Separate Appendix to:

Nonparametric Nonlinear Co-Trending Analysis, With an Application to Interest and Inflation in the U.S.

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Proof of Lemma 1

Lemma 1 follows straightforwardly from:

Lemma A.1: *Let Assumptions 1-2 be true, let* θ *be a common eigenvector of the matrices* $M_{1,n}$, M_1, $M_{2,n}$, *and* M_2 *corresponding to a zero eigenvalue, and let* θ_* *be any other conformable vector. Moreover, let* $m \to \infty$ *at rate* $o(n)$. *Then the following hold*:

$$n\theta^T\hat{M}_1\theta \to \theta^T C(1)\left(\int \bar{W}_k(x)\bar{W}_k(x)^T dx\right)C(1)^T\theta \ \ in\ \ distr., \tag{A.1}$$

$$n^{-p+1/2}\theta^T\hat{M}_1\theta_* \to \theta^T C(1)\left(\int \bar{W}_k(x)F(x)^T dx\right)\theta_* \ \ in\ \ distr., \tag{A.2}$$

$$n^{-2p}\theta_*^T\hat{M}_1\theta_* = \theta_*^T\left(\int F(x)F(x)^T dx\right)\theta_* + O_p(n^{-p-1/2}), \tag{A.3}$$

$$m\theta^T\hat{M}_2\theta = \theta^T C(1)C(1)^T\theta + O_p(\sqrt{m/n}) + O_p(1/\sqrt{m}). \tag{A.4}$$

$$n^{-p}\sqrt{m}\theta^T\hat{M}_2\theta_* = O_p(\sqrt{m/n}) + O_p(1/\sqrt{m}). \tag{A.5}$$

$$n^{-2p}\theta_*^T\hat{M}_2\theta_* = \theta_*^T\left(\int F'(x)F'(x)^T dx\right)\theta_* + O_p(n^{-p-1/2}) + O_p(n^{-p}/m). \tag{A.6}$$

Before we can prove Lemma A.1, we have to prove (12) and (13) first:

Proof of (12): Let

$$U_n(x) = (1/\sqrt{n})\sum_{t=1}^{[nx]} u_t \ \ if\ \ x \in [n^{-1},1], \quad U_n(x) = 0 \ \ if\ \ x \in [0,n^{-1})$$

Then it follows from Assumption 2, the decomposition (11), and the functional central limit theorem, that

$$U_n(x) \Rightarrow C(1)W_k(x), \tag{A.7}$$

where W_k is a k-variate standard Wiener process and "⇒" denotes weak convergence. Cf. Billingsley (1968). Next, observe that, with $g(t) = \beta_{0,n} + \beta_{1,n}t + f_n(t)$, where $f_n(t)$ satisfies the conditions in (7), we have

$$z_t - \hat{\beta}_0 - \hat{\beta}_1 t = u_t + f_n(t) - \left(\hat{\beta}_0 - \beta_{0,n}\right) - \left(\hat{\beta}_1 - \beta_{1,n}\right)t = \hat{u}_t + f_n(t),$$

where $\hat{u}_t$ is the OLS residual of the regression of u_t on an intercept and time t, for t = 1,..,n. Denoting the OLS coefficients involved by $\hat{\delta}_0$ and $\hat{\delta}_1$, respectively, and using the fact that by Lemma 9.6.3 in Bierens (1994, p. 200), $(1/\sqrt{n})\Sigma_{t=1}^{n}(t/n)u_t = U_n(1) - \int U_n(x)dx$, it follows that

$$\begin{pmatrix} \sqrt{n}\hat{\delta}_0 \\ n\sqrt{n}\hat{\delta}_1 \end{pmatrix} = \begin{pmatrix} 1 & (1/n)\Sigma_{t=1}^{n}(t/n) \\ (1/n)\Sigma_{t=1}^{n}(t/n) & (1/n)\Sigma_{t=1}^{n}(t/n)^2 \end{pmatrix}^{-1} \begin{pmatrix} U_n(1) \\ U_n(1) - \int U_n(x)dx \end{pmatrix}. \tag{A.8}$$

This result implies that for $x \in [0,1]$,

$$(1/\sqrt{n})\sum_{t=1}^{[nx]} \hat{u}_t = U_n(x) - xU_n(1) + 3(x^2 - x)\left(2\int U_n(y)dy - U_n(1)\right) + o_p(1), \tag{A.9}$$

where the $o_p(1)$ term is uniform in $x \in [0,1]$. Combining (7) and (9), the result (12) follows.

Proof of (13): It follows from (8) and the decomposition (11) that

$$\hat{F}_n^{\prime}(t/n) - n^{p} F_{p,n}^{\prime}(t/n) = (1/m)\sum_{j=0}^{m-1}\left(z_{t+1-j}-\hat{\beta}_0-\hat{\beta}_1(t+1-j)\right) - (1/m)\sum_{j=0}^{m-1} f_n(t+1-j) = (1/m)\sum_{j=0}^{m-1}\hat{u}_{t+1-j}$$

$$= (1/m)\sum_{j=0}^{m-1} u_{t+1-j} - \hat{\delta}_0 - \hat{\delta}_1\left(t+1 - (m+1)/2\right) = (1/m)\sum_{j=0}^{m-1} u_{t+1-j} + O_p\left(1/\sqrt{n}\right)$$

$$= C(1)(1/m)\sum_{j=0}^{m-1}\varepsilon_{t+1-j} + \frac{v_{t+1}-v_{t+1-m}}{m} + O_p\left(1/\sqrt{n}\right),$$

where the O_p term is uniform in t. This proves the result involved.

Proof of (1), (2), *and* (3): These parts of Lemma 1 follow directly from (12) and Assumption 1.

Proof of (4): It follows from (13) that

$$m\theta^T\hat{M}_2\theta = \frac{m}{n}\sum_{t=m}^{n}\left((1/m)\sum_{j=0}^{m-1}\theta^T\left(z_{t-j}-\hat{\beta}_0-\hat{\beta}_1(t-j)\right)\right)^2$$

$$= \frac{1}{n}\sum_{t=m}^{n}\left(\theta^T C(1)(1/\sqrt{m})\sum_{j=0}^{m-1}\varepsilon_{t-j} + \frac{\theta^T(v_t-v_{t-m})}{\sqrt{m}} + O_p\left(\sqrt{m/n}\right)\right)^2 \tag{A.10}$$

$$= \theta^T C(1)C(1)^T\theta + O_p(1/\sqrt{m}) + O_p(\sqrt{m/n}).$$

The last equality follows from the fact that under Assumption 3,

$$(1/n)\sum_{t=m}^{n}\left((1/\sqrt{m})\sum_{j=0}^{m-1}\varepsilon_{t-j}\right)\left((1/\sqrt{m})\sum_{j=0}^{m-1}\varepsilon_{t-j}\right)^T = I_k + O_p(\sqrt{m/n}),$$

which is proved in Lemma A.2 below, and that

$$(1/n)\sum_{t=m}^{n}(v_t-v_{t-m})(v_t-v_{t-m})^T = O_p(1).$$

Since the first O_p term is dominated by the last one, (4) follows from (10).

Proof of (5) *and* (6): It follows from (13) that

$$n^{-p}\sqrt{m}\theta^T\hat{M}_2\theta_* = \frac{1}{n}\sum_{t=m-1}^{n-1}\theta^T C(1)(1/\sqrt{m})\sum_{j=0}^{m-1}\varepsilon_{t+1-j}F_n'(t/n)^T\theta_*$$

$$+ \frac{1}{n}\sum_{t=m-1}^{n-1}\frac{v_{t+1}-v_{t+1-m}}{\sqrt{m}}F_n'(t/n)^T\theta_* + O_p\left(\sqrt{m/n}\right)$$

$$+ \frac{n^{-p}}{\sqrt{m}}\frac{1}{n}\sum_{t=m-1}^{n-1}\theta^T\left(C(1)(1/\sqrt{m})\sum_{j=0}^{m-1}\varepsilon_{t+1-j} + \frac{v_{t+1}-v_{t+1-m}}{\sqrt{m}} + O_p\left(\sqrt{m/n}\right)\right)$$

$$\times\left(C(1)(1/\sqrt{m})\sum_{j=0}^{m-1}\varepsilon_{t+1-j} + \frac{v_{t+1}-v_{t+1-m}}{\sqrt{m}} + O_p\left(\sqrt{m/n}\right)\right)^T\theta_*$$

$$= \frac{1}{\sqrt{m}}\sum_{j=0}^{m-1}\left(\frac{1}{n}\sum_{t=m-1}^{n-1}\theta^T C(1)\varepsilon_{t+1-j}F_n'(t/n)^T\theta_*\right) + O_p(1/\sqrt{m}) + O_p\left(\sqrt{m/n}\right) + O_p(n^{-p}/\sqrt{m})$$

$$= O_p(\sqrt{m/n}) + O_p(1/\sqrt{m}),$$

which proves (5). Part (6) can easily be proved similarly to the proofs of (4) and (5).

This completes the proof of Lemma A.1. Q.E.D.

Lemma A.2: *Let e_t be a sequence of independent standard normal distributed random variables. Then*

$$(1/n)\Sigma_{t=m}^{n}\left((1/\sqrt{m})\Sigma_{j=0}^{m-1}e_{t-j}\right)^2 = 1 + O_p(\sqrt{m/n}).$$

Proof: Let $N = [n/m]$. Then

$$(1/n)\sum_{t=m}^{n}\left((1/\sqrt{m})\sum_{j=0}^{m-1} e_{t-j}\right)^2 = (1/n)\sum_{i=0}^{m-1}\sum_{\tau=1}^{N}\left((1/\sqrt{m})\sum_{j=0}^{m-1} e_{\tau m+i-j}\right)^2 + (1/n)\sum_{t=mN}^{n}\left((1/\sqrt{m})\sum_{j=0}^{m-1} e_{t-j}\right)^2$$

$$= \frac{mN}{n}\frac{1}{m}\sum_{i=0}^{m-1}\left[\frac{1}{N}\sum_{\tau=1}^{N}\left((1/\sqrt{m})\sum_{j=0}^{m-1} e_{\tau m+i-j}\right)^2\right] + O_p(m/n)$$

$$= \frac{mN}{n}\frac{1}{m}\sum_{i=0}^{m-1}\left[1 + O_p(1/\sqrt{N})\right] + O_p(m/n) = 1 + O_p(\sqrt{m/n}),$$

due to the fact the expression between square brackets in the second line is a mean of N independent $\chi^2(1)$ distributed random variables.

Proof of Lemma 2

Proof of part (16): It follows from (14) that

$$\begin{pmatrix} 1/\sqrt{n} & 0^T \\ 0 & n^p I_{k-1} \end{pmatrix} Q^T \hat{M}_1^{-1} Q \begin{pmatrix} 1/\sqrt{n} & 0^T \\ 0 & n^p I_{k-1} \end{pmatrix}$$

$$\to \begin{pmatrix} \theta^T C(1)\int \overline{W}_k(x)\overline{W}_k(x)^T dx\, C(1)^T\theta & \theta^T C(1)^T \int \overline{W}_k(x) F(x)^T dx\, Q_* \\ Q_*^T \int F(x)\overline{W}_k(x)^T dx\, C(1)^T\theta & \Lambda_* \end{pmatrix}^{-1}$$

in distribution. It is a standard linear algebra exercise to verify that the latter matrix is

$$\begin{pmatrix} \tilde{\mu}^{-1} & -\theta^T C(1)\int \overline{W}_k(x)F(x)^T dx\, Q_* \Lambda_*^{-1}\tilde{\mu}^{-1} \\ -\tilde{\mu}^{-1}\Lambda_*^{-1} Q_*^T \int F(x)\overline{W}_k(x)^T dx\, C(1)^T\theta & \Lambda_*^{-1} + \tilde{\mu}^{-1}\Lambda_*^{-1} Q_*^T \int F(x)\overline{W}_k(x)^T dx\, C(1)^T\theta\theta^T C(1)\int \overline{W}_k(y)F(y)^T dy\, Q_*\Lambda_*^{-1} \end{pmatrix}.$$

Part (16) now follows from:

$$
\begin{pmatrix} 1/\sqrt{m} & 0^T \\ 0 & n^p I_{k-1} \end{pmatrix} Q^T \left(\frac{m}{n} \hat{M}_1^{-1} \right) Q \begin{pmatrix} 1/\sqrt{m} & 0^T \\ 0 & n^p I_{k-1} \end{pmatrix}
$$

$$
= \begin{pmatrix} 1/\sqrt{n} & 0^T \\ 0 & n^p \sqrt{m/n} I_{k-1} \end{pmatrix} Q^T \hat{M}_1^{-1} Q \begin{pmatrix} 1/\sqrt{n} & 0^T \\ 0 & n^p \sqrt{m/n} I_{k-1} \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{\mu}-1 & 0^T \\ 0 & O \end{pmatrix} \ \textit{in distr.}
$$

Proof of part (18): This part follows trivially from (15), realizing that by Assumption 3, $\theta^T C(1) C(1)^T \theta$ is positive, and that by Assumption 1 and the hypothesis $\mathfrak{C}(1)$, $Q_*^T M_2 Q_*$ has rank k-1, and is therefore nonsingular.

Proof of Theorem 2

The proof of part (24) of Theorem 2 for the case $r = 1$ is based on Mercer's theorem. Cf. Dunford and Schwartz (1963, p. 1088), and Bierens and Ploberger (1997). Let

$$
\Gamma(x,y) = E\left(\bar{W}_1(x)\bar{W}_1(y)\right).
$$

This function is real valued symmetric positive semi-definite, and it follows from (12) that Γ is continuous on $[0,1]\times[0,1]$. Now Mercer's theorem states that there exists a sequence λ_j of nonnegative eigenvalues and corresponding sequence $\psi_j(x)$ of real valued continuous eigenfunctions such that

$$
\int \Gamma(x,y)\psi_j(y)dy = \lambda_j \psi_j(x),\ j = 1,2,..,\ \Sigma_{j=1}^{\infty}\lambda_j < \infty,\ \int \psi_i(x)\psi_j(x)dx = I(i=j),
$$

where $I()$ is the indicator function, and

$$
\Gamma(x,y) = \sum_{j=1}^{\infty} \lambda_j \psi_j(x)\psi_j(y).
$$

If follows straightforwardly from (12) that $\int \bar{W}_1(x)dx = 0$ a.s., hence $\int \Gamma(x,y)dy = 0$. This implies that 1 is an eigenfunction corresponding to a zero eigenvalue. Since the eigenfunctions are orthogonal, all the other eigenfunctions are orthogonal to 1, and therefore satisfy $\int\psi_j(x)dx = 0$.

Moreover, if follows trivially from (12) that $\bar{W}_1(0) = \bar{W}_1(1) = 0$ a.s., hence $\Gamma(0,0) = \sum_{j=1}^{\infty}\lambda_j\psi_j(0)^2$ $= 0$ and $\Gamma(1,1) = \sum_{j=1}^{\infty}\lambda_j\psi_j(1)^2 = 0$. Consequently, the eigenfunction $\psi_j(x)$ corresponding to the positive eigenvalues satisfy $\psi_j(0) = \psi_j(1) = 0$.

We can now find sub-sequences $j_m(1),...,j_m(k\text{-}1)$ such that for $i,i_1,i_2 = 1,..,k\text{-}1$,

$$\int\psi_{j_m(i_1)}(x)\psi_{j_m(i_2)}(x)dx = I(i_1=i_2),$$

$$\int\psi_{j_m(i)}(x)dx = 0,\ \int x\psi_{j_m(i)}(x)dx = 0,\ \psi_{j_m(i)}(0) = \psi_{j_m(1)}(1) = 0,$$

$$\lim_{m\to\infty}\max_{1\le i\le k-1}\lambda_{j_m(i)} = 0.$$

hence, denoting

$$F_m(x) = Q_*\Lambda_*^{1/2}\begin{pmatrix}\psi_{j_m(1)}(x)\\ :\\ \psi_{j_m(k-1)}(x)\end{pmatrix}$$

we have

$$\int F_m(x)F_m(x)^T dx = M_1,\ F_m(0) = F_m(1) = 0,\ \int F_m(x)dx = 0,$$

and

$$\lim_{m\to\infty}E\left[\int\bar{W}_1(x)F_m(x)^T dx Q_*\Lambda_*^{-1}Q_*^T\int\bar{W}_1(y)F_m(y)dy\right]$$

$$= \lim_{m\to\infty}\sum_{i=1}^{k-1}\int\int\psi_{j_m(i)}(x)\Gamma(x,y)\psi_{j_m(i)}(y)dxdy = \lim_{m\to\infty}\sum_{i=1}^{k-1}\lambda_{j_m(i)} = 0.$$

Part (24) of Theorem 2 in the case $r = 1$ follows now from Chebishev's inequality.

Since the components of $\bar{W}_r(x)$ are independent, and each component is distributed as $\bar{W}_1(x)$, the general case follows straightforwardly from the proof in the case $r = 1$.

Part (25) of Theorem 2 follows directly from part (24). Q.E.D.

Proof of Theorem 6

Observe that

$$\frac{1}{\sqrt{n}}(\hat{\beta}_0 - \beta_0) = 4(1/n)\sum_{t=1}^{n} U_n(t/n) - 6(1/n)\sum_{t=1}^{n}(t/n)U_n(t/n) + o_p(1),$$

$$\sqrt{n}(\hat{\beta}_1 - \beta_1) = -6(1/n)\sum_{t=1}^{n} U_n(t/n) + 12(1/n)\sum_{t=1}^{n}(t/n)U_n(t/n) + o_p(1),$$

hence

$$\begin{aligned}\frac{z_{[nx]} - \hat{\beta}_0 - \hat{\beta}_1[nx]}{\sqrt{n}} &= U_n(x) - \frac{1}{\sqrt{n}}(\hat{\beta}_0 - \beta_0) - \sqrt{n}(\hat{\beta}_1 - \beta_1)\frac{[nx]}{n} \\ &= U_n(x) - 4(1/n)\sum_{t=1}^{n} U_n(t/n) + 6(1/n)\sum_{t=1}^{n}(t/n)U_n(t/n) \\ &+6x(1/n)\sum_{t=1}^{n} U_n(t/n) - 12x(1/n)\sum_{t=1}^{n}(t/n)U_n(t/n) + o_p(1), \\ &\Rightarrow C(1)\left(W_k(x) + (6x-4)\int W_k(y)dy - (12x-6)\int y\, W_k(y)dy\right) \\ &= C(1)W_k^*(x),\end{aligned} \tag{A.11}$$

say. Thus,

$$\hat{F}(x)/\sqrt{n} \Rightarrow C(1)\int_0^x W_k^*(y)dy = C(1)W_k^{**}(x),$$

say, and consequently

$$\frac{1}{n}\hat{M}_1 \rightarrow C(1)\int W_k^{**}(x)W_k^{**}(x)^T dx\, C(1)^T \tag{A.12}$$

in distr.

Now suppose that z_t is cointegrated with one cointegrating vector θ. Then $\theta^T C(1) = 0^T$. Since by (11), $z_t = \beta_0 + \beta_1 t + C(1)\Sigma_{j=1}^{t}\varepsilon_j + v_t - v_0$, with $v_t = D(L)\varepsilon_t$, we how have

$$\theta^T z_t = \theta^T(\beta_0 - v_0) + \theta^T\beta t + \theta^T v_t = \theta^T(\beta_0 - v_0) + \theta^T\beta t + \theta^T D(L)\varepsilon_t,$$

and consequently, similarly to (12),

$$\sqrt{n}\theta^T\hat{F}(x) \Rightarrow \theta^T D(1)\overline{W}_k(x).$$

Let $Q = (\theta, Q_*)$ be the orthogonal matrix of eigenvectors of $C(1)C(1)^T$, corresponding to the increasingly ordered eigenvalues. It is now easy to verify that under the unit root hypothesis with one cointegrated vector,

$$\frac{1}{n}\begin{pmatrix} n & 0^T \\ 0 & I_{k-1} \end{pmatrix} Q^T\hat{M}_1 Q \begin{pmatrix} n & 0^T \\ 0 & I_{k-1} \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} \theta^T D(1)\int\overline{W}_k(x)\overline{W}_k(x)^T dx D(1)^T\theta & \theta^T D(1)\int\overline{W}_k(x)W_k^{**}(x)^T dx C(1)^T Q_* \\ Q_*^T C(1)\int W_k^{**}(x)\overline{W}_k(x)^T D(1)^T\theta & Q_*^T C(1)\int W_k^{**}(x)W_k^{**}(x)^T dx C(1)^T Q_* \end{pmatrix}$$

in distr. Therefore, similarly to Lemma 2, it follows that for every nonnegative sequence $m = o(n)$ we have:

$$\begin{aligned} &\begin{pmatrix} 1/\sqrt{n} & 0^T \\ 0 & \sqrt{n}\sqrt{m/n}I_{k-1} \end{pmatrix} Q^T\hat{M}_1^{-1} Q \begin{pmatrix} 1/\sqrt{n} & 0^T \\ 0 & \sqrt{n}\sqrt{m/n}I_{k-1} \end{pmatrix} \\ &= \begin{pmatrix} 1/\sqrt{m} & 0^T \\ 0 & \sqrt{n}I_{k-1} \end{pmatrix} Q^T\frac{m}{n}\hat{M}_1^{-1} Q \begin{pmatrix} 1/\sqrt{m} & 0^T \\ 0 & \sqrt{n}I_{k-1} \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{\mu}_*^{-1} & 0^T \\ 0 & O \end{pmatrix} \end{aligned} \tag{A.13}$$

in distr., where

$$\tilde{\mu}_* = \theta^T D(1)\int \bar{W}_k(x)\bar{W}_k(x)^T dx D(1)^T\theta$$

$$- \theta^T D(1)\int \bar{W}_k(x)W_k^{**}(x)^T dx C(1)^T Q_*\left(Q_*^T C(1)\int W_k^{**}(x)W_k^{**}(x)dx C(1)^T Q_*\right)^{-1}$$

$$\times Q_*^T C(1)\int W_k^{**}(x)\bar{W}_k(x)^T dx D(1)^T\theta$$

$$\sim \theta^T D(1)D(1)^T\theta$$

$$\times\left(\int\left(\bar{W}_1(x)\right)^2 dx - \int \bar{W}_1(x)W_{k-1}^{**}(x)^T dx\left(\int W_{k-1}^{**}(x)W_{k-1}^{**}(x)dx\right)^{-1}\int W_{k-1}^{**}(x)\bar{W}_1(x)dx\right)$$

Next we investigate the asymptotic properties of the matrix $\hat{M}_2$ under the unit root with cointegration hypothesis. Let again m be a sequence of natural numbers converging to infinity at rate $o(n)$. Then it follows from (11) that for $x \in [0,1]$,

$$\frac{1}{m\sqrt{n}}\sum_{j=0}^{m-1}\left(z_{[nx]-j} - \hat{\beta}_0 - \hat{\beta}_1([nx]-j)\right) =$$

$$= \frac{1}{m}\sum_{j=0}^{m-1} U_n\left(\frac{[nx]-j}{n}\right) - 4(1/n)\sum_{t=1}^{n} U_n(t/n) + 6(1/n)\sum_{t=1}^{n}(t/n)U_n(t/n)$$

$$+6\frac{[nx]}{n}(1/n)\sum_{t=1}^{n} U_n(t/n) - 12\frac{[nx]}{n}(1/n)\sum_{t=1}^{n}(t/n)U_n(t/n) + o_p(1).$$

Moreover,

$$\frac{1}{m}\sum_{j=0}^{m-1} U_n\left(\frac{[nx]-j}{n}\right) = \frac{1}{m}\sum_{j=0}^{m-1} U_n\left(\frac{[nx]-m+1+j}{n}\right) = \frac{1}{m}\sum_{j=0}^{m-1}\int_j^{j+1} U_n\left(\frac{[nx]-m+1+y}{n}\right)dy$$

$$= \frac{1}{m}\int_0^m U_n\left(\frac{[nx]-m+1}{n}+\frac{m}{n}\frac{y}{m}\right)dy = \int_0^1 U_n\left(\frac{[nx]}{n}-\frac{m-1}{n}+\frac{m}{n}y\right)dy = U_n(x) + o_p(1)$$

Therefore

$$\frac{1}{m\sqrt{n}}\sum_{j=0}^{m-1}\left(z_{[nx]-j} - \hat{\beta}_0 - \hat{\beta}_1([nx]-j)\right) \Rightarrow C(1)W_k^*(x).$$

and consequently,

$$\frac{\hat{M}_2}{n} \to C(1)\int W_k^*(x)W_k^*(x)^T dx\, C(1)^T \tag{A.14}$$

in distr. Furthermore, it is easy to verify that

$$\frac{m}{n}\theta^T\hat{M}_2\theta = \theta^T D(1)D(1)^T\theta + o_p(1),$$

and

$$\frac{\sqrt{m}}{n}\hat{M}_2\theta = o_p(1).$$

Thus,

$$\begin{pmatrix} \sqrt{m} & 0^T \\ 0 & \frac{1}{\sqrt{n}}I_{k-1} \end{pmatrix} Q^T\hat{M}_2 Q \begin{pmatrix} \sqrt{m} & 0^T \\ 0 & \frac{1}{\sqrt{n}}I_{k-1} \end{pmatrix} \to$$
$$\begin{pmatrix} \theta^T D(1)D(1)^T\theta & 0^T \\ 0 & Q_*^T C(1)\int W_k^*(x)W_k^*(x)^T dx\, C(1)^T Q_* \end{pmatrix}$$

in distr, and therefore, similarly to Lemma 2, we have

$$\begin{pmatrix} 1/\sqrt{m} & 0^T \\ 0 & \sqrt{n}I_{k-1} \end{pmatrix} Q^T\hat{M}_2^{-1} Q \begin{pmatrix} 1/\sqrt{m} & 0^T \\ 0 & \sqrt{n}I_{k-1} \end{pmatrix}$$
$$\to \begin{pmatrix} \left(\theta^T D(1)D(1)^T\theta\right)^{-1} & 0^T \\ 0 & \left(Q_*^T C(1)\int W_k^*(x)W_k^*(x)^T dx\, C(1)^T Q_*\right)^{-1} \end{pmatrix} \tag{A.15}$$

in distr.

Comparing (13) and (15) with eigenvalue problem (20), we see that under the unit root hypothesis with single cointegration and $m = [n^{\alpha}]$, with $0 < \alpha < 1$, the minimum solution $\hat{\lambda}_1$ of eigenvalue problem (19) satisfies:

$$n^{1-\alpha}\hat{\lambda}_1 \rightarrow \frac{\tilde{\mu}_*}{\theta^T D(1)D(1)^T\theta}$$

$$= \int\left(\overline{W}_1(x)\right)^2 dx - \int \overline{W}_1(x) W_{k-1}^{**}(x)^T dx\left(\int W_{k-1}^{**}(x)W_{k-1}^{**}(x)dx\right)^{-1}\int W_{k-1}^{**}(x)\overline{W}_1(x)dx$$

in distr.

Note that the limiting random variable involved has the same upperbound as in the case of co-trending. Moreover, it follows from (12) and (14) that under the unit root hypothesis without cointegration, $\hat{\lambda}_1$ converges in distribution to the minimum solution of the generalized eigenvalue problem

$$\det\left[\int W_{k-1}^{**}(x)W_{k-1}^{**}(x)^T dx - \lambda\int W_{k-1}^{*}(x)W_{k-1}^{*}(x)^T dx\right] = 0.$$

This completes the proof of Theorem 6 for the case of a k-variate unit root process with drift and one cointegrating vector. The general case with multiple cointegrating vectors can be shown along similar lines.

In the case of a k-variate unit root process z_t without drift with one cointegrating vector θ and z_t demeaned rather than detrended, we have $z_t = z_0 + \Sigma_{j=1}^{t} u_j = z_0 - v_0 + C(1)\Sigma_{j=1}^{t}\varepsilon_j + v_t$, where $\bar{z} = (1/n)\Sigma_{t=1}^{n} z_t$ and again $v_t = D(L)\varepsilon_t$. Hence

$$(z_{nx} - \bar{z})/\sqrt{n} = U_n(x) - (1/n)\sum_{t=1}^{n} U_n(t/n) + o_p(1) \Rightarrow C(1)\left(W_k(x) - \int W_k(y)dy\right) = C(1)\tilde{W}_k^{*}(x),$$

and $\theta^T(z_t - \bar{z}) = v_t - \bar{v}$, where $\bar{v} = (1/n)\Sigma_{t=1}^{n} v_t$. The latter implies

$$\sqrt{n}\theta^T\hat{F}(x) \Rightarrow \theta^T D(1) W_k^0(x).$$

The rest of the proof is now similar to the previous case. Q.E.D.