Separate Appendix to: "Econometric Analysis of Linearized Singular Dynamic Stochastic General Equilibrium Models"

Herman J. Bierens
Pennsylvania State University and Tilburg University

Derivation of (18):

$$
\begin{aligned}
& \frac{1}{n}\sum_{t=1}^{n}\ln\left[p_{t-1}(\Pi,\Theta,\beta_1,Q,\Lambda_1|\tau)\right] \\
= \; & \frac{k-m}{2}\ln(\tau)-\frac{k}{2}\ln(1+\tau)+\frac{1}{2}\ln\left[\det\left(\Lambda_1+\tau I_m\right)\right] \\
& -\frac{1}{2}\frac{1}{n}\sum_{t=1}^{n}\left(\eta_{t-1}-\mu_{t-1}(\beta_1)\right)^T\Theta^{-1/2}Q\left(I_k-\Lambda\right)^{-1}Q^T\Theta^{-1/2} \\
& \times\left(\eta_{t-1}-\mu_{t-1}(\beta_1)\right) \\
= \; & \frac{k-m}{2}\ln(\tau)-\frac{k}{2}\ln(1+\tau)+\frac{1}{2}\ln\left[\det\left(\Lambda_1+\tau I_m\right)\right] \\
& -\frac{1}{2}\mathrm{trace}\left[\left(I_k-\Lambda\right)^{-1}Q^T\Gamma_n(\beta_1)Q\right] \\
= \; & \frac{k-m}{2}\ln(\tau)-\frac{k}{2}\ln(1+\tau)+\frac{1}{2}\sum_{j=1}^{m}\ln\left(\lambda_j+\tau\right) \\
& -\frac{1}{2}\sum_{j=1}^{m}\frac{q_j^T\Gamma_n(\beta_1)q_j}{1-\lambda_j}-\frac{1}{2}\sum_{j=m+1}^{k}q_j^T\Gamma_n(\beta_1)q_j,
\end{aligned}
$$

Derivation of (24):

$\Phi(\tau)$

$$
\begin{aligned}
&= \Theta^{-1/2}\left[\left(\left(\frac{1}{1+\tau}\Theta^{-1/2}\Sigma\Theta^{-1/2}+\frac{\tau}{1+\tau}I_k\right)^{-1}-I_k\right)^{-1}+I_k\right]\Theta^{-1/2} \\
&= \Theta^{-1/2}\left[\left((1+\tau)\begin{pmatrix}\Theta_{11}^*\Sigma_1\Theta_{11}^*+\tau I_m & O \\ O & \tau I_{k-m}\end{pmatrix}^{-1}-I_k\right)^{-1}+I_k\right]\Theta^{-1/2} \\
&= \Theta^{-1/2}\left[\left(\begin{pmatrix}(1+\tau)\left(\Theta_{11}^*\Sigma_1\Theta_{11}^*+\tau I_m\right)^{-1}-I_m & O \\ O & \frac{1}{\tau}I_{k-m}\end{pmatrix}\right)^{-1}+I_k\right]\Theta^{-1/2} \\
&= \Theta^{-1/2}\left[\begin{pmatrix}\left[(1+\tau)\left(\Theta_{11}^*\Sigma_1\Theta_{11}^*+\tau I_m\right)^{-1}-I_m\right]^{-1}+I_m & O \\ O & (1+\tau)I_{k-m}\end{pmatrix}\right]\Theta^{-1/2}
\end{aligned}
$$

Derivation of (25):

$$
\begin{aligned}
& \Phi(\tau) \\
&= \Theta^{-1/2}\left[\begin{pmatrix}Q_{11}\left[(1+\tau)\left(\Lambda_1+\tau I_m\right)^{-1}-I_m\right]^{-1}Q_{11}^T+I_m & O \\ O & (1+\tau)I_{k-m}\end{pmatrix}\right]\Theta^{-1/2} \\
&= \Theta^{-1/2}Q\left[\begin{pmatrix}\left[(1+\tau)\left(\Lambda_1+\tau I_m\right)^{-1}-I_m\right]^{-1}+I_m & O \\ O & (1+\tau)I_{k-m}\end{pmatrix}\right]Q^T\Theta^{-1/2} \\
&= (1+\tau)\,\Theta^{-1/2}Q\left(I_k-\Lambda\right)^{-1}Q^T\Theta^{-1/2},
\end{aligned}
$$

Derivation of (83):
Substitute

$$(Q_t/A_t)(Q_t/Q_{t-1})^{-1}\exp(-v_t+\gamma)=(Q_{t-1}/A_{t-1})$$

in (81). Then,

$$
\begin{aligned}
& (Q_t/A_t)^{1/(1-\alpha)}\nu^{-\alpha/(1-\alpha)}-(1-\overline{x})(Q_t/A_t) \\
&= (1-\delta)(Q_t/A_t)^{1/(1-\alpha)}(Q_t/Q_{t-1})^{-1/(1-\alpha)}(\exp(v_t-\gamma))^{-\alpha/(1-\alpha)}\nu^{-\alpha/(1-\alpha)}
\end{aligned}
$$

$$\Leftrightarrow$$

$$
\begin{aligned}
& (Q_t/A_t)^{\alpha/(1-\alpha)}\nu^{-\alpha/(1-\alpha)}-(1-\overline{x}) \\
&= (1-\delta)(Q_t/A_t)^{\alpha/(1-\alpha)}(Q_t/Q_{t-1})^{-1/(1-\alpha)}(\exp(v_t-\gamma))^{-\alpha/(1-\alpha)}\nu^{-\alpha/(1-\alpha)}
\end{aligned}
$$

$$\Leftrightarrow$$

$$\begin{aligned}
& (Q_t/A_t)^{\alpha/(1-\alpha)}\left[1-(1-\delta)(Q_t/Q_{t-1})^{-1/(1-\alpha)}(\exp(v_t-\gamma))^{-\alpha/(1-\alpha)}\right] \\
= \; & (1-\overline{x})\nu^{-\alpha/(1-\alpha)}
\end{aligned}$$

$$\Leftrightarrow$$

$$(Q_t/A_t)=\nu\left[\frac{(1-\overline{x})}{1-(1-\delta)(Q_t/Q_{t-1})^{-1/(1-\alpha)}(\exp(v_t-\gamma))^{-\alpha/(1-\alpha)}}\right]^{(1-\alpha)/\alpha}$$

Derivation of (86):
Substituting (85) in (83) yields

$$(Q_t/Q_{t-1})=(1-\delta)^{(1-\alpha)}\exp(\alpha\gamma-\alpha v_t)\exp(w_t),$$

$$\Leftrightarrow$$

$$(Q_t/Q_{t-1})^{1/\alpha}=(1-\delta)^{(1-\alpha)/\alpha}\exp(\gamma-v_t)\exp(w_t/\alpha)$$

$$\Leftrightarrow$$

$$(Q_t/Q_{t-1})^{1/(1-\alpha)}=(1-\delta)\exp(\alpha(\gamma-v_t)/(1-\alpha))\exp(w_t/(1-\alpha)),$$

$$\Leftrightarrow$$

$$\begin{aligned}
& \ln(Q_t/A_t)=\frac{1-\alpha}{\alpha}\ln(1-\overline{x})+\ln(\nu)+w_t/\alpha \\
& -\frac{1-\alpha}{\alpha}\ln\left[\exp(w_t/(1-\alpha))-1\right],
\end{aligned}$$

Derivation of (87):
Combining (80), (85) and (86) yields

$$\begin{aligned}
& w_t/\alpha-\frac{1-\alpha}{\alpha}\ln\left[\exp(w_t/(1-\alpha))-1\right] \\
& -w_{t-1}/\alpha+\frac{1-\alpha}{\alpha}\ln\left[\exp(w_{t-1}/(1-\alpha))-1\right] \\
= \; & (1-\alpha)\ln(1-\delta)-(1+\alpha)\gamma+(1-\alpha)v_t+w_t
\end{aligned}$$

$$\Leftrightarrow$$

$$\begin{aligned}
& (1-\alpha)w_t - (1-\alpha)\ln\left[\exp(w_t/(1-\alpha)) - 1\right] \\
& -w_{t-1} + (1-\alpha)\ln\left[\exp(w_{t-1}/(1-\alpha)) - 1\right] \\
= \quad & \alpha(1-\alpha)\ln(1-\delta) - \alpha(1+\alpha)\gamma + \alpha(1-\alpha)v_t
\end{aligned}$$

$$\Leftrightarrow$$

$$\begin{aligned}
& w_t - \ln\left[\exp(w_t/(1-\alpha)) - 1\right] \\
= \quad & w_{t-1}/(1-\alpha) - \ln\left[\exp(w_{t-1}/(1-\alpha)) - 1\right] \\
& +\alpha\ln(1-\delta) - \alpha\gamma + \alpha v_t
\end{aligned}$$

$$\Leftrightarrow$$

$$\begin{aligned}
& \frac{\exp(w_t)}{\exp(w_t/(1-\alpha)) - 1} \\
= \quad & \frac{\exp(w_{t-1}/(1-\alpha))}{\exp(w_{t-1}/(1-\alpha)) - 1} \\
& \times \exp\left(\alpha\ln(1-\delta) - \alpha\gamma + \alpha v_t\right) \\
= \quad & \frac{(\exp(w_{t-1}))^{1/(1-\alpha)}}{\exp(w_{t-1}/(1-\alpha)) - 1} \\
& \times \exp\left(\alpha\ln(1-\delta) - \alpha\gamma + \alpha v_t\right) \\
= \quad & \frac{\exp(w_{t-1})(\exp(w_{t-1}))^{\alpha/(1-\alpha)}}{\exp(w_{t-1}/(1-\alpha)) - 1} \\
& \times \exp\left(\alpha\ln(1-\delta) - \alpha\gamma + \alpha v_t\right)
\end{aligned}$$

$$\Leftrightarrow$$

$$\begin{aligned}
& \frac{\exp(w_t)}{\exp(w_t/(1-\alpha)) - 1} \\
= \quad & \frac{\exp(w_{t-1}/(1-\alpha))}{\exp(w_{t-1}/(1-\alpha)) - 1} \times (1-\delta)\alpha\exp\left[\alpha(v_t - \gamma)\right]
\end{aligned}$$

Derivation of (93) and (94):
It follows from (87) that

$$\frac{\partial}{\partial w_t}\left(\frac{\exp(w_t)}{\exp(w_t/(1-\alpha)) - 1}\right) \tag{108}$$

$$
\begin{aligned}
&= \frac{\exp(w_t)}{\exp(w_t/(1-\alpha))-1} - \frac{\exp(w_t)\exp(w_t/(1-\alpha))}{(1-\alpha)\left(\exp(w_t/(1-\alpha))-1\right)^2} \\
&= \frac{(1-\alpha)\exp(w_t)\exp(w_t/(1-\alpha)) - (1-\alpha)\exp(w_t) - \exp(w_t)\exp(w_t/(1-\alpha))}{(1-\alpha)\left(\exp(w_t/(1-\alpha))-1\right)^2} \\
&= -\frac{\exp(w_t)\left(\alpha\exp(w_t/(1-\alpha)) + (1-\alpha)\right)}{(1-\alpha)\left(\exp(w_t/(1-\alpha))-1\right)^2} \\
&= \frac{-1}{1-\alpha}\left(\frac{\exp(w_t)}{\exp(w_t/(1-\alpha))-1}\right)\left(\frac{\alpha\exp(w_t/(1-\alpha)) + (1-\alpha)}{\exp(w_t/(1-\alpha))-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial}{\partial w_{t-1}}\left(\frac{\exp(w_t)}{\exp(w_t/(1-\alpha))-1}\right) \qquad (109) \\
&= \frac{\exp(w_{t-1}/(1-\alpha))}{\exp(w_{t-1}/(1-\alpha))-1} \times \frac{(1-\delta)\alpha\exp\left[\alpha(v_t-\gamma)\right]}{1-\alpha} \\
&\quad - \left(\frac{\exp(w_{t-1}/(1-\alpha))}{\exp(w_{t-1}/(1-\alpha))-1}\right)^2 \times \frac{(1-\delta)\alpha\exp\left[\alpha(v_t-\gamma)\right]}{1-\alpha} \\
&= \frac{1}{1-\alpha}\left(\frac{\exp(w_t)}{\exp(w_t/(1-\alpha))-1}\right) \\
&\quad \times \left[1 - \frac{1}{(1-\delta)\alpha\exp\left[\alpha(v_t-\gamma)\right]}\left(\frac{\exp(w_t)}{\exp(w_t/(1-\alpha))-1}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial}{\partial v_t}\left(\frac{\exp(w_t)}{\exp(w_t/(1-\alpha))-1}\right) \qquad (110) \\
&= \alpha\frac{\exp(w_{t-1}/(1-\alpha))}{\exp(w_{t-1}/(1-\alpha))-1} \times (1-\delta)\alpha\exp\left[\alpha(v_t-\gamma)\right] \\
&= \alpha\left(\frac{\exp(w_t)}{\exp(w_t/(1-\alpha))-1}\right)
\end{aligned}
$$

Dividing (109) by (108) yields

$$
\begin{aligned}
\frac{\partial w_t}{\partial w_{t-1}} &= \frac{\partial \varphi_{\alpha,\delta,\gamma}(w_{t-1}, v_t)}{\partial w_{t-1}} \qquad (111) \\
&= -\frac{1 - \frac{1}{(1-\delta)\alpha\exp[\alpha(v_t-\gamma)]}\left(\frac{\exp(w_t)}{\exp(w_t/(1-\alpha))-1}\right)}{\frac{\alpha\exp(w_t/(1-\alpha))+(1-\alpha)}{\exp(w_t/(1-\alpha))-1}}
\end{aligned}
$$

$$= \quad -\frac{\exp(w_t/(1-\alpha)) - 1 - \frac{\exp(w_t)}{(1-\delta)\alpha\exp[\alpha(v_t-\gamma)]}}{\alpha\exp(w_t/(1-\alpha)) + (1-\alpha)} \tag{112}$$

and dividing (110) by (108) yields

$$\frac{\partial w_t}{\partial v_t} = \frac{\partial \varphi_{\alpha,\delta,\gamma}(w_{t-1}, v_t)}{\partial v_t} \tag{113}$$

$$= \quad \frac{\alpha\left(\frac{\exp(w_t)}{\exp(w_t/(1-\alpha))-1}\right)}{\frac{-1}{1-\alpha}\left(\frac{\exp(w_t)}{\exp(w_t/(1-\alpha))-1}\right)\left(\frac{\alpha\exp(w_t/(1-\alpha))+(1-\alpha)}{\exp(w_t/(1-\alpha))-1}\right)}$$

$$= \quad \frac{-\alpha(1-\alpha)\left(\exp(w_t/(1-\alpha)) - 1\right)}{\alpha\exp(w_t/(1-\alpha)) + (1-\alpha)}$$

Finally, substituting (92) in (111) and (113) yield

$$\left.\frac{\partial \varphi_{\alpha,\delta,\gamma}(w_{t-1}, v_t)}{\partial w_{t-1}}\right|_{w_{t-1}=\overline{w}, v_t=0}$$

$$= \quad -\frac{\exp\left(\gamma - \ln(1-\delta)\right) - 1 - \frac{\exp[(1-\alpha)(\gamma-\ln(1-\delta))]}{(1-\delta)\alpha\exp[-\alpha\gamma]}}{\alpha\exp\left(\gamma - \ln(1-\delta)\right) + (1-\alpha)}$$

$$= \quad -\frac{\exp\left(\gamma - \ln(1-\delta)\right) - 1 - \frac{\exp(\gamma)}{1-\delta}}{\alpha\exp\left(\gamma - \ln(1-\delta)\right) + (1-\alpha)}$$

$$= \quad \frac{1-\delta}{\alpha\exp(\gamma) + (1-\alpha)(1-\delta)}$$

$$\left.\frac{\partial \varphi_{\alpha,\delta,\gamma}(w_{t-1}, v_t)}{\partial v_t}\right|_{w_{t-1}=\overline{w}, v_t=0}$$

$$= \quad \frac{-\alpha(1-\alpha)\left(\exp\left(\gamma - \ln(1-\delta)\right) - 1\right)}{\alpha\exp\left(\gamma - \ln(1-\delta)\right) + (1-\alpha)}$$

$$= \quad \frac{-\alpha(1-\alpha)\left(\frac{\exp(\gamma)}{1-\delta} - 1\right)}{\alpha\frac{\exp(\gamma)}{1-\delta} + (1-\alpha)}$$

$$= \quad \frac{-\alpha(1-\alpha)\left(\exp(\gamma) - (1-\delta)\right)}{\alpha\exp(\gamma) + (1-\alpha)(1-\delta)}$$

Derivation of (96):

It follows from (85) that

$$w_t = \ln(Q_t/Q_{t-1}) - (1-\alpha)\ln(1-\delta) - \alpha\gamma + \alpha v_t$$

Substituting this in (95) yields:

$$\begin{aligned}
&\ln(Q_t/Q_{t-1}) - (1-\alpha)\ln(1-\delta) - \alpha\gamma + \alpha v_t \\
= &\left(1 - \frac{1-\delta}{\alpha\exp(\gamma) + (1-\delta)(1-\alpha)}\right)(1-\alpha)(\gamma - \ln(1-\delta)) \\
&+ \frac{1-\delta}{\alpha\exp\gamma + (1-\delta)(1-\alpha)}(\ln(Q_{t-1}/Q_{t-2}) - (1-\alpha)\ln(1-\delta) - \alpha\gamma + \alpha v_{t-1}) \\
&- \frac{\alpha(1-\alpha)(\exp(\gamma) - (1-\delta))}{\alpha\exp(\gamma) + (1-\alpha)(1-\delta)}v_t
\end{aligned}$$

hence

$$\begin{aligned}
&\ln(Q_t/Q_{t-1}) \\
= &\left(1 - \frac{1-\delta}{\alpha\exp(\gamma) + (1-\delta)(1-\alpha)}\right)((1-\alpha)\gamma - (1-\alpha)\ln(1-\delta)) + (1-\alpha)\ln(1-\delta) + \alpha\gamma \\
&- \left(\frac{1-\delta}{\alpha\exp\gamma + (1-\delta)(1-\alpha)}\right)((1-\alpha)\ln(1-\delta) + \alpha\gamma) \\
&+ \frac{1-\delta}{\alpha\exp\gamma + (1-\delta)(1-\alpha)}\ln(Q_{t-1}/Q_{t-2}) \\
&- \alpha\left(\frac{\exp(\gamma)}{\alpha\exp(\gamma) + (1-\alpha)(1-\delta)}\right)v_t + \alpha\left(\frac{1-\delta}{\alpha\exp\gamma + (1-\delta)(1-\alpha)}\right)v_{t-1} \\
= &(1-\alpha)\gamma - (1-\alpha)\ln(1-\delta) + (1-\alpha)\ln(1-\delta) + \alpha\gamma \\
&- \left(\frac{1-\delta}{\alpha\exp\gamma + (1-\delta)(1-\alpha)}\right)((1-\alpha)\ln(1-\delta) + \alpha\gamma + (1-\alpha)\gamma - (1-\alpha)\ln(1-\delta)) \\
&+ \frac{1-\delta}{\alpha\exp\gamma + (1-\delta)(1-\alpha)}\ln(Q_{t-1}/Q_{t-2}) \\
&- \alpha\left(\frac{\exp(\gamma)}{\alpha\exp(\gamma) + (1-\alpha)(1-\delta)}\right)v_t + \alpha\left(\frac{1-\delta}{\alpha\exp\gamma + (1-\delta)(1-\alpha)}\right)v_{t-1} \\
= &\alpha\gamma\left[\frac{\exp\gamma - (1-\delta)}{\alpha\exp\gamma + (1-\delta)(1-\alpha)}\right] + \frac{1-\delta}{\alpha\exp\gamma + (1-\delta)(1-\alpha)}\ln(Q_{t-1}/Q_{t-2}) \\
&- \alpha\left(\frac{\exp(\gamma)}{\alpha\exp(\gamma) + (1-\alpha)(1-\delta)}\right)v_t + \alpha\left(\frac{\exp(\gamma)}{\alpha\exp(\gamma) + (1-\alpha)(1-\delta)}\right)\left(\frac{1-\delta}{\exp(\gamma)}\right)v_{t-1}
\end{aligned}$$