Multiple Object Auctions with Budget Constrained Bidders

SUPPLEMENTARY NOTES

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Abstract

These notes contain some detailed proofs that were omitted from the body of the paper, "Multiple Object Auctions with Budget Constrained Bidders," *Review of Economic Studies*, **68** (No. 1), 2001, pp. 155-179.

0.1 Proofs of Properties 1 and 2

We begin by deriving some properties of the functions π_i.

Lemma A1 *If* $\alpha \geq 0$ *then* $\pi_i\left(V^A, y_i; V^A+\alpha, y_j-p\right) \geq \pi_i\left(V^B, y_i; V^B+\alpha, y_j-p\right)$.

Proof. Suppose $\pi_i\left(V^B, y_i; V^B+\alpha, y_j-p\right) > 0$. Then it must be that $\min\left\{V^B, y_i\right\} > \min\left\{V^B+\alpha, y_j-p\right\}$ and $\pi_i\left(V^B, y_i; V^B+\alpha, y_j-p\right) = V^B - \min\left\{V^B+\alpha, y_j-p\right\}$. Now since $\alpha \geq 0$, this is positive only if $y_j - p < V^B + \alpha$. But then $y_j - p < V^A + \alpha$ and so $\min\left\{V^B, y_i\right\} > y_j - p$ implies that $\min\left\{V^A, y_i\right\} > y_j - p$ also.

Thus

$$\begin{aligned}\pi_i\left(V^A, y_i; V^A+\alpha, y_j-p\right) &= V^A - (y_j - p) \\ &\geq V^B - (y_j - p) \\ &= \pi_i\left(V^B, y_i; V^B+\alpha, y_j-p\right).\end{aligned}$$

■

Lemma A2 *If* $\pi_i\left(V^A, y_i; V^A+\alpha, y_j-p\right) > 0$ *then*

$$\pi_i\left(V^A, y_i; V^A+\alpha, y_j-p\right) \geq \pi_i\left(V^B, y_i; V^B+\alpha, y_j-p\right).$$

Proof. If $\pi_i\left(V^B, y_i; V^B+\alpha, y_j-p\right)=0$ there is nothing to prove. So suppose that $\pi_i\left(V^B, y_i; V^B+\alpha, y_j-p\right)>0$. Then, we have:

$$\begin{aligned}
\pi_i\left(V^A, y_i; V^A+\alpha, y_j-p\right) &= V^A-\min\left\{V^A+\alpha, y_j-p\right\} \\
&= \max\left\{-\alpha, V^A-(y_j-p)\right\} \\
&\geq \max\left\{-\alpha, V^B-(y_j-p)\right\} \\
&= V^B-\min\left\{V^B+\alpha, y_j-p\right\} \\
&= \pi_i\left(V^B, y_i; V^B+\alpha, y_j-p\right).
\end{aligned}$$

■

Property 1 $\overline{p}_i^A \geq \overline{p}_i^B$.

Proof. It is convenient to consider the case of complements and substitutes separately.

Case 1. $\alpha \geq 0$.

If $\overline{p}_i^B = y_3$ the inequality obviously holds. If $\overline{p}_i^B > y_3$ then for any p such that $y_3 < p < \overline{p}_i^B$ we have:

$$\left(V^B-p\right)+\pi_i\left(V^A+\alpha, y_i-p; V^A, y_j\right) > \pi_i\left(V^A, y_i; V^A+\alpha, y_j-p\right).$$

Now suppose the order is AB.

If $\pi_i\left(V^A+\alpha, y_i-p; V^A, y_j\right)=0$, then:

$$\begin{aligned}
\left(V^A-p\right)+\pi_i\left(V^B+\alpha, y_i-p; V^B, y_j\right) &\geq \left(V^A-p\right) \\
&\geq \left(V^B-p\right) \\
&= \left(V^B-p\right)+\pi_i\left(V^A+\alpha, y_i-p; V^A, y_j\right) \\
&> \pi_i\left(V^A, y_i; V^A+\alpha, y_j-p\right) \\
&\geq \pi_i\left(V^B, y_i; V^B+\alpha, y_j-p\right),
\end{aligned}$$

where the last inequality follows from Lemma A1. So i still wants to stay in at p in the order AB.

If $\pi_i\left(V^A+\alpha, y_i-p, V^A, y_j\right)>0$, then $y_i-p>\min\left\{V^A, y_j\right\}$ and

$$\begin{aligned}
\pi_i\left(V^A+\alpha, y_i-p, V^A, y_j\right) &= V^A+\alpha-\min\left\{V^A, y_j\right\} \\
&> 0.
\end{aligned}$$

Clearly, then $y_i-p>\min\left\{V^B, y_j\right\}$ also and so

$$\pi_i\left(V^B+\alpha, y_i-p; V^B, y_j\right)=\left(V^B+\alpha-\min\left\{V^B, y_j\right\}\right)_+.$$

We have:

$$\begin{aligned}
\left(V^A - p\right) + \pi_i\left(V^B, y_i - p, y_j\right) &= \left(V^A - p\right) + \left(V^B + \alpha - \min\left\{V^B, y_j\right\}\right)_+ \\
&\geq \left(V^A - p\right) + \left(V^B + \alpha - \min\left\{V^B, y_j\right\}\right) \\
&= \left(V^B - p\right) + \left(V^A + \alpha - \min\left\{V^B, y_j\right\}\right) \\
&\geq \left(V^B - p\right) + \left(V^A + \alpha - \min\left\{V^A, y_j\right\}\right) \\
&= \left(V^B - p\right) + \pi_i\left(V^A + \alpha, y_i - p; V^A, y_j\right) \\
&> \pi_i\left(V^A, y_i; V^A + \alpha, y_j - p\right) \\
&\geq \pi_i\left(V^B, y_i; V^B + \alpha, y_j - p\right),
\end{aligned}$$

where the last inequality again follows from Lemma A1. So again i still wants to stay in at p in the order AB.

This completes the consideration of Case 1.

Case 2. $\alpha < 0$.

If $\pi_i\left(V^A, y_i; V^A + \alpha, y_j - p\right) > 0$ then Lemma A2 implies that the arguments made in Case 1 can be duplicated.

So the only remaining case is where $\alpha < 0$ and $\pi_i\left(V^A, y_i; V^A + \alpha, y_j - p\right) = 0$.

If $\pi_i\left(V^A, y_i; V^A + \alpha, y_j - p\right) = 0$, then $\min\left\{V^A + \alpha, y_j - p\right\} > \min\left\{V^A, y_i\right\}$. Since $\alpha < 0$ this cannot hold unless $\min\left\{V^A, y_i\right\} = y_i$. So we have: $y_j - p > y_i$ and $V^A > V^A + \alpha > y_i \geq p$.

In order AB, at p staying in yields:

$$\begin{aligned}
V^A - p + \pi_i\left(V^B + \alpha, y_i - p; V^B, y_j\right) &\geq V^A - p \\
&> -\alpha.
\end{aligned}$$

If i drops out at p then since $y_j - p > y_i$, i can win B only if $V^B + \alpha < y_j - p$. Thus

$$\begin{aligned}
\pi_i\left(V^B, y_i; V^B + \alpha, y_j - p\right) &\leq V^B - \min\left\{V^B + \alpha, y_j - p\right\} \\
&= V^B - \left(V^B + \alpha\right) \\
&= -\alpha
\end{aligned}$$

since i is choosing not to win the second object. Thus it pays to stay in at p in the order AB. ■

Property 2 Either $\overline{p}_1^I \leq \overline{p}_2^I \leq V^I$ or $\overline{p}_1^I \geq \overline{p}_2^I \geq V^I$.

Proof. Suppose p is such that $y_3 < p < \overline{p}_1^I$ and $p < V^I$. Then

$$\left(V^I - p\right) + \pi_1\left(V^{II} + \alpha, y_1 - p; V^{II}, y_2\right) > \pi_1\left(V^{II}, y_1; V^{II} + \alpha, y_2 - p\right).$$

If $\pi_1 \left(V^{II} + \alpha, y_1 - p; V^{II}, y_2\right) = 0$, then $\pi_2 \left(V^{II} + \alpha, y_2 - p; V^{II}, y_1\right) = 0$ also and we have:

$$\begin{aligned}\left(V^I - p\right) + \pi_2 \left(V^{II} + \alpha, y_2 - p; V^{II}, y_1\right) &= \left(V^I - p\right) + \pi_1 \left(V^{II} + \alpha, y_1 - p; V^{II}, y_2\right) \\ &> \pi_1 \left(V^{II}, y_1; V^{II} + \alpha, y_2 - p\right) \\ &\geq \pi_2 \left(V^{II}, y_2; V^{II} + \alpha, y_1 - p\right)\end{aligned}$$

so that bidder 2 does not want to drop out at p either.

If $\pi_1 \left(V^{II} + \alpha, y_1 - p; V^{II}, y_2\right) > 0$ then $\min \left\{V^{II} + \alpha, y_1 - p\right\} > \min \left\{V^{II}, y_2\right\}$ so that $\pi_2 \left(V^{II}, y_2; V^{II} + \alpha, y_1 - p\right) = 0$ and since $V^I - p > 0$, again 2 does not want to drop out. Thus we have shown that any $p < V^I$ such that bidder 1 wants to stay in, bidder 2 also wants to stay in.

This implies that if $\overline{p}_1^I \leq V^I$ then $\overline{p}_1^I \leq \overline{p}_2^I$. It also implies that if $\overline{p}_1^I > V^I$ then $\overline{p}_2^I \geq V^I$.

Now suppose $\overline{p}_2^I > V^I$. Then for p such that $V^I < p < \overline{p}_2^I$:

$$\left(V^I - p\right) + \pi_2 \left(V^{II} + \alpha, y_2 - p; V^{II}, y_1\right) > \pi_2 \left(V^{II}, y_2; V^{II} + \alpha, y_1 - p\right)$$

and since $V^I - p < 0$ it must be that $\pi_2 \left(V^{II} + \alpha, y_2 - p; V^{II}, y_1\right) > 0$. But this implies that $\pi_1 \left(V^{II}, y_1; V^{II} + \alpha, y_2 - p\right) = 0$ and since $\pi_1 \left(V^{II} + \alpha, y_1 - p; V^{II}, y_2\right) \geq \pi_2 \left(V^{II} + \alpha, y_2 - p; V^{II}, y_1\right)$ we have

$$\left(V^I - p\right) + \pi_1 \left(V^{II} + \alpha, y_1 - p; V^{II}, y_2\right) > \pi_1 \left(V^{II}, y_1; V^{II} + \alpha, y_2 - p\right)$$

so that if bidder 2 wants to stay in at p then bidder 1 also wants to stay in.

Thus if $\overline{p}_2^I > V^I$ then $\overline{p}_1^I \geq \overline{p}_2^I \geq V^I$.

This completes the proof. ■

Remark 1 *Lemmas A1 and A2 also constitute a proof of inequality (6) on page 34 of the text.*

0.2 Proof of Lemma 5

Lemma 5 *Suppose $\alpha > 0$. The budgets $(\overline{y}_i, \overline{y}_j)$ and the prices $(\overline{p}^I, \overline{p}^{II})$ constitute an equilibrium outcome of the game with endogenous budgets if and only if:*

$$\begin{array}{lll}\overline{y}_i \geq V^I + 2\alpha & \overline{p}^I = V^I - V^{II} + \alpha & \text{if } \frac{1}{2}V^I \geq V^{II} - \frac{1}{2}\alpha \\ \overline{y}_j = V^I - V^{II} + \alpha & \overline{p}^{II} = 0 & \\ & & \\ \overline{y}_i \geq \frac{1}{2}V^I + V^{II} + \frac{3}{2}\alpha & \overline{p}^I = \frac{1}{2}\left(V^I + \alpha\right) & \text{if } \frac{1}{2}V^I < V^{II} - \frac{1}{2}\alpha \\ \overline{y}_j = \frac{1}{2}\left(V^I + \alpha\right) & \overline{p}^{II} = 0 & \end{array}$$

All equilibria result in the same payoffs (except for a relabelling of the bidders).

Proof. Considering the two cases separately, we first verify that $(\overline{y}_i, \overline{y}_j)$ constitute equilibrium budget choices.

Case 1: $\frac{1}{2}V^I \geq V^{II} - \frac{1}{2}\alpha$. Given budgets $\overline{y}_i \geq V^I + 2\alpha$ and $\overline{y}_j = V^I - V^{II} + \alpha$, it is an equilibrium outcome of the auction subgame for bidder j to win I for a price of $V^I - V^{II} + \alpha$ and for bidder i to win II for free. The equilibrium payoff of bidder j is $V^{II} - \alpha$ and that of i is V^{II}.

Now, suppose $\overline{y}_i \geq V^I + 2\alpha$. If bidder j chooses a $y_j \neq \overline{y}_j$ and wins both objects then bidder i will bid up to at least $V^I + 2\alpha - V^{II}$ on the first object since $\overline{y}_i \geq V^I + 2\alpha$ and $V^I + 2\alpha - V^{II} < V^I + \alpha$. Bidder j's payoff is less than

$$(V^I + V^{II} + \alpha) - \left(V^I + 2\alpha - V^{II}\right) - V^{II} = V^{II} - \alpha.$$

If j wins only the second object his payoff is at most $V^{II} - 2\alpha$. Finally, i will not let j win the first object alone for less than $V^I - V^{II} + \alpha$. Thus, bidder j cannot profitably deviate if $\overline{y}_i \geq V^I + 2\alpha$. Similarly, if $\overline{y}_j = V^I - V^{II} + \alpha$, bidder i cannot profitably deviate.

Case 2: $\frac{1}{2}V^I < V^{II} - \frac{1}{2}\alpha$. With budgets $\overline{y}_i \geq \frac{1}{2}V^I + V^{II} + \frac{3}{2}\alpha$ and $\overline{y}_j = \frac{1}{2}\left(V^I + \alpha\right)$, it is an equilibrium outcome of the auction subgame for bidder j to win I for a price of $\frac{1}{2}\left(V^I + \alpha\right)$ and for bidder i to win II for free. The equilibrium payoff of bidder j is $\frac{1}{2}V^I - \frac{1}{2}\alpha$ and that of bidder i is V^{II}.

Suppose $\overline{y}_i \geq \frac{1}{2}V^I + V^{II} + \frac{3}{2}\alpha$. If bidder j chooses a $y_j < \frac{1}{2}\left(V^I + \alpha\right)$, then i will win both objects. Therefore, suppose that $y_j > \frac{1}{2}\left(V^I + \alpha\right)$. If j wins I and only I then he pays at least $\frac{1}{2}\left(V^I + \alpha\right)$. (This is because bidder i can force the price up to $\frac{1}{2}\left(V^I + \alpha\right)$ when j's budget is $\frac{1}{2}\left(V^I + \alpha\right)$, and i can still force the price to at least this level when j's budget is larger.) If bidder j wins only II then he pays at least

$$\begin{aligned} \min\{V^{II} + \alpha, \overline{y}_i - V^I\} &\geq \min\{V^{II} + \alpha, V^{II} + \frac{3}{2}\alpha - \frac{1}{2}V^I\} \\ &= V^{II} + \frac{3}{2}\alpha - \frac{1}{2}V^I \end{aligned}$$

and so his net gain is no greater than $\frac{1}{2}V^I - \frac{3}{2}\alpha$. Bidder j can win both objects then he pays a total of at least $\frac{1}{2}V^I + V^{II} + \frac{3}{2}\alpha$ and thus his gain is at most

$$V^I + V^{II} + \alpha - \left(\frac{1}{2}V^I + V^{II} + \frac{3}{2}\alpha\right) = \frac{1}{2}V^I - \frac{1}{2}\alpha.$$

Thus $\overline{y}_j = \frac{1}{2}\left(V^I + \alpha\right)$ is a best response to any $\overline{y}_i \geq \frac{1}{2}V^I + V^{II} + \frac{1}{2}\alpha$.

Suppose $\overline{y}_j = \frac{1}{2}\left(V^I + \alpha\right)$ and bidder i chooses a $y_i \neq \overline{y}_i$. If i wins both objects then he pays a total of $2\overline{y}_j = V^I + \alpha$ for a net gain of V^{II}, which is his equilibrium payoff. If he wins only I, then this is a profitable deviation only if I is obtained at a price below $V^I - V^{II} < \frac{1}{2}V^I + \frac{1}{2}\alpha$. But i cannot obtain I at such a price, since bidder j would continue to raise the bid. Finally, i is already winning II for a price of 0. Thus $\overline{y}_i \geq \frac{1}{2}V^I + V^{II} + \frac{3}{2}\alpha$ is a best response to $\overline{y}_j$.

Thus, we have shown that in both cases $(\overline{y}_i, \overline{y}_j)$ as specified are equilibrium budget choices. The fact that in each case the equilibrium payoffs are the same from all such equilibria is immediate.

Suppose $(\overline{y}_i, \overline{y}_j)$ are equilibrium budget choices with $\overline{y}_j \leq \overline{y}_i$. Now, since j is not choosing a budget $y_j < \overline{y}_j$, it must be the case that doing so causes him to not win object I, since if he still won I it would perforce be at a price below $\overline{y}_j$, and from Claim 3 this is better for him. Thus, in the subgame with budgets $(\overline{y}_i, y_j)$ either :

(i) there is an equilibrium such that bidder i wins the first object instead of the second; or

(ii) there is an equilibrium such that bidder i wins both objects.

CLAIM 1: *(i) is impossible.*

If any choice of $y_j < \overline{y}_j$ causes i to win the first object instead of the second, it must be that when j chooses a budget of $\overline{y}_j$, bidder i is just indifferent between winning II and winning I. Hence, $V^{II} = V^I - \overline{y}_j$ or equivalently, $\overline{y}_j = V^I - V^{II}$, and by construction $\overline{y}_i \geq \overline{y}_j = V^I - V^{II}$.

Since bidder i does not prefer to win both objects rather than just II, we have:

$$\begin{aligned} V^I + V^{II} + \alpha - \overline{y}_j - \min\left\{V^{II}, \overline{y}_j\right\} &\leq V^{II} \\ \alpha + V^{II} - \min\left\{V^{II}, V^I - V^{II}\right\} &\leq 0 \end{aligned}$$

If $V^{II} \leq V^I - V^{II}$ then $\alpha \leq 0$. If $V^{II} > V^I - V^{II}$ then $\frac{1}{2}V^I \geq V^{II} + \frac{1}{2}\alpha$ and again $\alpha \leq 0$. Since $\alpha > 0$ we have a contradiction. □

CLAIM 2: *(ii) holds only if $\frac{1}{2}V^I \leq V^{II} + \frac{1}{2}\alpha$ and $\overline{y}_j = \frac{1}{2}\left(V^I + \alpha\right)$.*

If any choice of $y_j < \overline{y}_j$ causes i to win both objects instead of the second, it must be that when j chooses a budget of $\overline{y}_j$, bidder i is just indifferent between winning II and winning both objects. Hence, $V^{II} = V^I + V^{II} + \alpha - 2\overline{y}_j$, or equivalently,

$$V^{II} = V^I + V^{II} + \alpha - \overline{y}_j - \min\left\{V^{II}, \overline{y}_j\right\} \tag{1}$$

Now there are two cases to consider.

If $\overline{y}_j \leq V^{II}$, then from (1), $\overline{y}_j = \frac{1}{2}\left(V^I + \alpha\right)$ and thus:

$$\frac{1}{2}V^I \leq V^{II} - \frac{1}{2}\alpha.$$

If $\overline{y}_j > V^{II}$, then from (1), $\overline{y}_j = V^I - V^{II} + \alpha$ and thus:

$$\frac{1}{2}V^I > V^{II} - \frac{1}{2}\alpha. \ \square$$

Now suppose $(\overline{y}_i, \overline{y}_j)$ are equilibrium budget choices.

Claim 2 determines $\overline{y}_j$ in both cases.

The prices are then determined by Lemma 3 in the Appendix to the paper and this implies that the payoffs are unique. ■

1 Reserve price generalization

1. Suppose

$$\begin{aligned} y_1 &> 2y_2 \\ V^B &> y_2 > \frac{V^A}{2} \end{aligned}$$

Then 2 wins A for y_2 while 1 wins B for 0 since $V^B > V^A + V^B - 2y_2 > V^A - y_2$. $TR = y_2$.

Let r be such that

$$2y_2 - V^A < r < y_2$$

Then 1will win both goods for y_2 since $V^A + V^B - 2y_2 > V^B - r$. $TR = 2y_2$.

2 Suppose

$$\begin{aligned} y_1 &< 2y_2 \\ y_1, y_2 &< V^A \\ 2y_2 - y_1 &< V^A - V^B < y_2 \end{aligned}$$

Then 2 wins A for y_2 while 1 wins B for 0 since $V^B > V^A - y_2$ and $V^A - y_2 > V^B - (y_1 - y_2)$. $TR = y_2$.

Let r be such that

$$y_2 - \left(V^A - V^B\right) < r < y_1 - y_2$$

Then 1 wins A for y_2 since $V^A - y_2 > V^B - r$ and 2 wins B for $y_1 - y_2$. $TR = y_1$.