The general-equilibrium part is a complete description of a non monetary economy; it describes preferences, resources, technologies, what people start out owning, and how they interact. When the quantity-theory equation is appended, we are led to ask the following kinds of questions which have no answers: who owns the money? why is the money valuable?

The second defect is more subtle, but is related. For about two thousand years, it has been asserted that the use of money is helpful in overcoming trading difficulties—the difficulties associated with what is generally labeled barter. Because the general-equilibrium part of the classical dichotomy model has complete competitive markets and, therefore, does not depict any trading difficulties, it cannot display any sense in which monetary exchange is helpful. Money really is a veil in that model.

One approach to modelling economies with money that deals with the first defect is to assume that the real value of money is productive in one way or another. One version assumes that people like not only goods, but that they also like the real value of money. Another is that the real value of money is productive in the more usual sense: having it frees labor that can be used either to produce other things or for leisure. These approaches, however, have never been viewed as serious attempts to overcome the second defect; that is, they were never intended to depict the sense in which monetary exchange is helpful. Instead, they are short-cuts which users of them hope are consistent with other specifications which do depict the beneficial role of monetary exchange. In part because such consistency has never been demonstrated, we will look directly at some specifications in which monetary exchange is helpful.

9 A Closer look at Credit

In the actual economy, credit and money are alternative ways of accomplishing transactions. In order to find a role for money, we should identify aspects of economies that inhibit the use of credit. Let's start by considering credit cards. Most retail establishments accept either payment by cash or payment with a credit card and some people have credit cards, while others do not. To get a credit card, you have to show some evidence of being a good risk. In particular, you cannot be someone who is known to no one. Next consider a simpler example, which is closer to credit in the inter-temporal models set out above. Suppose you find yourself without a pencil and you notice a person who has several. You can either ask to borrow a pencil from that person or you can offer to buy one with cash. If the person is a friend, then you are likely to ask to borrow a pencil; however, if you do not know the person—or, more to the point, if you are not known to the person—then you are likely to offer to buy one with cash. In general, someone who is not known to anyone will not be able to get a credit card or any other form of credit.

This feature of borrowing and lending does not show up in the CE model. One interpretation of what goes on in the CE model is that people can somehow commit to repay. If we, instead, suppose, as is very reasonable, that people cannot commit to anything, then we have taken one step toward making the use of credit difficult. But we need an additional assumption if we are to sufficiently inhibit credit so that there is a role for money. The additional assumption is that people are to some extent strangers in the following sense. There is no record-keeping device that allows people at one date to perfectly recall what was done earlier.

For the first part of this course, we will rule out credit completely by assuming that *nothing* about the past can be recalled. Then, after studying models with that assumption, we will weaken it somewhat to show how we can get a mix of credit and money. The first model we will use is a special case of the inter-temporal model set out above with one good per date and an infinite horizon.

Exercise 29 *Consider the K-date version of the K-good pure exchange model. If actions at one date cannot be recalled at later dates, what do you think would* *happen in this economy?*

10 Appendix: Some math

Here are some math results we will use.

Definition 5 *A function is a* ***rule*** *that assigns to each element in a* ***domain set*** *a single element in a* ***range set****.*

For us, the domain set will most often be a set of real numbers or a set of vectors. The range will most often be real numbers. We begin with the case in which both the sets are sets of real numbers. Such functions are called real functions of a single variable or are said to *map* real numbers into real numbers. An example is the positive square root function, $x^{1/2}$, for $x \geq 0$.

Let f map real numbers into real numbers. We often consider the question: at what elements in the domain is the function f a maximum? We answer it for functions f which satisfy two further properties: differentiability and strict concavity.

Definition 6 *Let f map real numbers into real numbers. The function f is said to be differentiable if it has a unique slope at each element in the domain. The function that associates the slope of f to each element in the domain is called the derivative function and is denoted f'.*

For example, if $f(x) = x^{1/2}$ for $x \geq 0$, where, again, we mean the positive square root, then $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}}$. (You will not be asked to compute derivatives. You will be asked to interpret them in the sense of understanding the above definition.)

Definition 7 *Let f map real numbers into real numbers. Let x_1 and x_2 be two elements (points) in the domain with $x_1 < x_2$ and consider the line segment that connects the point $(x_1, f(x_1))$ and $(x_2, f(x_2))$. This line segment is a function, call it g, whose domain is the interval $[x_1, x_2]$ and which is defined by*

$$g(x) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1). \tag{19}$$

The function f is said to be strictly concave if $f(x) > g(x)$ for any x satisfying $x_1 < x < x_2$.

There is an equivalent and more useful way to state the definition of strict concavity. If x is between x_1 and x_2, then $x = \lambda x_1 + (1 - \lambda)x_2$ for some λ

satisfying $0 < \lambda < 1$. The converse is also true. If $x = \lambda x_1 + (1 - \lambda)x_2$ is substituted for x on the right-hand side of (19), then,

$$g(x) = \lambda f(x_1) + (1 - \lambda)f(x_2). \quad (20)$$

Therefore, the alternative definition says,

Definition 8 *Let f map real numbers into real numbers. Let x_1 and x_2 be two different elements (points) in the domain and let $x = \lambda x_1 + (1 - \lambda)x_2$ for some λ satisfying $0 < \lambda < 1$. The function f is said to be strictly concave if $f(x) > \lambda f(x_1) + (1 - \lambda)f(x_2)$.*

Exercise 30 *Consider the function $f(x) = x^2$ on the domain $[0, 1]$. Show that f is not strictly concave.*

Claim 2 *Let the domain of f be an interval, including the endpoints, and suppose that f is differentiable and strictly concave. Then there is a unique point in the domain at which f is a maximum. Call this point x^*. If there is a point x such that $f'(x) = 0$, then $x = x^*$. If there is no point x such that $f'(x) = 0$, then x^* is an endpoint of the interval.*

Claim 3 *Let f and g be two functions that map real numbers to real numbers and that are differentiable and strictly concave. Let*

$$h(x) \equiv a_1 f(b_1 + b_2 x) + a_2 g(b_3 + b_4 x), \quad (21)$$

where a_1 and a_2 are non negative real numbers and b_1, b_2, b_3, b_4 are any real numbers, including negative real numbers. Then h is strictly concave and

$$h'(x) \equiv a_1 b_2 f'(b_1 + b_2 x) + a_2 b_4 g'(b_3 + b_4 x). \quad (22)$$

The result in (22) is an application of some calculus results: (i) the derivative of a constant times a function is the constant times the derivative of the function; (ii) the derivative of a sum is the sum of the derivatives; and (iii) the chain rule—if $h(x) \equiv f(g(x))$, then $h'(x) = f'(g'(x))$.

We now consider real functions of two variables, functions that map points in the plane, ordered pairs, into real numbers. Here the domain is a subset of the plane. For the moment, denote a generic point in the plane by (x, y). The definition of a function is not affected. A real function whose domain is a subset of the plane will be called a function that maps points in the plane to real numbers. Such a function can be represented in a 3-dimensional diagram. If we let the first two dimensions be the plane that contains the domain, then the third dimension assigns to each point in the domain the value of the function. In general you can think of the function as a surface "over" the plane.

Definition 9 *Let f map points in the plane into real numbers. The function f is said to be differentiable if it has a unique tangent plane at each element in the domain.*

Definition 10 *Let f map points in the plane into real numbers and denote it $f(x, y)$. Assume that f is differentiable. The partial derivative of f with respect to x (the first argument) at (x, y) is denoted $f_1(x, y)$ and is the slope of the tangent plane at (x, y) in the x direction. The partial derivative of f with respect to y (the second argument) at (x, y) is denoted $f_2(x, y)$ and is the slope of the tangent plane at (x, y) in the y direction.*

Definition 11 *Let f map points in the plane into real numbers and denote it $f(x, y)$. Let (x_1, y_1) and (x_2, y_2) be two different elements (points) in the domain and let $x = \lambda x_1 + (1 - \lambda)x_2$ and $y = \lambda y_1 + (1 - \lambda)y_2$ for some λ satisfying $0 < \lambda < 1$. The function f is said to be strictly concave if $f(x, y) > \lambda f(x_1, y_1) + (1 - \lambda)f(x_2, y_2)$.*

Claim 4 *Let the domain of f be a closed subset of the plane, and suppose that f is differentiable and strictly concave. Then there is a unique point in the domain at which f is a maximum. Call this point (x^*, y^*). If there is a point (x, y) such that $f_1(x, y) = 0$ and $f_2(x, y) = 0$, then $(x, y) = (x^*, y^*)$. If there is no point (x, y) such that $f_1(x, y) = 0$ and $f_2(x, y) = 0$, then (x^*, y^*) is on the boundary of the domain. More generally, if (x', y') is a point in the domain and neither small variations around (x', y') in the x direction nor small variations around (x', y') in the y direction increase the function, then $(x', y') = (x^*, y^*)$.*

Versions of the last three definitions and of the last claim also generalize to functions of more than two variables. In particular, we will use the generalization to many variables in the following way. We will often be constructing a candidate to be a maximum of a function of many variables. Given that the function is strictly concave, to check whether the candidate is, in fact, a maximum, it is enough to check departures one variable at a time and to check only for "small" departures.