

VECTOR TIME SERIES AND INNOVATION RESPONSE ANALYSIS

Herman J. Bierens

Pennsylvania State University

(Revised: October 15, 2007)

1. *The Wold decomposition theorem, and the vector auto-regressive (VAR) and vector moving average (VMA) representations*

Let X_t be a k -variate covariance stationary time series process: $X_t \in \mathbb{R}^k$, $E(X_t) = \mu$, $E[(X_t - \mu)(X_{t-m} - \mu)^T] = \Xi(m)$, where the expectation vector μ and the covariance matrices $\Xi(m)$ do not depend on the time index t . The multivariate Wold decomposition theorem states that the process X_t has the Wold representation

$$X_t = \sum_{s=0}^{\infty} \Gamma_s U_{t-s} + W_t, \quad (1)$$

where the $k \times k$ matrices Γ_s are such that

$$\Gamma_0 = I_k, \quad \sum_{s=1}^{\infty} \Gamma_s \Gamma_s^T \text{ converges}, \quad (2)$$

the process U_t is k -variate white noise:

$$U_t \in \mathbb{R}^k, \quad E(U_t) = 0, \quad E(U_t U_t^T) = \Sigma, \quad E(U_t U_{t-m}^T) = O \text{ for } m \neq 0, \quad (3)$$

and $W_t \in \mathbb{R}^k$ is a linear deterministic process: there exists a k -vector c_0 and $k \times k$ matrices C_s such that without error,

$$W_t = c_0 + \sum_{s=1}^{\infty} C_s W_{t-s}, \quad \text{and } E[U_t W_{t-m}^T] = O \text{ for } m = 0, \pm 1, \pm 2, \dots \quad (4)$$

Assuming that the deterministic process W_t is constant, it follows from (1) that W_t must be equal to the expectation of X_t : $W_t = E(X_t) = \mu$. Then the Wold decomposition theorem implies that the covariance stationary process X_t is a Vector Moving Average (VMA) process, possibly of infinite order:

$$X_t = \mu + \sum_{m=0}^{\infty} \Gamma_m U_{t-m} = \mu + \Gamma(L)U_t, \text{ where } \Gamma_0 = I_k, \Gamma(L) = I_k + \sum_{m=1}^{\infty} \Gamma_m L^m. \quad (5)$$

Now suppose that there exists a matrix valued lag polynomial $C(L)$ such that $C(L)\Gamma(L) = I_k$. Note that $C(0) = I_k$ because $\Gamma(0) = I_k$, so that we can write

$$\Gamma(L)^{-1} = C(L) = I_k - \sum_{m=1}^{\infty} C_m L^m. \quad (6)$$

Then

$$X_t = \eta + \sum_{s=1}^{\infty} C_s X_{t-s} + U_t, \quad (7)$$

where $\eta = C(1)\mu = \Gamma(0)^{-1}\mu$. This is the Vector Auto-Regressive (VAR) representation.

Next, assume in addition to the covariance stationarity condition that X_t is a Gaussian process: for arbitrary $m \geq 1$ and arbitrary indices $t_1 < t_2 < \dots < t_m$, the vector $(X_{t_1}^T, \dots, X_{t_m}^T)^T$ is jointly normally distributed. Then U_t is a Gaussian process, and since the U_t 's are serially uncorrelated, they are independent [*Exercise: Why?*] and normally distributed: U_t is i.i.d. $N_k[0, \Sigma]$. Consequently, X_t is then strictly stationary: the distribution of $(X_{t_1}^T, \dots, X_{t_m}^T)^T$ only depends on the differences of the indices t_j , and not on their levels, and moreover,

$$E(X_t | X_{t-1}, X_{t-2}, \dots) = \eta + \sum_{s=1}^{\infty} C_s X_{t-s}, \quad (8)$$

because $E(U_t) = 0$ and U_t is independent of X_{t-m} for $m > 0$, so that

$$E(U_t | X_{t-1}, X_{t-2}, \dots) = E(U_t) = 0. \quad (9)$$

2. VAR(p) models

All linear vector time series models are approximations of model (7). In particular, the assumption of the VAR(p) model is that in model (7), $C_s = O$ for $s > p$:

$$X_t = \eta + \sum_{s=1}^p C_s X_{t-s} + U_t. \quad (10)$$

A necessary condition for the strict stationarity of the $VAR(p)$ model is that the error process U_t is strictly stationary, and the lag polynomial

$$C(L) = I_k - C_1L - \dots - C_pL^p \quad (11)$$

can be inverted: $C(L)^{-1} = \Gamma(L)$, where $\Gamma(L)$ is the same as before, because then $X_t = \mu + \Gamma(L)U_t$, where the right-hand side is a moving average of a stationary process and therefore stationary itself.

As is well-known, the inverse of $C(L)$ is:

$$C(L)^{-1} = [c^{ij}(L)] = \frac{1}{\det C(L)} [\text{cof}_{j,i}\{C(L)\}]. \quad (12)$$

Thus, denoting

$$C^*(L) = [\text{cof}_{j,i}\{C(L)\}] \quad (13)$$

we have

$$C^*(L)C(L) = \det(C(L))I \quad (14)$$

hence:

$$\det(C(L))X_t = C^*(1)\eta + C^*(L)U_t. \quad (15)$$

Note that $C^*(L)$ consists of finite-order lag polynomials, and that $\det(C(L))$ is a finite-order lag polynomial. Now it follows from the condition for stationarity of univariate $AR(p)$ processes that:

PROPOSITION 1: *A necessary condition for the (strict) stationarity of the $VAR(p)$ process (10) is that all the roots of $\det(C(L))$ are located outside the complex unit circle.*

3. Granger causality

Consider the strictly stationary bi-variate vector time series process $Z_t = (x_t, y_t)^T$. As is well-known (or it ought to be), the best one-step ahead forecast of Z_t given the whole past of the

process Z_t is the conditional expectation

$$E[Z_t | Z_{t-1}, Z_{t-2}, \dots] = \begin{pmatrix} E[x_t | Z_{t-1}, Z_{t-2}, \dots] \\ E[y_t | Z_{t-1}, Z_{t-2}, \dots] \end{pmatrix}. \quad (16)$$

Now if the past of the y_t process does not contribute to the best forecast of x_t , one says that y_t does *not* Granger cause x_t :

DEFINITION 1: y_t does not Granger-cause x_t if

$$E [x_t - E(x_t | x_{t-1}, y_{t-1}, x_{t-2}, y_{t-2}, \dots)]^2 = E [x_t - E(x_t | x_{t-1}, x_{t-2}, \dots)]^2. \quad (17)$$

If y_t Granger-causes x_t , then one can predict x_t better using the whole past of the x_t and y_t processes than using only the past of x_t :

$$E [x_t - E(x_t | x_{t-1}, y_{t-1}, x_{t-2}, y_{t-2}, \dots)]^2 < E [x_t - E(x_t | x_{t-1}, x_{t-2}, \dots)]^2. \quad (18)$$

Suppose that Z_t is a Gaussian VAR(p) process:

$$Z_t = \eta + C_1 Z_{t-1} + \dots + C_p Z_{t-p} + U_t, \quad U_t \text{ is i.i.d. } N_2(0, \Sigma). \quad (19)$$

Then

$$E(Z_t | Z_{t-1}, Z_{t-2}, \dots) = \eta + \sum_{s=1}^p C_s Z_{t-s} \quad (20)$$

hence

$$E(x_t | Z_{t-1}, Z_{t-2}, \dots) = \eta_1 + \sum_{s=1}^p (c_{1,1,s} x_{t-s} + c_{1,2,s} y_{t-s}), \quad (21)$$

where η_1 is the first component of η and

$$C_s = \begin{pmatrix} c_{1,1,s} & c_{1,2,s} \\ c_{2,1,s} & c_{2,2,s} \end{pmatrix}. \quad (22)$$

Now y_t does *not* Granger cause x_t if $c_{1,2,s} = 0$ for $s = 1, \dots, p$, so that then the matrices C_s are lower-triangular, as then the lagged y_t 's disappear from the right-hand side of (21).

Finally, the above argument easily extends to the case where y_t and/or x_t are vectors themselves.

4. *Sims' nonstructural VAR innovation response analysis*

In his seminal paper "Macroeconomics and Reality", Sims (1980) works with a vector X_t of macro-economics variables (dimension $k = 6$). He assumes for X_t a stationary VAR(p) process:

$$C(L)X_t = X_t - C_1X_{t-1} - \dots - C_pX_{t-p} = \eta + U_t, \text{ with } U_t \text{ i.i.d. } N(0, \Sigma). \quad (23)$$

Since by stationarity all the roots of $\det(C(L))$ are located outside the unit circle, the lag polynomial $C(L)$ is invertible: $C(L)^{-1} = \Gamma(L)$. We can now write the VAR(p) process as a VMA(∞) process:

$$X_t = \mu + \sum_{s=0}^{\infty} \Gamma_s U_{t-s}, \quad \Gamma_0 = I, \quad (24)$$

which is just the Wold representation.

Since the innovations U_t are *i.i.d.* $N(0, \Sigma)$ we have for $m \geq 0$

$$E(X_{t+m} | U_t) = \Gamma_m U_t + \mu. \quad (25)$$

The expected "impact" of the first component $u_{1,t}$ of U_t on X_{t+m} is:

$$E(X_{t+m} | u_{1,t}) = E[E(X_{t+m} | U_t) | u_{1,t}] = \Gamma_m E(U_t | u_{1,t}) + \mu. \quad (26)$$

The *innovation response* of $u_{1,t}$ on X_{t+m} is now defined as

$$E(X_{t+m} | u_{1,t}) - E(X_{t+m}) = \Gamma_m E(U_t | u_{1,t}) = \Gamma_m \begin{pmatrix} u_{1,t} \\ E(u_{2,t} | u_{1,t}) \\ \vdots \\ E(u_{k,t} | u_{1,t}) \end{pmatrix}. \quad (27)$$

We see that the innovation $u_{1,t}$ has two effects on X_{t+m} : a direct effect and an indirect effect via $E(u_{j,t} | u_{1,t})$ for $j > 1$.

Sims (1980, 1982) proposes to calculate $E(U_t | u_{1,t})$ as follows. Observe that $\text{Var}(U_t) = \Sigma$. Since Σ is positive definite we can write $\Sigma = \Delta \Delta^T$, where Δ is a lower triangular matrix:

$$\Delta = \begin{pmatrix} \delta_{1,1} & 0 & 0 & \dots & 0 \\ \delta_{2,1} & \delta_{2,2} & 0 & \dots & 0 \\ \delta_{3,1} & \delta_{3,2} & \delta_{3,3} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \delta_{k,1} & \delta_{k,2} & \delta_{k,3} & \dots & \delta_{k,k} \end{pmatrix} \quad (28)$$

with positive diagonal elements. Let $e_t = \Delta^{-1}U_t$. Then e_t is i.i.d. $N(0,I)$. Note that $u_{1,t} = \delta_{1,1}e_{1,t}$ with $e_{1,t}$ the first element of e_t . We can now write:

$$E(U_t|u_{1,t}) = \Delta E(e_t|e_{1,t}) = \Delta \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} e_{1,t} = \delta_1 e_{1,t} \quad (29)$$

with δ_1 the first column of Δ . Thus the response of $u_{1,t}$ on X_{t+m} is $\Gamma_m \delta_1 e_{1,t}$. Sims replaces in his analysis $e_{1,t}$ by its standard error (=1). He calls it a unit shock. Thus the *innovation response* of a unit shock in the first component of X_t on X_{t+m} is given by $\Gamma_m \delta_1$, for $m = 0,1,2,\dots$

We have seen that U_t can be written as $U_t = \Delta e_t$, with Δ a lower triangular matrix and e_t i.i.d. $N(0,I)$. Thus:

$$U_t = \begin{pmatrix} u_{1,t} \\ u_{2,t} \\ \vdots \\ u_{i,t} \\ \vdots \\ u_{k,t} \end{pmatrix} = \begin{pmatrix} \delta_{1,1}e_{1,t} \\ \delta_{2,1}e_{1,t} + \delta_{2,2}e_{2,t} \\ \vdots \\ \sum_{j=1}^i \delta_{i,j}e_{j,t} \\ \vdots \\ \sum_{j=1}^k \delta_{k,j}e_{j,t} \end{pmatrix}, \quad (30)$$

hence

$$u_{i,t} = \sum_{j=1}^i \delta_{i,j} e_{j,t} \quad (i = 1,\dots,k). \quad (31)$$

Rather than identifying the innovation response of a shock in variable 2 on the future values of X_t ,

by $E(X_{t+m}|u_{2,t})$, Sims considers the *net* effect of $u_{2,t}$ on X_{t+m} :

$$E(X_{t+m}|u_{1,t}, u_{2,t}) - E(X_{t+m}|u_{1,t}) = E(X_{t+m}|e_{1,t}, e_{2,t}) - E(X_{t+m}|e_{1,t}) \quad (32)$$

The equality follows from the fact

$$\begin{pmatrix} u_{1,t} \\ u_{2,t} \end{pmatrix} = \begin{pmatrix} \delta_{1,1} & 0 \\ \delta_{2,1} & \delta_{2,2} \end{pmatrix} \begin{pmatrix} e_{1,t} \\ e_{2,t} \end{pmatrix} \quad (33)$$

is a one-to-one mapping:

$$\begin{pmatrix} e_{1,t} \\ e_{2,t} \end{pmatrix} = \frac{1}{\delta_{1,1}\delta_{2,2}} \begin{pmatrix} \delta_{2,2} & 0 \\ -\delta_{2,1} & \delta_{2,2} \end{pmatrix} \begin{pmatrix} u_{1,t} \\ u_{2,t} \end{pmatrix}. \quad (34)$$

Since

$$E(X_{t+m}|e_{1,t}, e_{2,t}) = \Gamma_m(\delta_1 e_{1,t} + \delta_2 e_{2,t}), \quad E(X_{t+m}|e_{1,t}) = \Gamma_m \delta_1 e_{1,t} \quad (35)$$

with δ_1 and δ_2 the first two columns of Δ , the *net* effect of $u_{2,t}$ on X_{t+m} is:

$$E(X_{t+m}|u_{1,t}, u_{2,t}) - E(X_{t+m}|u_{1,t}) = \Gamma_m \delta_2 e_{2,t} \quad (36)$$

Again, Sims replaces $e_{2,t}$ by its standard error 1, so that $\Gamma_m \delta_2$ is the innovation response of a unit shock in variable 2 on X_{t+m} .

In general we have for $i > 1$:

$$\begin{aligned} & E(X_{t+m}|u_{1,t}, \dots, u_{i,t}) - E(X_{t+m}|u_{1,t}, \dots, u_{i-1,t}) \\ &= E(X_{t+m}|e_{1,t}, \dots, e_{i,t}) - E(X_{t+m}|e_{1,t}, \dots, e_{i-1,t}) = \Gamma_m \delta_i e_{i,t} \end{aligned} \quad (37)$$

with δ_i the i -th column of Δ . Again, Sims replaces $e_{i,t}$ by its standard error 1. Thus the innovation response of a unit shock in the i -th component of X_t on X_{t+m} is $\Gamma_m \delta_i$.

Summarizing:

DEFINITION 2: Denoting $R_m = (r_{ij}(m)) = \Gamma_m \Delta$, the innovation response of a unit shock in variable j on variable i is given by $r_{ij}(m)$, $m = 0, 1, 2, \dots$, where $r_{ij}(m)$ is the element of R_m in the i -th row and j -th column.

The graphs in Sims' (1980) article are the plots of the functions $r_{ij}(m)$ for $m = 0, 1, 2, \dots$

Remark 1. Since the innovations e_t are the errors of the VAR model for $\Delta^{-1}X_t$, it is in general

not possible to measure the components of e_t in the unit of measurement of the corresponding components of X_t . For example, let X_t be bivariate, with component 1 measured in US dollars and component 2 measured in euros. Then the same applies to the corresponding VAR errors $u_{1,t}$ and $u_{2,t}$ in (33). Hence it follows from (34) that the unit of measurement of $e_{1,t}$ is the same as for $u_{1,t}$ (US \$), but the unit of measurement of $e_{2,t}$ is a combination of US dollars and euros. On the other hand, the units of measurement of the innovation responses are the same as for the variables involved.

Note that the matrix Δ is not unique. It depends on the *order* of the components of X_t . Therefore, in conducting innovation response analysis one has to determine the order of the shocks to the system in advance.

The matrixes C_1, \dots, C_p and the vector η of intercepts can be estimated consistently by OLS. Let $\hat{C}_1, \dots, \hat{C}_p, \hat{\eta}$ be these OLS estimators and let

$$\hat{U}_t = X_t - \hat{\eta} - \sum_{i=1}^p \hat{C}_i X_{t-i} \quad (t = p+1, \dots, n). \quad (38)$$

be the vectors of OLS residuals. Then

$$\hat{\Sigma} = \frac{1}{n-p} \sum_{t=p+1}^n \hat{U}_t \hat{U}_t^T \quad (39)$$

is a consistent estimator of Σ (the variance matrix of U_t). Under the normality assumption, these estimators are just the maximum likelihood estimators.

Given the order of the variables in X_t we can calculate the lower triangular matrix $\hat{\Delta}$ such that $\hat{\Sigma} = \hat{\Delta} \hat{\Delta}^T$. Thus the main problem is how to estimate the matrices Γ_m for $m = 0, 1, 2, \dots$. This can be done as follows.

First, note that $\Gamma_0 = I$. Next, observe that the matrices Γ_m can be obtained from backwards substitution of the recursive relation $\Gamma_m = C_1 \Gamma_{m-1} + \dots + C_p \Gamma_{m-p} + E_m$, where $\Gamma_j = O$ for $j < 0$, $\Gamma_0 = I$, $E_j = O$ for $j \neq 0$, and $E_0 = I$. Replacing C_j by its OLS estimate \hat{C}_j , we then get consistent estimates $\hat{\Gamma}_m$ of Γ_m . Denoting $\hat{R}_m = (\hat{r}_{i,j}(m)) = \hat{\Gamma}_m \hat{\Delta}$, the estimated innovation response of a unit shock in variable j on variable i is $\hat{r}_{i,j}(m)$.

For each m , $\hat{r}_{i,j}(m)$ is a nonlinear but continuously differentiable function of the elements of the matrices \hat{C}_i , $i = 1, \dots, p$, and $\hat{\Sigma}$, and the vector $\hat{\beta}$ of these stacked elements has asymptotically a multivariate normal distribution:

$$\sqrt{n}(\hat{\beta} - \beta) \rightarrow N(0, \Omega) \text{ in distr.}, \quad (40)$$

where β is the vector of corresponding elements of the matrices C_i , $i = 1, \dots, p$, and Σ . Therefore, it follows by the so-called delta method that for each i, j , and m ,

$$\sqrt{n}(\hat{r}_{i,j}(m) - r_{i,j}(m)) \rightarrow N(0, \xi_{i,j,m}^T \Omega \xi_{i,j,m}) \text{ in distr.}, \quad (41)$$

where

$$\xi_{i,j,m} = \partial r_{i,j}(m) / \partial \beta^T. \quad (42)$$

Hence

$$\frac{\sqrt{n}(\hat{r}_{i,j}(m) - r_{i,j}(m))}{\sqrt{\hat{\xi}_{i,j,m}^T \hat{\Omega} \hat{\xi}_{i,j,m}}} \rightarrow N(0, 1) \text{ in distr.}, \quad (43)$$

where

$$\hat{\xi}_{i,j,m} = \partial \hat{r}_{i,j}(m) / \partial \hat{\beta}^T, \quad (44)$$

and $\hat{\Omega}$ is a consistent estimator of Ω . [*Exercise: Assuming normality, how would you determine $\hat{\Omega}$?*] On the basis of this result it is possible to endow each of the estimated innovation responses $\hat{r}_{i,j}(m)$ with asymptotic confidence intervals. See Baillie (1987).

5. *Bernanke-Sims' structural VAR innovation response analysis*

A disadvantage of Sims' (1980, 1982) nonstructural VAR approach is that the only way to incorporate economic theory into the VAR model is through the order of the variables in X_t . Therefore, Bernanke (1986) and Sims (1986) propose the following structural VAR(p) model:

$$BX_t = a_0 + \sum_{s=1}^p A_s X_{t-s} + e_t, \quad e_t \sim i.i.d. N_k(0, I). \quad (45)$$

The matrix B is the matrix of structural coefficients. This matrix represents the contemporaneous interaction between the variables in X_t , similar to the classical structural equations model.

Assuming that the matrix B is nonsingular, this structural model is related to the non-structural $\text{Var}(p)$ model (10):

$$X_t = B^{-1}a_0 + \sum_{s=1}^p B^{-1}A_s X_{t-s} + B^{-1}e_t = \eta + \sum_{s=1}^p C_s X_{t-s} + U_t. \quad (46)$$

Therefore, the matrix B links the nonstructural errors U_t to the innovations e_t :

$$BU_t = e_t, \text{ where } e_t \sim N_k(0, I). \quad (47)$$

Note that there are many matrices B for which this is true, for example let $B = \Sigma^{-1/2}$.

Given that B is specified such that B is invertible, (47) reads

$$U_t = B^{-1}e_t, \quad (48)$$

hence B^{-1} now takes over the role of the lower triangular matrix Δ in non-structural VAR innovation response analysis. Thus,

DEFINITION 3: Denoting $R_m^{(s)} = (r_{ij}^{(s)}(m)) = \Gamma_m B^{-1}$, the (structural) innovation response of a unit shock in variable j on variable i is given by $r_{ij}^{(s)}(m)$, $m = 0, 1, 2, \dots$, where $r_{ij}^{(s)}(m)$ is the element of $R_m^{(s)}$ in the i -th row and j -th column.

Remark 2. Note that Remark 1 applies to the structural case as well.

Since

$$\Sigma = B^{-1}(B^T)^{-1} = (B^T B)^{-1}, \text{ hence } B^T B = \Sigma^{-1}, \quad (49)$$

and Σ contains only $k + (k^2 - k)/2$ different elements, we can identify no more than $k + (k^2 - k)/2$ elements of B . Therefore, we have to set at least $(k^2 - k)/2$ elements of B equal to given constants (usually zeros). This is where economic theory comes in the picture.

However, even if the matrix B contains $k + (k^2 - k)/2$ or less non-zero entries, identification is not guaranteed. For example, consider the case

$$B = \begin{pmatrix} b_1 & 0 & b_5 \\ 0 & b_2 & b_4 \\ b_6 & 0 & b_3 \end{pmatrix}. \quad (50)$$

Then

$$B^T B = \begin{pmatrix} b_1 & 0 & b_6 \\ 0 & b_2 & 0 \\ b_5 & b_4 & b_3 \end{pmatrix} \begin{pmatrix} b_1 & 0 & b_5 \\ 0 & b_2 & b_4 \\ b_6 & 0 & b_3 \end{pmatrix} = \begin{pmatrix} b_1^2 + b_6^2 & 0 & b_1 b_5 + b_3 b_6 \\ 0 & b_2^2 & b_2 b_4 \\ b_1 b_5 + b_3 b_6 & b_2 b_4 & b_3^2 + b_5^2 \end{pmatrix} = \Sigma^{-1}, \quad (51)$$

which effectively consists of five different nonlinear equations in six unknowns.

Given appropriate restrictions on B , we can estimate B by solving the nonlinear system of equations

$$\hat{B}^T \hat{B} = \hat{\Sigma}^{-1} \quad (52)$$

in the just-identified case, or by maximum likelihood in the over-identified case, where there are less unknown elements of B than $k + (k^2 - k)/2$. Replacing the lower triangular matrices Δ and $\hat{\Delta}$ in Sims' approach by B^{-1} and \hat{B}^{-1} , respectively, now yields the structural innovation responses.

REFERENCES

- Baillie, R. T. (1987): "Inference in Dynamic Models Containing 'Surprise' Variables", *Journal of Econometrics* 35, 101-117.
- Bernanke, B.S. (1986): "Alternative Explanations of the Money-Income Correlation", *Carnegie-Rochester Conference Series on Public Policy* 25, 49-100
- Sims, C.A. (1980): "Macroeconomics and Reality", *Econometrica* 48, 1-48
- Sims, C.A. (1982): "Policy Analysis with Econometric Models", *Brookings Papers on Economics Activity* 1, 107-152
- Sims, C.A. (1986): "Are Forecasting Models Usable for Policy Analysis?", *Federal Reserve Bank of Minneapolis Quarterly Review*, 1-16.