VAR models with exogenous variables

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February 21, 2004

Abstract

In this note I will address the question how to include exogenous variables in a VAR model, and what the consequences are for the innovation response analysis.

1 VARX models

Consider a VAR model with exogenous variables:

$$\begin{aligned} Y_t &= a_0 + A_1 Y_{t-1} + ... + A_p Y_{t-p} \\ &\quad + B_1 X_{t-1} + ... + B_q X_{t-q} + U_t, \end{aligned} \tag{1}$$

where $Y_t \in \mathbb{R}^k$, $X_t \in \mathbb{R}^m$ is a vector of exogenous variables, $a_0 \in \mathbb{R}^k$ is a vector of intercepts, the A_j's are $k \times k$ coefficient matrices, the B_i's are $k \times m$ coefficient matrices, and $U_t \in \mathbb{R}^k$ is the vector of errors. This is an VARX model. The crucial condition for the correctness of this model is that

$$E\left[U_t | \{Y_{t-j}\}_{j=1}^{\infty}, \{X_{t-i}\}_{i=1}^{\infty}\right] = 0 \ (\in \mathbb{R}^k) \tag{2}$$

with probability 1.

Next, assuming a VAR model for X_t itself, say

$$\begin{aligned} &X_t = c_0 + C_1 X_{t-1} + ... + C_r X_{t-r} + V_t, \\ &E\left[V_t | \{Y_{t-j}\}_{j=1}^{\infty}, \{X_{t-i}\}_{i=1}^{\infty}\right] = 0 \ (\in \mathbb{R}^m). \end{aligned} \tag{3}$$

Note that model (3) implies that Y_t does not Granger-cause X_t, which is a weak form of exogeneity.

In principle one could also include a contempaneous X_t in (3):

$$\begin{aligned} Y_t &= a_0 + A_1 Y_{t-1} + ... + A_p Y_{t-p} \\ &\quad + B_0 X_t + B_1 X_{t-1} + ... + B_q X_{t-q} + U_t. \end{aligned} \tag{4}$$

However, if $B_0 \neq O$ it may be possible that Y_t has an indirect impact on X_t via possible mutual dependence of U_t and V_t. If so then (4) and (3) together form a (block-triangular) system of simultaneous equations. However, we can write (4) as a reduced form VARX model by substituting (3) in (4),

$$\begin{aligned} Y_t &= a_0 + B_0 c_0 + A_1 Y_{t-1} + ... + A_p Y_{t-p} \\ &\quad + B_0 \left(C_1 X_{t-1} + ... + C_r X_{t-r} \right) \\ &\quad + B_1 X_{t-1} + ... + B_q X_{t-q} + U_t + B_0 V_t, \end{aligned}$$

which is of the form (1). Thus, the absence of a contemporaneous X_t in (1) is no loss of generality.

2 Exogeneity

By allowing some of the coefficient matrices A_j, B_j and C_j to be zero matrices we may assume that $p = q = r$:

$$\begin{aligned} Y_t &= a_0 + A_1 Y_{t-1} + ... + A_p Y_{t-p} + B_1 X_{t-1} + ... + B_p X_{t-p} + U_t, \\ X_t &= c_0 + C_1 X_{t-1} + ... + C_p X_{t-p} + V_t. \end{aligned} \tag{5}$$

Then we can write model (5) as a VAR(p) model with Granger-causality restrictions imposed:

$$\begin{aligned} \begin{pmatrix} Y_t \\ X_t \end{pmatrix} &= \begin{pmatrix} a_0 \\ c_0 \end{pmatrix} + \begin{pmatrix} A_1 & B_1 \\ O & C_1 \end{pmatrix} \begin{pmatrix} Y_{t-1} \\ X_{t-1} \end{pmatrix} + ... \\ &\quad + \begin{pmatrix} A_p & B_p \\ O & C_p \end{pmatrix} \begin{pmatrix} Y_{t-p} \\ X_{t-p} \end{pmatrix} + \begin{pmatrix} U_t \\ V_t \end{pmatrix}. \end{aligned} \tag{6}$$

As said before, the Granger-causality restriction involved is a weak form of exogeneity:

Weak exogeneity of X_t:
Y_t does not Granger-cause X_t.

The usual assumption in VAR analysis is that the errors are i.i.d. normally distributed:

$$\begin{pmatrix} U_t \\ V_t \end{pmatrix} \sim \text{ i.i.d. } N_{k+m}\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\right]. \qquad (7)$$

A stronger exogeneity condition, in addition to the condition that Y_t does not Granger-cause X_t, is now that $\Sigma_{12} = O$ and thus $\Sigma_{21} = \Sigma_{12}' = O$. This implies independence of U_t and V_t:

Strong exogeneity of X_t:
Y_t does not Granger-cause X_t, and the error vectors U_t and V_t are independent.

3 Testing weak exogeneity

Setup the VAR model as an unrestricted VAR:

$$\begin{aligned} \begin{pmatrix} Y_t \\ X_t \end{pmatrix} &= \begin{pmatrix} a_0 \\ c_0 \end{pmatrix} + \begin{pmatrix} A_1 & B_1 \\ D_1 & C_1 \end{pmatrix}\begin{pmatrix} Y_{t-1} \\ X_{t-1} \end{pmatrix} + \dots \\ &\quad + \begin{pmatrix} A_p & B_p \\ D_p & C_p \end{pmatrix}\begin{pmatrix} Y_{t-p} \\ X_{t-p} \end{pmatrix} + \begin{pmatrix} U_t \\ V_t \end{pmatrix}, \end{aligned} \qquad (8)$$

say, and test the joint null hypothesis

$$H_0\colon\ D_1 = D_2 = \dots = D_p = 0,$$

using the Wald test option in EasyReg. If you do not reject this weak exogeneity hypothesis, and if you are not interested in the strong exogeneity hypothesis, you may re-estimate the VAR with the Granger causality restrictions imposed, in the form (6), and conduct VAR innovation response analysis, similarly to the example Y_t = DIF1[LN[Income Sweden]], X_t = DIF1[LN[nominal GDP]] in the section *Granger-causality testing in practice* of EasyReg's *Guided tour on VAR innovation response analysis.*

4 Testing strong exogeneity

The variance matrix in (7) can be written as

$$\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} = \begin{pmatrix} L_{11} & O \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} L_{11} & O \\ L_{21} & L_{22} \end{pmatrix}' = L.L',$$

say, where L_{11} and L_{22} are lower-triangular matrices, so that L is a lower-triangular matrix. EasyReg provides the option to test the joint significance of elements of the matrix L. Thus, test the null hypothesis

$$H_0\colon\ L_{21} = O.$$

This hypothesis is equivalent to the hypotheses that $\Sigma_{12} = O$ and $\Sigma_{21} = O$.

You can conduct this test after having re-estimated the VAR in the form (6), but it is better to conduct this test jointly with the test of weak exogeneity, using the unrestricted VAR (8), by testing the joint hypotheses

$$H_0\colon\ D_1 = D_2 = ... = D_p = 0,\ L_{21} = O.$$

Of course, if you have rejected the weak exogeneity hypothesis it makes no sense to test the strong exogeneity hypothesis.

5 Innovation response analysis under the strong exogeneity condition

In structural VAR analysis, the non-structural errors are related to the innovation via a matrix B,[1] say. In our case,

$$B \begin{pmatrix} U_t \\ V_t \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} U_t \\ V_t \end{pmatrix} = \begin{pmatrix} e_{1,t} \\ e_{2,t} \end{pmatrix} = e_t,$$

[1] Not related to the matrices B_i in models (1), (6) and (8).

say, where

$$e_t = \begin{pmatrix} e_{1,t} \\ e_{2,t} \end{pmatrix} \sim N_{k+m}[0, I_{k+m}],$$

and the errors U_t and V_t are those of the restricted model (6).

Now specify B as

$$B = \begin{pmatrix} \Delta_{11} & O \\ O & \Delta_{22} \end{pmatrix},$$

where Δ_{11} and Δ_{22} are lower-triangular matrices. Then under the strong exogeneity condition

$$\begin{aligned} I_{k+m} &= \operatorname{Var}\left[B\begin{pmatrix} U_t \\ V_t \end{pmatrix}\right] \\ &= \begin{pmatrix} \Delta_{11} & O \\ O & \Delta_{22} \end{pmatrix}\begin{pmatrix} \Sigma_{11} & O \\ O & \Sigma_{22} \end{pmatrix}\begin{pmatrix} \Delta_{11} & O \\ O & \Delta_{22} \end{pmatrix}' \\ &= \begin{pmatrix} \Delta_{11} & O \\ O & \Delta_{22} \end{pmatrix}\begin{pmatrix} L_{11} & O \\ O & L_{22} \end{pmatrix}\begin{pmatrix} L_{11} & O \\ O & L_{22} \end{pmatrix}'\begin{pmatrix} \Delta_{11} & O \\ O & \Delta_{22} \end{pmatrix}' \\ &= \begin{pmatrix} \Delta_{11}L_{11}L_{11}'\Delta_{11}' & O \\ O & \Delta_{22}L_{22}L_{22}'\Delta_{22}' \end{pmatrix}, \end{aligned}$$

which holds for $\Delta_{11} = L_{11}^{-1}$ and $\Delta_{22} = L_{22}^{-1}$.

Innovation response analysis can now be conducted on the basis of this structural model. Because now

$$\begin{pmatrix} U_t \\ V_t \end{pmatrix} = \begin{pmatrix} \Delta_{11}^{-1}e_{1,t} \\ \Delta_{22}^{-1}e_{2,t} \end{pmatrix}, \tag{9}$$

where $e_{1,t}$ and $e_{2,t}$ are independent, and because the VAR model is restricted such that Y_t does not Granger-cause X_t, a shock in one of the innovations in $e_{1,t}$ will have no effect on X_t and its future values.

To see this, write model (6) in VMA(∞) form:

$$\begin{aligned} \begin{pmatrix} Y_t \\ X_t \end{pmatrix} &= \left(I_{k+m} - \sum_{j=1}^{p}\begin{pmatrix} A_j & B_1 \\ O & C_j \end{pmatrix}\right)^{-1}\begin{pmatrix} a_0 \\ c_0 \end{pmatrix} \\ &\quad \left(I_{k+m} - \sum_{j=1}^{p}\begin{pmatrix} A_j & B_1 \\ O & C_j \end{pmatrix}L^j\right)^{-1}\begin{pmatrix} U_t \\ V_t \end{pmatrix} \end{aligned} \tag{10}$$

$$= \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} + \sum_{i=0}^{\infty} \begin{pmatrix} \Gamma_{11,i} & \Gamma_{12,i} \\ \Gamma_{21,i} & \Gamma_{22,i} \end{pmatrix} \begin{pmatrix} U_{t-i} \\ V_{t-i} \end{pmatrix},$$

say, where L is now the lag operator, and

$$\sum_{i=0}^{\infty} \begin{pmatrix} \Gamma_{11,i} & \Gamma_{12,i} \\ \Gamma_{21,i} & \Gamma_{22,i} \end{pmatrix} L^i = \left(I_{k+m} - \sum_{j=1}^{p} \begin{pmatrix} A_j & B_1 \\ O & C_j \end{pmatrix} L^j \right)^{-1}.$$

Similarly, X_t can be written in VMA(∞) form as

$$X_t = \gamma_2 + \sum_{i=0}^{\infty} \Gamma^*_{2,i} V_{t-i}. \tag{11}$$

Comparing (10) and (11), it follows that $\Gamma_{21,i} = O$ and $\Gamma^*_{2,i} = \Gamma_{22,i}$, so that after substituting (9) in (10) the latter becomes

$$\begin{pmatrix} Y_t \\ X_t \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} + \sum_{i=0}^{\infty} \begin{pmatrix} \Gamma_{11,i}\Delta_{11}^{-1} & \Gamma_{12,i}\Delta_{22}^{-1} \\ O & \Gamma_{22,i}\Delta_{22}^{-1} \end{pmatrix} \begin{pmatrix} e_{1,t-i} \\ e_{2,t-i} \end{pmatrix}.$$

The innovation responses are now defined by

$$E\left[\begin{pmatrix} Y_{t+h} \\ X_{t+h} \end{pmatrix} \middle| e_{1,t-i}, e_{2,t-i} \right] - E\left[\begin{pmatrix} Y_{t+h} \\ X_{t+h} \end{pmatrix} \right] \tag{12}$$
$$= \begin{pmatrix} \Gamma_{11,h}\Delta_{11}^{-1} & \Gamma_{12,h}\Delta_{22}^{-1} \\ O & \Gamma_{22,h}\Delta_{22}^{-1} \end{pmatrix} \begin{pmatrix} e_{1,t} \\ e_{2,t} \end{pmatrix},\ h = 0, 1, 2, ...$$

Finally, consider the response of Y_t and X_t to a unit shock in say the innovation in the first component of Y_t. Thus, replace $e_{1,t}$ in (12) by $\iota_1 = (1, 0, 0, ..., 0)' \in \mathbb{R}^k$ and $e_{2,t}$ by $0 \in \mathbb{R}^m$. Then the resulting innovation responses are

$$E\left[\begin{pmatrix} Y_{t+h} \\ X_{t+h} \end{pmatrix} \middle| e_{1,t-i} = \iota_1, e_{2,t-i} = 0 \right] - E\left[\begin{pmatrix} Y_{t+h} \\ X_{t+h} \end{pmatrix} \right]$$
$$= \begin{pmatrix} \Gamma_{11,h}\Delta_{11}^{-1}\iota_1 \\ 0 \end{pmatrix},\ h = 0, 1, 2, ...$$