

The matrix norm $\|A\| = \sqrt{\text{trace}(AA')}$

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Let A be an $k \times m$ matrix. Define the matrix norm $\|\cdot\|$ by

$$\|A\| = \sqrt{\text{trace}(AA')} \quad (1)$$

Recall that a norm needs to satisfy three conditions

$$\|A\| = 0 \text{ if and only if } A = O, \quad (2)$$

$$\|c \cdot A\| = |c| \cdot \|A\| \text{ for any scalar } c, \quad (3)$$

$$\|A + B\| \leq \|A\| + \|B\| \text{ (triangular inequality)}, \quad (4)$$

where of course in the latter case the matrix B has the same size as A . Conditions (2) and (3) follow trivially from (1), but condition (4) is not obvious and will be shown below. We also have

$$\|A\| = \|A'\| \quad (5)$$

because $\text{trace}(AA') = \text{trace}(A'A)$. Moreover, it will be shown that for conformable matrices A and B ,

$$\|AB\| \leq \|A\| \cdot \|B\| \quad (6)$$

and thus for $b \in \mathbb{R}^m$,

$$\|Ab\| \leq \|A\| \cdot \|b\|.$$

Proof of (4): To check the triangular inequality, let B be a $k \times m$ matrix. Then

$$\|A + B\|^2 = \text{trace}((A + B)(A' + B'))$$

$$\begin{aligned}
&= \text{trace}(AA') + \text{trace}(AB') + \text{trace}(BA') + \text{trace}(BB') \\
&= \|A\|^2 + 2 \cdot \text{trace}(B'A) + \|B\|^2 \\
&\leq \|A\|^2 + 2 \cdot \|A\| \cdot \|B\| + \|B\|^2 \\
&= (\|A\| + \|B\|)^2
\end{aligned}$$

which implies (4). The inequality follows from Schwarz inequality, applied twice, as follows. Let $a_{i,j}$ and $b_{i,j}$ be the typical elements of A and B , respectively. Then

$$\begin{aligned}
|\text{trace}(B'A)| &= \left| \sum_{j=1}^m \sum_{i=1}^k b_{j,i} a_{i,j} \right| \leq \sum_{j=1}^m k \left| \frac{1}{k} \sum_{i=1}^k b_{j,i} a_{i,j} \right| \\
&\leq \sum_{j=1}^m k \sqrt{\frac{1}{k} \sum_{i=1}^k b_{j,i}^2} \sqrt{\frac{1}{k} \sum_{i=1}^k a_{i,j}^2} = \sum_{j=1}^m \sqrt{\sum_{i=1}^k b_{j,i}^2} \sqrt{\sum_{i=1}^k a_{i,j}^2} \\
&\leq \sqrt{\sum_{j=1}^m \sum_{i=1}^k b_{j,i}^2} \sqrt{\sum_{j=1}^m \sum_{i=1}^k a_{i,j}^2} = \sqrt{\text{trace}(BB')} \sqrt{\text{trace}(AA')} \\
&= \|B\| \cdot \|A\|
\end{aligned}$$

Proof of (6): Let B be an $m \times \ell$ matrix. Then

$$\|AB\|^2 = \text{trace}(ABB'A') = \text{trace}((BB')(A'A))$$

We can write $BB' = \sum_{i=1}^m \lambda_i q_i q_i'$, where the λ_i 's are the (nonnegative) eigenvalues of BB' and the q_i 's are the corresponding orthonormal eigenvectors. Note that $\sum_{i=1}^m q_i q_i' = I_m$ and $\text{trace}(BB') = \sum_{i=1}^m \lambda_i \text{trace}(q_i q_i') = \sum_{i=1}^m \lambda_i (q_i' q_i) = \sum_{i=1}^m \lambda_i$. Then

$$\begin{aligned}
\|AB\|^2 &= \text{trace}((BB')(A'A)) \\
&= \sum_{i=1}^m \lambda_i \text{trace}(q_i q_i' A' A) = \sum_{i=1}^m \lambda_i q_i' A' A q_i \\
&\leq \sum_{i=1}^m \lambda_i \sum_{i=1}^m q_i' A' A q_i \\
&= \text{trace}(BB') \sum_{i=1}^m \text{trace}(A' A q_i q_i')
\end{aligned}$$

$$\begin{aligned} &= \text{trace}(BB') \cdot \text{trace} \left(A' A \sum_{i=1}^m q_i q_i' \right) \\ &= \text{trace}(BB') \cdot \text{trace}(A' A) \\ &= \text{trace}(BB') \cdot \text{trace}(AA') = \|B\|^2 \cdot \|A\|^2 \end{aligned}$$