Let $\{X_j\}_{j=1}^n$be random sample from the normal distribution with unknown expectation θ_0 and unknown variance σ^2. However, it is known that θ_0 is contained in the interval $\Theta = [-10, 10]$. An alternative to using the sample mean as an estimator of θ_0 is to use as estimator for θ_0 the solution $\widehat{\theta}_n$ of the maximization problem $\max \frac{1}{n} \sum_{j=1}^n \exp\left(-(X_j - \theta)^2\right)$ subject to $\theta \in \Theta$, i.e.,

$$\widehat{\theta}_n = \operatorname*{argmax}_{\theta \in \Theta} \frac{1}{n} \sum_{j=1}^{n} \exp\left(-(X_j - \theta)^2\right).$$

Prove that

$$\operatorname*{plim}_{n\to\infty} \widehat{\theta}_n = \theta_0$$

by verifying the conditions of Theorems 6.10 and 6.11.

Hint: Denote $g(X_j, \theta) = \exp\left(-(X_j - \theta)^2\right)$. Show first that

$$E\left[g(X_j, \theta)\right] = \kappa. \exp\left(-\kappa^2(\theta - \theta_0)^2\right)$$

for some constant $\kappa > 0$.