Conditional expectations as integrals with respect to conditional probability measures

According to Definition 3.2 the conditional probability measure relative to a sub-σ-algebra $\mathcal{F}_0$ of the σ-algebra $\mathcal{F}$ of the probability space $\{\Omega, \mathcal{F}, P\}$ can be defined as

$$P(A|\mathcal{F}_0) = E\left[I_A|\mathcal{F}_0\right] \text{ for all } A \in \mathcal{F}, \tag{1}$$

where $I_A = I(\omega \in A)$.

The question is: Can we represent the conditional expectation $E[Y|\mathcal{F}_0]$ as an integral of $Y(\omega)$ with respect to the conditional probability $P(.|\mathcal{F}_0)$, i.e., can we write

$$E[Y|\mathcal{F}_0] = \int Y(\omega) dP(\omega|\mathcal{F}_0),$$

where the latter integral is defined similar to the unconditional expectation $E[Y] = \int Y(\omega)dP(\omega)$? The answer is Yes.

The simple case

Suppose first that Y is simple:

$$Y(\omega) = \sum_{j=1}^{n} b_j I(\omega \in A_j),$$

where the A_j's are disjoint sets in $\mathcal{F}$. Hence, denoting $I_{A_j} = I(\omega \in A_j)$, we can write Y as

$$Y = \sum_{j=1}^{n} b_j I_{A_j}.$$

Then by (1),

$$Z = \sum_{j=1}^{n} b_j E[I_{A_j}|\mathcal{F}_0] = \sum_{j=1}^{n} b_j P(A_j|\mathcal{F}_0) \overset{def.}{=} \int Y(\omega) dP(\omega|\mathcal{F}_0).$$

To check whether $Z = E\left[Y|\mathcal{F}_0\right]$ a.s.,[1] denote $Z_j = E[I_{A_j}|\mathcal{F}_0]$, and observe from Definition 3.1 that the Z_j's are measurable w.r.t. $\mathcal{F}_0$. Consequently,

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[1] Here and in the sequel "a.s." stands for "almost surely", which means that the property involved holds with probability 1.

$Z = \sum_{j=1}^{n} b_j Z_j$ is measurable w.r.t. $\mathcal{F}_0$. Now pick an arbitrary set A in $\mathcal{F}_0$. Then

$$\begin{aligned}\int_A (Z(\omega) - Y(\omega))\, dP(\omega) &\overset{def.}{=} \int I(\omega \in A)\, (Z(\omega) - Y(\omega))\, dP(\omega) \\ &= \sum_{j=1}^{n} b_j \int I(\omega \in A)\, (Z_j(\omega) - I(\omega \in A_j))\, dP(\omega) \\ &= \sum_{j=1}^{n} b_j \left(\int I(\omega \in A) Z_j(\omega) dP(\omega) - P(A \cap A_j) \right) \\ &= \sum_{j=1}^{n} b_j \left(E\left(I_A Z_j\right) - P(A \cap A_j) \right) \\ &= \sum_{j=1}^{n} b_j \left(E\left(I_A E[I_{A_j} | \mathcal{F}_0] \right) - P(A \cap A_j) \right) \\ &= 0\end{aligned}$$

because by Theorems 3.1 and 3.9,

$$\begin{aligned}E\left(I_A E[I_{A_j} | \mathcal{F}_0] \right) &= E\left(E[I_A . I_{A_j} | \mathcal{F}_0] \right) = E[I_A . I_{A_j}] = E\left[I_{A \cap A_j} \right] \\ &= \int I\left(\omega \in A \cap A_j\right) dP(\omega) = P(A \cap A_j)\end{aligned}$$

Thus indeed

$$Z = E\left[Y | \mathcal{F}_0\right] = \int Y(\omega) dP(\omega | \mathcal{F}_0).$$

The nonnegative case

Next, consider the case that Y is non-negative. Define similar to Definition 2.9,

$$\int Y(\omega) dP(\omega | \mathcal{F}_0) \overset{def.}{=} \sup_{Y_* \in S_Y} \int Y_*(\omega) dP(\omega | \mathcal{F}_0) = \sup_{Y_* \in S_Y} E\left[Y_* | \mathcal{F}_0\right] \qquad (2)$$

where S_Y is the collection of simple random variables Y_* satisfying $0 \leq Y_*(\omega) \leq Y(\omega)$ for all ω in a set C in $\mathcal{F}$ with $P(C) = 1$. Similar to Theorem 2.6 we can construct a nondecreasing sequence of simple random variables $Y_n(\omega)$ in S_Y such that pointwise in $\omega \in C$, $Y_n(\omega) \uparrow Y(\omega)$. Then by Theorem 3.8,

$$\lim_{n \to \infty} E\left[Y_n | \mathcal{F}_0\right] = E\left[Y | \mathcal{F}_0\right] \text{ a.s.}$$

and therefore by (2),

$$E\left[Y|\mathcal{F}_0\right] \leq \sup_{Y_* \in S_Y} E\left[Y_*|\mathcal{F}_0\right] \text{ a.s.}$$

On the other hand, for each $Y_* \in S_Y$ we have, by Theorem 3.2, that $E\left[Y_*|\mathcal{F}_0\right] \leq E\left[Y|\mathcal{F}_0\right]$ a.s. and therefore

$$\sup_{Y_* \in S_Y} E\left[Y_*|\mathcal{F}_0\right] \leq E\left[Y|\mathcal{F}_0\right] \text{ a.s.}$$

Thus,

$$E\left[Y|\mathcal{F}_0\right] = \sup_{Y_* \in S_Y} \int Y_*(\omega) dP(\omega|\mathcal{F}_0) \overset{def.}{=} \int Y(\omega) dP(\omega|\mathcal{F}_0).$$

The general case

Finally, denoting $Y^+ = \max(0, Y),\ Y^- = \max(0, -Y)$, we have

$$\begin{aligned} E\left[Y|\mathcal{F}_0\right] &= E\left[Y^+|\mathcal{F}_0\right] - E\left[Y^-|\mathcal{F}_0\right] \\ &= \int Y^+(\omega) dP(\omega|\mathcal{F}_0) - \int Y^-(\omega) dP(\omega|\mathcal{F}_0) \overset{def.}{=} \int Y(\omega) dP(\omega|\mathcal{F}_0). \end{aligned}$$

provided that at least one of the two integrals is a.s. finite.