Semi-Nonparametric Estimation of First-Price Auctions Models with Auction-Specific Heterogeneity via an Integrated Simulated Conditional Moments Method

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Abstract

In this paper we propose to estimate first-price auctions models with observed auction-specific heterogeneity via a semi-nonparametric integrated simulated conditional moments method. The auction-specific heterogeneity will be incorporated via a median regression model for the log values with unknown error distribution. The latter distribution will be modeled semi-nonparametrically using orthonormal Legendre polynomials, similar to the approach in Bierens (2007). Given a parametric specification of the median function, the semi-nonparametric conditional value distribution involved can be estimated consistently by minimizing the integrated square distance between the empirical characteristic function of the actual bids and the simulated bids, together with the covariates, via an integrated conditional moment criterion. This approach yields as a by-product an integrated conditional moment test for the validity of the model.

Keywords: Auction-specific heterogeneity, empirical characteristic function, first-price auction, integrated conditional moment, semi-nonparametric estimation, sieve estimation, simulated method of moments, selection of truncation order, weak convergence.

JEL codes: C14, C21, C51, D44

1 Introduction

In many repeated auctions the objects to be auctioned off are different across auctions. Consequently, the value distributions are then different across auctions. However, if we observe the auction-specific characteristics in the form of covariates, and the value distributions conditional on these covariates have the same functional form, the conditional bid distribution given the auction-specific covariates will be the same for all auctions. The question then arises how to incorporate the observable characteristics into the auction model. Laffont, Ossard and Vuong (1995) incorporate covariates in the value distribution by specifying a linear regression model for the log of values with zero-mean normal errors. Donald and Paarsch (1996) parameterize the upper bound of the values as a function of covariates. Li (2005) specifies the value distribution as the exponential distribution with mean a linear function of covariates. Guerre, Perrigne and Vuong (2000) propose a two-stage nonparametric kernel density estimation approach, where in the first stage the bid distribution and density conditional on the covariates are estimated nonparametrically, which then is used in inverse form to generate values given the actual bids and the covariates. The generated values are then used to estimate the conditional value distribution nonparametrically.

In this paper, we propose an alternative semi-nonparametric approach to estimate first-price auction models with observed auction-specific heterogeneity and private, symmetric and independent values conditional on the auction specific covariates.[1] This approach extends the semi-nonparametric integrated simulated moments estimation method of Bierens and Song (2007) to the heterogenous case with observable auction-specific covariates. We consider a first-price auction model where the log value takes the form of a median regression model conditional on covariates, with unknown error distribution. The latter distribution is modeled semi-nonparametrically using orthonormal Legendre polynomials, similar to the approach in Bierens (2007). Given a parametric specification of the median function, we generate for each auction artificial bids conditional on the auction-specific covariates. Next, we take the difference of the empirical characteristic functions of the actual bids and the simulated bids, both jointly with the covariates, as the moment conditions. Integrating the squared difference of these empirical characteristic functions yields an integrated conditional moment (ICM) ob-

[1] Thus, asymmetry and risk-aversion is beyond the scope of this paper.

jective function, similar to the ICM test statistic proposed by Bierens (1982) and Bierens and Ploberger (1997). Minimizing this ICM objective function to the median regression parameters and the corresponding semi-nonparametric error distribution via a sieve method then yields a consistent estimator of the conditional value distribution. Similar to Bierens and Song (2007) we propose a data-driven sieve order selection procedure based on an information criterion. Moreover, the minimum value of the ICM objective function can be used as a test statistic of a consistent ICM test for the validity of the model, similar to Bierens (1990) and Bierens and Ploberger (1997).

The parametric specification of the median regression function of the log values is a matter of convenience rather than a necessity. Bierens and Song (2006) have shown that this median regression function is nonparametrically identified, provided that the errors of the model are independent of the covariates. Therefore, in principle it is possible to estimate the median regression function semi-nonparametrically as well.

Throughout the paper, we denote a random variable in upper-case and a non-random variable in lower-case. The indicator function is denoted by $\mathbf{I}(\cdot)$.[2] Almost sure (a.s.) convergence is denoted by $X_n \to X$ a.s.[3] Similarly, convergence in probability will be denoted by $X_n \to_p X$ or $p\lim_{n\to\infty} X_n = X$, and $X_n \to_d X$ indicates that X_n converges in distribution to X. In the case that X_n and X are random functions we use the notation $X_n \Rightarrow X$ to indicate that $X_n(\cdot)$ converges weakly to $X(\cdot)$. See for example Billingsley (1999) for the meaning of the notion of weak convergence.

2 Model and Data-Generating Process

2.1 The Equilibrium Bid Function

Given a vector X of auction-specific characteristics, let $F(v|X)$ be the conditional distribution of the private value V that each potential bidder has for the object to be auctioned off, and let

$$\underline{v}(X) = \inf_{F(v|X)>0} v$$

[2] $\mathbf{I}(True) = 1$, $\mathbf{I}(False) = 0$.

[3] This means that $P\left[\lim_{n\to\infty} X_n = X\right] = 1$.

be the lower bound of the support of $F(v|X)$. We do not restrict $\underline{v}(X)$ to be positive.[4] As is well-known, the equilibrium bid function of first-price sealed bid auctions where values are independent and private, and bidders are symmetric and risk-neutral, takes the form

$$\begin{aligned}\beta(v|X) &= v - \frac{1}{F(v|X)^{I(X)-1}} \int_{\max(p_0(X),\underline{v}(X))}^{v} F(y|X)^{I(X)-1} dy \qquad (1)\\ \text{for } v &> \max\left(p_0(X), \underline{v}(X)\right),\end{aligned}$$

where $I(X) \geq 2$ is the number of potential bidder[5] and $p_0(X)$ is the seller's reservation price. This is a unique symmetric Nash equilibrium for an actual bidder, i.e., a potential bidder whose private value V is greater or equal to the reservation price $p_0(X)$. See for example Riley and Samuelson (1981) or Krishna (2002). Note that we allow the number of potential bidders and the reservation price to depend on the auction-specific characteristics. We will assume that $p_0(X)$ and $I(X)$ are observed. Since we condition on X, we therefore do not need to bother about the functional form of $p_0(X)$ and $I(X)$.

Note that in the binding reservation price case, $p_0(X) > \underline{v}(X)$, the equilibrium bid function (1) can also be written as

$$\begin{aligned}\beta(v|X) &= v - \frac{v - p_0(X)}{F(v|X)^{I(X)-1}} \int_0^1 F\left(p_0(X) + u(v - p_0(X))|X\right)^{I(X)-1} du \qquad (2)\\ v &> p_0(X).\end{aligned}$$

Only those potential bidders whose private values V are greater or equal to the reservation price $p_0(X)$ will issue a bid $B = \beta(V|X)$. However, since the number of potential bidders $I(X)$ in each auction is considered to be known, we may without loss of generality assume that the non-bidders issue a zero bid:

$$B = \begin{cases} \beta(V|X) & \text{if } V \geq p_0(X), \\ 0 & \text{if } V < p_0(X). \end{cases}$$

Note that the number of zero bids has a Binomial $(I(X), F(p_0(X)|X))$ distribution, conditional on X.

[4] In the empirical auction literature it is usually assumed that the value distribution $F(v)$ has bounded support $[\underline{v}, \overline{v}]$, with $\underline{v} > 0$, $\overline{v} < \infty$. However, for our approach we only need the condition that the expectation of the values is finite. See Bierens and Song (2006, 2007).

[5] Which is assumed to be known to each potential bidder as well to the econometrician.

If the reservation price is not binding, that is, $p_0(X) \leq \underline{v}(X)$ or equivalently, $F(p_0(X)|X) = 0$, the equilibrium bid function simplifies to

$$\begin{aligned}
\beta(v|X) &= v - \frac{1}{F(v|X)^{I(X)-1}} \int_0^v F(y|X)^{I(X)-1} dy \\
&= v - \frac{v}{F(v|X)^{I(X)-1}} \int_0^1 F(u.v|X)^{I(X)-1} du, \\
v &> \underline{v}(X)
\end{aligned} \tag{3}$$

In this case each potential bidder with private value V will issue a bid $B = \beta(V|X)$.

2.2 Data-Generating Process

As argued before, we may without loss of generality assume that the potential bidders with a lower value than the reservation price issue a zero bid. Thus, for each auction $\ell = 1, ..., L$ which auction-specific covariates X_ℓ we observe $I_\ell = I(X_\ell)$ bids (including zero bids)

$$B_{\ell,j} = \begin{cases} \beta_0(V_{\ell,j}|X_\ell) & \text{if} \quad V_{\ell,j} \geq p_0(X_\ell) \\ 0 & \text{if} \quad V_{\ell,j} < p_0(X_\ell) \end{cases}, \ j = 1, ..., I_\ell, \tag{4}$$

where the values $V_{\ell,j}$, $j = 1, ..., I_\ell$ are independent random drawings from the true value distribution $F_0(v|X_\ell)$, and $\beta_0(v|X_\ell)$ is the corresponding true bid function. Conditional on X_ℓ the bids $B_{\ell,j}$, $j = 1, ..., I_\ell$ are independent.

We will also assume that the auctions themselves are independent. In particular,

Assumption 1. *The covariate vectors X_ℓ are independently and identically distributed as $X \in \mathbb{X} \subset \mathbb{R}^d$, where $\mathbb{X}$ is the support of X,*

so that all the bids $B_{\ell,j}$ are independently distributed.

2.3 Conditional Boundedness of the Bids

It has been shown by Bierens and Song (2006, Lemma 1) that if the value distribution is absolutely continuous then the support of the bid distribution is bounded if and only if the value distribution has a finite expectation. This result carries over to our case, conditional on X:

Lemma 1. *If conditional on the vector X of the auction-specific covariates the value distribution $F_0(v|X)$ is absolutely continuous then* $\sup_{v>0} \beta_0 (v|X) < \infty$ *if and only if* $\int_0^\infty v dF_0(v|X) < \infty$.

Proof: It follows from (1) and integration by parts that for $v \to \infty$,

$$\begin{aligned}\beta_0 (v|X) \quad \to \quad & \max (p_0 (X), \underline{v}(X)) F_0(\max (p_0 (X), \underline{v}(X)) |X)^{I(X)-1} \\ & + (I(X) - 1) \int_{\max(p_0(X),\underline{v}(X))}^{\infty} v F_0(v|X)^{I(X)-2} dF(v|X)\end{aligned}$$

The integral involved is bounded from above by $\int_0^\infty v dF_0(v|X)$ and bounded from below by $\left(\int_0^\infty v dF_0(v|X) - \int_0^M y dF_0(y|X)\right) .F_0(M|X)^{I(X)-2}$, for any $M > \max (p_0 (X), \underline{v}(X))$. Q.E.D.

The conditional boundedness of the actual bids is crucial for our approach, because the conditional bid distribution $\Lambda_0(b|X)$ has then finite conditional moments $\int_0^\infty b^n d\Lambda_0(b|X)$ of any order n. As is well-known, this implies that $\Lambda_0(b|X)$ is completely identified by the shape of its conditional characteristic function

$$\varphi_0(t|X) = \int_0^\infty \exp (i.t.b) \, d\Lambda_0(b|X), \; i = \sqrt{-1}, \tag{5}$$

in an arbitrary neighborhood of $t = 0$.

2.4 The conditional value distribution

To incorporate auction-specific heterogeneity in the conditional value distribution we need to put some structure on $F(v|X)$. We will do that by assuming that

Assumption 2. *There exists a function $\gamma(X)$ such that*

$$\ln V = \gamma(X) + \varepsilon, \tag{6}$$

where V is a random drawing from the true conditional value distribution $F_0(v|X)$. The random variable ε in (6) *is independent of X, and its distribution is absolutely continuous.*

Then

$$\begin{aligned} F_0(v|X) &= P\left[V \leq v|X\right] \\ &= P\left[\exp(\gamma(X) + \varepsilon) \leq v|X\right] \\ &= P\left[\exp(\varepsilon) \leq v\exp(-\gamma(X))|X\right] \\ &= \Gamma\left(v\exp(-\gamma(X))\right) \end{aligned} \tag{7}$$

for example, where Γ is the distribution of $\exp(\varepsilon)$: $\Gamma(x) = P[\exp(\varepsilon) \leq x]$.

Since without loss of generality we may add a constant to $\gamma(X)$ and subtract this constant from ε, we need to pin down the location of ε. For example, assume that $E[\varepsilon] = 0$, or that the median of ε is zero, $P\left[\varepsilon \leq 0\right] = 0.5$. Moreover it follows from Bierens and Song (2006) that a necessary condition for the nonparametric identification of the first-price auction model is that $E[V|X] = \exp\left(\gamma\left(X\right)\right) E\left[\exp\left(\varepsilon\right)\right] < \infty$, so that we need to require that $E\left[\exp\left(\varepsilon\right)\right] < \infty$. However, the latter condition does not guarantee that $E[\varepsilon]$ is finite; it is possible that $E[\varepsilon] = -\infty$ whereas $E\left[\exp\left(\varepsilon\right)\right] < \infty$.[6] Therefore we assume that

Assumption 3. $E\left[\exp\left(\varepsilon\right)\right] < \infty$, *and the median of* ε *in* (6) *is zero.*

Thus $\exp\left(\gamma\left(X\right)\right)$ is now the conditional median of V given X.

It follows trivially that

Lemma 2. *Under Assumptions 2 and 3 the true conditional value distribution* $F_0(v|X)$ *is absolutely continuous with density* $f_0(v|X)$ *and finite expectation* $\int v f_0(v|X)dv < \infty$.

Given the median function $\gamma\left(X\right)$, $F_0(v|X)$ is now determined by the distribution function $\Gamma(x) = P[\exp(\varepsilon) \leq x]$:

$$F_0(v|X) = \Gamma\left(v\exp\left(-\gamma\left(X\right)\right)\right)$$

Since by Assumption 2, $\Gamma(x)$ is absolutely continuous, it follows that, given an a priori chosen absolutely continuous distribution function $G(x)$ with support $(0, \infty)$, there exists an absolutely continuous distribution function

[6] For example, let $\varepsilon = \min\left(Z_1, Z_2/|Z_3|\right)$, where Z, Z_2 and Z_3 are independent $N(0,1)$ distributed. Note that $Z_2/|Z_3|$ is standard Cauchy distributed.

$H_0(u)$ on $[0,1]$ such that $\Gamma(x) = H_0(G(x))$, namely $H_0(u) = \Gamma(G^{-1}(u))$. Then

$$F_0(v|X) = H_0\left(G\left(v.\exp\left(-\gamma\left(X\right)\right)\right)\right)$$

For example, if we choose for $G(x)$ the standard exponential distribution

$$G(x) = 1 - \exp(-x),\ x \geq 0, \tag{8}$$

then

$$\begin{aligned} F_0(v|X) &= H_0\left(1 - \exp(-v\exp(-\gamma(X)))\right) & (9)\\ &= H_0(\overline{G}(v|X)) & (10) \end{aligned}$$

where

$$\overline{G}(v|X) = 1 - \exp(-v\exp(-\gamma(X))). \tag{11}$$

We may consider $\overline{G}(v|X)$ as an initial guess for $F(v|X)$, which is right if $H_0(u)$ is the uniform $[0,1]$ distribution: $H_0(u) = u$, and if wrong we can correct that by estimating $H_0(u)$ semi-nonparametrically, similar to the approach in Bierens (2007).

The median restriction $F_0\left(\exp\left(\gamma(X)\right)|X\right) = 1/2$ can be implemented by imposing the quantile restriction

$$H_0\left(G\left(1\right)\right) = 1/2. \tag{12}$$

If the reservation price is non-binding, this quantile restriction identifies $\gamma(X)$ nonparametrically because then $F_0(v|X)$ is completely identified by the conditional distribution of the bids. See Bierens and Song (2006). In the binding reservation price case $F_0(v|X)$ is only identified on $(p_0(X), \infty)$. However, if $P\left[p_0(X) < \exp\left(\gamma(X)\right)\right] = 1$ then $\gamma(X)$ is also nonparametrically identified. Thus, in principle, it is possible to estimate $\gamma(X)$ nonparametrically. Nevertheless, we will use a parametric specification for $\gamma(X)$, for example the linear specification $\gamma(X) = (1, X')\,\theta = \gamma_0\left(X, \theta\right)$, say. Thus,

Assumption 4. *Given an a priori chosen absolutely continuous distribution function G with density g and support $(0, \infty)$, the true conditional value distribution $F_0(v|X)$ and its density $f_0(v|X)$ take the forms*

$$\begin{aligned} F_0(v|X) &= H_0\left(G\left(v\exp\left(-\gamma_0\left(X,\theta_0\right)\right)\right)\right), & (13)\\ f_0(v|X) &= h_0\left(G\left(v\exp\left(-\gamma_0\left(X,\theta_0\right)\right)\right)\right) g\left(v\exp\left(-\gamma_0\left(X,\theta_0\right)\right)\right) \\ &\quad \times \exp\left(-\gamma_0\left(X,\theta_0\right)\right),\ \ \theta_0 \in \Theta, & (14) \end{aligned}$$

respectively, where

- $\gamma_0(x,\theta)$ *with* $(x,\theta) \in \mathbb{X} \times \Theta$ *is a parametric specification of the conditional median* $\gamma(X)$ *of* $\ln(V)$, *satisfying* $P[\gamma(X) = \gamma_0(X,\theta_0)] = 1$.
- $\Theta \subset \mathbb{R}^p$ *is a given compact parameter space for* θ *containing* θ_0.
- *For each* $x \in \mathbb{X}$, $\gamma_0(x,\theta)$ *is continuous on* Θ, *and for each* $\theta \in \Theta$, $\gamma_0(x,\theta)$ *is Borel measurable on* $\mathbb{X}$,
- θ_0 *is unique*: $P[\gamma(X) = \gamma_0(X,\theta)] < 1$ *for all* $\theta = \Theta \backslash \{\theta_0\}$.
- H_0 *is an absolutely continuous distribution function on* $[0,1]$ *with density* h_0, *satisfying the quantile restriction* (12).

3 Integrated Simulated Moments Estimation

3.1 Characteristic Functions as Moment Conditions

Recall that, since by Lemmas 1-2 the bids $B_{\ell,j}$ are bounded random variables, the actual conditional bid distribution

$$\Lambda_0(b|X_\ell) = P[B_{\ell,j} \le b|X_\ell]$$

is completely identified by the shape if its conditional characteristic function

$$\begin{aligned}\varphi_0(t|X_\ell) &= E\left[\frac{1}{I_\ell}\sum_{j=1}^{I_\ell}\exp(i.t.B_{\ell,j})\middle| X_\ell\right] \\ &= \int_{p_0(X_\ell)}^{\infty}\exp(i.t.\beta_0(v|X_\ell))\,dF_0(v|X_\ell) + F_0(p_0(X_\ell)|X_\ell)\end{aligned}$$

in an arbitrary neighborhood of $t = 0$.

Let $F(v|X)$ be a potential candidate (henceforth called a *candidate conditional value distribution*) for the true conditional value distribution $F_0(v|X)$. Similar to Assumption 4 we assume that

Assumption 5. *The candidate conditional value distributions take the form*

$$F(v|X,H,\theta) = H(G(v\exp(-\gamma_0(X,\theta)))),\quad (H,\theta) \in \mathcal{H} \times \Theta, \tag{15}$$

where

- *G, $\gamma_0(x,\theta)$, Θ are the same as in Assumption 4.*
- *$\mathcal{H}$ is a compact metric space of absolutely continuous distribution functions H on $[0,1]$ endowed with the* sup *metric*

$$\|H_1 - H_2\| = \sup_{0 \le u \le 1} |H_1(u) - H_2(u)|,$$

  containing H_0.
- *Each $H \in \mathcal{H}$ satisfies the quantile restriction*

$$H(G(1)) = 1/2. \tag{16}$$

For each auction ℓ, draw a random sample $\widetilde{V}_{\ell,1}, ..., \widetilde{V}_{\ell,I_\ell}$ from $F(v|X_\ell, H, \theta)$,[7] and generate simulated bids $\widetilde{B}_{\ell,j}$ similar to (4) by:

$$\widetilde{B}_{\ell,j} = \begin{cases} \beta(\widetilde{V}_{\ell,j}|X_\ell) & \text{if } \widetilde{V}_{\ell,j} \ge p_0(X_\ell) \\ 0 & \text{if } \widetilde{V}_{\ell,j} < p_0(X_\ell) \end{cases}, \quad j = 1, ..., I_\ell. \tag{17}$$

Then

$$\begin{aligned} \varphi(t|X_\ell, H, \theta) &= E\left[\frac{1}{I_\ell} \sum_{j=1}^{I_\ell} \exp\left(i.t.\widetilde{B}_{\ell,j}\right) \middle| X_\ell \right] \\ &= \int_{p_0(X_\ell)}^{\infty} \exp(i.t.\beta(v|X_\ell))\, dF(v|X_\ell, H, \theta) + F(p_0(X_\ell)|X_\ell, H, \theta) \end{aligned}$$

is the conditional characteristic function of $\widetilde{B}_{\ell,j}$, which depend on $(H,\theta) \in \mathcal{H} \times \Theta$ via (15).

Lemma 3. *Let Assumptions 1-5 hold. Then $\|H - H_0\| = 0$ and $\theta = \theta_0$ if and only if $\varphi(t|X, H, \theta) = \varphi_0(t|X)$ a.s. for all t in an arbitrary open interval.*

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[7] The method for generating these simulated values will be discussed in the next section.

Proof: First, let the arbitrary open interval involved be $(-\kappa, \kappa)$ for some $\kappa > 0$. Since conditional on X_ℓ, $\varphi_0(t|X_\ell)$ is the characteristic function of a bounded random variable $B_{\ell,1}$ we can write

$$\varphi_0(t|X_\ell) = E\left[\exp(i.t.B_{\ell,1})|X_\ell\right] = \sum_{m=0}^{\infty} \frac{i^m t^m}{m!} E\left[B_{\ell,1}^m|X_\ell\right].$$

Recall that $\varphi(t|X_\ell, H, \theta)$ is the conditional characteristic function of the simulated bid $\widetilde{B}_{\ell,1}$. Then the equality $\varphi(t|X_\ell, h, \theta) = \varphi_0(t|X_\ell)$ a.s. for all $t \in (-\kappa, \kappa)$ implies that

$$\begin{aligned} \partial^m \varphi(t|X_\ell, H, \theta) / (\partial t)^m\big|_{t=0} &= \partial^m \varphi_0(t|X_\ell)(\partial t)^m\big|_{t=0} \\ &= i^m E\left[B_{\ell,1}^m|X_\ell\right] \end{aligned}$$

a.s. for all $m \geq 0$, which in its turn implies that $E\left[\widetilde{B}_{\ell,1}^m|X_\ell\right] = E\left[B_{\ell,1}^m|X_\ell\right]$ a.s. for all $m \geq 0$, so that

$$\varphi_0(t|X_\ell) = \varphi(t|X_\ell, H, \theta) = \sum_{m=0}^{\infty} \frac{i^m t^m}{m!} E\left[B_{\ell,1}^m|X_\ell\right]$$

for all $t \in \mathbb{R}$.

Next, suppose that $\varphi(t|X_\ell, H, \theta) = \varphi_0(t|X_\ell)$ a.s. for all $t \in (t_* - \kappa, t_* + \kappa)$, where $t_* \in \mathbb{R}$ and $\kappa > 0$ are arbitrary. Then by the same argument as for the case $t_* = 0$,

$$\varphi_0(t|X_\ell) = \varphi(t|X_\ell, H, \theta) = \sum_{m=0}^{\infty} \frac{i^m (t - t_*)^m}{m!} E\left[\exp(i.t_* B_{\ell,1}) B_{\ell,1}^m|X_\ell\right]$$

for all $t \in \mathbb{R}$. The result involved now follows straightforwardly from the well-known fact that distributions are equal if and only if their characteristic functions are equal. Q.E.D.

The next result is a straightforward corollary of Theorem 1 in Bierens and Ploberger (1997):

Lemma 4. *Let $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ be a bounded one-to-one mapping. Suppose that for some (possibly random) $t \in \mathbb{R}$, $P[\varphi(t|X, H, \theta) = \varphi_0(t|X)] < 1$. Then the set $\left\{\varsigma \in \mathbb{R}^d : E\left[(\varphi(t|X, H, \theta) - \varphi_0(t|X)) \exp(i.\varsigma'\Phi(X))\right] = 0\right\}$ has Lebesgue measure zero and is nowhere dense in $\mathbb{R}^d$.*

Of course, if X is already bounded we may choose for Φ the identity matrix I_d.

3.2 The Objective Function

Denote

$$\overline{Q}(H,\theta)=\frac{1}{\mu(\Xi)}\int_{\Xi}|\psi(\xi|H,\theta)|^{2}d\xi, \tag{18}$$

where

$$\begin{aligned}\psi(\xi|H,\theta) &= E\left[\left(\varphi(t|X,H,\theta)-\varphi_{0}(t|X)\right)\exp\left(i.\varsigma'\Phi(X)\right)\right],\\ \xi &= \begin{pmatrix} t\\ \varsigma\end{pmatrix}\in\Xi\subset\mathbb{R}^{d+1},\ \mu(\Xi)=\int_{\Xi}1.d\xi>0.\end{aligned}$$

with $\Xi\subset\mathbb{R}^{d+1}$ a set with positive Lebesgue measure $\mu(\Xi)$. Then Lemmas 3 and 4 imply that

$$(H_{0},\theta_{0})=\arg\min_{(H,\theta)\in\mathcal{H}\times\Theta}\overline{Q}(H,\theta)\ \text{ is unique.}$$

This suggests to estimate (H_0,θ_0) by the simulated integrated conditional moments method:

$$\left(\widehat{H},\widehat{\theta}\right)=\arg\min_{(H,\theta)\in\mathcal{H}\times\Theta}\widehat{Q}(H,\theta). \tag{19}$$

where

$$\widehat{Q}(H,\theta)=\frac{1}{\mu(\Xi)}\int_{\Xi}\left|\widehat{\psi}(\xi|H,\theta)\right|^{2}d\xi \tag{20}$$

with

$$\begin{aligned}\widehat{\psi}(\xi|H,\theta) &= \frac{1}{L}\sum_{\ell=1}^{L}\left(\widetilde{\varphi}_{\ell}(t|H,\theta)-\widehat{\varphi}_{\ell}(t)\right)\exp\left(i.\varsigma'\Phi(X_{\ell})\right),\\ \widehat{\varphi}_{\ell}(t) &= \frac{1}{I_{\ell}}\sum_{j=1}^{I_{\ell}}\exp\left(i.t.B_{\ell,j}\right),\\ \widetilde{\varphi}_{\ell}(t|H,\theta) &= \frac{1}{I_{\ell}}\sum_{j=1}^{I_{\ell}}\exp\left(i.t.\widetilde{B}_{\ell,j}\right).\end{aligned}$$

Note that if we choose

$$\Xi=[-\kappa,\kappa]^{d+1}\ \text{ for a }\kappa>0, \tag{21}$$

the function $\widehat{Q}(H,\theta)$ has a closed form. See Bierens and Song (2007).

Of course, the estimator (19) is not feasible because the metric space $\mathcal{H}$ is infinite-dimensional. Therefore, the actual estimation will be done by sieve estimation, discussed in the next subsection.

3.3 Sieve Estimation

The idea of sieve estimation[8] is to construct an increasing sequence of subspaces $\mathcal{H}_n$ of $\mathcal{H}$ such that the computation of the sieve estimator

$$\left(\widetilde{H}_n, \widetilde{\theta}_n\right) = \arg \min_{(H,\theta)\in\mathcal{H}_n\times\Theta} \widehat{Q}\left(h,\theta\right). \tag{22}$$

is feasible, and with $n = n_L \to \infty$ as $L \to \infty$, $\left(\widetilde{H}_{n_L}, \widetilde{\theta}_{n_L}\right)$ is strongly consistent.

The latter can be shown by proving that the conditions (23), (24) and (25) in the following theorem hold:

Theorem 1. *Let n_L be an arbitrary subsequence of L such that $\lim_{L\to\infty} n_L = \infty$. Under Assumptions 1-5 and the conditions*

$$\overline{Q}\left(H,\theta\right) \text{ is continuous on } \mathcal{H}\times\Theta, \tag{23}$$

$$P\left[\lim_{L\to\infty} \sup_{(H,\theta)\in\mathcal{H}\times\Theta} \left|\widehat{Q}\left(H,\theta\right) - \overline{Q}\left(H,\theta\right)\right| = 0\right] = 1, \tag{24}$$

$$\left\{\mathcal{H}_n\right\}_{n=1}^{\infty} \text{ is dense in } \mathcal{H}, \tag{25}$$

the sieve estimator $\left(\widetilde{H}_{n_L}, \widetilde{\theta}_{n_L}\right)$ is strongly consistent:

$$\sup_{0\leq u\leq 1} \left|\widetilde{H}_{n_L}(u) - H_0(u)\right| \to 0 \text{ a.s.}, \widetilde{\theta}_{n_L} \to \theta_0 \text{ a.s.}$$

Proof: Bierens and Song (2007, Theorem 4).

Note that condition (25) means that $\mathcal{H} = \overline{\cup_{n=1}^{\infty}\mathcal{H}_n}$, where the bar indicates the closure. This condition holds if for each $H \in \mathcal{H}$ there exists a sequence of distribution functions $H_n \in \mathcal{H}_n$ such that

$$\lim_{n\to\infty} \sup_{0\leq u\leq 1} \left|H_n\left(u\right) - H\left(u\right)\right| = 0. \tag{26}$$

[8] See for example Chen (2004) and the references therein.

4 Continuity of $\overline{Q}(H,\theta)$

4.1 Generation of Simulated Values and Bids

There are various ways to generate random drawing $\widetilde{V}_{\ell,j}$ from a candidate value distribution $F(v|X_\ell,H,\theta)$. A convenient way is the well-known accept-reject method. See, for example, Devroye (1986) and Rubinstein (1981). However, it is difficult to prove that then condition (23) holds. Following Bierens and Song (2007), we will therefore assume that the random drawing $\widetilde{V}_{\ell,j}$ are generated as follows.

Assumption 6. *For $\ell = 1,....,L$, draw a random sample $\widetilde{U}_{\ell,1},...,\widetilde{U}_{\ell,I_\ell}$ from the uniform $[0,1]$ distribution, and compute for each candidate value distribution $F(v|X_\ell,H,\theta)$ defined in Assumption 5,*

$$\widetilde{V}_{\ell,j} = \exp\left(\gamma_0\left(X,\theta\right)\right).G^{-1}\left(H^{-1}\left(\widetilde{U}_{\ell,j}\right)\right). \tag{27}$$

Then it follows from (15) that $\widetilde{U}_{\ell,j} = F\left(\widetilde{V}_{\ell,j}|X_\ell,H,\theta\right)$, which implies that $\widetilde{V}_{\ell,j}$ is a random drawing from $F(v|X_\ell,H,\theta)$. The computation of $H^{-1}\left(\widetilde{U}_{\ell,j}\right)$ can be done numerically, and G can be chosen such that G^{-1} has a closed form.

4.2 Continuity of the Simulated Values and Bids

The simulation procedure in Assumption 6 has the advantage that it is easier to prove that the simulated values and bids involved are continuous in H and θ, in the following sense:

Lemma 5. *Let $F\left(v|X,H_n,\theta_n\right)$ and $F\left(v|X,H,\theta\right)$ be candidate value distributions conditional on X such that*

$$\lim_{n\to\infty}\sup_{0\le u\le 1}\left|H_n\left(u\right)-H\left(u\right)\right| = 0,\ \ \lim_{n\to\infty}\theta_n = \theta. \tag{28}$$

For a given random drawing $\widetilde{U}$ from the uniform $[0,1]$ distribution, let $\widetilde{V}_n$ and $\widetilde{V}$ be the solutions of $F\left(\widetilde{V}_n|X,H_n,\theta_n\right) = \widetilde{U}$ and $F\left(\widetilde{V}|X,H,\theta\right) = \widetilde{U}$, *respectively. Then under Assumptions 1-5,*

$$P\left[\lim_{n\to\infty}\widetilde{V}_n=\widetilde{V}\right]=1 \tag{29}$$

Consequently, the corresponding simulated bids $\widetilde{B}_n$ *and* $\widetilde{B}$ *satisfy*

$$P\left[\lim_{n\to\infty}\widetilde{B}_n=\widetilde{B}\right]=1 \tag{30}$$

as well.

Proof: It follows from (27) that

$$\begin{aligned}\widetilde{V}_n &= -\exp\left(\gamma_0\left(X,\theta_n\right)\right).G^{-1}\left(H_n^{-1}\left(\widetilde{U}\right)\right),\\ \widetilde{V} &= -\exp\left(\gamma_0\left(X,\theta\right)\right).G^{-1}\left(H^{-1}\left(\widetilde{U}\right)\right)\end{aligned}$$

Let $U_n=H_n^{-1}\left(\widetilde{U}\right)$ and $U=H^{-1}\left(\widetilde{U}\right)$,so that

$$H_n\left(U_n\right)=H(U)=\widetilde{U}. \tag{31}$$

Since by (28),

$$\left|H_n\left(U_n\right)-H(U_n)\right|\to 0 \text{ a.s.}$$

it follows from (31) that $H(U_n)\to H(U)$ a.s., which by the continuity of $H(u)$ implies that $U_n\to U$ a.s., hence

$$G^{-1}\left(H_n^{-1}\left(\widetilde{U}\right)\right)\to G^{-1}\left(H^{-1}\left(\widetilde{U}\right)\right) \text{ a.s.} \tag{32}$$

Moreover, it follows trivially from (28) and the continuity of $\gamma_0\left(X,\theta\right)$ that

$$\exp\left(\gamma_0\left(X,\theta_n\right)\right)\to\exp\left(\gamma_0\left(X,\theta\right)\right) \text{ a.s.} \tag{33}$$

The result (29) now follows from (32) and (33).

It follows from (2) and (3) that the simulated bids $\widetilde{B}_n$ and $\widetilde{B}$ can be generated by, respectively,

$$\begin{aligned}\widetilde{B}_n &= \mathbf{I}\left(\widetilde{U}>F\left(p_0(X)|X,H_n,\theta_n\right)\right).\left[\widetilde{V}_n-\left(\widetilde{V}_n-p_0(X)\right)\widetilde{U}^{1-I(X)}\right.\\ &\quad\left.\times\int_0^1 F\left(p_0(X)+u\left(\widetilde{V}_n-p_0(X)\right)|X,H_n,\theta_n\right)^{I(X)-1}du\right]\\ \widetilde{B} &= \mathbf{I}\left(\widetilde{U}>F\left(p_0(X)|X,H,\theta\right)\right)\left[\widetilde{V}-\left(\widetilde{V}-p_0(X)\right)\widetilde{U}^{1-I(X)}\right. .\\ &\quad\left.\times\int_0^1 F\left(p_0(X)+u\left(\widetilde{V}-p_0(X)\right)|X,H,\theta\right)^{I(X)-1}du\right]\end{aligned}$$

in the binding reservation price case, and by

$$\begin{aligned}\widetilde{B}_n &= \widetilde{V}_n.\left(1-\widetilde{U}^{1-I}\int_0^1 F\left(u.\widetilde{V}_n|X,H_n,\theta_n\right)^{I(X)-1}du\right),\\ \widetilde{B} &= \widetilde{V}.\left(1-\widetilde{U}^{1-I}\int_0^1 F\left(u.\widetilde{V}|X,H,\theta\right)^{I(X)-1}du\right),\end{aligned}$$

in the non-binding case. Moreover, it follows straightforwardly from (28) and (29) that

$$F\left(p_0(X)|X,H_n,\theta_n\right) \to F\left(p_0(X)|X,H,\theta\right) \text{ a.s.}$$

and pointwise in $u \in [0,1]$,

$$\begin{aligned}&F\left(p_0(X)+u\left(\widetilde{V}_n-p_0(X)\right)|X,H_n,\theta_n\right)\\ &\to F\left(p_0(X)+u\left(\widetilde{V}-p_0(X)\right)|X,H,\theta\right) \text{ a.s.}\end{aligned}$$

so that (30) follows from the bounded convergence theorem. Q.E.D.

The results of Lemma 5 now imply that:

Theorem 2. *Under Assumptions 1-6 the conditions* (23) *and* (24) *in Theorem 1 hold.*

Proof: The continuity condition (23) follows straightforwardly from Lemma 5, and condition (24) is not too hard to verify from Lemma 5 and Bierens and Song (2007, Theorem 1). Q.E.D.

5 The Compact Metric Space $\mathcal{H}$ and its Sieve Spaces

5.1 The Space $\mathcal{H}$

It has been shown by Bierens (2007) that any density function $h(u)$ on $[0,1]$ can be written as

$$h(u) = \frac{(1+\sum_{k=1}^{\infty}\delta_k\rho_k(u))^2}{1+\sum_{k=1}^{\infty}\delta_k^2} \text{ a.e. on } [0,1] \qquad (34)$$

where $\sum_{k=1}^{\infty} \delta_k^2 < \infty$ and the $\rho_k(u)$'s are orthonormal Legendre polynomials of order k. These polynomials can be constructed recursively by the three-term recursive relation

$$\rho_k(u) = \frac{\sqrt{2k-1}\sqrt{2k+1}}{n}(2u-1)\rho_{k-1}(u) - \frac{(k-1)\sqrt{2k+1}}{k\sqrt{2k-3}}\rho_{k-2}(u)$$

for $k \geq 2$, starting from $\rho_0(u) = 1$, $\rho_1(u) = \sqrt{3}(2u-1)$.

The standard consistency proof for parameter estimators of nonlinear parametric models requires that the parameters are confined to a compact subset of a Euclidean space. Since the density h in (18) plays the role of unknown parameter, we need to construct a compact metric space of densities on the unit interval. This can be done by imposing restrictions on the parameters δ_k in (34), as follows.

Theorem 3. *Let $\mathcal{D}$ be the space of density function $h(u)$ on $[0,1]$ of the form (34), where the parameters δ_k are restricted by the inequality*

$$|\delta_k| \leq c\left(1 + \sqrt{k}\ln k\right)^{-1}, \; k = 1, 2, 3, ... \tag{35}$$

for an a priori chosen constant $c > 0$. If we endow $\mathcal{D}$ with the L^1 metric

$$||h_1 - h_2||_1 = \int_0^1 |h_1(u) - h_2(u)|\, du, \tag{36}$$

then $\mathcal{D}$ is a compact metric space. Consequently, the corresponding space of absolutely continuous distribution functions on $[0,1]$,

$$\mathcal{H} = \left\{ H(u) = \int_0^u h(x)dx, \; h \in \mathcal{D} \right\},$$

endowed with the "sup" metric $\sup_{0 \leq u \leq 1} |H_1(u) - H_2(u)|$, is a compact metric space as well.

Proof: Bierens (2007).

Of course, we need to assume that

Assumption 7. *The constant c in (35) is chosen so large that $h_0 \in \mathcal{D}$, so that $H_0 \in \mathcal{H}$.*

5.2 The Sieve Spaces $\mathcal{H}_n$

For a density function $h(u)$ (34) and its associated parameter sequence $\{\delta_k\}_{k=1}^{\infty}$, let

$$h_n(u) = h(u|\boldsymbol{\delta}_\mathbf{n}) = \frac{(1 + \sum_{k=1}^{n} \delta_k \rho_k(u))^2}{1 + \sum_{k=1}^{n} \delta_k^2}, \ \boldsymbol{\delta}_\mathbf{n} = (\delta_1, \ldots, \delta_n)', \tag{37}$$

be the n-th order truncation of $h(u)$. The case $n = 0$ corresponds to the uniform density: $h_0(u) = 1$. Following Gallant and Nychka (1987) we will call this truncated density a SNP density function.

It has been shown by Bierens (2007) that

$$\lim_{n\to\infty} \int_0^1 |h_n(u) - h(u)|\, du = 0. \tag{38}$$

Therefore,

Theorem 4. *Let $\mathcal{D}_n$ be the space of all densities of the type* (37), *subject to the same restrictions on the δ_k's as in Theorem 3. Then $\{\mathcal{D}_n\}_{n=1}^{\infty}$ is dense in $\mathcal{D}$. Consequently, defining*

$$\mathcal{H}_n = \left\{ H_n(u) = \int_0^u h_n(v)du, \ h_n \in \mathcal{D}_n \right\} \tag{39}$$

it follows that $\{\mathcal{H}_n\}_{n=1}^{\infty}$ is dense in $\mathcal{H}$.

Note that the distribution functions $H_n(u)$ can be computed by the method proposed in Bierens (2007).

Since the density functions in $\mathcal{D}_n$ and distributions functions in $\mathcal{H}_n$ are parametric, with parameters $\boldsymbol{\delta}_\mathbf{n} = (\delta_1, \ldots, \delta_n)'$, the computation of the sieve estimator (22) is feasible. In particular, the parameter vector $\widetilde{\boldsymbol{\delta}}_\mathbf{n}$ for which $\widetilde{h}_n(u) = h\left(u|\widetilde{\boldsymbol{\delta}}_\mathbf{n}\right)$, together with $\widetilde{\theta}_n$, can be computed via the simplex method of Nelder and Mead (1965), penalized for violations of the restrictions (35) and the quantile restriction (16).

5.3 Strong Consistency of the Sieve Estimator

Summarizing, we have shown that

Theorem 5. *Under Assumptions 1-7 all the conditions of Theorem 1 hold, so that the sieve estimator* $\left(\widetilde{H}_{n_L}, \widetilde{\theta}_{n_L}\right)$ *is strongly consistent.*

6 An ICM Test of the Validity of First-Price Auction Models with Heterogeneity

If the assumptions of symmetric independent private values with risk neutral bidders do not hold, the bid functions (1) and (3) no longer apply to the actual bids. The same applies if the functional form of the median function $\gamma_0(X, \theta)$ is misspecified. If so,

$$\widehat{Q}\left(\widetilde{H}_{n_L}, \widetilde{\theta}_{n_L}\right) \to \inf_{(H,\theta)\in\mathcal{H}\times\Theta} \overline{Q}(H, \theta) > 0 \text{ a.s.}, \tag{40}$$

This suggests to use $\widehat{Q}\left(\widetilde{H}_{n_L}, \widetilde{\theta}_{n_L}\right)$ as a basis for a consistent ICM test of the null hypothesis that

H_0: *the actual bids come from a first-price sealed bid auctions with auction-specific heterogeneity where values are symmetric, independent, private and bidders are risk-neutral, and the functional specification of the median function* $\gamma_0(X, \theta)$ *is correct,*

against the general alternative that

H_1: *the null hypothesis* H_0 *is false.*

The ICM test we will propose is based on the fact that similar to the results in Bierens (1990) and Bierens and Ploberger (1997), the following results hold.

Theorem 6. *Let* $\Xi \subset \mathbb{R}^{d+1}$ *be compact. Under* H_0,

$$\widehat{W}_L(.) = \sqrt{L}\widehat{\psi}(\xi|H_0, \theta_0) \Rightarrow W(.)$$

on Ξ, *hence*[9]

$$L.\widehat{Q}(H_0,\theta_0) = \frac{1}{\mu(\Xi)}\int_{\Xi}\left|\widehat{W}_L(\xi)\right|^2 d\xi \to_d \frac{1}{\mu(\Xi)}\int_{\Xi}|W(\xi)|^2 d\xi,$$

where $W(\zeta)$ *is a complex-valued zero-mean Gaussian process on* Ξ *with covariance function* $\Gamma(\xi_1,\xi_2) = E\left[\widehat{W}_L(\xi_1)\overline{\widehat{W}_L(\xi_2)}\right]$. *Under* H_1, (40) *holds.*

Proof: Similar to Bierens and Song (2007, Theorem 5).

Note that the result in Theorem 6 does **not** imply that

$$L.\widehat{Q}\left(\widetilde{H}_{n_L},\widetilde{\theta}_{n_L}\right) \to_d \frac{1}{\mu(\Xi)}\int_{\Xi}|W(\xi)|^2 d\xi.$$

However, if

Assumption 8. *The distribution* H_0 *is of the SNP type itself*: $H_0 \in \bigcup_{n=0}^{\infty}\mathcal{H}_n$,

then for $L \to \infty$,

$$L.\widehat{Q}\left(\widetilde{H}_{n_L},\widetilde{\theta}_{n_L}\right) \leq L.\widehat{Q}(H_0,\theta_0)$$

so that upper bounds of the critical values of the test can be based on the limiting distribution of the upper bound $L.\widehat{Q}(H_0,\theta_0)$. These critical values can be derived by a bootstrap approach, similar to Bierens and Song (2007).

7 Determination of the Sieve Order via an Information Criterion

Under Assumption 8 there exists a smallest natural number n_0 such that $H_0 \in \mathcal{H}_{n_0}$. Similar to Bierens and Song (2007) we propose to estimate n_0 by minimizing a criterion function of the type

$$\begin{aligned}\widetilde{C}_L(n) &= \inf_{(H,\theta)\in\mathcal{H}_n\times\Theta}\widehat{Q}(H,\theta) + \Phi(n).\frac{\phi(L)}{L}, \\ \phi(L) &= o(L),\ \lim_{L\to\infty}\phi(L) = \infty.\end{aligned} \tag{41}$$

[9] By the continuous mapping theorem.

where $\Phi(n)$ is an increasing but bounded function of n. For example, let for some $\alpha \in (0,1)$,

$$\Phi(n) = 1 - (n+1)^{-\alpha}.$$

Then similar to the Hannan-Quinn and Schwarz information criteria we have:

Theorem 6. *Let* $\widetilde{n}_L = \max_{s.t.\ \widetilde{C}_L(n) \leq \widetilde{C}_L(n-1)} n$ *and*

$$\left(\widetilde{H}, \widetilde{\theta}\right) = \arg \min_{(H,\theta) \in \mathcal{F}_{\widetilde{n}_L} \times \Theta} \widehat{Q}(H, \theta).$$

Under Assumption 8, $\lim_{L\to\infty} P\left[\widetilde{n}_L = n_0\right] = 1$, *hence*

$$\sup_{0 \leq u \leq 1} \left|\widetilde{H}(u) - H_0(u)\right| \to 0 \ a.s. \ and \ \widetilde{\theta} \to \theta_0 \ a.s.$$

If Assumption 8 is not true then $p \lim_{L\to\infty} \widetilde{n}_L = \infty$, *hence*

$$p \lim_{L\to\infty} \sup_{0 \leq u \leq 1} \left|\widetilde{H}(u) - H_0(u)\right| = 0 \ and \ p \lim_{L\to\infty} \widetilde{\theta} = \theta_0.$$

Proof: Bierens and Song (2007, Theorem 6).

8 Application to the USFS Timber Auctions

To be completed....

9 Concluding Remarks

To be completed....

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