Semi-Nonparametric Estimation of First-Price Auctions Models with Auction-Specific Heterogeneity via an Integrated Simulated Conditional Moments Method

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1 The equilibrium bid function

Given a vector X of auction-specific characteristics, let

$$F_0(v|X)$$

be the conditional distribution of the private value V that each potential bidder has for the object to be auctioned off.

The equilibrium bid function of first-price sealed bid auctions where values are independent and private, and bidders are symmetric and risk-neutral, takes the form

$$\beta_0\left(v|X\right) = v - F_0(v|X)^{1-I(X)} \int_{p_0(X)}^{v} F_0(y|X)^{I(X)-1} dy$$

for $v > p_0\left(X\right)$, where $I(X) \geq 2$ is the (observable) number of potential bidder and $p_0(X)$ is the seller's reservation price.

However, in this presentation I will assume that there is no reservation price, so that

$$\beta_0\left(v|X\right) = v - F_0(v|X)^{1-I(X)} \int_{0}^{v} F_0(y|X)^{I(X)-1} dy.$$

2 Data-generating process

Thus, for each auction $\ell = 1, ..., L$ which auction-specific covariates X_ℓ we observe $I_\ell = I\left(X_\ell\right)$ bids

$$\begin{aligned} B_{\ell,j} &= \beta_0(V_{\ell,j}|X_\ell) \\ &= V_{\ell,j} - F_0(V_{\ell,j}|X)^{1-I_\ell}\int_0^{V_{\ell,j}} F_0(y|X_\ell)^{I_\ell - 1}dy, \\ j &= 1, ..., I_\ell, \end{aligned}$$

where the values $V_{\ell,j},\ j = 1, ..., I_\ell$ are independent random drawings from the true value distribution $F_0(v|X_\ell)$, and $\beta_0(v|X_\ell)$ is the corresponding true bid function.

Moreover, we will assume that the covariate vectors $X_\ell \in \mathbb{R}^d$ are independently and identically distributed, so that all the bids $B_{\ell,j}$ are independently distributed.

3 The conditional value distribution

To incorporate auction-specific heterogeneity in the conditional value distribution we need to put some structure on $F_0(v|X)$.

We will do that by assuming that the values $V_{\ell,j}$ in auction ℓ are related to the auction-specific covariates X_ℓ by a median regression model,

$$\ln\left(V_{\ell,j}\right) = \gamma\left(X_\ell\right) + \varepsilon_{\ell,j},,\ j = 1, ..., I_\ell,$$

where the error terms $\varepsilon_{\ell,j}$ are independent random drawings from an absolutely continuous distribution with median zero, and are independent of X_ℓ.

Then

$$F_0(v|X) = \Gamma\left(v \exp(-\gamma(X))\right)$$

where Γ is the distribution function of $\exp(\varepsilon_{\ell,j})$:

$$\Gamma\left(x\right) = \Pr\left[\exp(\varepsilon_{\ell,j}) \leq x\right],\ \Gamma\left(e\right) = 1/2.$$

The condition $\Gamma\left(e\right) = 1/2$ is the median restriction.

Given an a priori chosen absolutely continuous distribution function $G(x)$ with support $(0, \infty)$, there exists an absolutely continuous distribution function $H_0(u)$ on $[0, 1]$ such that

$$\Gamma(x) = H_0 \left(G(x) \right).$$

Then

$$F_0(v|X) = H_0 \left(G \left(v. \exp \left(-\gamma \left(X \right) \right) \right) \right)$$

The median restriction then becomes

$$H_0 \left(G(e) \right) = 1/2.$$

It has been shown in the paper that under this median restriction, and the condition that the reservation price is non-binding, the distribution function $H_0 \left(u \right)$ and the median function $\gamma \left(X \right)$ are nonparametrically identified, hence, given the a priori chosen distribution function $G(x)$ the conditional value distribution $F_0(v|X)$ is nonparametrically identified.

Thus, in principle, it is possible to estimate $\gamma(X)$ nonparametrically.

Nevertheless, we will use a parametric specification for $\gamma(X)$, in particular the linear specification

$$\begin{aligned}\gamma(X) &= (1, X')\,\theta_0 = \gamma_0\,(X, \theta_0)\,,\\ \theta_0 &\in \Theta \subset \mathbb{R}^{d+1},\end{aligned}$$

where Θ is a given parameter space.

Thus, it will be assumed that the true conditional value distribution takes the form

$$F_0(v|X) = H_0\,(G\,(v.\exp\,(-\gamma_0\,(X, \theta_0))))$$

where the distribution function G on $(0, \infty)$ and the conditional median function $\gamma_0\,(x, \theta)$ are chosen.

The vector θ_0 and the distribution function $H_0\,(u)$ on $[0, 1]$ are the unknown ”parameters” to be estimated.

4 Moment conditions

If the values have a finite conditional expectation,

$$\int_0^\infty v dF_0(v|X_\ell) < \infty,$$

then conditional on X_ℓ the bids $B_{\ell,j}$ in auction ℓ are bounded random variables. The actual conditional bid distribution

$$\Lambda_0\left(b|X_\ell\right) = P\left[B_{\ell,j} \leq b|X_\ell\right]$$

is then completely identified by the shape if its conditional characteristic function

$$\begin{aligned}\varphi_0\left(t|X_\ell\right) &= E\left[\left.\frac{1}{I_\ell}\sum_{j=1}^{I_\ell}\exp\left(i.t.B_{\ell,j}\right)\right| X_\ell\right] \\ &= \int_0^\infty \exp\left(i.t.\beta_0(v|X_\ell)\right) dF_0\left(v|X_\ell\right)\end{aligned}$$

in an arbitrary neighborhood of $t = 0$.

Recall that

$$F_0(v|X) = H_0\left(G\left(v.\exp\left(-\gamma_0\left(X,\theta_0\right)\right)\right)\right)$$

where $\gamma_0\left(X,\theta_0\right)$ is the conditional median of $\ln V$. Let

$$\begin{aligned} F(v|X,H,\theta) &= H\left(G\left(v\exp(-\gamma_0\left(X,\theta\right))\right)\right), \\ (H,\theta) &\in \mathcal{H}\times\Theta, \end{aligned}$$

be a candidate conditional value distribution, where

- Θ is a compact parameter space containing θ_0,
- $\mathcal{H}$ is a compact metric space of absolutely continuous distribution functions H on $[0,1]$ endowed with the $\sup$ metric
$$\|H_1 - H_2\| = \sup_{0\le u\le 1}\left|H_1\left(u\right) - H_2\left(u\right)\right|,$$
containing H_0,
- Each $H\in\mathcal{H}$ satisfies the quantile restriction
$$H\left(G(1)\right) = 1/2.$$

Given a pair $(H,\theta)\in\mathcal{H}\times\Theta$, for each auction ℓ draw a random sample $\widetilde{V}_{\ell,1}, ..., \widetilde{V}_{\ell,I_\ell}$ from $F(v|X_\ell,H,\theta)$, and generate simulated bids $\widetilde{B}_{\ell,j}$ by:

$$\begin{aligned}
\widetilde{B}_{\ell,j} &= \beta\left(\widetilde{V}_{\ell,j}|X_\ell,H,\theta\right) \\
&= \widetilde{V}_{\ell,j} - F(\widetilde{V}_{\ell,j}|X_\ell,H,\theta)^{1-I_\ell}\int_0^{\widetilde{V}_{\ell,j}} F(y|X_\ell,H,\theta)^{I_\ell-1}dy, \\
j &= 1,...,I_\ell.
\end{aligned}$$

Then the conditional characteristic function of $\widetilde{B}_{\ell,j}$ is

$$\begin{aligned}
\varphi\left(t|X_\ell,H,\theta\right) &= E\left[\left.\frac{1}{I_\ell}\sum_{j=1}^{I_\ell}\exp\left(i.t.\widetilde{B}_{\ell,j}\right)\right| X_\ell\right] \\
&= \int_0^{\infty}\exp\left(i.t.\beta\left(v|X_\ell,H,\theta\right)\right)dF\left(v|X_\ell,H,\theta\right)
\end{aligned}$$

Denote

$$\psi(\xi|H,\theta) = E\left[\left(\varphi(t|X,H,\theta) - \varphi_0(t|X)\right)\exp\left(i.\varsigma'\Phi(X)\right)\right],$$
$$\xi = \begin{pmatrix} t \\ \varsigma \end{pmatrix} \in \mathbb{R}^{d+1},$$

where $\Phi : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a bounded one-to-one mapping.

Note that $\psi(\xi|H,\theta)$ is the difference between the unconditional characteristic functions of $(B_{\ell,j}, \Phi(X_\ell))$ and $\left(\widetilde{B}_{\ell,j}, \Phi(X_\ell)\right)$.

Under the moment condition

$$\int_0^\infty v dF_0(v|X) < \infty,$$

and some additional regularity conditions,

$$H = H_0 \text{ and } \theta = \theta_0$$

if and only if

$$\psi(\xi|H,\theta) = 0$$

in an arbitrary open neighborhood of the origin of $\mathbb{R}^{d+1}$.

5 Simulated integrated conditional moments estimation

Consequently,

$$(H_0, \theta_0) = \arg \min_{(H,\theta)\in\mathcal{H}\times\Theta} \overline{Q}\left(H, \theta\right)$$

is unique, where

$$\overline{Q}\left(H, \theta\right) = \int_{\Xi} \left|\psi\left(\xi|H, \theta\right)\right|^2 d\xi,$$

with $\Xi \subset \mathbb{R}^{d+1}$ a set with positive Lebesgue measure containing the origin of $\mathbb{R}^{d+1}$.

For example, let

$$\Xi = \left[-\kappa, \kappa\right]^{d+1} \text{ for some } \kappa > 0.$$

This suggests to estimate (H_0, θ_0) by the simulated integrated conditional moments method:

$$\left(\widehat{H}, \widehat{\theta}\right) = \arg \min_{(H,\theta)\in\mathcal{H}\times\Theta} \widehat{Q}\left(H, \theta\right).$$

where

$$\widehat{Q}\left(H, \theta\right) = \int_{\Xi} \left|\widehat{\psi}\left(\xi|H, \theta\right)\right|^2 d\xi$$

with

$$\begin{aligned}
\widehat{\psi}\left(\xi|H, \theta\right) &= \frac{1}{L}\sum_{\ell=1}^{L}\left(\widetilde{\varphi}_\ell\left(t|H, \theta\right) - \widehat{\varphi}_\ell\left(t\right)\right)\exp\left(i.\varsigma'\Phi\left(X_\ell\right)\right), \\
\widehat{\varphi}_\ell\left(t\right) &= \frac{1}{I_\ell}\sum_{j=1}^{I_\ell}\exp\left(i.t.B_{\ell,j}\right), \\
\widetilde{\varphi}_\ell\left(t|H, \theta\right) &= \frac{1}{I_\ell}\sum_{j=1}^{I_\ell}\exp\left(i.t.\widetilde{B}_{\ell,j}\right).
\end{aligned}$$

Of course, the estimator

$$\left(\widehat{H},\widehat{\theta}\right)=\arg\min_{(H,\theta)\in\mathcal{H}\times\Theta}\widehat{Q}\left(H,\theta\right).$$

is not feasible because the metric space $\mathcal{H}$ is infinite-dimensional.

The actual estimation can be done by sieve estimation, in the same way as proposed in:

Bierens, H. J., and H. Song (2008), ”Semi-Nonparametric Estimation of Independently and Identically Repeated First-Price Auctions via an Integrated Simulated Moments Method”.