Theoretical homework 1

The following are midterm and final exam questions of ECON 490, Spring 2004. At the end of the exam you will find the table of the standard normal distribution function and the table of critical values of the two-sided *t* test. You have to determine yourself which one to use.

1. Let X and Y be independent normally distributed with $E(X) = 2$, $var(X) = 9$, $E(Y) = 2$, $var(Y) = 4$, and let $Z = X - 2Y$.

(a) Compute $\Pr[-7 < Z \leq 8]$. [10 points]

(b) Compute $E[Z|X = 2]$. [10 points]

2. Roll a fair die. The outcome of this statistical experiment is one of the numbers 1 through 6 (i.e., the number of dots on the side of the die facing up). Let Y be this outcome, and let $X = 1$ if the outcome is even, and $X = 0$ if the outcome is odd.

(a) Compute for $y = 1,2,...,6$ and $x = 0,1$ the conditional probabilities $P[Y = y|X = x]$. [10 points]

(b) Compute $E[Y|X = 0]$ and $E[Y|X = 1]$. [10 points]

(c) Recall that we can always write $E[Y|X] = g(X)$ for some function g. Determine $g(X)$. [5 points]

(d) Let $U = Y - E[Y|X]$. Compute[1] $cov(U,X)$. (Motivate your answer: just a number does not suffice). [10 points]

[1] You may need to use the law of iterated expectations, $E[E(Y|X)] = E[Y]$, and the fact that $E[X.Y|X] = X.E[Y|X]$.

3. Let Y_j be the monthly expenditures on food of household j, and let X_j be the monthly income of household j. Suppose you have estimated the regression model $\ln(Y_j) = \alpha + \beta\ln(X_j) + U_j$ on the basis of a random sample of size $n = 25$, with results

$$\ln(Y) = \underset{(1.5)}{2.5} + \underset{(2.5)}{0.5}\ln(X)$$

The numbers between brackets are the t-values. Thus, the OLS estimators $\hat{\alpha}$ and $\hat{\beta}$ of α and β, respectively, are $\hat{\alpha} = 2.5$ and $\hat{\beta} = 0.5$. You may assume that $U_j \sim N(0,\sigma^2)$ and treat the X_j's as being nonrandom.

(a) Compute the standard errors of $\hat{\alpha}$ and $\hat{\beta}$. [5 points]

(b) Test the significance of each of the parameters at the 5% significance level. [10 points]

(c) Test the null hypothesis $\beta = 1$ against the alternative hypothesis $\beta \neq 1$ at the 5% significance level. [10 points]

(d) The sum of squared residuals[2] is SSR = 23. How would you compute an unbiased[3] estimator $\hat{\sigma}^2$ of the error variance σ^2? [10 points]

(e) The total sum of squares[4] of this regression is TSS = 29.25. Compute the overall F statistic (i.e., the test statistic of the F test[5] that $\ln(X_j)$ does not affect $\ln(Y_j)$). [10 points]

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[2] $\text{SSR} = \sum_{j=1}^{n}\hat{U}_j^2 = \sum_{j=1}^{n}(\ln(Y_j) - \hat{\alpha} - \hat{\beta}\ln(X_j))^2$.

[3] Recall that an estimator of a parameter is unbiased if the expectation of the estimator is equal to the parameter to be estimated.

[4] $\text{TSS} = \sum_{j=1}^{n}(\ln(Y_j) - \overline{\ln(Y)})^2$, where $\overline{\ln(Y)} = (1/n)\sum_{j=1}^{n}\ln(Y_j)$.

[5] In general, if SSR is the sum of squared residuals of a linear regression model with k parameters and SSR_0 is the sum of squared residuals of this regression after imposing m linear restrictions on the parameters ($m < k$), then under the imposed parameter restrictions and some regularity conditions the F-test statistic

$$\hat{F} = \frac{(\text{SSR}_0 - \text{SSR})/m}{\text{SSR}/(n-k)}$$

has an F distribution with m and n-k degrees of freedom, where n is the sample size.

4. In the fall of 1995 more than 15 million students were enrolled in institutions of higher education in the United States, divided among 4-year institutions, 2-year institutions, and less than 2-year institutions. Draw randomly four students from this population. It turns out that two of these students are males enrolled in a 2-year institution, and the other two students are females enrolled in a 4-year institution. Let Y_j be the tuition fee paid by student j, and let X_j =1 if student j is a female, and X_j = 0 is student j is a male. Thus, your random sample is

j	Y_j	X_j
1	2800	0
2	2800	0
3	7000	1
4	7000	1

Estimate the regression model $Y_j = \alpha + \beta X_j + U_j$, $j = 1,2,3,4,$ by OLS:

(a) Compute the OLS estimates $\hat{\alpha}$ and $\hat{\beta}$ of α and β, respectively. [10 points]

(b) Estimate the standard error σ of U_j and the standard errors of $\hat{\alpha}$ and $\hat{\beta}$. [10 points]

(c) Compute the R^2 of this regression. [5 points]

(d) Are the results in (b) and (c) surprising? Explain your answer. [5 points]

Value of the distribution function $F(x)$ of the standard normal distribution for x the sum of a value in the first column and the first row.

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

Critical values of the two-sided t test

with m degrees of freedom

% significance levels

m	5%	10%
1	12.704	6.313
2	4.303	2.920
3	3.183	2.353
4	2.776	2.132
5	2.571	2.015
6	2.447	1.943
7	2.365	1.895
8	2.306	1.859
9	2.262	1.833
10	2.228	1.813
11	2.201	1.796
12	2.179	1.782
13	2.160	1.771
14	2.145	1.761
15	2.131	1.753
16	2.120	1.746
17	2.110	1.740
18	2.101	1.734
19	2.093	1.729
20	2.086	1.725
21	2.080	1.721
22	2.074	1.717
23	2.069	1.714
24	2.064	1.711
25	2.059	1.708
26	2.056	1.706
27	2.052	1.703
28	2.048	1.701
29	2.045	1.699
30	2.042	1.697

m = degrees of freedom