Econ 497 Spring 2007
Assignment 1: Complete Markets: Answers

The due date for this assignment is Tuesday, January 30.

1. *In the PSU-Michigan football game, the possible outcomes are as follows. In the first half, PSU can go ahead by 10 points or Michigan can go ahead by 10 points. In the second half, the change in the score can be 10 points in PSU's favor or 10 points in Michigan's favor. PSU has the better team, so in the first half there is a 70% probability that PSU will take the lead. However PSU's team gets too conservative when ahead, so if it ends the first half in the lead, there is a 50% probability that the score will move in Michigan's favor in the second half. If Michigan's team ends the first half in the lead, the probability remains 70% that the score will move in PSU's favor in the second half.*

   (a) *Draw the event tree that describes the outcomes in this game. At each node of the tree, enter the PSU lead (the number of points that PSU is ahead, a negative number if Michigan is ahead). What are the probabilities of each final PSU lead (i.e., of each final score, PSU's lead can be positive or negative)?*

   brief answer Here is the tree, with $H = PSU$ lead.

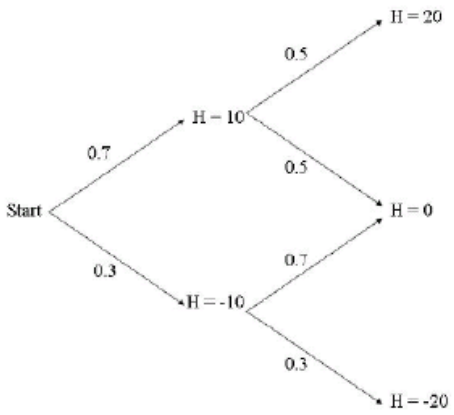

Figure 1: Event Tree

   There are three final outcomes: (i) PSU up by 20 has a probability of $70\% \times 50\% = 35\%$; (ii) a tie has a probability of $70\% \times 50\% + 30\% \times 70\% = 56\%$; (iii) PSU loses by 20 has a probability of $30\% \times 30\% = 9\%$.

   (b) *Suppose there are bookmakers who allow you to bet at the start of the game on the halftime score. If the bookmakers have no costs and make no profits on average, how much do you have to pay at the start of the game to receive \$1 if PSU is ahead at half-time? How much do you have to pay to receive \$1 if Michigan is ahead?*

   brief answer Since the bookmakers make no profits, whatever they take in has to be what they expect to pay out. For someone buying a "PSU Ahead" bet,

$P = 70\% \times \$1 + 30\% \times \$0 = \$0.70$, i.e., you pay the expected value. Similarly, for someone buying a "Michigan Ahead" bet, $P = 70\% \times \$0 + 30\% \times \$1 = \$0.30$.

(c) *Suppose the bookmakers also operate at half-time, and allow you to bet on the final score. If PSU is ahead at half-time, how much do you have to pay at half-time to receive \$1 if PSU wins the game, and how much do you have to pay at half-time to receive \$1 if the game is a tie? What if Michigan is ahead at half-time?*

brief answer The logic is similar to part b). If PSU is ahead (conditional on...), "PSU Wins" bets are \$0.50 and "PSU Ties" bets are \$0.50. If PSU is behind (conditional on...), "PSU Ties" bets are \$0.70 and "PSU Loses" bets are \$0.30.

(d) *Show that even if the bookmakers do not offer bets at the start of the game on the final outcome of the game, you can still use a sequence of bets, starting at the beginning of the game, to obtain \$1 if PSU wins the game, or \$1 if Michigan wins, or \$1 if the game is a tie. How much does it cost to obtain each of these payments?*

brief answer Note that what you are buying is a riskless asset—an asset that pays \$1 no matter what the outcome is. Let's look at this going backwards for intuition. If PSU wins the game and we get a dollar, we must have bought a "PSU Wins" ticket, which costs \$0.50. But to do that, we must have had \$0.50 at the "PSU Ahead" point at halftime. To have that, we must have bet \$0.35 on "PSU Ahead" at the start of the game. If PSU loses the game and we get a dollar, we must have bought a "PSU Loses" ticket, which costs \$0.30. But to do that, we must have had \$0.30 at the "PSU Behind" point at halftime. To have that, we must have bet \$0.09 at the start on "PSU Behind." If PSU ties the game and we get a dollar, one of two things happened. Either we bet "PSU Ties" after "PSU Ahead" or we bet "PSU Ties" after "PSU Behind." The first one costs \$0.50, requiring \$0.35 of a bet on "PSU Ahead" at the start. The second bet costs \$0.70, requiring a bet of \$0.21 at the start on "PSU Behind."

Result Bet \$0.70 on "PSU Ahead" at the start and \$0.30 on "PSU Behind" at the start. This costs you one dollar. Then no matter what, you end up with one dollar at halftime (look at part b). If PSU is ahead, bet \$0.50 on both "PSU Wins" and "PSU Ties," using up your dollar. If PSU is behind, bet \$0.70 on the tie and \$0.30 on the loss, using up your dollar. Then, as we calculated in part c), you end up with a dollar no matter what, because whatever the outcome is, you bought the bet that pays off one dollar.

Conclusion So it costs you one dollar to get a riskless one dollar in the future. It also is unnecessary for bookies to offer direct bets on the end of the game (note if they did, you'd pay \$0.35 for "PSU Wins," \$0.09 for "PSU Loses," and \$0.56 for "PSU Ties," as we calculated in part a)). By using a sequence of bets, you can construct in effect a bet on the final outcome. Note the price of the bet sequence on any outcome is the expected value of that outcome, because the bookies are not trying to make any money.

(e) *How do these costs relate to the probabilities in part a)? Under what conditions would you expect this relationship to hold in other more general examples?*

brief answer The costs are the probabilities. What if the bookies were trying to make a profit or recoup costs? Then they'd have to "charge" more, which means you'd pay a little more for each bet. You couldn't pay a dollar in initial bets and then get a dollar back in the end as you are now. Another consideration is the riskless rate of interest. As we know from intermediate economics, investment in a riskless asset returns the riskless rate of interest. Here we invest in a riskless asset for the duration of the game. If the riskfree rate of interest is positive, we'd expect that investing a dollar at the start of the game would return slightly more than a dollar by the end of it. In other sorts of bets people make, such as in the financial markets, transaction costs and interest rates will affect the costs of assets.

Comment What is neat here is that we have three outcomes here—PSU wins, ties, or loses. But we only have two assets, really—a bet that PSU moves up or falls behind. At first glance, it might seem markets cannot be complete. If they are not complete, we cannot construct a riskless asset (convince yourself of this: A riskless asset is the sum of all Arrow-Debreu securities. If you don't have complete markets, you don't have all Arrow-Debreu securities). But we did construct a riskless asset, because markets are complete. The fact that this is a multiperiod setup allows one more factor - time - to give us the tools to complete markets. There are (at most) only two branches leaving each node on the tree above, so we need (at most) only two assets to complete markets.