Semi-Nonparametric Estimation of Independently and Identically Repeated First-Price Auctions via an Integrated Simulated Moments Method

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2 First-price auctions

A first-price auction is a sealed bid auction, where values are independent and private, bidders are symmetric and risk-neutral, and the highest bidder is the winner:

A single object is auctioned off. The seller sets a reservation price p_0: no bid below p_0 will be accepted.

The number of potential bidders is $I \geq 2$. The number I of potential bidders and the reservation price p_0 are known in advance to all potential bidders.

Private values

Each potential bidder j has a private value $V_j > 0$ for this object, and issues a sealed bid

$$\begin{array}{lll} B_j & \in & [p_0, V_j] \text{ if } V_j \geq p_0 \\ B_j & = & 0 \qquad\quad \text{ if } V_j < p_0 \end{array}$$

Independence and symmetry

The private values $V_i, i \neq j$ are not known to potential bidder j, but he/she knows that the other private values are independently distributed with the same distribution function $F(v)$ as bidder j (= independence and symmetry conditions).

Risk-neutrality

The utility of winning the auction is the difference between the value and the winning bid:

$U_j = V_j - B_j$ if B_j is the highest bid,

$U_j = 0$ if not.

Equilibrium bid function

If the bidders set their bids such that the expected utility of winning the auction is maximized, then the equilibrium bid function takes the form

$$\beta\left(v|F\right) = v - \frac{1}{F(v)^{I-1}} \int_{p_0}^{v} F(x)^{I-1} dx, \ v > p_0.$$

3 Nonparametric kernel approach

In their seminal paper, Guerre, Perrigne and Vuong (2000) propose an indirect nonparametric kernel estimation approach, based on the inverse bid function

$$V = B + (I-1)^{-1}\,\Lambda(B)/\lambda(B),$$

where I is the number of potential bidders, V is a private value, B is a corresponding bid, Λ is the distribution function of the bids and λ is the associated density function.

The latter two functions are estimated via nonparametric kernel methods, as $\hat{\Lambda}(b)$ and $\hat{\lambda}(b)$, respectively.

Using the pseudo-private values

$$\widetilde{V} = B + (I-1)^{-1}\,\hat{\Lambda}(B)/\hat{\lambda}(B),$$

the density of the private value distribution can now be estimated via a nonparametric kernel density estimation method.

However, the ratio $\hat{\Lambda}(b)/\hat{\lambda}(b)$ may be an unreliable estimate of $\Lambda(b)/\lambda(b)$ near the boundary of the support of $\lambda(b)$.

To solve this problem, Guerre, Perrigne and Vuong use a trimming procedure which amounts to discarding pseudo-private values $\widetilde{V}$ corresponding to bids B that are too close to the boundary of the (known) support of the bid distribution.

Our semi-nonparametric approach

The nonparametric identification of first-price auction models carries over to value distributions with support $(0, \infty)$ or smaller, as long as the value distribution is absolutely continuous. Moreover, if the values have a finite expectation then the corresponding bid distribution has bounded support.

To estimate these more general value distributions semi-nonparametrically, we propose in this paper a direct semi-nonparametric integrated simulated moments sieve estimation approach, as an alternative to the two-step nonparametric kernel estimation approach of Guerre, Perrigne and Vuong (2000), for independently and identically repeated first-price auctions.

Admittedly, this type of data is non-existing. However, this paper merely serves as a pilot study for the more realistic case of auction with observed auction-specific heterogeneity.

4 Data-generating process

In this paper we will consider the case where a first-price auction is repeated independently L times, with the same true value distribution $F_0(v)$, the same fixed number of potential bidders I, and the same reservation price p_0. Thus, we observe $N = I \times L$ bids B_j generated independently according to

$$B_j = \begin{cases} \beta(V_j|F_0) & \text{if } V_j > p_0, \\ 0 & \text{if } V_j \le p_0, \end{cases},$$
$$j = 1, 2, ..., N = I \times L,$$

where

$$\beta(v|F) = v - \frac{1}{F(v)^{I-1}} \int_{p_0}^{v} F(x)^{I-1} dx$$

The private values V_j are independent random drawings from the unknown true value distribution $F_0(v)$:

Assumption. The true value distribution $F_0(v)$ is absolutely continuous with density $f_0(v)$ and finite expectation,

$$\int_0^\infty v f_0(v) dv < \infty.$$

The latter condition is necessary for the existence of a unique Nash equilibrium, as otherwise the expected utility of winning the auction is not well-defined.

5 Identification via characteristic functions

Under this assumption the value distribution $F_0(v)$ is identified on (p_0, ∞) from the distribution $\Lambda_0(b) = P[B_j \leq b]$ of the bids B_j.

Moreover, the bid distribution $\Lambda_0(b)$ has then bounded support: $P[B_j \leq \bar{b}_0] = 1$, where

$$\begin{aligned}\bar{b}_0 &= \sup_{v>0} \beta(v|F_0) \leq (I-1)\int_0^\infty x f_0(x)dx + p_0 \\ &< \infty\end{aligned}$$

The significance of the boundedness of the bids is that then the bid distribution

$$\Lambda_0(b) = P[B_j \leq b]$$

is completely determined by the shape of its characteristic function

$$\begin{aligned} \varphi(t) &= E\left[\exp(i.t.B_j)\right] = \int_0^\infty \exp\left(i.t.b\right) d\Lambda_0(b), \\ i &= \sqrt{-1}, \end{aligned}$$

in an arbitrary neighborhood of $t = 0$.

Lemma. Let B be a bounded random variable with distribution function $\Lambda_0(b)$ and characteristic function $\varphi(t)$. Let $\psi(t)$ be the characteristic function of a distribution function $\Lambda(b)$. Then

$$\Lambda(b) = \Lambda_0(b) \text{ for all } b \in \mathbb{R}$$

if and only if for an arbitrary $\kappa > 0$,

$$\varphi(t) = \psi(t) \text{ for all } t \in (-\kappa, \kappa).$$

6 Integrated simulated moments sieve estimation

Let F be a potential candidate (henceforth called a **candidate value distribution**) for the true value distribution F_0, and let $\left\{\widetilde{V}_j\right\}_{j=1}^{N}$ be a random sample drawn from F.

In particular, $\widetilde{V}_j = F^{-1}\left(\widetilde{U}_j\right)$, where the $\widetilde{U}_j$'s are i.i.d. Uniform[0,1].

Next, generate simulated bids $\widetilde{B}_j$ according to

$$\widetilde{B}_j = \begin{cases} \beta(\widetilde{V}_j|F) & \text{if } \widetilde{V}_j > p_0, \\ 0 & \text{if } \widetilde{V}_j \leq p_0, \end{cases}, \; j = 1, 2, ..., N = I \times L,$$

where

$$\beta\left(v|F\right) = v - \frac{1}{F(v)^{I-1}} \int_{p_0}^{v} F(x)^{I-1} dx$$

Let

$$\widehat{\Psi}(t|F) = \widehat{\varphi}(t) - \widehat{\psi}(t|F),\ t \in \mathbb{R},$$

where

$$\widehat{\varphi}(t) = \frac{1}{N}\sum_{j=1}^{N}\exp(i.t.B_j),\ \widehat{\psi}(t|F) = \frac{1}{N}\sum_{j=1}^{N}\exp(i.t.\widetilde{B}_j)$$

are the empirical characteristic functions of the actual bids and the simulated bids, respectively.

Choose a constant $\kappa > 0$ and let

$$\overline{Q}(F) = \frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\left|E\left[\widehat{\Psi}(t|F)\right]\right|^2 dt,\ \widehat{Q}(F) = \frac{1}{2\kappa}\int_{-\kappa}^{\kappa}\left|\widehat{\Psi}(t|F)\right|^2 dt$$

Then

$$\widehat{Q}(F) \to \overline{Q}(F) \text{ a.s., pointwise in } F,$$

by bounded convergence, and

$$F_0 = \arg\min_F \overline{Q}(F) \text{ is unique on } [p_0, \infty)$$

by the uniqueness of characteristic functions.

We will construct a compact metric space $\mathcal{F}$ of distribution functions F on $(0, \infty)$ containing F_0, endowed with the sup metric

$$\|F_1 - F_2\| = \sup_{v>0} |F_1(v) - F_2(v)|,$$

such that

$$\sup_{F\in\mathcal{F}} \left|\widehat{Q}(F) - \overline{Q}(F)\right| \to 0 \text{ a.s,}$$

Moreover, it can be shown that

$$\overline{Q}(F) \text{ is continuous on } \mathcal{F}$$

By a generalization of the standard consistency proof to non-Euclidean parameters in a compact metric space it then follows that

$$\widehat{F} = \arg\min_{F\in\mathcal{F}} \widehat{Q}(F)$$

is a uniform strongly consistent estimator of F_0:

$$\sup_{v>p_0} \left|\widehat{F}(v) - F_0(v)\right| \to 0 \text{ a.s.}$$

Of course, the computation of

$$\widehat{F} = \arg\min_{F \in \mathcal{F}} \widehat{Q}(F)$$

is not feasible in practice. To solve that problem, we will use sieve estimation:

Let $\mathcal{F}_n$ be an increasing sequence of subspaces of $\mathcal{F}$ such that the computation of

$$\widetilde{F}_n = \arg\min_{F \in \mathcal{F}_n} \widehat{Q}(F)$$

is feasible.

Choose this sequence $\mathcal{F}_n$ such that

$$\{\mathcal{F}_n\}_{n=1}^{\infty} \text{ is dense in } \mathcal{F}\text{: } \mathcal{F} = \overline{\cup_{n=1}^{\infty} \mathcal{F}_n}$$

where the bar denotes the closure.

Then it can be shown that for any subsequence n_N of $N = I \times L$,

$$\sup_{v > p_0} \left| \widetilde{F}_{n_N}(v) - F_0(v) \right| \to 0 \text{ a.s.}$$

7 The compact metric space of distribution functions

Choose an absolutely continuous distribution function $G(v)$ with density $g(v)$, finite expectation $\int_0^\infty vg(v)dv < \infty$, and support $(0, \infty)$, as initial guess of the true value distribution $F_0(v)$.

Any distribution function $F(v)$ on $(0, \infty)$ can be written as

$$F(v) = H\left(G(v)\right),$$

where $H(u) = F\left(G^{-1}(v)\right)$ is a distribution function on $[0, 1]$.

If $F(v)$ is absolutely continuous then its density $f(v)$ can be written as

$$f(v) = h\left(G(v)\right)g(v),$$

where $h(u) = H'(u)$.

Therefore, it suffices to model the density $h(u)$ on [0,1] semi-nonparametrically.

In particular, any density function $h(u)$ on $[0,1]$ can be represented by

$$h(u) = \frac{(1+\sum_{k=1}^{\infty}\delta_k\rho_k(u))^2}{1+\sum_{k=1}^{\infty}\delta_k^2}, \quad \sum_{k=1}^{\infty}\delta_k^2 < \infty,$$

where the sequence $\{\rho_k(u)\}_{k=1}^{\infty}$ together with $\rho_0(u) = 1$ is an orthonormal basis for the Hilbert space of square-integrable functions on [0,1].

In this paper we will use for $\rho_k(u)$ the Legendre polynomials.

Moreover, we will impose restrictions on the δ_k's such that the space of these density functions and the space of corresponding distribution functions become compact.

Legendre Polynomials

Legendre polynomials of order $n \geq 2$ on the unit interval $[0, 1]$ can be constructed recursively by

$$\begin{aligned}\rho_n(u) \;=\; & \frac{\sqrt{2n-1}\sqrt{2n+1}}{n}(2u-1)\rho_{n-1}(u) \\ & -\frac{(n-1)\sqrt{2n+1}}{n\sqrt{2n-3}}\rho_{n-2}(u)\end{aligned}$$

starting from

$$\rho_0(u) = 1,\ \rho_1(u) = \sqrt{3}(2u-1).$$

They are orthonormal, in the sense that

$$\int_0^1 \rho_m(u)\rho_k(u)du = \begin{cases} 1 & \text{for } m = k \\ 0 & \text{otherwise} \end{cases}$$

The Legendre polynomials $\rho_k(u)$ form a complete orthonormal basis for the Hilbert space $L^2_B(0,1)$ of square-integrable Borel measurable real functions on $[0,1]$, endowed with the inner product

$$\langle f, g\rangle = \int_0^1 f(u)g(u)du$$

and associated norm $||f||_2 = \sqrt{\langle f, f\rangle}$ and metric $||f-g||_2$.

Hence, any square-integrable Borel measurable real function $q(u)$ on $[0,1]$ can be represented by

$$q(u) = \sum_{k=0}^{\infty} \gamma_k \rho_k(u) \text{ a.e. on } [0,1]$$

where the γ_k's are the Fourier coefficients: $\gamma_k = \int_0^1 \rho_k(u)q(u)du$, satisfying

$$\int_0^1 q(u)^2 du = \sum_{k=0}^{\infty} \gamma_k^2 < \infty.$$

Therefore, every density function $h(u)$ on $[0,1]$ can be written as

$$h(u) = q(u)^2$$

where $q \in L_B^2(0,1)$, with

$$\int_0^1 q(u)^2 du = \sum_{k=0}^{\infty} \gamma_k^2 = 1.$$

The restriction $\sum_{k=0}^{\infty} \gamma_k^2 = 1$ can be imposed by reparametrizing the γ_k's as

$$\begin{aligned} \gamma_0 &= \frac{1}{\sqrt{1+\sum_{k=1}^{\infty}\delta_k^2}}, \\ \gamma_k &= \frac{\delta_k}{\sqrt{1+\sum_{k=1}^{\infty}\delta_k^2}} \text{ for } k = 1,2,3,\ldots \end{aligned}$$

Thus, any density function $h(u)$ on $[0,1]$ can be represented by

$$h(u) = \frac{(1+\sum_{k=1}^{\infty}\delta_k\rho_k(u))^2}{1+\sum_{k=1}^{\infty}\delta_k^2}, \quad \sum_{k=1}^{\infty}\delta_k^2 < \infty.$$

Compactness

Lemma. Let $\mathcal{D}$ be the space of density function $h(u)$ on $[0,1]$ of the form

$$h(u) = \frac{(1 + \sum_{k=1}^{\infty} \delta_k \rho_k(u))^2}{1 + \sum_{k=1}^{\infty} \delta_k^2},$$

where the parameters δ_k are restricted by the inequality

$$|\delta_k| \leq c \left(1 + \sqrt{k} \ln k\right)^{-1}, \; k = 1, 2, 3, ...$$

for an a priori chosen constant $c > 0$. If we endow $\mathcal{D}$ with the L^1 metric

$$||h_1 - h_2||_1 = \int_0^1 |h_1(u) - h_2(u)| \, du,$$

then $\mathcal{D}$ is a compact metric space.

Consequently, the corresponding space of absolutely continuous distribution functions on $[0, 1]$,

$$\mathcal{H} = \left\{ H(u) = \int_0^u h(x)dx,\ h \in \mathcal{D} \right\},$$

endowed with the "sup" metric $\sup_{0 \leq u \leq 1} |H_1(u) - H_2(u)|$, is a compact metric space as well.

It follows straightforwardly from the previous lemma that

Lemma. The space

$$\mathcal{D}(G) = \{f(v) = h(G(v))g(v),\ h \in \mathcal{D}\}$$

of densities on $(0, \infty)$, endowed with the L^1 metric

$$||f_1 - f_2|| = \int_0^\infty |f_1(v) - f_2(v)|\, dv.$$

is a compact metric space.

Moreover, the corresponding space

$$\mathcal{F} = \left\{ F(v) = \int_0^v f(x)dx,\ f \in \mathcal{D}(G) \right\}$$

of absolutely continuous distribution functions on $(0, \infty)$, endowed with the sup metric

$$||F_1 - F_2|| = \sup_{v>0} |F_1(v) - F_2(v)|$$

is a compact metric space as well.

Now $\mathcal{F}$ is the "parameter" space of candidate value distributions $F(v)$, provided that:

Assumption. The constant $c > 0$ in the parameter bounds

$$|\delta_k| \leq c\left(1 + \sqrt{k}\ln k\right)^{-1}, \ k = 1, 2, 3, ...$$

for δ_k is chosen so large that the density f_0 of the true value distribution F_0 is contained in

$$\mathcal{D}(G) = \{f(v) = h(G(v))g(v), \ h \in \mathcal{D}\}$$

8 The sieve spaces

For a density function $h(u)$ and its associated parameter sequence $\{\delta_k\}_{k=1}^{\infty}$, let

$$\begin{aligned} h_n(u) &= h(u|\boldsymbol{\delta}_{\mathbf{n}}) = \frac{(1+\sum_{k=1}^{n}\delta_k\rho_k(u))^2}{1+\sum_{k=1}^{n}\delta_k^2}, \\ \boldsymbol{\delta}_{\mathbf{n}} &= (\delta_1,\ldots,\delta_n)', \end{aligned}$$

be the n-th order truncation of $h(u)$. It can be shown that

$$\lim_{n\to\infty}\int_0^1 |h_n(u)-h(u)|\,du = 0.$$

Define the space $\mathcal{D}_n$ of n-th order truncations of $h(u)$ by

$$\begin{aligned} \mathcal{D}_n &= \Bigg\{ h_n(u) = \frac{(1+\sum_{k=1}^{n}\delta_k\rho_k(u))^2}{1+\sum_{k=1}^{n}\delta_k^2}, \\ &\qquad |\delta_k| \le c\left(1+\sqrt{k}\ln k\right)^{-1} \text{ for } k \ge 1 \Bigg\}. \end{aligned}$$

Then it follows that for each $h \in \mathcal{D}$ there exists a sequence $h_n \in \mathcal{D}_n$ of SNP densities such that

$$\lim_{n\to\infty} \int_0^1 |h_n(u) - h(u)|\, du = 0$$

Consequently, defining

$$\mathcal{H}_n = \left\{ H_n(u) = \int_0^u h_n(v) du,\ h_n \in \mathcal{D}_n \right\}$$

it follows that for each distribution function $H \in \mathcal{H}$ there exists a sequence of SNP distribution functions $H_n \in \mathcal{H}_n$ such that

$$\lim_{n\to\infty} \sup_{0\le u\le 1} |H_n(u) - H(u)| = 0.$$

Next, define

$$\mathcal{F}_n = \{F_n(v) = H_n(G(v)),\ H_n \in \mathcal{H}_n\}\,.$$

Then

Lemma. For each distribution function $F \in \mathcal{F}$ there exists a sequence of distribution functions $F_n \in \mathcal{F}_n$ such that

$$\lim_{n\to\infty} \sup_{v>0} |F_n(v) - F(v)| = 0.$$

Consequently, $\{\mathcal{F}_n\}_{n=1}^{\infty}$ is dense in $\mathcal{F}$.

The subspaces $\mathcal{F}_n$ now form the sieve spaces.

9 Uniform consistency

The sieve estimator of the true value distribution F_0 is

$$\widetilde{F}_n = \arg \min_{F \in \mathcal{F}_n} \widehat{Q}(F)$$

where

$$\widehat{Q}(F) = \frac{1}{2\kappa} \int_{-\kappa}^{\kappa} \left| \widehat{\varphi}(t) - \widehat{\psi}(t|F) \right|^2 dt$$

with

$$\widehat{\varphi}(t) = \frac{1}{N} \sum_{j=1}^{N} \exp(i.t.B_j), \; \widehat{\psi}(t|F) = \frac{1}{N} \sum_{j=1}^{N} \exp(i.t.\widetilde{B}_j)$$

the empirical characteristic functions of the actual bids and the simulated bids from F, respectively, and $\kappa > 0$ is an a priori chosen constant. Then for any subsequence n_N of the sample size N,

$$\sup_{v > p_0} \left| \widetilde{F}_{n_N}(v) - F_0\left(v\right) \right| \to 0 \text{ a.s.}$$

10 Determination of the sieve order

For nested likelihood models the most parsimonious model can be determined via information criteria, for example the Hannan-Quinn or Schwarz information criteria.

These information criteria are of the form

$$C_N(n) = \frac{-2}{N} \ln\left(L_N(n)\right) + n.\frac{\phi(N)}{N}$$

where $L_N(n)$ is the maximum likelihood of a models with n parameters, with

$$\phi(N) = \ln(N)$$

for the Schwarz criterion and

$$\phi(N) = 2.\ln(\ln(N))$$

for the Hannan-Quinn criterion.

In the paper we propose to use an "information" criterion of the form

$$\widetilde{C}_N(n) = \inf_{F \in \mathcal{F}_n} \widehat{Q}(F) + \Phi(n).\frac{\phi(N)}{N},$$
$$\phi(N) = o(N), \quad \lim_{N \to \infty} \phi(N) = \infty.$$

where $\Phi(n)$ is an increasing but bounded function of n.

For example, let for some $\alpha \in (0, 1)$,

$$\Phi(n) = 1 - (n+1)^{-\alpha}.$$

The reason for $\Phi(n)$ instead of n is that $\widehat{Q}(F)$ is bounded:

$$\sup_F \widehat{Q}(F) \leq 4$$

Using n instead of $\Phi(n)$ as in the case of the Hannan-Quinn and Schwarz information criteria would make the penalty term too dominant.

Similar to the Hannan-Quinn and Schwarz information criteria we have:

Theorem. Let

$$\widetilde{n}_N = \max_{s.t.\ \widetilde{C}_N(n) \leq \widetilde{C}_N(n-1)} n$$

and

$$\widetilde{F} = \arg \min_{F \in \mathcal{F}_{\widetilde{n}_N}} \widehat{Q}(F).$$

If $F_0 \in \cup_{n=1}^{\infty} \mathcal{F}_n$, so that there exists a smallest natural number n_0 such that $F_0 \in \mathcal{F}_{n_0}$, then

$$\lim_{N \to \infty} P\left[\widetilde{n}_N = n_0\right] = 1,$$

and

$$\sup_{v > p_0} \left| \widetilde{F}(v) - F_0(v) \right| \to 0 \; a.s.$$

If $F_0 \notin \cup_{n=1}^{\infty} \mathcal{F}_n$ but $F_0 \in \overline{\cup_{n=1}^{\infty} \mathcal{F}_n}$ then

$$p \lim_{N \to \infty} \widetilde{n}_N = \infty,$$

and

$$p \lim_{N \to \infty} \sup_{v > p_0} \left| \widetilde{F}(v) - F_0(v) \right| = 0.$$

11 An integrated moment test

If the independent private values and/or the risk neutrality assumptions do not hold, we then have

$$\widehat{Q}(\widetilde{F}) \to \inf_{F \in \mathcal{F}} \overline{Q}(F) > 0 \text{ a.s.},$$

where $\widetilde{F}$ is the sieve estimator. This suggests to use $\widehat{Q}(\widetilde{F})$ as a basis for an Integrated Moment (IM) test of the null hypothesis that

H_0: the actual bids come from a first-price sealed bid auction where values are independent, private, and bidders are symmetric and risk-neutral,

against the general alternative that

H_1: the null hypothesis H_0 is false.

The test we propose is based on the fact that similar to the results for the Integrated Conditional Moment (ICM) test in Bierens (1990) and Bierens and Ploberger (1997),

Theorem. Under H_0,

$$N.\widehat{Q}(F_0) \xrightarrow{d} \frac{1}{2\kappa}\int_{-\kappa}^{\kappa} |W(t)|^2 \, dt,$$

where W is a complex-valued zero-mean Gaussian process on $[-\kappa, \kappa]$, whereas under H_1,

$$N.\widehat{Q}(F_0) \to \infty \text{ a.s.}$$

Note that this result does not imply that

$$N.\widehat{Q}(\widetilde{F}) \xrightarrow{d} \frac{1}{2\kappa}\int_{-\kappa}^{\kappa} |W(t)|^2 \, dt.$$

because this requires that

$$\sup_{v>0}\left|\widetilde{F}(v) - F_0(v)\right| = o_p\left(1/\sqrt{N}\right)$$

However, if

Assumption. The true value distribution F_0 is of the SNP type itself: $F_0 \in \cup_{n=1}^{\infty}\mathcal{F}_n$,

then there exists a smallest natural number n_0 such that $F_0 \in \mathcal{F}_{n_0}$, so that

$$N.\widehat{Q}(\widetilde{F}) \leq N.\widehat{Q}(F_0) \text{ for } n_N \geq n_0.$$

This suggests that upper bounds of the critical values of the test can be based on the limiting distribution of $N.\widehat{Q}(F_0)$.

12 Bootstrap critical values

To approximate the limiting process $W(t)$, generate for large M simulated bids $\widetilde{B}_j$ for $j = 1, 2, ..., 2.M$ from the bid distribution corresponding to the sieve estimator $\widetilde{F}$ of F_0.

Denote

$$\widetilde{W}_M\left(t|\widetilde{F}\right) = \frac{1}{\sqrt{M}} \sum_{j=1}^{M} \left(\exp\left(i.t.\widetilde{B}_j\right) - \exp\left(i.t.\widetilde{B}_{M+j}\right)\right)$$

It can be shown that

Lemma. Let $M \to \infty$ first, and then $N \to \infty$. Under the null hypothesis,

$$\frac{1}{2\kappa} \int_{-\kappa}^{\kappa} \left|\widetilde{W}_M\left(t|\widetilde{F}\right)\right|^2 dt \to_d \frac{1}{2\kappa} \int_{-\kappa}^{\kappa} |W(t)|^2 dt.$$

13 The fit

To check how our approach works we have conducted three experiments.

In each case we generate independently 200 auctions without a reservation price, where each auction consists of 5 bids whose private values come from a chi-square distribution, so that in each case we have 1000 i.i.d. sample bids.

The three cases only differ with respect to the degrees of freedom r of the chi-square distribution, namely $r = 3, 4, 5$, respectively.

The initial guess of the value density is

$$g(v) = \exp(-v/3)/3$$

which is quite different from the actual chi-square densities, in particular in the left tails.

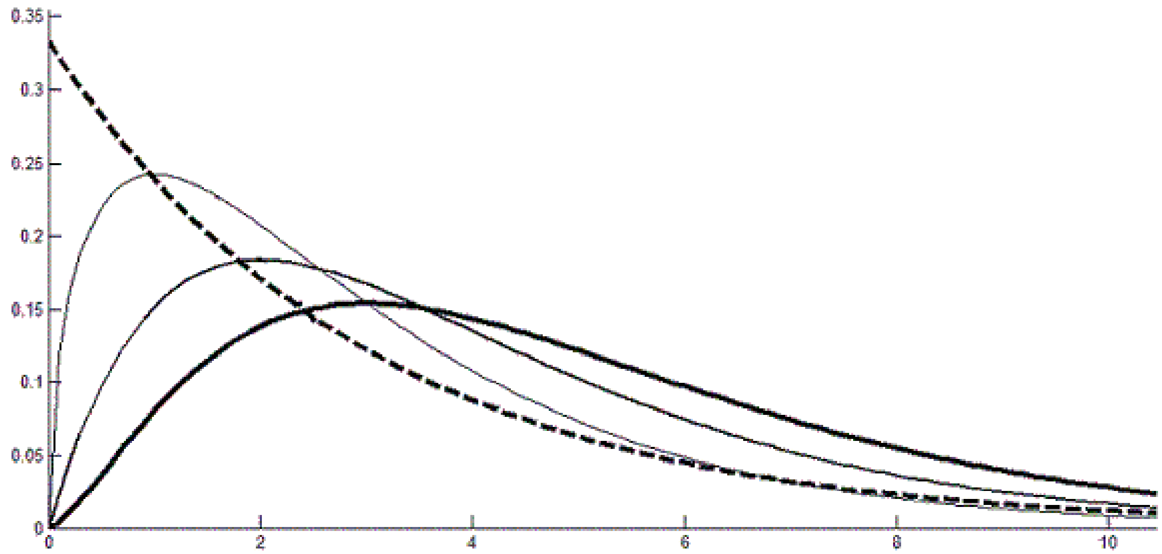

Figure 1. $g(v) = \exp(-v/3)/3$ (dashed curve) compared with the χ_r^2 densities for $r = 3, 4, 5$

The SNP density $h_n(u)$ needs to convert the exponential density $g(v)$ into an approximation $\widetilde{f}_n(v) = h_n(G(v))\,g(v)$ of a χ_r^2 density, so that $h_n(u)$ needs to bend down the left tail of $g(v)$ towards zero. This seems challenging.

However, it appears that the SNP density $h_n(u)$ has no problem doing that.

Using the information criterion specification

$$\widetilde{C}_N(n) = \inf_{F\in\mathcal{F}_n(G)} \widehat{Q}(F) + \left(1-(n+1)^{-1/3}\right).\frac{\ln(\ln(N))}{N},$$

the following estimates of the sieve orders were obtained:

r:	3	4	5
$\widetilde{n}$:	4	2	4

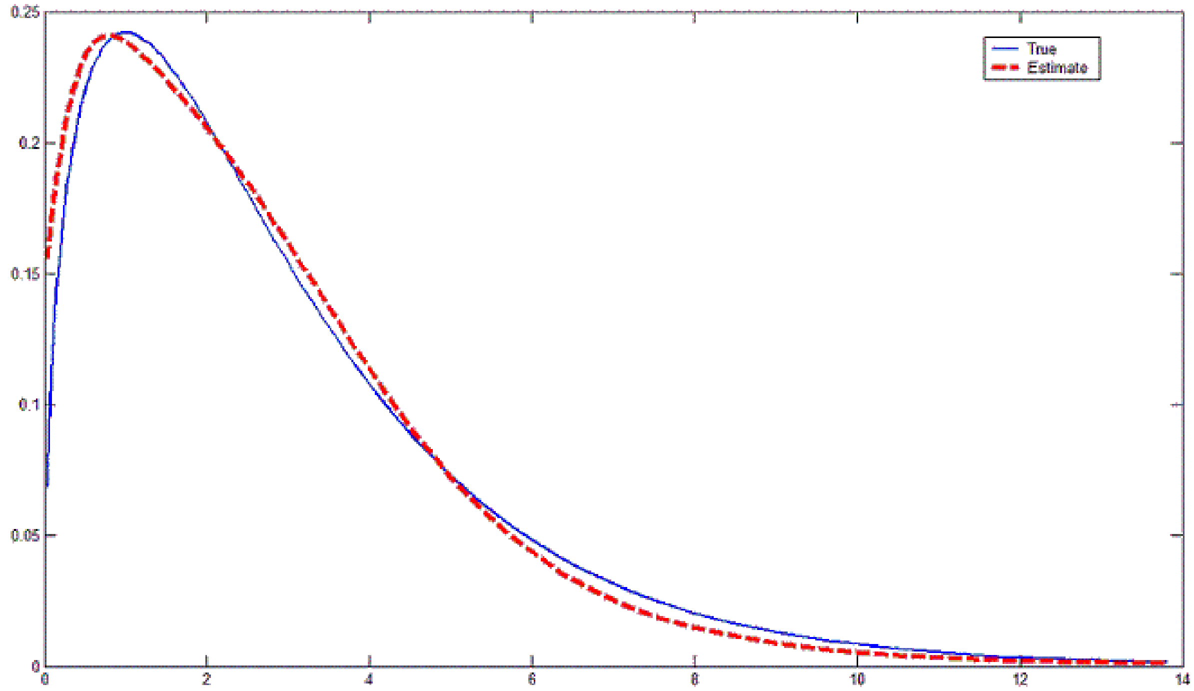

Figure 2. $\widetilde{f}_4(v)$ (dashed curve) compared with the true χ^2_3 density $f_0(v)$

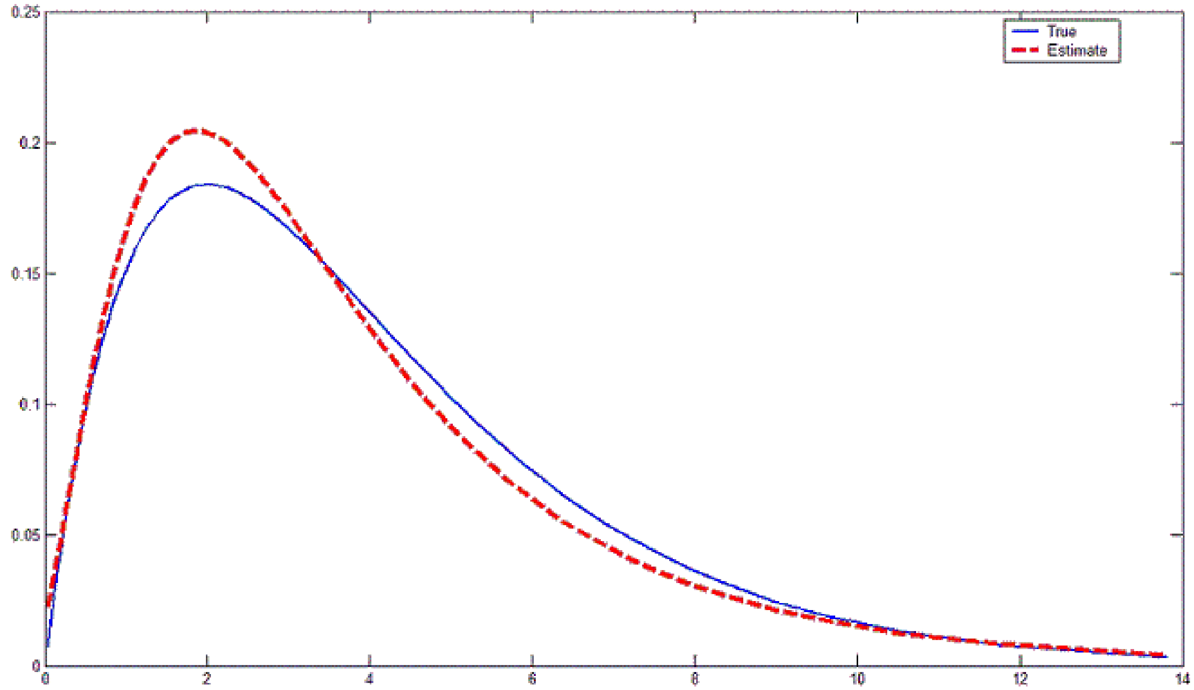

Figure 3. $\widetilde{f}_2(v)$ (dashed curve) compared with the true χ^2_4 density $f_0(v)$

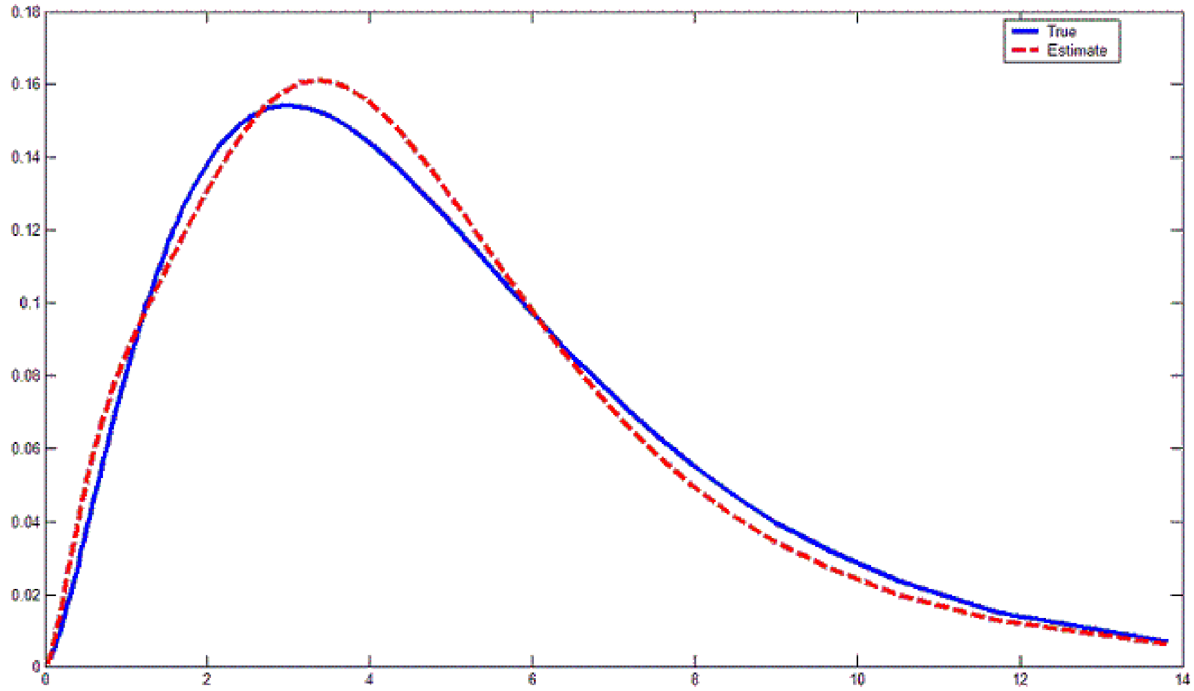

Figure 4. $\widetilde{f}_4(v)$ (dashed curve) compared with the true χ^2_5 density $f_0(v)$

14 Conclusions

In this paper we have proposed a new semi-nonparametric estimation method for the value distribution of a first-price auction, based on a comparison of the empirical characteristic functions of the actual bids and simulated bids.

Our approach differs fundamentally from the nonparametric estimation approaches in the literature in that we estimate the value distribution directly, whereas in the nonparametric auction literature, the value distribution is estimated indirectly via kernel estimation of the inverse bid function.

Another novelty of our approach is that it yields as by-product an integrated moment test for the validity of the first-price auction model.

A limitation is that our results have only been derived for independently and identically repeated first-price auctions.

Admittedly, this type of data is non-existing. However, this paper merely serves as a pilot study for the more realistic case of auction with observed auction-specific heterogeneity:

Bierens, H. J., and H. Song (2007), ”Semi-Non-parametric Estimation of First-Price Auctions Models with Auction-Specific Heterogeneity via an Integrated Simulated Conditional Moments Method”, working paper.