1 The CES production function

The homogenous **C**onstant **E**lasticity of **S**ubstitution (CES) production function takes the form

$$Q=\gamma\left\{\alpha K^{-\rho}+(1-\alpha)L^{-\rho}\right\}^{-1/\rho}\exp(U),\ \rho\geq-1,\ 0\leq\alpha\leq1,\ \gamma>0, \quad (1)$$

where K is capital, L is labor, Q is output, and U is an error term satisfying $E[U|K,L]=0$.

The CES production function was introduced by Arrow, Chenery, Minhas and Solow[1] in 1961 Formally, the elasticity of substitution measures the percentage change in factor proportions due to a percentage change in the marginal rate of technical substitution. In particular, for a canonical production function $Q=f(K,L)$ with marginal products $f_K=\partial f(K,L)/\partial K$ and $f_L=\partial f(K,L)/\partial L$, the marginal rate of technical substitution is f_L/f_K , i.e., minus the slope of the isoquant in point (L,K), hence the elasticity of substitution between capital and labor is given by:

$$\sigma=-d\ln(L/K)/d\ln(f_L/f_K).$$

In the case of the deterministic homogenous CES production function $Q=\gamma\left\{\alpha K^{-\rho}+(1-\alpha)L^{-\rho}\right\}^{-1/\rho}$ we have

$$\begin{aligned} f_L &= \gamma\left\{\alpha K^{-\rho}+(1-\alpha)L^{-\rho}\right\}^{-(1+\rho)/\rho}(1-\alpha)L^{-\rho-1}, \\ f_K &= \gamma\left\{\alpha K^{-\rho}+(1-\alpha)L^{-\rho}\right\}^{-(1+\rho)/\rho}\alpha K^{-\rho-1}, \end{aligned}$$

hence $d\ln(f_L/f_K)=-(\rho+1)d\ln(L/K)$ and thus $\sigma=1/(\rho+1)$.

In order to estimate the parameters of the CES production function via EasyReg, rewrite (1) as

$$\begin{aligned} \ln(Q/L) &= \ln(\gamma)-\frac{1}{\rho}\ln\left\{\alpha\left[\exp\left(\rho\ln(L/K)\right)-1\right]+1\right\}+U \quad (2) \\ &= \ln(\gamma)-\frac{\ln\left\{\alpha\left[\exp\left(\rho\ln(L/K)\right)-1\right]+1\right\}}{\alpha\left[\exp\left(\rho\ln(L/K)\right)-1\right]} \\ &\quad \times\frac{\exp\left(\rho\ln(L/K)\right)-1}{\rho\ln(L/K)}\times\left(\alpha\ln(L/K)\right)+U. \end{aligned}$$

[1] Arrow, K.J., H.B. Chenery, B.S. Minhas and R.M. Solow (1961), "Capital-Labor Substitution and Economic Efficiency", *Review of Economics and Statistic.*

It is an elementary calculus exercise to verify that

$$\begin{aligned}\lim_{\rho\to 0}\frac{\ln\left\{\alpha\left[\exp\left(\rho\ln(L/K)\right)-1\right]+1\right\}}{\alpha\left[\exp\left(\rho\ln(L/K)\right)-1\right]} &= \lim_{\delta\to 0}\frac{\ln(1+\delta)-\ln(1)}{\delta}\\ &= \left.\frac{d\ln(x)}{dx}\right|_{x=1}=1,\end{aligned}$$

$$\lim_{\rho\to 0}\frac{\exp\left(\rho\ln(L/K)\right)-1}{\rho\ln(L/K)}=\lim_{\delta\to 0}\frac{\exp(\delta)-\exp(0)}{\delta}=\left.\frac{d\exp(x)}{dx}\right|_{x=0}=1,$$

hence it follows that for $\rho\to 0$ (2) becomes $\ln(Q/L)=\ln(\gamma)-\alpha\ln(L/K)+U$. Thus, for $\rho=0$ the CES production function becomes a Cobb-Douglas production function:

$$\ln(Q)=\ln(\gamma)+\alpha\ln(K)+(1-\alpha)\ln(L)+U. \tag{3}$$

The CES production function (2) is now a nonlinear regression model,

$$Y=g(x,b)+U,$$

where $Y=\ln(Q/L)$, $x=\ln(L/K)$, $b=(b(1),b(2),b(3))'=(\ln(\gamma),\rho,\alpha)'$, and

$$\begin{aligned}g(x,b) &= b(1)-\frac{\ln\left\{b(3)\left[\exp\left(b(2)x\right)-1\right]+1\right\}}{b(3)\left[\exp\left(b(2)x\right)-1\right]}\\ &\quad\times\frac{\exp\left(b(2)x\right)-1}{b(2)x}\times\left(b(3)x\right).\end{aligned} \tag{4}$$

2 Specifying the CES production function in EasyReg

2.1 Selection of the initial X variables

In EasyReg a nonlinear regression function is build up recursively by augmenting the list of X variables with nonlinear transformations and linear and/or multiplicative combinations of previous X variables. In order to build up (4), we have to select two initial X variables, $X(1)=\ln(L/K)$ and $X(2)=1$. The latter is necessary for two reasons: First, EasyReg does not allow to specify constants directly. Second the parameters $b(2)$ and $b(3)$ are common to different transformations. The constant $X(2)=1$ enables us to associate them to new X variables.

2.2 Storing the parameters in X variables

Thus, we need to specify three new X variables first, each associated with a parameter, by selecting $X(2)$ and choosing the "Linear combination" option three times: $X(3) = b(1)X(2)$, $X(4) = b(2)X(2)$, $X(5) = b(3)X(2)$. Then (4) becomes:

$$\begin{aligned} g(x,b) &= X(3) - \frac{\ln\left\{X(5)\left[\exp\left(X(1)X(4)\right) - X(2)\right] + 1\right\}}{X(5)\left[\exp\left(X(1)X(4)\right) - X(2)\right]} \\ &\times \frac{\exp\left(X(1)X(4)\right) - 1}{X(1)X(4)} \times X(1)X(5), \end{aligned}$$

Now the CES production function involved is build up recursively as follows.

$X(.)$		*Transformation*	*Actual transformation*	*Transformation option*
$X(1)$	=	$\ln(L/K)$		
$X(2)$	=	1		
$X(3)$	=	$b(1)X(2) = b(1)$	$\ln(\gamma)$	Linear combination
$X(4)$	=	$b(2)X(2) = b(2)$	ρ	Linear combination
$X(5)$	=	$b(3)X(2) = b(3)$	α	Linear combination
$X(6)$	=	$X(1)X(4)$	$\rho\ln(L/K)$	Multiply
$X(7)$	=	$\exp(X6))$	$\exp\left(\rho\ln(L/K)\right)$	EXP(z) [z = $X(6)$]
$X(8)$	=	$X(7) - X(2)$	$\exp\left(\rho\ln(L/K)\right) - 1$	Subtract
$X(9)$	=	$X(5)X(8)$	$\alpha\left(\exp\left(\rho\ln(L/K)\right) - 1\right)$	Multiply
$X(10)$	=	$\ln\left(X(9)+1\right)/X(9)$	$\frac{\ln(\alpha(\exp(\rho\ln(L/K))-1)+1)}{\alpha(\exp(\rho\ln(L/K))-1)}$	LOG(z+1)/z [z = $X(9)$]
$X(11)$	=	$\left(\exp\left(X(6)\right) - 1\right)/X(6)$	$\frac{\exp(\rho\ln(L/K))-1}{\rho\ln(L/K)}$	(EXP(z)-1)/z [z = $X(6)$]
$X(12)$	=	$X(1)X(5)X(10)X(11)$	$\alpha\times\ln(L/K) \times\frac{\ln(\alpha(\exp(\rho\ln(L/K))-1)+1)}{\alpha(\exp(\rho\ln(L/K))-1)} \times\frac{\exp(\rho\ln(L/K))-1}{\rho\ln(L/K)}$	Multiply
$X(13)$	=	$X(3) - X(12)$	$\ln(\gamma) - \alpha\times\ln(L/K) \times\frac{\ln(\alpha(\exp(\rho\ln(L/K))-1)+1)}{\alpha(\exp(\rho\ln(L/K))-1)} \times\frac{\exp(\rho\ln(L/K))-1}{\rho\ln(L/K)} = g(b,x)$	Subtract

As you see, this is actually a simple computer program, like a macro in MS Word or Corel Wordperfect.

2.3 The non-homogenous CES production function

The general CES production function takes the form

$$Q = \gamma \left\{ \alpha K^{-\rho} + (1-\alpha) L^{-\rho} \right\}^{-\tau/\rho} \exp(U),\ \rho \geq -1,\ 0 \leq \alpha \leq 1,\ \gamma > 0,\ \tau > 0, \tag{5}$$

where τ is the degree of homogeneity: if K and L are both increased with a factor λ, then Q increases with a factor $\tau\lambda$. Thus, if $\tau > 1$ we have increasing return to scale, and if $\tau < 1$ we have decreasing returns to scale. In this case (2) becomes

$$\begin{aligned} \ln(Q/L) &= \ln(\gamma) - \frac{\tau}{\rho} \ln \left\{ \alpha \left[\exp\left(\rho \ln(L/K) \right) - 1 \right] + 1 \right\} + U \qquad (6) \\ &= \ln(\gamma) - \tau \frac{\ln \left\{ \alpha \left[\exp\left(\rho \ln(L/K) \right) - 1 \right] + 1 \right\}}{\alpha \left[\exp\left(\rho \ln(L/K) \right) - 1 \right]} \\ &\quad \times \frac{\exp\left(\rho \ln(L/K) \right) - 1}{\rho \ln(L/K)} \times \left(\alpha \ln(L/K) \right) + U. \end{aligned}$$

The specification of (6) in EasyReg is the same as steps $X(1)$ through $X(12)$, with $X(13)$ replaced with the following three steps:

X		*Transformation*	*Transformation option*
$X(13)$	$=$	$b(4)X(2) = b(4)\ (= \tau)$	Linear combination
$X(14)$	$=$	$X(12)X(13)$	Multiply
$X(15)$	$=$	$X(3) - X(14)$	Subtract