

Constrained Efficiency with Search and Information Frictions

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Abstract

I characterize the constrained efficient (or planner's) allocation in a directed search model with adverse selection. Guerrieri, Shimer and Wright (2010) analyze equilibrium in this environment and show, through three examples, that the equilibrium is Pareto dominated by some pooling or semi-pooling allocations. I formally define a planner whose objective is to maximize social welfare subject to the information and matching frictions of the environment. The planner can impose taxes and subsidies on agents that vary across sub-markets while being subject to an overall budget-balance condition. (The special case of no such taxes and transfers is the equilibrium in Guerrieri, et al. (2010).) I show that if the equilibrium is not first best efficient, then the equilibrium is not constrained efficient. I also derive conditions under which the planner can achieve the first best. I present examples in the context of financial and labor markets, explicitly solve for the efficient tax and transfer schemes and compare the planner's allocation with the equilibrium allocation.

Keywords: Directed search, constrained efficiency, adverse selection, free entry, cross-subsidization.

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1 introduction

This paper studies the constrained efficient allocation in an economy with search frictions and private information. There is a large number of homogeneous buyers on one side of the market whose population is endogenously determined through free entry. There is a fixed population of sellers on the other side of the market who have private information about their types. Buyers and sellers match bilaterally and trade in different locations, called sub-markets. In each sub-market, there are search frictions, in the sense that buyers and sellers on both sides get matched generally with probability less than 1.

Guerrieri, Shimer and Wright [10] characterize equilibrium in such an environment and show through three examples that the equilibrium allocation is Pareto dominated by pooling or semi-pooling allocations under some conditions. I define and characterize the constrained efficient allocation in this environment and show that the equilibrium is generally constrained inefficient (when the equilibrium fails to achieve the first best). I also derive sufficient conditions under which the planner can achieve the first best.

To define constrained efficient allocation, I consider a planner whose objective is to maximize social welfare. The planner chooses a set of sub-markets for agents as locations for trade. He might choose some sub-markets which would be inactive in the market economy, or he might shut down some sub-markets which would be active in the market economy. He can also impose taxes and subsidies on the agents. The planner, however, is subject to a budget balance condition. That is, the net amount of transfers that the planner makes to agents over all sub-markets must be non-positive. Note that the equilibrium allocation is a feasible allocation for the planner, because the revenues that the planner makes over **each** sub-market is zero under the equilibrium allocation.

The planner faces the same information and search frictions of the market economy. The planner cannot observe types of sellers and also cannot force sellers to participate in the planner's allocation. Also, the net revenue of buyers (after tax) from entry to those sub-markets that are chosen by the planner must be exactly 0. (This condition is equivalent to the free entry condition in the market economy.) In the language of mechanism design, the planner faces incentive compatibility of sellers and individual rationality of sellers and buyers and his own budget balance condition.

To understand how the planner can achieve strictly higher welfare than the market economy, let's consider the first example that I study in section 4. Sellers have one indivisible asset which is of 2 types: high and low. The high type asset is more valuable both to buyers and sellers. Guerrieri et al. [10] show that there exists a unique separating equilibrium in

which different types trade in different sub-markets. High type sellers prefer the higher price sub-market with lower probability of matching (sub-market two), because the asset is worth more to them in case of being unmatched. Low type sellers are just indifferent between the two sub-markets. Probability of matching here is used as a screening device.

The planner can do better than equilibrium in the following way: Beginning from the equilibrium allocation (which is feasible for the planner), the planner subsidizes sellers in sub-market one (low type sellers) just a little bit so that their incentive compatibility constraint for choosing sub-market two becomes slack. Now more buyers enter sub-market two to get matched with already unmatched high type sellers. Therefore the welfare increases due to the formation of new matches. To finance subsidies to sellers in sub-market one (low type sellers), the planner decreases the payment to sellers in sub-market two (high type sellers). The planner keeps doing that until he achieves the first best (high type sellers get matched with probability 1) or participation constraint of high type sellers binds. Even if there are more than two types, increase in the utility of low types makes incentive compatibility constraint of higher types less restrictive, so higher types will also generate weakly higher surplus.

To understand the nature of inefficiency in the market economy, consider the externalities implied by having one more buyer in a sub-market. First, it decreases the probability that other buyers are matched in that sub-market. This effect is present generally in all environments with search frictions. It is a well established result in the literature that the buyers who enter the market in a directed search setting, as opposed to a random search one, can internalize these externalities by choosing the "right" price (contract), if sellers' types are observable and contractible and if the firms can commit¹.

Second, having one more buyer in a sub-market changes the payoff of sellers at that sub-market. This change will affect the set of feasible sub-markets that other buyers can enter to attract **other types** of sellers, because the change in the payoff of one type affects the incentive compatibility constraints of other types. Buyers in the market economy do not take into account the effect of their entry on the payoff of sellers in other sub-markets (and subsequently on the contracts posted on those sub-market). The planner takes this effect into account and therefore can do better than the market economy. The extent to which the planner can improve efficiency depends on the details of the environment. In my second result, I derive sufficient conditions under which the planner can eliminate distortions

¹There are a lot of papers with this message. For example, see the following: [14], [1], [17], [18], [19] and [6].

completely and achieve the first best.

In the second example in section 5, I characterize the constrained efficient allocation in a version of the rat race (Akerlof [2]) and compare my results with Guerrieri et al. [10] who solve for the equilibrium allocation in this environment. There are two types of workers. Type two workers incur less cost for working longer hours and generate higher output compared to type one workers. Also the marginal output (with respect to hours of work) that type two workers generate is higher. In equilibrium, type two workers works inefficiently for longer hours (than they would work under complete information) and get matched with inefficiently higher probability. The planner, in contrast to the market economy, achieves the first best. He pays type one workers higher wages and type two workers lower wages than what they would get under complete information. These subsidies (to type one) and taxes (on type two) are needed to ensure that type one workers do not have any incentive to apply to the sub-market that type two workers apply to. Moreover, if the share of type two workers in the population is sufficiently small, then the planner's allocation even Pareto dominates the equilibrium allocation.

Recently, Chang [4] and Gurrieri and Shimer [9] have used a directed search model with adverse selection to explain the sharp declines in asset prices and liquidity of assets in the recent financial crises. I study in section 6 an asset market which is a static version of Chang [4]. I derive conditions under which the planner can achieve the first best by taxing the sellers in high price sub-markets and subsidizing sellers in low-price sub-markets (similar to the simple two-type example). I characterize the parameter region in which the planner's allocation is separating. I show that in general this region is different than that in which the equilibrium allocation is separating. In particular, I show that the planner might want to separate types in the region of parameters that fire-sales (pooling of high type sellers who need liquidity and low types together) occur in Chang [4].

Guerrier and Shimer [9] study the effect of asset subsidy programs by governments, or in general some entity with a deep pocket, and show that these programs can increase liquidity and prices of assets, and therefore, these programs can save market from liquidity crisis. However, they do not consider the funding sources of such programs. I show that the planner can strictly improve welfare by taxing sellers in sub-markets with higher prices and subsidizing sellers in sub-markets with lower prices.

Shi and Delacroix [5] study a model in which sellers with private information post contracts (in contrast to Guerrieri et al. [10] in which the uninformed side of the market posts contracts). They investigate the potentially conflicting roles of prices: the signaling role

and the search directing role. Aside from some details², the notion of constrained efficiency defined in this paper and the ideas behind that (that the planner chooses some sub-markets and makes appropriate transfers to agents) apply to their model as well, because the environments are similar. They just have different trading mechanism.

The paper is organized as follows. In section 2, I develop the environment of the model and define the planner's problem³. In section 3, I characterize the planner's allocation and state my main results. In section 4, I study a two-type asset market example, characterize the planner's allocation and compare it with the equilibrium allocation. I explain the nature of inefficiency in the market economy and discuss why and how the planner can allocate more efficiently than the market economy. In section 5, I study a version of the rat race. In section 6, I study an asset market with a continuous type space and compare my results with Chang [4]. Section 7 concludes. All proofs appear in the appendix.

2 The Model

2.1 Environment

Consider an economy with two types of agents: buyers and sellers. There is a measure 1 of sellers. A fraction π_i of sellers are of type $i \in \{1, 2, \dots, I\}$. Type is seller's private information. On the other side of the market, there is a large continuum of homogenous buyers who can enter the market by incurring cost $k > 0$. After buyers enter the market, buyers and sellers are allocated to different sub-markets (which are just some locations for trade). Matching is bilateral. After they match, they trade.

There are $n + 1$ goods in this economy. Goods 1 to n are produced by sellers. There is also a numeraire good which is produced by both. Sellers' and buyers' payoff functions are quasi-linear in the numeraire good⁴. Let $a \equiv (a^1, a^2, \dots, a^n) \in \mathbb{A} \subset \mathbb{R}^n$ be a vector where \mathbb{A}

²For example in their model, sellers choose the quality of their products. The quality then becomes their private information.

³Guerrieri [8] and Moen and Rosen [15] study constrained efficient allocation in environments with directed search and private information. Specifically, Guerrieri [8] shows that the competitive search equilibrium is constrained inefficient in a dynamic setting, if the economy is not on the steady state path. However in both papers, the agents who search (workers) do not have ex-ante private information. After they match with firms, they learn their types which become their private information.

⁴The difference between payoff functions here and in [10] is that I assume quasi-linear preferences for both sellers and buyers, while they do not make such an assumption. The reason that I impose quasi-linearity assumption is that I want to do welfare analysis and I want to use taxes and subsidies. If the preferences

is compact, convex and non-empty. Component k of this vector, a^k , denotes the quantity of good k . For example in a labor market, a can be just a real number denoting the hours of work. When I say an agent produces (or consumes) a , I mean that the agent produces (or consumes) a^1 units of good 1, a^2 units of good 2 and so on. The payoff of a buyer, who enters the market, from consuming a and producing $t \in \mathbb{R}$ units of the numeraire good is $v_i(a) - t - k$ if matched with a type i seller and is $-k$ if unmatched⁵. The payoff of a type i seller from producing a and consuming $t \in \mathbb{R}$ units of the numeraire good is $u_i(a) + t$ if matched with a buyer and is 0 otherwise. For all i , $v_i : \mathbb{A} \rightarrow \mathbb{R}$ and $u_i : \mathbb{A} \rightarrow \mathbb{R}$ are continuous.

There are search frictions in this environment. By search frictions I mean that sellers generally get to match with the buyers they have chosen with probability less than one. Matching occurs in sub-markets which are simply some locations for trades. Matching technology determines the probability that sellers and buyers in each sub-market get matched. If the ratio of buyers to sellers in one sub-market is $\theta \in [0, \infty]$, then the buyers are matched with probability $q(\theta)$. Symmetrically, matching probability for sellers is $m(\theta) \equiv \theta q(\theta)$. As is standard in the literature, I assume that m is non-decreasing and q is non-increasing. Both are continuous.

2.2 Planner's Problem

We define a planner whose objective is to maximize the sum of welfare of sellers and buyers in this economy. This planner faces the same information and search frictions present in the market economy. Before we explain what is feasible and what is not for this planner, let's first understand how the market economy works in this environment.

Sub-markets in the market economy are characterized by (a, p) where $a \in \mathbb{A}$ denotes the vector of goods 1 to n to be produced by sellers in this sub-market and $p \in \mathbb{R}$ is the amount of the numeraire good to be transferred from buyers to sellers. No sub-market which can deliver buyers a strictly positive payoff is inactive in the equilibrium. If there was such a sub-market, some buyers would have already entered that sub-market to exploit that opportunity. On the other side of the market, sellers observe all (a, p) pairs posted in

are not quasi-linear, the weight that the planner assigns to buyers and sellers might become important.

⁵Buyers may transfer the numeraire good both to sellers or to the planner. In contrast in the market economy, the amount of the numeraire good produced by buyers is the exactly the same amount that the sellers in that sub-markets consume. Also, t can be positive or negative. If t is negative, it means that the buyer is the net consumer of the numeraire good.

the market, anticipate the market tightness at each sub-market, and then direct their search toward one which delivers them the highest expected payoff.

What exactly the planner does in this environment is to choose a set of sub-markets which are characterized by 4 elements, $(a, p, t_m, t_u) \in Y \equiv \mathbb{A} \times \mathbb{R}^3$, where a denotes the vector of goods to be produced by sellers, p denotes the amount of the numeraire good to be transferred from buyers to sellers, t_m denotes the amount of the numeraire good to be transferred from buyers to the planner conditional on trade, and t_u denotes the amount of the numeraire good to be transferred from buyers to the planner if they want to enter this sub-market. Note that any post in the market economy is a special case of this description with $t_m = t_u = 0$.⁶

The planner, in contrast to the market economy, is assumed to have the power to pick any selection of sub-markets for agents as possible locations for trade. He might select some sub-markets which would be inactive in the market economy, or he might shut down some sub-markets which would be active in market economy. He is also assumed to have the power to impose taxes and subsidies on agents. Aside from these two powers (picking sub-markets and possibility of transfers), the market economy and the planner face the same restrictions: None can condition amounts of goods to be produced or payments of the numeraire good on the types of sellers. Ex-ante payoff of buyers in both cases should be 0 to ensure that buyers want to participate and also to ensure that there is no excess entry into any sub-market. Also in both cases sellers choose sub-markets which maximize their expected payoff or stay out. (If they get a negative expected payoff, they will not participate.)

The planner faces a budget constraint, or a budget balance condition as called in the mechanism design literature. This condition states that the net amount of transfers that the planner makes to the buyers should not exceed 0. Notice that in the market economy, it is not possible to transfer funds (the numeraire good) from one sub-market to another. In other words, in the market economy all the surplus generated in any sub-market belongs to sellers in that sub-market. The planner, on the other hand, might give the sellers more or less than the surplus they generate. The planner imposes these taxes and subsidies to achieve higher welfare, as long as the net amount of transfers that the planner makes does not exceed 0.

I could assume alternatively that the planner sets a direct mechanism. Then sellers report their types to the planner and the planner assigns them to different sub-markets. It can be

⁶I show in the appendix that t_u can be assumed to be 0 without loss of generality. I include t_u in the description of sub-markets to ensure that I consider a framework as general as possible for the planner.

shown that the outcome would not have been different if I had formulated the problem this way. The reason that I did not choose that language is that implementing such a mechanism in the real world requires a large amount of communication. Rather, the way I formulated the problem is closer to real world applications. The planner's allocation in my model can be implemented in the market by imposing appropriate scheme of taxes and subsidies on the agents.

2.3 Formal definition of the planner's problem

First I formally define an allocation and then a feasible allocation. Then I define the constrained efficient allocation or the planner's allocation.

Let $y \equiv (a, p, t_m, t_u)$. To make the notation clearer, the first component of y is denoted by a , rather than y_1 . The second, the third and the fourth components are respectively denoted by p , t_m and t_u . Similarly if the sub-market is denoted by y' , then the first, the second, the third and the fourth components of y' are denoted by a' , p' , t'_m and t'_u . Let also $\gamma_i(y)$ denote the share of sellers that are type i in the sub-market denoted by y , with $\Gamma(y) \equiv \{\gamma_1(y), \dots, \gamma_i(y), \dots, \gamma_I(y)\} \in \Delta^I$. Δ^I is an I -dimensional simplex, that is, for all y , $0 \leq \gamma_i(y) \leq 1$ and $\sum_{i=1}^I \gamma_i(y) = 1$. Any allocation describes a set of open sub-markets Y^P , distribution of buyers over open sub-markets λ , the ratio of buyers to sellers for each open sub-market θ and finally the distribution of types in each open sub-market Γ .

Definition 1. *A planner's allocation is a measure λ over the set of sub-markets Y with support Y^P , a function $\theta : Y^P \rightarrow [0, \infty]$, and a function $\Gamma : Y^P \rightarrow \Delta^I$.*

Because the planner faces some constraints, only some allocations are feasible for the planner:

Definition 2. *A planner's allocation $\{\lambda, Y^P, \theta, \Gamma\}$ is feasible if it satisfies the following conditions:*

1. (Sellers' maximization) Let $U_i \equiv \max_{y' \in Y^P} \{m(\theta(y'))(u_i(a') + p')\}$. Given i ,

(a) if $U_i < 0$, then $\gamma_i(y) = 0$ for all $y \in Y^P$,

(b) if $U_i \geq 0$, for any $y \in Y^P$ and i such that $\gamma_i(y) > 0$ and $\theta(y) < \infty$,

$$y \in \arg \max_{y' \in Y^P} \left\{ m(\theta(y'))(u_i(a') + p') \right\},$$

(c) and $\int_{Y^P} \frac{\gamma_i(y)}{\theta(y)} d\lambda(\{y\}) \leq \pi_i$, with equality if $U_i > 0$.

2. (Buyers' zero profit) For any $y \in Y^P$,

$$q(\theta(y)) \sum_i \gamma_i(y)(v_i(a) - p - t_m) = k + t_u.$$

3. (Planner's budget constraint)

$$\int_{Y^P} [q(\theta(y))t_m + t_u] d\lambda(\{y\}) \geq 0.$$

With respect to the first condition, sellers' maximization problem summarizes two conditions. The first condition is participation (individual rationality) constraint: If type i sellers get a negative payoff, they will choose their outside option (with 0 payoff). If they get a positive payoff, they apply to some sub-market, unless they get exactly 0 payoff in which case they are indifferent between the allocation and their outside option. The second condition is incentive compatibility (IC) constraint. For any allocation, let X_i be defined as follows: $X_i \equiv \{(\theta(y), a) \mid y \equiv (a, p, t_s, t_m) \in Y^P, \gamma_i(y) > 0, \theta(y) < \infty\}$. Denote elements of X_i by (θ_i, a_i) . In words, θ_i is the market tightness of a sub-market to which type i applies with positive probability and a_i is the production level at that sub-market. Sellers' maximization condition implies that for any i, j , $(\theta_i, a_i) \in X_i$ and $(\theta_j, a_j) \in X_j$:

$$m(\theta_i)(u_i(a_i) + p_i) \geq m(\theta_j)(u_i(a_j) + p_j) \text{ (IC)}.$$

With respect to the second condition, if the expected payoff in one sub-market is strictly negative, no buyer enters that sub-market. If it is strictly positive, more buyers will enter that sub-market. Therefore, for all markets that the planner wants to be open, buyers must get exactly 0 expected payoff. A buyer has to incur entry cost k and has to pay t_u units of the numeraire good to the planner, if he wants to enter sub-market y . Then, he gets matched with a type i seller with probability $\gamma_i(y)$ from which he gets a payoff of $v_i(a)$ in terms of the numeraire good, and pays p units of the numeraire good to the seller and t_m to the planner.

The last constraint states that the net amount of the numeraire good that the planner collects from buyers in different sub-markets must be non-negative. The planner charges each buyer in sub-market y amount t_u when they enter and charges each matched buyer amount t_m in terms of the numeraire good.

Among all feasible allocations, the planner chooses one that maximizes his objective, which is just sum of welfare of buyers and sellers in this economy.

Definition 3. A constrained efficient allocation is a feasible allocation $\{\lambda, Y^P, \theta, \Gamma\}$ which maximizes the planner's objective among all feasible allocations:

$$\max \int_{Y^P} \left[q(\theta(y)) \sum_i \gamma_i(y) [u_i(a) + v_i(a) - t_m] - (k + t_u) \right] d\lambda(\{y\})$$

Subject to: $\{\lambda, Y^P, \theta, \Gamma\}$ is feasible.

In sub-market y , a buyer gets matched with a seller with probability $q(\theta(y))$. If the buyer matches with a type i , then the buyer gets $v_i(a) - p - t_m$ and the seller gets $u_i(a) + p$. The buyer already incurred $k + t_u$. By integrating sum of buyers' and sellers' payoffs over all sub-markets, we can calculate the value of the planner's objective.

2.4 Equilibrium definition

I define the equilibrium here in order to be able to compare the equilibrium allocation with the planner's allocation. The definition of equilibrium is taken from Guerrieri et al. [10]. Let \bar{Y} be defined as $\bar{Y} \equiv \cup_i \bar{Y}_i$, where

$$\bar{Y}_i \equiv \{(a, p, 0, 0) \mid (a, p) \in \mathbb{A} \times R, q(0)(v_i(a) - p) \geq k, \text{ and } u_i(a) + p \geq 0\}.$$

Note that for any $y \in \bar{Y}$, the third and fourth elements are 0, because there is no transfers between the planner and the agents in the equilibrium. If $(a, p, 0, 0) \notin \bar{Y}$, then no type will be attracted to this sub-market. Therefore, we restrict our attention to only $(a, p, 0, 0)$ in \bar{Y} .

Definition 4 (Guerrieri et al. [10]). An equilibrium $\{Y^{eq}, \lambda^{eq}, \theta^{eq}, \Gamma^{eq}\}$, is a measure λ^{eq} on \bar{Y} with support Y^{eq} , a function $\theta^{eq} : \bar{Y} \rightarrow [0, \infty]$, and a function $\Gamma^{eq} : \bar{Y} \rightarrow \Delta^I$ which satisfies the following conditions:

1. (Sellers' optimal search) Let $U_i^{eq} = \max \left\{ 0, \max_{y' \in Y^{eq}} \{m(\theta^{eq}(y'))(u_i(a') + p')\} \right\}$ and $U_i^{eq} = 0$ if $Y^P = \emptyset$. Then for any $y \in Y$ and i , $U_i^{eq} \geq m(\theta^{eq}(y))(u_i(a) + p)$ with equality if $\theta^{eq}(y) < \infty$ and $\gamma_i^{eq}(y) > 0$. Moreover, if $u_i(a) + p < 0$, either $\theta^{eq}(y) = \infty$ or $\gamma_i^{eq}(y) > 0$.

2. (Buyers' profit maximization and free entry) For any $y \in \bar{Y}$,

$$q(\theta^{eq}(y)) \sum_i \gamma_i^{eq}(y) (v_i(a) - p) \leq k,$$

with equality if $y \in Y^{eq}$.

3. (Market clearing) For all i , $\int_{Y^{eq}} \frac{\gamma_i^{eq}(y)}{\theta^{eq}(y)} d\lambda^{eq}(\{y\}) \leq \pi_i$, with equality if $U_i^{eq} > 0$.

Given any equilibrium $\{Y^{eq}, \lambda^{eq}, \theta^{eq}, \Gamma^{eq}\}$, I construct an **equilibrium allocation** $\{Y^{eq}, \lambda^{eq}, \theta, \Gamma\}$ where $\theta : Y^{eq} \rightarrow [0, \infty]$, $\Gamma : Y^{eq} \rightarrow \Delta^I$ and $\theta(y) = \theta^{eq}(y)$, $\Gamma(y) = \Gamma^{eq}(y)$, for all $y \in Y^{eq}$. The difference between equilibrium and equilibrium allocation is that in equilibrium, the market tightness and the composition of types must be defined over all sub-markets (for all $y \in \bar{Y}$), because when buyers want to enter the market, they need to form beliefs about all sub-markets, either in the support of λ^{eq} or not. In contrast for the equilibrium allocation, we just need to define them over active sub-markets ($y \in Y^{eq}$), because buyers cannot enter any other sub-market.

To construct this allocation, we took all equilibrium objects on the equilibrium path (over the support Y^{eq}) and set $t_m = t_u = 0$. This allocation is feasible, because sellers' maximization condition and buyers' zero profit condition are satisfied following their counterparts in the definition of equilibrium. The planner's budget constraint is also clearly satisfied, because $t_m = t_u = 0$ for all $y \in Y^P$. When I say **equilibrium allocation**, I mean the feasible allocation which is constructed from equilibrium objects as above. Finally, when I refer to equilibrium in the paper, I mean the notion of equilibrium that I defined in this section. I do not mean a notion of equilibrium with signaling in which the informed side of the market posts contracts, unless otherwise noted.

3 Characterization

I present my main results in this section. In the first part of lemma 1, we show that we can assume without loss of generality that $t_u = 0$. The idea is that the if the planner wants to raise some given amount of the numeraire good from buyers in sub-market y , he can do that just through one transfer upon trade in that sub-market. He does not need to impose two different types of transfers. This result is easy to show, but it is important. Note that we have implicitly made an assumption in the definition of a feasible allocation that buyers at the beginning commit to participate in the matching stage (especially after they incur the entry cost and after they pay t_u to the planner). Suppose for now that $t_u + k < 0$. Then we need to use the assumption about commitment, for otherwise, buyers refuse to participate in the matching stage after they receive $-t_u$ (because their expected payoff after entry is negative). Thanks to this lemma, we do not need to require buyers to commit to participate, because after buyers incur cost k , they have incentives to recover the entry cost through participation in the matching stage. According to the second part of the lemma,

the planner's budget constraint must be binding.

Lemma 1. *In any constrained feasible allocation, the following holds.*

- For all $y \in Y^P$, $t_u = 0$ without loss of generality.
- The budget constraint must hold with equality.

Following the second part of the lemma, the planner's objective can be written as

$$\max \int_{Y^P} \left[q(\theta(y)) \sum_i \gamma_i(y) [u_i(a) + v_i(a)] - k \right] d\lambda(\{y\})$$

which is the net amount of surplus generated in this economy. Transfers naturally do not appear here, because they are just transfers between agents who have quasi-linear preferences.

3.1 Complete information allocation

As a benchmark and for the future reference, consider the market economy when the type of sellers is common knowledge. The sub-markets, therefore, are also indexed by the type of sellers the buyers want to meet. Consider type i sellers. The buyers, contemplating what sub-market to enter to attract type i , enter a sub-market which maximizes the payoff of type i subject to the free entry condition. (See Moen [14] for further explanation.) If there is any sub-market that would deliver type i sellers a higher payoff, some buyers would enter that sub-market and then, sellers would strictly prefer that sub-market. Therefore buyers who attract type i solves the following problem in the market economy with complete information:

$$\begin{aligned} & \max_{\theta, a, p} \{m(\theta)(p + u_i(a))\} \\ \text{s.t. } & q(\theta)(v_i(a) - p) = k. \end{aligned}$$

Denote the solution to this problem by $(\theta_i^{CI}, a_i^{CI}, p_i^{CI})$. Eliminating p from the maximization problem, one can write the payoff of type i sellers from participating in the market in the complete information case as follows:

$$U_i^{CI} = \max_{\theta, a} \{m(\theta)(v_i(a) + u_i(a)) - k\theta\}.$$

Notice that the function in the maximization problem, $m(\theta)(v_i(a) + u_i(a)) - k\theta$, is exactly equal to the welfare created by type i sellers. Thus, the planner who observes types of the sellers solves exactly the same problem as buyers in the market economy. (If $U_i^{CI} < 0$,

the agent does not participate in the market and the planner also does not want him to participate.) If $U_i^{CI} \geq 0$, the planner wants type i to get matched with probability $m(\theta_i^{CI})$ and produce a_i^{CI} . This is the core of the argument in the literature which states that the market economy decentralizes the planner's allocation under complete information⁷. In this paper, when I say that the planner achieves the first best or achieves the complete information allocation, I mean that there exists a feasible allocation in which type i sellers get matched with probability $m(\theta_i^{CI})$ and produce a_i^{CI} .

3.2 Results

We have already seen that the equilibrium allocation is feasible for the planner. It is immediately followed that the planner can achieve at least the level of welfare in the market economy. The following proposition states that the planner can achieve strictly higher welfare.

Assumption 1. *Assumptions of Guerrieri et al. [10]:*

- *Monotonicity:* For all $a \in \mathbb{A}$, $v_1(a) \leq v_2(a) \leq \dots \leq v_I(a)$.
- *Sorting:* For all i , $a \in \mathbb{A}$ and $\epsilon > 0$, there exists a $a' \in B_\epsilon(a) \equiv \{a' \in \mathbb{A} \mid \|a - a'\|_2 < \epsilon\}$ such that $u_j(a') - u_j(a) < u_{j'}(a') - u_{j'}(a)$ for all $j < i \leq j'$.

Proposition 1. *Suppose assumption 1 holds. Also assume that for all i , $U_i^{CI} > 0$ and that the market economy fails to achieve the first best. Then, the planner achieves **strictly** higher welfare than the market economy.*

The idea of the proof is as follows. Since the equilibrium allocation is feasible for the planner, I begin from that allocation and propose a set of transfers to improve upon that allocation. We need first to understand how the equilibrium is constructed. Under assumption 1, Guerrieri et al. [10] prove that the equilibrium for type i is characterized by maximizing the payoff of type i , subject to free entry condition and incentive compatibility (IC) constraints of lower types, that is, type $j < i$ does not get a higher payoff if he chooses the sub-market that type i chooses. They prove that this equilibrium is unique in terms of payoffs. Let $\{\lambda^{eq}, Y^{eq}, \theta^{eq}, \Gamma^{eq}\}$ denote the equilibrium allocation where $Y^{eq} \equiv \{y_1^{eq}, y_2^{eq}, \dots, y_I^{eq}\}$.

If any IC constraint is binding in the equilibrium, say type j is indifferent between y_j^{eq} and y_i^{eq} with $j < i$, then the planner can do the following to improve the welfare. The planner first

⁷There are many papers with this message in the literature, such as Moen [14], Acemoglu and Shimer [1], Shi [17], [18] and Shimer [19].

increases the expected payoff of all lower types by some small amount $\epsilon > 0$ in the following way: He increases p at each sub-market y_j ($j < i$) by $\frac{\epsilon}{m(\theta(y_j))}$ amount. Now constraints of the maximization problem for type i become slack, so the planner can find (θ'_i, a'_i) which increases the welfare generated by type i . Therefore, the payoff of type i increases. Now consider type $i + 1$. The planner solves the maximization problem for type $i + 1$ again. Since all lower types including type i get a strictly higher payoff than the equilibrium allocation now, the maximization problem for type $i + 1$ is now less constrained, so the planner can achieve higher welfare for type $i + 1$ as well. The planner keeps doing the same thing for all types above i and assigns them new (θ, a) pairs. So far the welfare of the population has increased. Then the planner, given (θ, a) of every sub-market, adjusts payments to buyers as well to make zero profit condition satisfied. Finally, the planner distributes the transfers back to all agents in a way which does not distort IC. Adjusting payments to buyers and making transfers to all agents does not change the welfare of the population, therefore, the welfare now is strictly higher than the level of welfare in the equilibrium.

In the next proposition, we provide sufficient conditions for the planner to achieve the first best.

Assumption 2. Let $a \equiv (a^1, a^2, \dots, a^n)$ and $i \in \{1, 2, \dots, I\}$ and $k \in \{1, 2, \dots, n\}$.

1. *Monotonicity of u in i : $u_1(a) \leq u_2(a) \leq \dots \leq u_I(a)$ for all $a \in \mathbb{A}$.*
2. *Single crossing of u in $(a; i)$: $u_i(a) - u_{i-1}(a)$ is increasing in a^k for all $a \in \mathbb{A}$, $i \geq 2$ and k .*
3. *Single crossing of $u + v$ in $(a; i)$: $u_i(a) + v_i(a) - (u_{i-1}(a) + v_{i-1}(a))$ is increasing in a^k for all $a \in \mathbb{A}$, $i \geq 2$ and k .*
4. *Supermodularity of $f_i \equiv u_i + v_i$ in a :*

$$f_i(a''') + f_i(a) \geq f_i(a') + f_i(a'') \text{ for all } i \text{ and } a, a', a'', a''' \in \mathbb{A},$$

where $a \equiv (a^1, a^2, \dots, a^n)$, $a' \equiv (a^1, \dots, a^{k-1}, b^k, a^{k+1}, \dots, a^n)$, $a'' \equiv (a^1, \dots, a^{l-1}, b^l, a^{l+1}, \dots, a^n)$, and $a''' \equiv (a^1, \dots, a^{k-1}, b^k, a^{k+1}, \dots, a^{l-1}, b^l, a^{l+1}, \dots, a^n)$, and $b^k \geq a^k$, $b^l \geq a^l$ and $k < l$.

5. *Either (a) holds or (b) and (c) hold, where (a), (b) and (c) are defined as follows.*

(a) *Monotonicity of v in i : $v_1(a) \leq v_2(a) \leq \dots \leq v_I(a)$ for all $a \in \mathbb{A}$.*

(b) *Monotonicity of $u + v$ in i : $u_1(a) + v_1(a) \leq u_2(a) + v_2(a) \leq \dots \leq u_I(a) + v_I(a)$ for all $a \in \mathbb{A}$.*

(c) *Sufficient gains from trade for all types:*

$$U_i^{CI} \geq m(\theta_{i-1}^{CI})(u_i(a_{i-1}^{CI}) - u_{i-1}(a_{i-1}^{CI})) \frac{\sum_{k=i}^n \pi_k}{\pi_i}, \text{ for } i > 1 \text{ and } U_1^{CI} \geq 0.$$

Proposition 2. *Under assumption 2, the planner achieves the first best.*

To understand this result, first suppose the planner does not face any private information. The planner wants type i to get matched with probability $m(\theta_i^{CI})$ and produce a_i^{CI} . Since the planner actually faces private information, he wants to find a set of transfers which together with the complete information allocation satisfies IC constraints, if possible. If the payoff function of agents satisfies single crossing condition⁸, which is guaranteed by part 2 of assumption 2, then monotonicity of (θ_i, a_i) in i is necessary and sufficient condition for the allocation to satisfy IC. (See Theorem 7.1 and 7.3 in [7] or section 3.1 in [11] for reference.) In other words, if single crossing condition is satisfied and (θ_i, a_i) is increasing in i , then there exists a set of transfers $\{p_i\}_{i \in \{1, 2, \dots, I\}}$ which together with $\{(a_i, \theta_i)\}_{i \in \{1, 2, \dots, I\}}$ satisfies IC.

But a_i^{CI} is increasing in i , if $u_i + v_i$ satisfies part 4 and 5 of assumption 2. (See Theorem 4 in Milgrom and Shannon [13].) If $u_i + v_i$ is increasing in i (part 3 of assumption 2), then $\arg \max_{\theta} \{m(\theta)(u_i(a_i^{CI}) + v_i(a_i^{CI})) - k\theta\}$ will be also increasing in i , so $(\theta_i^{CI}, a_i^{CI})$ is increasing in i . Finally, we used monotonicity of u in i (part 1 of assumption 2) together with part 2 of the assumption to ensure that if local IC constraints hold, then the global IC constraints hold as well. This property helps us to calculate the amount of taxes and subsidies that the planner needs to impose on different sub-markets. Then the planner adjusts payments to the agents such that zero profit condition and the planner's budget constraint are satisfied. In future sections, we will make it clear by a couple of examples the mechanism through which the planner can improve the welfare upon the market economy and how he might achieve the first best.

4 Example 1: Asset Market with Lemons

So far we have considered a general framework. In the following two sections, we study two examples from Guerrieri et al. [10] and characterize constrained efficient allocation for them and compare them with the associated equilibrium allocations. At the end of this section, I describe how the planner can increase welfare by using appropriate transfers. Also, I explain the nature of inefficiency in models of directed search with private information.

⁸Single crossing condition simply states that indifference curves of different types should cross only once.

In the first example, I consider an asset market with lemons, similar to Akerlof [3]. There are two types of assets, with value c_i to the seller and h_i to the buyer. Both c_i and h_i are in terms of a numeraire good. The payoff of a buyer matched with a type i seller is $\alpha h_i - t - k$ where α is the probability that the buyer gets the asset from the seller and t is the amount of transfer that he pays (either to the planner or to sellers) in terms of the numeraire good. The payoff of a type i seller matched with a buyer is $-\alpha c_i + t$ where α is the probability that the seller gives the asset to the buyer and t is the amount of the numeraire good he consumes. The buyer's payoff is $-k$ if unmatched. As a special case of the original setting, here: $I = 2$, $n = 1$, $a \equiv \alpha$, $u_i(\alpha) = -\alpha c_i$ and $v_i(\alpha) = \alpha h_i$. The matching function is $m(\theta) = \min\{1, \theta\}$, that is, the short side of the market gets matched for sure. Following Guerrieri et al. [10], we also make the following assumptions:

Assumption 3. *In the asset market with lemons,*

1. $0 < h_1 < h_2$ and $0 < c_1 < c_2$.
2. For $i = 1, 2$, $c_i < b_i \equiv h_i - k$.

Proposition 3. *In the asset market with lemons,*

- If $\pi_1 b_1 + \pi_2 b_2 \geq c_2$, then the planner achieves the first best.
- If $\pi_1 b_1 + \pi_2 b_2 < c_2$, then the planner cannot achieve the first best. However, the planner can allocate resources more efficiently than the market and also than any feasible allocation without transfers. See full details of allocation in table 1.

In the second and third columns of table 1 the equilibrium outcome under the complete information and under the incomplete information are described respectively. In the fourth and fifth columns, I describe the planner's allocation under different conditions.

Since there are positive gains from trade for both types according to part 2 of assumption 3, under the complete information the planner wants both types to get matched with probability 1 ($\theta_1 = \theta_2 = 1$) and also trade with probability 1 ($\alpha_1 = \alpha_2 = 1$). As already discussed under complete information, the market decentralizes the efficient allocation.

In the equilibrium with incomplete information, different types trade in different sub-markets. In sub-market 1, price (p_1) is lower, but probability of matching is higher compared to sub-market 2. The market tightness is used as a screening device here. Putting it differently, the probability of matching for type 2 is distorted so that type 1 does not want to apply to sub-market 2, although the price is higher there. The equilibrium allocation is independent of the distribution of types.

	Complete information	Equilibrium	Constrained efficient if $\pi_1 b_1 + \pi_2 b_2 \geq c_2$	Constrained efficient if $\pi_1 b_1 + \pi_2 b_2 < c_2$
α_1, α_2	1	1	1	1
θ_1	1	1	1	1
θ_2	1	$\frac{b_1 - c_1}{b_2 - c_1} < 1$	—	$\frac{b_1 - c_1}{b_2 - c_1} < \frac{\pi_1(b_1 - c_1)}{c_2 - \pi_1 c_1 - \pi_2 b_2} < 1$
p_1	b_1	b_1	$\pi_1 b_1 + \pi_2 b_2$	$b_1 < \frac{\pi_1 b_1 (c_2 - c_1) + \pi_2 c_1 (c_2 - b_2)}{c_2 - \pi_1 c_1 - \pi_2 b_2} < c_2$
p_2	b_2	b_2	—	c_2
$t_{m,1}$	0	0	0	$-\frac{\pi_2(b_2 - c_2)(b_1 - c_1)}{c_2 - \pi_1 c_1 - \pi_2 b_2} < 0$
$t_{m,2}$	0	0	—	$b_2 - c_2 > 0$
\bar{U}_1	$b_1 - c_1$	$b_1 - c_1$	$\pi_1 b_1 + \pi_2 b_2 - c_1$	$\frac{\pi_1(b_1 - c_1)(c_2 - c_1)}{c_2 - \pi_1 c_1 - \pi_2 b_2}$
\bar{U}_2	$b_2 - c_2$	$\frac{b_1 - c_1}{b_2 - c_1}(b_2 - c_2)$	$\pi_1 b_1 + \pi_2 b_2 - c_2$	0

Table 1: Comparison between different allocations

If $\pi_1 b_1 + \pi_2 b_2 \geq c_2$, then the planner achieves the first best through a pooling allocation. The planner's allocation in this case is reported in the fourth column of table 1. In this allocation, the planner does not need to use any transfers. All he needs to do is to restrict entry of buyers to other sub-markets. This allocation cannot be sustained as equilibrium, because buyers would have incentives to open a new sub-market with a higher price to attract only high type sellers from the pool. The planner does not open such a sub-market, because in that case, the probability that high quality sellers get matched will be reduced compared to the complete information allocation.

Now assume that $\pi_1 b_1 + \pi_2 b_2 < c_2$. The planner's allocation in this case is reported in the fifth column of table 1. Type 2 would get less than 0 under the pooling allocation, so pooling both types is not feasible. Therefore, the first best is not achievable via a pooling allocation. The first best is not achievable through any separating allocation as well, because if $\alpha_1 = \alpha_2 = \theta_1 = \theta_2 = 1$, then the payment to sellers in both sub-markets should be the same to satisfy IC condition. If the payments in both sub-markets are equal, then this allocation is pooling, but it is already shown that the pooling allocation is not feasible.

4.1 Explanation of the results

To understand how the planner improves efficiency, assume as a thought experiment that the planner begins from the equilibrium allocation and wants to increase welfare. We have

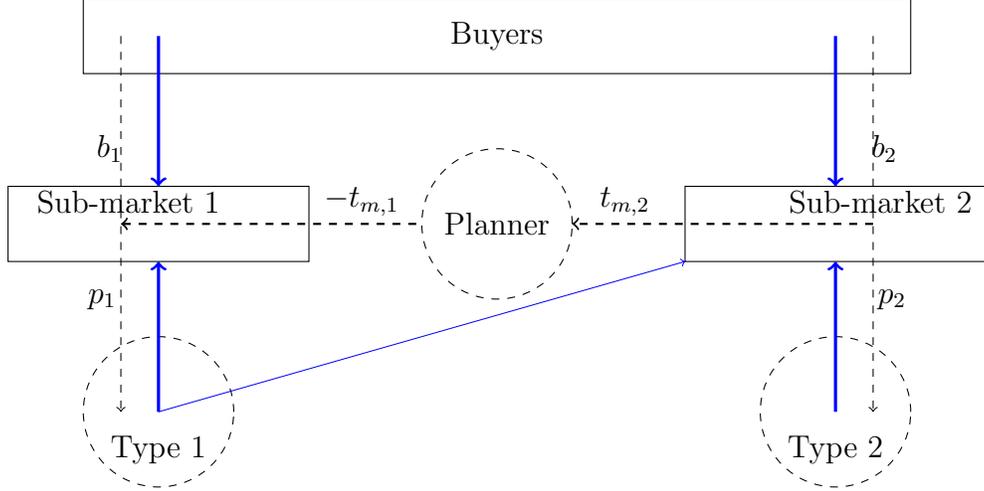


Figure 1: This schematic diagram illustrates how the planner allocates resources. The planner increases the payment to type 1 and decreases the payment to type 2 relative to the equilibrium allocation so that incentive compatibility of type 1 becomes slack. Now, more buyers enter sub-market 2 and the outcome approaches the complete information allocation.

already seen that the market allocation is feasible for the planner. In equilibrium type 1 is indifferent between choosing sub-market 1 and sub-market 2. Although there are some type 2 sellers unmatched in sub-market 2, buyers do not enter sub-market 2 any more, because more entry will make sub-market 2 strictly preferable for type 1, thus leading to entry of type 1 to sub-market 2. Nevertheless, matching with type 1 sellers in sub-market 2 with positive probability is not worthwhile for buyers given the high price that buyers need to pay in sub-market 2.

To improve efficiency, the planner increases the payment to type 1 (p_1) so that the IC of type 1 for choosing sub-market 2 becomes slack. Now more buyers have incentives to enter sub-market 2 to get matched with already unmatched sellers of type 2. To finance subsidies to sellers in sub-market 1 (type 1 sellers), the planner decreases the payment to sellers in sub-market 2 (type 2 sellers). The planner keeps increasing p_1 and decreasing p_2 until one of the following happens. Either he achieves the first best, which is the case in the pooling allocation, or participation constraints of type 2 sellers bind, that is, type 2 sellers get exactly 0 payoff. The former happens if $\pi_1 b_1 + \pi_2 b_2 \geq c_2$ and the latter happens if $\pi_1 b_1 + \pi_2 b_2 < c_2$. Figure 1 illustrates this point. Although we explained the main idea through a two-type example, the intuition is the same in the general n-type setting, or even in a continuous type one which will be discussed in section 6.

The main difference between planner's allocation and the equilibrium allocation is that in the equilibrium, the payment to sellers is exactly equal to the payment that buyers make.

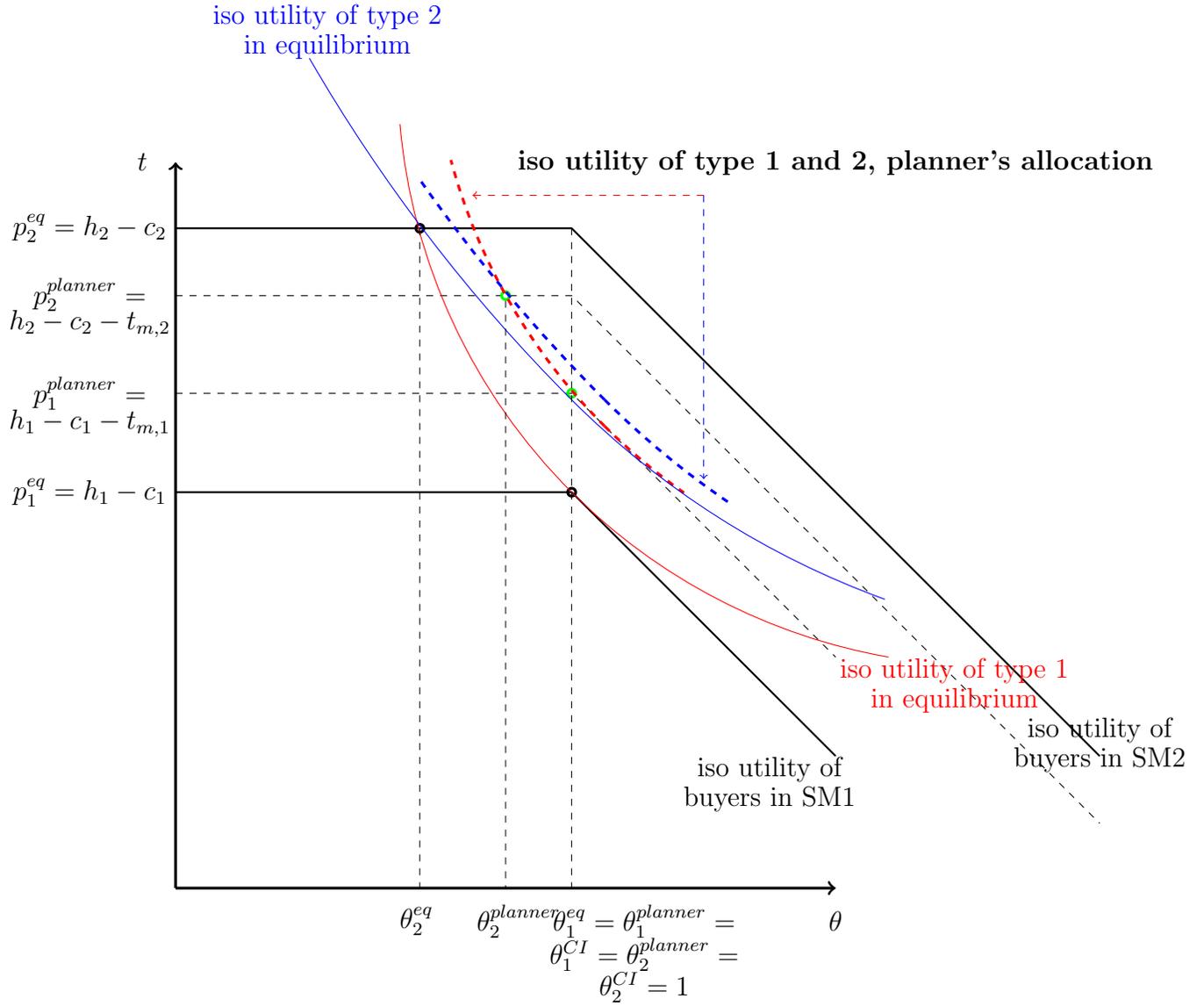


Figure 2: The indifference curves of buyers and sellers are illustrated here. In the equilibrium allocation, the market tightness for type 2 is less than 1. Intersection of indifference curve of type 1 and indifference curve of buyers in sub-market 2 determines θ_2^{eq} . At this point, type 1 is indifferent between both sub-markets. The planner makes subsidies to buyers at sub-market 1 ($t_{m,1} < 0$), thus pushing buyers' indifference curves in that sub-market upward. Because of zero profit condition for buyers, eventually type 1 sellers get a higher payoff than equilibrium. The planner taxes buyers in sub-market 2 ($t_{m,2} > 0$) to raise funds for subsidies made to type 1. Now, the market tightness that the planner assigns to type 2 is increased compared to that in equilibrium.

Also, because of free entry condition, the buyers get 0, so the sellers get the whole surplus in every sub-market. But it is feasible for the planner to give sellers in one sub-market more and sellers in another sub-market less than the surplus they generate. The only constraint that the planner faces is the budget constraint over **all sub-markets**. That is, the amount of transfers that buyers pay must be equal to amount of transfers that sellers receive over all sub-markets.

Entry of buyers in a sub-market creates two types of externalities on others. First, entry of new buyers decreases the probability of matching of other buyers in that sub-market. Second, the composition of sellers applying to this and other sub-markets will be changed, because the change in the market tightness might make this market more attractive for other types of sellers. Therefore, the composition of types in this sub-market and/or other sub-markets might be changed. The first type of externality is present in many models with search frictions. The second type exists only if buyers can not post contingent contracts on the types of sellers. The buyers cannot internalize these externalities in the presence of incomplete information, because the share of the surplus that they get from the match is already fixed with free entry condition.

The inability of buyers to internalize the externalities they create in equilibrium is similar to the situation in random search models with ex-post bargaining (like Mortensen and Pissarides [16]) in which the share of the surplus that buyers get is exogenously fixed, so the outcome is generally inefficient. Here, although the division of the surplus to buyers and sellers is not exogenously fixed, it is endogenously pinned down by two constraints that private information and the free entry impose on the allocation, so it is generally unlikely that the efficiency is achieved. The planner can internalize these externalities, because he is not constrained by the free entry condition at each sub-market, so he can make the buyers' share of the surplus satisfy Hosios condition.

5 Example 2: The Rat Race

In this section we study another example from Guerrieri et al. [10], the rat race, which was originally discussed in Akerlof [2]. The main point of this example is that the first best is achievable only through a separating allocation, in contrast to the previous example (asset market with lemons) where the first best was achievable through a pooling allocation (if $\pi_1 b_1 + \pi_2 b_2 \geq c_2$). The planner here achieves the first best by separating different types and then using appropriate transfers.

	Complete information	Equilibrium allocation	Planner's allocation
θ_1	θ_1^{CI}	θ_1^{CI}	θ_1^{CI}
θ_2	θ_2^{CI}	$\theta_2 > \theta_2^{CI}$	θ_2^{CI}
h_1	h_1^{CI}	h_1^{CI}	h_1^{CI}
h_2	h_2^{CI}	$h_2 > h_2^{CI}$	h_2^{CI}
p_1	$p_1^{CI} \equiv v_1(h_1^{CI}) - \frac{k}{q(\theta_1^{CI})}$	$v_1(h_1^{CI}) - \frac{k}{q(\theta_1^{CI})}$	$p_1^{CI} - t_{m,1}$
p_2	$p_2^{CI} \equiv v_2(h_2^{CI}) - \frac{k}{q(\theta_2^{CI})}$	$v_2(h_2) - \frac{k}{q(\theta_2)}$	$p_2^{CI} - t_{m,2}$
$t_{m,1}$	0	0	$-\frac{\pi_2}{m(\theta_1^{CI})}(U_2^{CI} - U_1^{CI}) + \pi_2(\tau - 1)\phi_2(h_1^{CI}) < 0$
$t_{m,2}$	0	0	$\frac{\pi_1}{m(\theta_2^{CI})}(U_2^{CI} - U_1^{CI}) - \pi_1(\tau - 1)\frac{m(\theta_1^{CI})}{m(\theta_2^{CI})}\phi_2(h_1^{CI}) > 0$

Table 2: The rat race results.

There are two types of workers (as sellers) on one side and firms (as buyers) on the other side of the market. The payoff of a type i worker matched with a firm from h hours of work and consuming t units of the numeraire good is $t - \phi_i(h)$. The worker's payoff is 0 if unmatched. The payoff of a firm matched with a type i worker when the worker works for h hours and the firm pays t units of the numeraire good (either to the worker or to the planner) is $v_i(h) - t - k$. The firm's payoff is $-k$ if unmatched. As a special case of the original setting, here $I = 2$, $n = 1$, $a \equiv h$ and $u_i(h) = -\phi_i(h)$. Matching function $m(\theta)$ is strictly concave and twice differentiable. We make the following assumptions on the payoff functions of the agents:

Assumption 4. *In the rat race example,*

1. ϕ_i is differentiable, increasing, strictly convex and $\phi_i(0) = \phi_i'(0) = 0$.
2. For all h , $\phi_1(h) = \tau\phi_2(h)$ where $\tau > 1$.
3. v_i is differentiable, increasing, strictly concave.
4. For all h , $v_1(h) - \phi_1(h) < v_2(h) - \phi_2(h)$ and $v_1'(h) - \phi_1'(h) < v_2'(h) - \phi_2'(h)$.
5. Also, $\pi_1 U_1^{CI} + \pi_2 U_2^{CI} \geq \pi_2(\tau - 1)m(\theta_1^{CI})\phi_2(h_1^{CI})$.

The assumptions that Guerrieri et al. [10] make for this example are different. They impose monotonicity of $v_i(h)$ in i , while I don't. Rather, I mostly follow the assumptions that I made for proposition 2. Part 1 and 2 of assumption 2 are satisfied here following the fact that $\phi_i(h)$ is increasing in h and also $\phi_2(h) = \tau\phi_1(h)$. Part 3 and 4 of assumption 2 are satisfied here due to the fact that $v_i(h) - \phi_i(h)$ and $v_i'(h) - \phi_i'(h)$ are increasing in i . Part

5 of assumption 2 is automatically satisfied here, because h (or a in the original setting of section 2) is just one-dimensional. To satisfy the planner's budget constraint, here I use a weaker condition, (part 5 in assumption 4) compared to part 6 in assumption 2.

Proposition 4. *If assumption 4 holds and $U_i^{CI} > 0$ for all i , then the planner achieves the first best. See the fourth column of table 2 for the full description of the planner's allocation.*

Guerrieri et al. [10] shows that if $v_1(h) < v_2(h)$ and $U_2^{CI} - U_1^{CI} \leq (\tau - 1)m(\theta_2^{CI})\phi_2(h_2^{CI})$, then the equilibrium cannot achieve the complete information allocation. If these conditions and the conditions in the proposition are satisfied simultaneously (which is possible), then the market allocation is inefficient, but the planner can recover efficiency completely. The intuition behind this result is exactly the same as in the previous section. Here the planner subsidizes type 1 ($p_1 > p_1^{CI}$) and taxes type 2 ($p_2 < p_2^{CI}$) to achieve efficiency. By offering this scheme of transfers (allocating the low type workers higher wage and the high type workers lower wage than their wages in the equilibrium under complete information), the planner tries to discourage low type workers from applying to sub-market 2, thus reducing the cost of private information.

In the equilibrium allocation, $\theta_2^{eq} > \theta_2^{CI}$ and $h_2^{eq} > h_2^{CI}$. Guerrieri et al. [10] propose a pooling allocation which Pareto dominates the equilibrium allocation if π_1 is sufficiently small. This pooling allocation does not achieve the first best. As stated earlier, the planner can achieve the first best. Moreover, if π_1 is sufficiently small, then the planner's allocation Pareto dominates the equilibrium allocation.

6 Extension: Asset Market with a Continuous Type Space

The original setting in section 2 has a discrete type space. In this section, we extend our analysis to a continuous type space. This case is interesting, because it makes it possible for us to consider cases in which the value of assets to sellers does not have the same order as the value of assets to buyers. I could do it with a discrete type space with more than two types, but the analysis with continuous type is simpler. Also, studying this case makes it possible for us to compare the planner's allocation with the equilibrium allocation in Chang [4] (and somewhat with Guerrieri and Shimer [9]). Although Chang has a dynamic environment, but the main ideas are captured in our static case⁹. Since this case is not a special case of the

⁹ In the dynamic setting, the planner has some inter-temporal considerations, because the distribution of types in the population does not necessarily remain the same over time, because some types get matched

original setting in section 2, we need to define the constrained efficient allocation again. The main ideas discussed so far go through for this case as well, but the mathematical tools that we use to characterize the planner's problem are different.

6.1 Environment

There is a continuum of measure one of heterogeneous sellers indexed by $z \in Z \equiv [z_L, z_H] \subset \mathbb{R}$, with $F(z)$ denoting the measure of sellers with types below z . F is continuously differentiable and strictly increasing in z and f is its derivative (a density function). Type z is seller's private information. Similar to the original setting, buyers' and sellers' payoffs are quasi-linear. A buyer's payoff who enters the market and gets matched with a type z is $h(z) - t - k$ where t denotes the amount of a numeraire good that he produces and $h(z)$ is the value of the asset to the buyer in terms of the numeraire good. His payoff is $-k$ if unmatched. The payoff of a type z seller matched with a buyer is $t - c(z)$ where t denotes the amount of the numeraire good that he consumes and $c(z)$ is the value of the asset to the seller in terms of the numeraire good. His payoff is 0 if unmatched. Functions $h : Z \rightarrow R$ and $c : Z \rightarrow R$ are twice continuously differentiable. Matching function $m(\cdot)$ is increasing, strictly concave and twice differentiable with strictly decreasing elasticity.

6.2 Complete information case

Here, I mostly follow the discussion of the complete information case for the discrete type in section 3. Consider the market economy with the complete information. The buyers who attempt to attract type z sellers solve the following problem:

$$U^{CI}(z) = \max_{\theta, p} \{m(\theta)(p - c(z))\}$$

$$\text{s.t. } q(\theta)(h(z) - p) = k.$$

Let $\theta^{CI}(z)$ and $p^{CI}(z)$ denote the market tightness and the price that solve this problem. We make the assumption that $U^{CI}(z) > 0$ for all z , stating that there are positive gains from

more quickly than others. This fact raises a new and interesting tradeoff, whether the planner wants to get rid of low types early or he wants to have all types together all the way to the end. The analysis of the dynamic setting is beyond the scope of this paper. Since the equilibrium allocation is distribution free, the equilibrium analysis is much easier than the planner's analysis in the dynamic case, unless we assume when sellers sell their assets, they produce a new asset with the same quality, which is assumed in some papers. In that case, the same results can be obtained as in the static case.

trade for all types. Similar to the discrete type case, $U^{CI}(z) = \max_{\theta} \{m(\theta)(h(z) - c(z)) - k\theta\}$. Also $\theta^{CI}(z)$ solves

$$m'(\theta)(h(z) - c(z)) = k, \quad (1)$$

for both the planner and the market economy when there is complete information. The left hand side of equation 1 is the marginal benefit of adding one more buyer to the sub-market composed of z sellers. The right hand side is the marginal cost of doing that. The planner keeps adding buyers to each sub-market until the marginal cost and marginal benefit of doing that become equal. We calculate the share of the surplus that sellers get in equilibrium to verify that the Hosios condition is satisfied for that:

$$\begin{aligned} \frac{p^{CI}(z) - c(z)}{h(z) - c(z)} &= \frac{U^{CI}(z)}{m(\theta^{CI})(h(z) - c(z))} = \frac{m(\theta^{CI})(h(z) - c(z)) - k\theta^{CI}}{m(\theta^{CI})(h(z) - c(z))} \\ &= \frac{m(\theta^{CI})(h(z) - c(z)) - \theta^{CI}m'(\theta^{CI})(h(z) - c(z))}{m(\theta^{CI})(h(z) - c(z))} = -\frac{\theta^{CI}q'(\theta^{CI})}{q(\theta^{CI})} \equiv \eta(\theta^{CI}), \end{aligned}$$

where the third equality follows equation 1. The share of the surplus that type z sellers get from the match is $\frac{p-c(z)}{h(z)-c(z)}$. Hosios condition states that a necessary condition for efficiency of the allocation is that this share must be equal to the elasticity of matching function. The equilibrium allocation under complete information satisfies this property.

6.3 Definition of Planner's Problem

In order to define the planner's problem, we need to introduce new notations. Let $\phi \in \Phi \in \mathbb{R}$ denote the index of sub-markets. Denote by $G(\phi)$ the measure of buyers in sub-markets with indices less than ϕ . Denote by $H(\phi, z)$ the measure of sellers with types below z in sub-markets with indices less than ϕ . The fraction of sellers in sub-markets with indices less than ϕ is denoted by $H_{\Phi}(\phi)$. The fraction of sellers with types below z is denoted by $H_Z(z)$. We impose the restriction that H_{Φ} is absolutely continuous with respect to G . This restriction means that if ϕ is not in the support of G , then no seller can enter that sub-market. Now, we are ready to define the allocation for the continuous type space.

Definition 5. *An allocation is a set which consists of distributions G and H and functions $p : \Phi \rightarrow \mathbb{R}$, $t_m : \Phi \rightarrow \mathbb{R}$ and $t_u : \Phi \rightarrow \mathbb{R}$.*

Denote by $p(\phi)$ the amount of the numeraire good to be transferred from buyers to sellers conditional on trade in sub-market ϕ . Denote by $t_m(\phi)$ the amount of the numeraire good to be transferred from buyers to the planner in sub-market ϕ conditional on trade, and denote

by $t_u(\phi)$ the amount of the numeraire good to be transferred from buyers to the planner if they enter sub-market ϕ .

Definition 6. An allocation $\{G, H, p, t_m, t_u\}$ is feasible if it satisfies the following conditions:

1. (Sellers' maximization) $(\phi, z) \in \text{supp } H$ only if

$$\phi \in \arg \max_{\phi' \in \text{supp } G} \{m(\theta_{GH}(\phi'))(p(\phi') - c(z))\} \text{ and } m(\theta_{GH}(\phi))(p(\phi) - c(z)) \geq 0.$$

2. (Buyers' zero payoff) For any $\phi \in \text{supp } G$,

$$q(\theta_{GH}(\phi)) \left[\int_Z h(z) \theta_{GH}(\phi) \frac{dH}{dG} - p(\phi) - t_m(\phi) \right] = k + t_u(\phi),$$

3. (Planner's budget constraint)

$$\int_{\text{supp } G} [q(\theta_{GH}(\phi)) t_m(\phi) + t_u(\phi)] dG \geq 0,$$

where $\theta_{GH}(\phi) = \left(\frac{dH_\phi}{dG}\right)^{-1}$ (Radon-Nikodym derivative), and $H_Z(z) = F(z)$.

The conditions above are similar to the conditions in the definition of feasible allocation in the original setting of section 2. The first condition states that if a type z seller chooses sub-market ϕ , then this sub-market must maximize his payoff among all open sub-markets and must deliver him a positive payoff. The second condition states that every open sub-market must yield exactly zero profit to the buyers who choose that sub-market. The last condition states that the net transfer that buyers make to the planner must be non-negative. Market tightness at every sub-market is given by the inverse of Radon-Nikodym derivative of H_ϕ with respect to G . Finally, $H_Z(z) = F(z)$ states that the sum of measure of sellers of types below z in all sub-markets must exactly equal to the measure of sellers with types below z in the population.

Definition 7. A constrained efficient allocation is a feasible allocation $\{G, H, p, t_m, t_u\}$ which maximizes the planner's objective among all feasible allocations:

$$\max \int_{\text{supp } G} [q(\theta_{GH}(\phi))(h(z) - c(z) - t_m(\phi)) - (k + t_u(\phi))] dG$$

Subject to: $\{G, H, t_s, t_m, t_u\}$ is feasible.

6.4 Results

In order to solve the planner's problem, we use somewhat a backward approach. We first guess that the planner can achieve the first best. That is, the planner can maximize his objective function for each type separately. Then we find a set of transfers, $p(\cdot)$ and $t_m(z)$, such that sellers' maximization condition and buyers' zero profit condition are satisfied. Given these schemes of payments, we derive sufficient conditions under which the planner's budget condition is satisfied.

Assumption 5. *For all z , $c'(z) > 0$ and either*

1. $h'(z) \leq 0$, or
2. $h'(z) \leq c'(z)$ and $\psi\left(\frac{k}{h(z)-c(z)}\right)\left[\frac{h(z)-c(z)}{c'(z)}\right] \geq \frac{F(z)}{f(z)}$, where $\psi(\cdot) \equiv \eta(m'^{-1}(\cdot))$ and $\eta(\theta) \equiv -\frac{\theta q'(\theta)}{q(\theta)}$.

Proposition 5. *If assumption 5 holds and $U^{CI}(z) \geq 0$, different types trade in different sub-markets (so without loss of generality sub-markets can be indexed by type z). Then the planner achieves the first best, that is,*

$$\theta(z) = \theta^{CI}(z), \text{ for all } z.$$

Also, the payments at each sub-market is given by:

$$p(z) = c(z) + \frac{U(z_H) + \int_z^{z_H} m(\theta(z_0))c'(z_0)dz_0}{m(\theta(z))},$$

$$t_m(z) = h(z) - p(z) - \frac{k}{q(\theta(z))}, \text{ and } t_u(z) = 0,$$

where $U(z_H) = \int [m(\theta(z))(h(z) - c(z)) - k\theta(z) - m(\theta(z))c'(z)\frac{F(z)}{f(z)}]dF(z)$.

To have a rough idea how the planner can take care of private information, let's write the IC problem. We assume (without loss of generality) that sellers are allocated to different sub-markets through a direct mechanism: If type z reports \hat{z} , his payoff is given by $m(\theta(\hat{z}))(p(\hat{z}) - c(z))$. The agent chooses a \hat{z} which maximizes his payoff:

$$\max_{\hat{z}} \{m(\theta(\hat{z}))(p(\hat{z}) - c(z))\}. \quad (2)$$

I use the assumption that $c'(z) > 0$ throughout this section, so single crossing condition holds. As already discussed in section 2, $\theta(z)$ being decreasing and $c(z)$ being increasing in z together imply that there exists a set of transfers that satisfies IC. Now assume $h'(z) \leq 0$ for

all z or $h'(z) \leq c'(z)$ for all z . In either case, $\theta^{CI}(z)$ is decreasing in z according to equation 1. Therefore, such transfers exist. We provide sufficient conditions in the proof such that the planner's budget constraint holds. If $h'(z) \leq 0$, the planner has enough resources to distribute among agents regardless of the distribution. If $h'(z) \leq 0$ is not satisfied for some z but $h'(z) - c'(z) \leq 0$ still holds, we need another condition which relates the distribution of types to the payoff and matching functions (second part of assumption 5) to ensure that the planner's budget constraint is satisfied.

In the next proposition, we keep the assumption that $c'(z) > 0$ and $h'(z) - c'(z) \leq 0$, but the distribution of types is such that the planner cannot achieve the first best. Therefore, the probability of matching for almost all types must be distorted (relative to the complete information allocation) so that IC and budget constraint are both satisfied.

Assumption 6. For all z , $h(z) - c(z) - c'(z)\frac{F(z)}{f(z)} > 0$ and $\frac{d}{dz}[h(z) - c(z) - c'(z)\frac{F(z)}{f(z)}] \leq 0$.

Proposition 6. Assume $c'(z) > 0$, $h'(z) - c'(z) \leq 0$ and $U^{CI}(z) \geq 0$ for all z . Also, suppose assumption 6 holds. If the first best is not achievable, then there exists a $\mu > 0$ such that the market tightness $\theta(\cdot)$ for the constrained efficient solves the following equations:

$$m'(\theta(z)) \left[h(z) - c(z) - \frac{\mu}{1 + \mu} c'(z) \frac{F(z)}{f(z)} \right] = k, \quad (3)$$

$$\int \left[m(\theta(z)) \left[h(z) - c(z) - c'(z) \frac{F(z)}{f(z)} \right] - k\theta(z) \right] f(z) dz = 0.$$

Moreover, $p(z)$, $t_m(z)$ and $t_u(z)$ are obtained similarly as before:

$$p(z) = c(z) + \frac{U(z_H) + \int_z^{z_H} m(\theta(z_0)) c'(z_0) dz_0}{m(\theta(z))},$$

$$t_m(z) = h(z) - p(z) - \frac{k}{q(\theta(z))}, \quad t_u(z) = 0,$$

and $U(z_H) = 0$.

Proposition 5 requires that the virtual surplus of each type, $m(\theta(z))[h(z) - c(z) - c'(z)\frac{F(z)}{f(z)}] - k\theta(z)$, be positive (if $h'(z) \leq 0$ is not satisfied for some z). However, if the virtual surplus of some types is negative, then proposition 6 requires that at least $h(z) - c(z) - c'(z)\frac{F(z)}{f(z)}$ be positive for all z . Also this proposition requires $h(z) - c(z) - c'(z)\frac{F(z)}{f(z)}$ to be decreasing in z to ensure that the monotonicity constraint is satisfied.

6.5 Equilibrium allocation

I report the results of a static version of Chang [4] here and compare equilibrium allocation with the planner's one. Her model is dynamic, but my model is static. See footnote 9 why I consider a static model. Chang assumes that utility of holding the asset until finding a buyer is different across different types of assets. Similarly, I assume the value of higher type assets to sellers is higher than that of lower type assets.

Assumption 7. $c'(z) > 0$, $h'(z) \geq 0$ for all z .

Proposition 7 (From Chang [4]). *If assumption 7 holds and if $U^{CI}(z) \geq 0$ for all z , then there exists a unique equilibrium. The equilibrium is separating. The market tightness solves the differential equation 5. The initial condition is given by $\theta(z_L) = \theta^{CI}(z_L)$. Prices are given by $p(z) = h(z) - \frac{k}{q(\theta(z))}$.*

See her paper for the formal proof. I just explain the logic behind her result. First note that the IC constraints that agents face in the market economy is the same as ones that the planner faces, therefore we can use equation 2 to describe IC constraints in the equilibrium. The only difference is that the price is different in the market economy, because it is pinned down by the free entry condition. In the market economy, she shows that any equilibrium under assumption 7 is separating, so free entry implies that $p(z) = h(z) - \frac{k}{q(\theta(z))}$ for all z . Therefore, the payoff of type z in the market economy, denoted by $U^{eq}(z)$, is calculated as follows:

$$U^{eq}(z) = \max_{\hat{z}} \{m(\theta(\hat{z}))(h(\hat{z}) - c(z)) - k\theta(\hat{z})\}, \quad (4)$$

where the objective is the payoff of type z when he reports type \hat{z} . FOC with respect to \hat{z} yields the following equation:

$$[m'(\theta(z))(h(z) - c(z)) - k] \frac{d\theta}{dz} + m(\theta(z))h'(z) = 0, \quad (5)$$

where we use the fact that at the solution $\hat{z} = z$ due to IC. With respect to the initial condition, roughly speaking, the market delivers the complete information payoff to the type which has the most incentive to deviate. For example, when $h'(\cdot) \geq 0$, the lowest type has the most incentive to deviate, so his allocation is set to the complete information level. The results are summarized in table 3. The necessary and sufficient condition for IC and the initial conditions are depicted in the table under different assumptions, for example if $c' > 0$ and $h' \leq 0$. We maintain this assumption that there are positive gains from trade for all types.

Condition	Necessary condition for I.C	Initial conditions
$0 < c'(z)$ and $0 < h'(z)$	$\frac{d\theta}{dz} < 0$	$\theta(z_L) = \theta^{CI}(z_L)$
$0 < c'(z)$ and $0 > h'(z)$	$\frac{d\theta}{dz} < 0$	$\theta(z_H) = \theta^{CI}(z_H)$
$0 > c'(z)$ and $0 < h'(z)$	$\frac{d\theta}{dz} > 0$	$\theta(z_L) = \theta^{CI}(z_L)$
$0 > c'(z)$ and $0 > h'(z)$	$\frac{d\theta}{dz} > 0$	$\theta(z_H) = \theta^{CI}(z_H)$

Table 3: Equilibrium allocation in different cases is depicted here.

6.6 Two-dimensional private information

In another part of her paper, she assumes that sellers have another dimension of private information. Some sellers get liquidity shocks so they need to sell their assets quickly. What is relevant to our discussion is that following this extension, it is possible that function h has a local maximum (keeping the assumption $c'(z) > 0$ fixed). If so, she proves that full separation in the market is not possible. Also, she derives some conditions under which an equilibrium which resembles fire-sales exists. My characterization, in contrast, shows if $h'(z) \leq c'(z)$ and if the assumptions in part 2 of proposition 5 or assumption 6 holds, even if h has a local maximum or minimum, the planner wants different types to trade in different sub-markets. This case is depicted in figure 3 where $h'(z)$ is drawn in terms of $c'(z)$ for all z .

Now suppose $h'(z) - c'(z) \leq 0$ is violated for some z . For example, $h - c$ has one local minimum, but $h'(z) \geq 0$ and $c'(z) > 0$, as depicted in figure 4. The equilibrium in this case is separating. The planner's allocation, in contrast, involves some pooling, because monotonicity constraint (that $\theta(z)$ should be decreasing in z) cannot be satisfied through any pooling allocation¹⁰. The bottom line is that pooling of types takes place under different conditions in the planner's allocation and the equilibrium allocation.¹¹ The welfare under planner's allocation is strictly higher by the same argument as in proposition 1 even if the planner does not achieve the first best.

¹⁰Solving explicitly for the planner's allocation in this case does not give us much new insights, so I skip its analysis. The method used in this situation is called bunching and is discussed in mechanism design literature. For example, see the appendix of chapter 7 in [7].

¹¹Roughly speaking, the planner is concerned with the surplus from matches, so in the conditions regarding the planner's allocation, usually $h - c$ shows up. The buyers in the equilibrium are concerned with the value of the assets to themselves, so in the conditions regarding the allocation, usually h shows up.

7 Conclusion

We have characterized the constrained efficient allocation in an environment with directed search and private information. Under the same assumptions that [10] characterize the unique equilibrium, the planner can achieve strictly higher welfare than the equilibrium if the equilibrium fails to achieve the first best. Under a different assumption (assumption 2), the planner can even achieve the first best. The main idea is that the planner uses transfers rather than market tightness or production level to have IC satisfied as long as agents' participation constraints and the planner's budget constraint are satisfied. In the market economy, cross-subsidization between different types (or more precisely between different sub-markets) is not possible. Hence, when incentive compatibility constraints bind, the market tightness or production level is distorted in order to satisfy IC of agents.

I illustrated my results in different examples like asset markets and labor markets. In an asset market example in section 6, I showed that if the value of assets to sellers is increasing and the surplus generated by assets is decreasing in the type of assets ($c'(\cdot) > 0$ and $h'(\cdot) - c'(\cdot) \leq 0$), then the planner can achieve the first best by subsidizing low types and taxing high types. I compared my results with Chang [4] and showed, specifically, that the conditions under which the equilibrium allocation and planner's allocation are pooling are different. That is, with some payoff functions, both equilibrium allocation and planner's allocation are separating. With some payoff functions, both are pooling and with some others, one of them is pooling and one is separating. My results highlight the role of cross-subsidization or transfers between different sub-markets in economies with directed search and private information and show that there exist appropriate schemes of taxes and subsidies through which the planner can improve welfare.

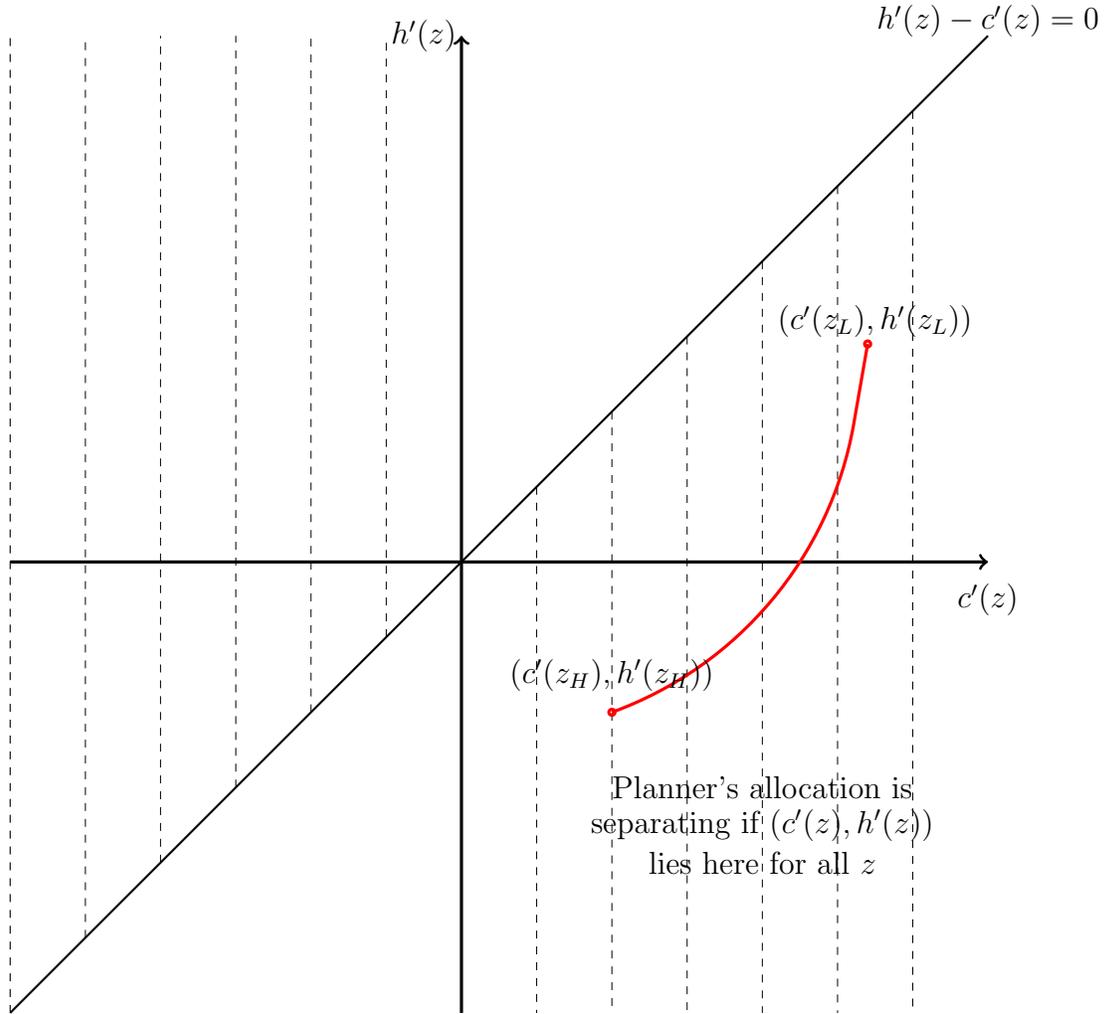


Figure 3: Assume that $h(z)$ has a local maximum but $h'(z) - c'(z) \leq 0$ for all z . Also assume that the distribution is such that part 2 of assumption 5 or assumption 6 holds. Because the value of assets with higher z to sellers is not monotone in z , the equilibrium will involve some pooling. (Chang [4] shows this point formally in her Proposition 5.) However, the planner's allocation is separating. The planner can actually achieve the first best according to proposition 5. Symmetrically, if $c'(z) < 0$ and $h'(z) - c'(z) \geq 0$, and if similar conditions to part 2 of assumption 5 or assumption 6 hold, then the planner will get a separating allocation.

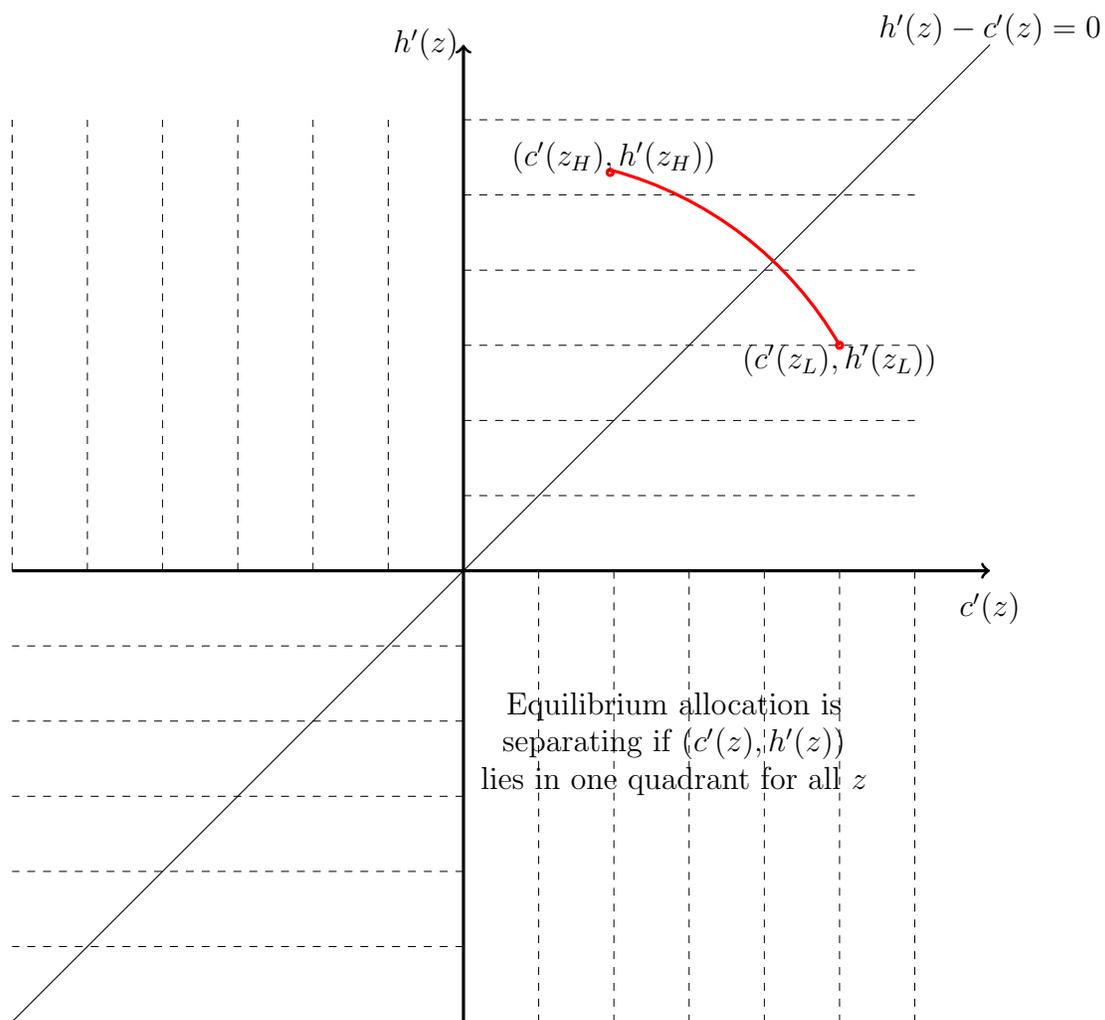


Figure 4: Assume that $h(z) - c(z)$ has one local minimum and $h'(z) \geq 0$ for all z . Since $h'(z) \geq 0$ for all z , then the equilibrium allocation is separating. However, the planner's allocation is pooling because monotonicity constraint is not satisfied. Indeed, the planner wants to pool all types higher than a threshold in one sub-market.

8 Appendix

8.1 Proof of lemma 1

Proof. First part:

Suppose for an allocation and for some $y \in Y^P$, $t_u \neq 0$. Consider another allocation with $t'_m = t_m + \frac{t_u}{q(\theta(y))}$ and $t'_u = 0$. In this new allocation, the maximization problem of sellers and zero profit of buyers do not change. The planner also makes the same amount of money over this sub-market. Therefore, we can always make another allocation with $t_u = 0$.

Second part:

Suppose budget constraint holds but is not binding, that is, $B \equiv \int_{Y^P} [q(\theta(y))t_m + t_u]d\lambda(\{y\}) > 0$. We propose another allocation which increases the objective and keep all other constraints satisfied: For any $y \in Y^P$, consider a new y' with the same a , but with a new t'_s and t'_m :

$$t'_s = t_s + \frac{B}{m(\theta(y))} \left(\int_{Y^P} \frac{d\lambda(\{y\})}{\theta(y)} \right)^{-1},$$

$$t'_m = t_m - \frac{B}{m(\theta(y))} \left(\int_{Y^P} \frac{d\lambda(\{y\})}{\theta(y)} \right)^{-1}.$$

All other elements of this new allocation is the same: for any y' , $\Gamma(y') = \Gamma(y)$ and $\theta(y') = \theta(y)$ and $\lambda(y') = \lambda(y)$. That is, the new allocation is just different from the old allocation in transfers (t_s and t_m) in sub-markets. Now we want to check that this allocation is also feasible.

Take any seller. The expected amount that the seller gets has increased by fixed amount of $B \left(\int_{Y^P} \frac{d\lambda(\{y\})}{\theta(y)} \right)^{-1}$ over all sub-markets, so the seller does not have any incentive to change the sub-markets he used to join under the old allocation. Therefore, seller's maximization condition is satisfied.

Regarding buyer's zero profit, note that that the expected amount that buyer pays remain the same because $t'_s + t'_m = t_s + t_m$.

Now we calculate planner's budget under new allocation:

$$\begin{aligned} & \int_{Y'^P} [q(\theta(y'))t'_m + t'_u]d\lambda(\{y'\}) \\ &= \int_{Y'^P} [q(\theta(y'))(t_m - \frac{B}{m(\theta(y'))} \left(\int_{Y^P} \frac{d\lambda(\{\tilde{y}\})}{\theta(\tilde{y})} \right)^{-1}) + t_u]d\lambda(\{y'\}) \\ &= \int_{Y'^P} [q(\theta(y'))t_m + t_u]d\lambda(\{y'\}) - B \left(\int_{Y'^P} \frac{d\lambda(\{\tilde{y}\})}{\theta(\tilde{y})} \right)^{-1} \int_{Y'^P} \frac{d\lambda(\{y\})}{\theta(y)} = B - B = 0. \end{aligned}$$

Therefore, the budget constraint holds. Note that the objective has increased exactly by amount B , because the planner have distributed all resources across sellers. Formally, the objective has increased by:

$$\begin{aligned} & \int_{Y^P} q(\theta(y'))(t_m - t'_m)d\lambda(\{y'\}) \\ &= \int_{Y^P} [q(\theta(y'))\frac{B}{m(\theta(y'))}(\int_{Y^P} \frac{d\lambda(\{\tilde{y}\})}{\theta(\tilde{y})})^{-1}]d\lambda(\{y'\}) = B. \end{aligned}$$

This concludes the proof. \square

Corollary 1. *For any feasible allocation where budget constraint is not binding, that is, $B \equiv \int_{Y^P} [q(\theta(y))t_m + t_u]d\lambda(\{y\}) > 0$, there exists another feasible allocation $\{\lambda', Y'^P, \theta', \Gamma'\}$ which increases the objective by exactly amount B and the budget constrains is now binding. New allocation is given by:*

$$Y'^P = \{y'|y' = (a, t_s + \frac{\Delta}{m(\theta(y))}, t_m - \frac{\Delta}{m(\theta(y))}, t_u), (a, t_s, t_m, t_u) \in Y^P\},$$

where $\Delta \equiv B(\int_{Y^P} \frac{d\lambda(\{y\})}{\theta(y)})^{-1}$.

Also, $\lambda'(\{y'\}) = \lambda(\{y\})$, $\theta'(\{y'\}) = \theta(\{y\})$ and $\Gamma'(\{y'\}) = \lambda(\{y\})$ where $y' = (a, t_s + \frac{\Delta}{m(\theta(y))}, t_m - \frac{\Delta}{m(\theta(y))}, t_u)$ for some $y \equiv (a, t_s, t_m, t_u) \in Y^P$.

8.2 Proof of result 1

Proof of result 1. We begin from equilibrium allocation and modify it to improve the welfare. Suppose type i , which gets strictly positive in the equilibrium, does not get his complete information level of a and θ . I define a set of problems, similar but not the same as one in GSW, and characterize its solution. From that solution, I construct an allocation and show that the constructed allocation is feasible and yields higher welfare for the planner than the equilibrium allocation.

Problem 1 ($P_i(\epsilon)$).

$$\max_{\theta \in [0, \infty], a \in A, p \in \mathbb{R}} \{m(\theta)(u_i(a) + p) + \delta_i\}$$

subject to

$$q(\theta)(v_i(a) - p) \geq k,$$

$$\delta_i \leq m(\theta)(u_i(a) + p) + \delta_i,$$

and

$$m(\theta)(u_j(a) + p) + \delta_i \leq \bar{U}_j \text{ for all } j < i,$$

where $\delta_i = \epsilon$ if $i < i_0$ and $\delta = 0$ otherwise.

Define problem $P(\epsilon)$ to be the set of problems $P_i(\epsilon)$ for all i .

Let \bar{U}_i denote the value of objective in problem $P_i(\epsilon)$ given $(\bar{U}_1, \bar{U}_2, \dots, \bar{U}_{i-1})$. For any i , let $(\bar{\theta}_i, \bar{a}_i, \bar{t}_i)$ denote the solution to problem $P_i(\epsilon)$ given $(\bar{U}_1, \bar{U}_2, \dots, \bar{U}_{i-1})$. Also consider the associated allocation obtained from this problem:

Suppose $\epsilon = 0$. Then this set of problems characterizes the equilibrium allocation as GSW show. From now on, we refer to the set of problems that describes equilibrium allocation by $P(0)$.

We want to construct an allocation which yields to strictly higher welfare than the equilibrium allocation. To do so, we begin from the equilibrium allocation and construct another one which leads to strictly higher welfare. By assumption the equilibrium does not achieve complete information allocation, thus some constraints in problem P_i should be binding. Let \tilde{i} denote the index of problem which does not achieve complete information allocation. The planner tries to relax this constraints without distorting the allocation for lower types. Consider the following allocation.

$$y_i(\epsilon) = \begin{cases} (\bar{a}_i(\epsilon), \bar{p}_i(\epsilon) + \frac{\epsilon}{m(\theta_i)}, 0, 0) & \text{if } 1 \leq i < \tilde{i} \\ (\bar{a}_i(\epsilon), \bar{p}_i(\epsilon), 0, 0) & \text{if } \tilde{i} \leq i \leq I. \end{cases} \quad (6)$$

$$Y^P = \{y_i(\epsilon)\}_{1 \leq i \leq I}, \theta(y_i) = \theta_i(\epsilon), \gamma_i(y_i(\epsilon)) = 1, \lambda(\{y_i(\epsilon)\}) = \theta_i(\epsilon)\pi_i. \quad (7)$$

Since the constraint of $P_i(\epsilon)$ is exactly the same as $P_i(0)$ for types below \tilde{i} , they get exactly $\bar{U}_i(\epsilon) = \bar{U}_i(0) + \epsilon$.

For type \tilde{i} , the planner maximizes the objective given $(\bar{U}_1, \bar{U}_2, \dots, \bar{U}_{\tilde{i}-1})$. Since the constraint is now looser, problem $P_i(\epsilon)$ yields to strictly higher objective relative to equilibrium allocation where all lower types had less \bar{U}_i .

For types above \tilde{i} , the objective is weakly higher than the equilibrium allocation. We show the latter claim by induction. Fix $j > \tilde{i}$ and assume that for all k such that $\tilde{i} \leq k < j$, then $\bar{U}_k(\epsilon) \geq \bar{U}_k(0)$, that is, the value of objective is higher than that in equilibrium. This implies that the constraints in problem $P - j(\epsilon)$ for all k with $\tilde{i} \leq k < j$ is looser. For $k < \tilde{i}$, we already know that each constraint is looser by amount ϵ . Hence, the objective for $P - j(\epsilon)$ should be weakly higher than that in equilibrium. For future reference:

$$\bar{U}_i(\epsilon) = \bar{U}_i(0) + \epsilon \text{ for all } i < \tilde{i}, \bar{U}_{\tilde{i}}(\epsilon) > \bar{U}_{\tilde{i}}(0), \text{ and } \bar{U}_i(\epsilon) \geq \bar{U}_i(0) \text{ for all } i > \tilde{i}. \quad (8)$$

We proceed by proving the following lemma, borrowed from GSW, stating that at the solution to problem $P(\epsilon)$ for any ϵ , the first constraint in $P - \tilde{i}(\epsilon)$ should be binding. Also, we show that sellers are not attracted to the sub-markets designed for higher types.

Lemma 2 (GSW, lemma 1 modified). *There exists $\{\bar{U}_i\}_{i \in \mathbb{I}}$ and $\{(\bar{\theta}_i, \bar{a}_i, \bar{t}_i)\}_{i \in \mathbb{I}}$ that solves problem $P(\epsilon)$ and $\bar{U}_i > \delta_i$ for all i . The following holds at any solution:*

$$q(\theta)(u_i(a) - p) = k,$$

and

$$m(\theta)(u_j(a) + p) + \delta_i \leq \bar{U}_j \text{ for all } i, j,$$

where $\delta_i = \epsilon$ if $i < \tilde{i}$ and $\delta = 0$ otherwise.

Proof. We mostly follow GSW lemma 1 for this proof. For $i = 1$, the objective is continuous and the constraint set is compact and non-empty. The constraint is non-empty because there exists a $p \in \mathbb{R}$ where the constraint is satisfied.

Now set $p = -u_1(a_1^{CI}) + \tau$ where $\tau > 0$ and set $\theta = \theta_1^{CI}$. Since $U_1^{CI} > 0$, there exists a sufficiently small $\tau > 0$ such that the objective is strictly positive for $(\theta_1^{CI}, a_1^{CI}, p)$ and the constraints are also satisfied. Therefore, $U_1 > 0$.

We proceed by induction. Fix i and suppose for $1 \leq j \leq i$, the solution exists and $U_j > \delta_j$. Consider problem P_i . the constraint is compact and non-empty. Now set $p = -u_i(a_i^{CI}) + \tau$ where $\tau > 0$ and set $\theta = \theta_i^{CI}$. Since $U_j > \delta_j$ for all $j < i$, there exists a sufficiently small $\tau > 0$ such that the objective is strictly positive for $(\theta_i^{CI}, a_i^{CI}, p)$ and the constraints are also satisfied.

Part 2:

Assume by way of contradiction that the constraint is not bonding. First note that $\theta > 0$ because $U_j > \delta_j$ for all j . According to the sorting assumption, for every $\tau > 0$, there exists an $a' \in B_\epsilon(a)$ such that

$$u_i(a') > u_i(a) \tag{9}$$

$$\text{and } u_j(a') < u_j(a). \tag{10}$$

Set $\tau > 0$ sufficiently small such that $q(\theta)v_i(a' - p) \geq k$ for all $B_\epsilon(a)$.

The first constraint is satisfied from choice of τ and other constraints are satisfied because for all $j < i$: $m(\theta)(u_j(a') + p) + \delta_j < m(\theta)(u_j(a) + p) + \delta_j \leq \bar{U}_j$. But the objective is now higher: $m(\theta)(u_i(a') + p) + \delta_i < m(\theta)(u_i(a) + p) + \delta_i$, which is a contradiction with (θ, a) being a solution to problem P_i .

Part 3: Assume by way of contradiction that there exists i, j such that $i < k$ and $m(\theta_i)(u_j(a_i) + p_i) + \delta_i > U_k$. Denote the smallest such k by k_0 . That is,

$$m(\theta_i)(u_j(a_i) + p_i) + \delta_i \leq U_j \text{ for all } i \leq j < k \tag{11}$$

$$\text{and } m(\theta_i)(u_{k_0}(a_i) + p_i) + \delta_i > U_k \text{ for all } i \leq j < k. \quad (12)$$

Now we show that θ_i, a_i, p_i is feasible for problem P_{k_0} . The first constraint is satisfied because $q(\theta_i)(v_{k_0}(a_i) - p_i) \geq q(\theta_i)(v_{i_0}(a_i) - p_i) \geq k$, where the first inequality follows from monotonicity assumption. Also, $m(\theta_i)(u_j(a_i) + p_i) + \delta_i \leq U_j$ holds true if $i < j < k$ according to equation 11 and is true for $j \leq i$, because (θ_i, a_i, p_i) is feasible for problem P_i . A contradiction is that (θ_i, a_i, p_i) is feasible for k , but equation 12 contradicts with U_k being the maximized value for problem P_k . □

Under allocation $\{\lambda, Y^P, \theta, \Gamma\}$, the planner needs to raise $\epsilon \sum_{i=1}^{\tilde{i}-1}$ money from the agents to have budget constraint satisfied. Now consider the following allocation which is derived from allocation $\{\lambda, Y^P, \theta, \Gamma\}$ by redistributing $-\epsilon \sum_{i=1}^{\tilde{i}-1}$ transfers across sellers. We show below that this allocation is feasible and yields to strictly higher welfare than equilibrium.

$$y'_i = \begin{cases} (\bar{a}_i, \bar{p}_i + \frac{\epsilon(1-\sum_1^{\tilde{i}-1} \pi_j)}{m(\theta(y))}, -\frac{\epsilon(1-\sum_1^{\tilde{i}-1} \pi_j)}{m(\theta(y))}, 0) & \text{if } 1 \leq i < \tilde{i} \\ (\bar{a}_i, \bar{p}_i - \frac{\epsilon \sum_1^{\tilde{i}-1} \pi_j}{m(\theta(y))}, \frac{\epsilon \sum_1^{\tilde{i}-1} \pi_j}{m(\theta(y))}, 0) & \text{if } \tilde{i} \leq i \leq I. \end{cases} \quad (13)$$

$$Y^{P'} = \{y'_i\}_{1 \leq i \leq I}, \theta'(y'_i) = \theta_i, \gamma'_i(y'_i) = 1, \lambda'(\{y'_i\}) = \theta_i \pi_i. \quad (14)$$

We calculate payoff of different types of sellers: $\bar{U}_i \equiv m(\theta_i)(u_i(a) + \bar{p}_i + \frac{\epsilon(1-\sum_1^{\tilde{i}-1} \pi_j)}{m(\theta(y))})$. We have assumed that all types get strictly positive payoff in the full information case. According to GSW proposition 4, all types get strictly positive payoff in equilibrium. Since θ_i and a_i are all continuous in ϵ , and for $\epsilon = 0$, $\bar{U}_i > 0$ for all i , then we can find a sufficiently small ϵ that all types still get strictly positive payoff. We fix epsilon at such level.

We argue that allocation $\{\lambda', Y^{P'}, \theta', \Gamma'\}$ is feasible.

For zero profit condition of buyers, note that buyers in any sub-market y_i pays $\bar{\pi}_i$ any way. By lemma ??, the constraint in problem $P_i(\epsilon)$ is binding. Therefore, the buyers get exactly zero payoff from any sub-market.

To show that seller's maximization condition is satisfied, first note that according to lemma ??, seller's maximization condition is satisfied at any solution to problem $P(\epsilon)$. Under allocation $\{\lambda', Y^{P'}, \theta', \Gamma'\}$, all types get the same a and the same market tightness as in allocation $\{\lambda, Y^P, \theta, \Gamma\}$. The only difference is that all types get an expected transfer of $-\epsilon \sum_1^{\tilde{i}-1} \pi_j$. Since this amount is the same for all types, sellers' maximization problems is not affected.

Now we check planner's budget constraint: $\int_{Y^P} [q(\theta(y))t_m + t_u] d\lambda(\{y\})$

$$\begin{aligned}
&= \sum_1^{\tilde{i}-1} q(\theta_i) \lambda(y_i) \frac{(-\epsilon(1-\sum_1^{\tilde{i}-1} \pi_j))}{m(\theta_i)} + \sum_i^I q(\theta_i) \lambda(y_i) \frac{\epsilon \sum_1^{\tilde{i}-1} \pi_j}{m(\theta_i)} \\
&= \sum_1^{\tilde{i}-1} q(\theta_i) \theta_i \pi_i \frac{(-\epsilon(1-\sum_1^{\tilde{i}-1} \pi_j))}{m(\theta_i)} + \sum_i^I q(\theta_i) \theta_i \pi_i \frac{\epsilon \sum_1^{\tilde{i}-1} \pi_j}{m(\theta_i)} \\
&= -\epsilon(1 - \sum_1^{\tilde{i}-1} \pi_j) \sum_1^{\tilde{i}-1} \pi_i + \epsilon \sum_1^{\tilde{i}-1} \pi_j = 0.
\end{aligned}$$

The first equality follows the fact that there are only I sub-markets, so we can change the integral to sum and also follows from construction of y_i in equation 16. The second equality comes from definition of λ .

Since allocation $\{\lambda', Y^{P'}, \theta', \Gamma'\}$ is derived from $\{\lambda, Y^P, \theta, \Gamma\}$ by just redistribution of transfers across types, the welfare level for both allocations are the same. This concludes the proof, because this implies that the welfare from allocation $\{\lambda', Y^{P'}, \theta', \Gamma'\}$ is strictly higher than that in equilibrium.

$$W\{\lambda', Y^{P'}, \theta', \Gamma'\} = \sum_{i=1}^{i=I} \pi_i \bar{U}_i = W(\{\lambda, Y^P, \theta, \Gamma\}) - \epsilon \sum_{i=1}^{i=I} \pi_i.$$

It only remains to show that the proposed allocation $\{\lambda', Y^{P'}, \theta', \Gamma'\}$ yields to higher welfare than the equilibrium allocation, that is, $W(\{\lambda, Y^P, \theta, \Gamma\}) - \epsilon \sum_{i=1}^{i=I} \pi_i > W^{eq}$. But according to equation 8, The welfare under $\{\lambda, Y^P, \theta, \Gamma\}$ is strictly more than the welfare under equilibrium at least by the following amount $\epsilon \sum_{i=1}^{\tilde{i}-1}$, because

$$\begin{aligned}
W\{\lambda, Y^P, \theta, \Gamma\} &= \sum_{i=1}^{i=I} \pi_i \bar{U}_i(\epsilon) \\
&= \sum_{i=1}^{i=\tilde{i}-1} \pi_i (\bar{U}_i(0) + \epsilon) + \sum_{i=\tilde{i}}^{i=I} \pi_i (\bar{U}_i(0)) \\
&= \sum_{i=1}^{i=\tilde{i}-1} \pi_i \bar{U}_i(0) + \sum_{i=\tilde{i}}^{i=I} \pi_i (\bar{U}_i(0) + \epsilon) + \epsilon \sum_{i=1}^{i=I} \pi_i \\
&> W^{eq} + \epsilon \sum_{i=1}^{i=I} \pi_i.
\end{aligned}$$

The first equality and the inequality directly follow 8. Also we used the fact that welfare is calculated by taking the sum of sellers surplus at different sub-markets. This is true because the buyers get zero payoff both in equilibrium and under allocation $\{\lambda, Y^P, \theta, \Gamma\}$. The proof is complete. \square

Thanks to this lemma, we can focus without loss of generality only on the separating allocations where different types choose different sub-markets.

Lemma 3. *For any feasible allocation which involves pooling, there exists another feasible allocation which involves no pooling, that is, for any $y \in Y^P$, $\gamma_i(y) = 1$ for some i , and the planner's objective remains unchanged.*

Proof. As proved earlier, we can assume without loss of generality that $t_u = 0$ for all $y \in Y^P$. We assume that there exists a sub-market \tilde{y} in which there are only two types, type i and type j . If there are more than two types pooling in one sub-market, we apply the same idea repeatedly to get a separating allocation (straight forward to check). Let $\tilde{y} \equiv (a, t_s, t_m, 0)$. Consider the following new allocation. In this allocation, we just separate types i and j and put them in two new sub-markets, y_i and y_j . Note that the rest of allocation remains the same.

$Y'^P = (Y^P \setminus \{\tilde{y}\}) \cup \{y_i, y_j\}$ where $y_i \equiv (a, t_s, t_{m,i}, 0)$, $y_j \equiv (a, t_s, t_{m,j}, 0)$, and $t_{m,i} \equiv v_i(a) - t_s - \frac{k}{q(\theta)}$ and $t_{m,j} \equiv v_j(a) - t_s - \frac{k}{q(\theta)}$.

$$\text{Also, } \lambda'(\{y\}) = \begin{cases} \lambda(\{y\}) & \text{if } y \in Y^P \text{ and } y \neq \tilde{y} \\ \gamma_i(\tilde{y})\lambda(\{\tilde{y}\}) & \text{if } y = y_i \\ \gamma_j(\tilde{y})\lambda(\{\tilde{y}\}) & \text{if } y = y_j \end{cases}$$

$$\text{For all } k: \gamma'_k(\{y\}) = \begin{cases} \gamma_k(\{y\}) & \text{if } y \in Y^P \text{ and } y \neq \tilde{y} \\ 1 & \text{if } y = y_i \text{ and } k = i \\ 1 & \text{if } y = y_j \text{ and } k = j \end{cases}$$

$$\theta'(\{y\}) = \begin{cases} \theta(\{y\}) & \text{if } y \in Y^P \text{ and } y \neq \tilde{y} \\ \theta(\{\tilde{y}\}) & \text{if } y = y_i \text{ or } y = y_j \end{cases}$$

Now we check that new allocation is feasible. Seller's profit maximization is satisfied, because nothing has changed for sellers (a , t_s and $\theta(y)$ that sellers face are the same). For buyers, zero profit condition is satisfied over all $Y^P \setminus \{\tilde{y}\}$, because nothing has changed. For sub-markets y_i and y_j , zero profit condition is satisfied by construction of $t_{m,i}$ and $t_{m,j}$. It remains to check planner's budget constraint. To do so, we just calculate the change in the planner's budget and show that this change is equal to zero. From sub-market y , the planner collects $q(\theta(y))t_m\lambda\{\tilde{y}\}$. With the new allocation, the planner collects: $\gamma_i(\tilde{y})q(\theta(\tilde{y}))t_{m,i}\lambda\tilde{y} + \gamma_j(\tilde{y})q(\theta(\tilde{y}))t_{m,j}\lambda\{\tilde{y}\}$ which is equal to $q(\theta(\tilde{y}))(\gamma_i(\tilde{y})v_i(a) + \gamma_j(\tilde{y})v_j(a) - t_s - \frac{k}{q(\theta(\tilde{y}))})\lambda\{\tilde{y}\} = q(\theta(y))t_m\lambda\{\tilde{y}\}$. This concludes the proof. \square

8.3 Proof of result 2: THIS SECTION HAS BEEN REVISED UNTIL THE END OF RAT RACE

Proof. I will construct an allocation in which type i sellers get matched with probability $m(\theta_i)$ and produce a_i^{CI} . Under assumption part 5 (a) or 5 (b), it can be easily shown that that U_i^{CI} is also increasing in i . Let \tilde{i} denote the highest type of sellers without gains from

trade. Then all types 1, 2, ..., \tilde{i} are inactive, that is, they are matched with probability 0. Given this point, I assume without loss of generality that there are positive gains from trade for all types and then I construct a feasible allocation that achieves the complete information allocation for all these types. It will become clear through our construction that due to our assumptions, no type has incentive to go to the sub-markets with higher types. Therefore, even if some types are inactive, they get enough transfers which discourage them from applying to any active sub-market. I NEED TO HAVE DIRECT TRANSFERS TO ALL SELLERS.

Consider the following allocation:

$$Y^P = \{y_i\}_{1 \leq i \leq I}, \theta(y_i) = \theta_i^{CI}, \gamma_i(y_i) = 1, \lambda(\{y_i\}) = \theta_i^{CI} \pi_i. \quad (15)$$

where y_i is defined as follows:

$$y_i = (a_i^{CI}, p_i, t_{m,i}, 0), \quad (16)$$

Moreover $p_1 = -u_1(a_1^{CI})$, and for $i \geq 2$, p_i is defined recursively as follows:

$$m(\theta_i^{CI})(p_i + u_i(a_i^{CI})) = m(\theta_{i-1}^{CI})(p_{i-1} + u_i(a_{i-1}^{CI})). \quad (17)$$

Finally, $t_{m,i}$ is defined from the following condition:

$$q(\theta_i^{CI})(v_i(a_i^{CI}) - p_i - t_{m,i}) = k.$$

I argue that this allocation is feasible. Then I construct another allocation derived from this allocation by just redistribution of transfers which maximizes the planner's objective.

Step 1:

According to assumption 2, θ_i^{CI}, a_i^{CI} is increasing in i . a_i^{CI} is increasing in i because $u_i(a) + v_i(a)$ satisfies increasing differences and supermodularity properties. Also, because $m(\theta)(u_i(a_i^{CI}) + v_i(a_i^{CI})) - k\theta$ satisfies increasing differences property in θ and i , therefore θ_i^{CI} is increasing in i .

Step 2: From now on in this proof, I drop the superscript CI to reduce the notation. (I MUST DO IT EVERY WHERE IN THIS PROOF) Note that in equation 17, p_i is set such that all local downward incentive compatibility constraints are binding. That is, for all $i \geq 2$ type i is indifferent between choosing y_i and y_{i-1} . Now, we want to show that sellers' maximization constraint is satisfied.

First, we show that type $i - 1$ weakly prefers y_{i-1} over y_i , that is,

$$m(\theta_{i-1})(p_{i-1} + u_{i-1}(a_{i-1})) \geq m(\theta_i)(p_i + u_{i-1}(a_i)). \quad (18)$$

We begin from the RHS and show that the inequality holds:

$$\begin{aligned}
& m(\theta_i)(p_i + u_i(a_i)) - m(\theta_i)(u_i(a_i) - u_{i-1}(a_i)) \\
&= m(\theta_{i-1})(p_{i-1} + u_{i-1}(a_{i-1})) + m(\theta_{i-1})(u_i(a_{i-1}) - u_{i-1}(a_{i-1})) - m(\theta_i)(u_i(a_i) - u_{i-1}(a_i)) \\
&\leq m(\theta_{i-1})(p_{i-1} + u_{i-1}(a_{i-1})) + m(\theta_i)(u_i(a_{i-1}) - u_{i-1}(a_{i-1}) - u_i(a_i) + u_{i-1}(a_i)) \\
&\leq m(\theta_{i-1})(p_{i-1} + u_{i-1}(a_{i-1}))
\end{aligned}$$

The first equality follows from construction of p_i (equation 17). The first inequality follows the fact that θ_i and $u_i(\cdot)$ are both increasing in i . The second inequality follows from single crossing property of u in $(a; i)$ and also from the fact that $a_{i-1} \leq a_i$ (component by component).

Second, we explicitly calculate p_i in terms of p_1 :

$$\begin{aligned}
& m(\theta_i^{CI})(p_i + u_i(a_i^{CI})) = m(\theta_{i-1}^{CI})(p_{i-1} + u_i(a_{i-1}^{CI})) \\
&= m(\theta_{i-1}^{CI})(p_{i-1} + u_{i-1}(a_{i-1}^{CI})) + m(\theta_{i-1}^{CI})(u_i(a_{i-1}^{CI}) - u_{i-1}(a_{i-1}^{CI})) \\
&= m(\theta_{i-1}^{CI})(p_{i-1} + u_{i-1}(a_{i-1}^{CI})) + K_i(\theta_{i-1}^{CI}, a_{i-1}^{CI}) - K_{i-1}(\theta_{i-1}^{CI}, a_{i-1}^{CI}), \tag{19}
\end{aligned}$$

where $K_i(\theta, a)$ is defined as follows: $K_i(\theta, a) = m(\theta)u_i(a)$.

Using telescoping technique yields the following for all $i \geq 2$:

$$m(\theta_i^{CI})(p_i + u_i(a_i^{CI})) = m(\theta_1^{CI})(p_1 + u_1(a_1^{CI})) + \sum_{j=2}^i [K_j(\theta_{j-1}^{CI}, a_{j-1}^{CI}) - K_{j-1}(\theta_{j-1}^{CI}, a_{j-1}^{CI})]$$

Now, we show that for all i and k with $k \leq i - 1$, type k does not gain by choosing sub-market y_i , that is,

$$m(\theta_k^{CI})(p_k + u_k(a_k^{CI})) \geq m(\theta_i^{CI})(p_i + u_k(a_i^{CI})).$$

Note that

$$\begin{aligned}
& m(\theta_k^{CI})(p_k + u_k(a_k^{CI})) - m(\theta_i^{CI})(p_i + u_k(a_i^{CI})) \\
&= \sum_{j=k}^{i-1} \left[m(\theta_j^{CI})(p_j + u_j(a_j^{CI})) - m(\theta_{j+1}^{CI})(p_{j+1} + u_j(a_{j+1}^{CI})) \right. \\
&\quad \left. + m(\theta_{j+1}^{CI})u_j(a_{j+1}^{CI}) - m(\theta_j^{CI})u_j(a_j^{CI}) \right. \\
&\quad \left. - m(\theta_{j+1}^{CI})u_k(a_{j+1}^{CI}) + m(\theta_j^{CI})u_k(a_j^{CI}) \right]
\end{aligned}$$

$$\begin{aligned}
&\geq \sum_{j=k}^{i-1} \left[m(\theta_{j+1}^{CI})u_j(a_{j+1}^{CI}) - m(\theta_j^{CI})u_j(a_j^{CI}) \right. \\
&\quad \left. - m(\theta_{j+1}^{CI})u_k(a_{j+1}^{CI}) + m(\theta_j^{CI})u_k(a_j^{CI}) \right] \\
&\geq \sum_{j=k}^{i-1} \left[m(\theta_j^{CI})(u_j(a_{j+1}^{CI}) - u_k(a_{j+1}^{CI}) - u_j(a_j^{CI}) + u_k(a_j^{CI})) \right] \geq 0.
\end{aligned}$$

The first equality is derived by doing some algebra and using telescoping technique. The first inequality uses the fact that type $i-1$ weakly prefers y_{i-1} over y_i (see equation 18). The second inequality uses $\theta_{j+1}^{CI} \geq \theta_j^{CI}$ and also the fact that u_i is increasing in i for every a . The last inequality is the implication of sorting assumption on u (the first part of assumption 1) and the fact that $a_{j+1}^{CI} \geq a_j^{CI}$.

If $i+1 \leq k$, we use the same technique as above:

$$\begin{aligned}
&m(\theta_k^{CI})(p_k + u_k(a_k^{CI})) - m(\theta_i^{CI})(p_i + u_k(a_i^{CI})) \\
&= \sum_{j=i+1}^k \left[m(\theta_j^{CI})(p_j + u_j(a_j^{CI})) - m(\theta_{j-1}^{CI})(p_{j-1} + u_j(a_{j-1}^{CI})) \right. \\
&\quad \left. + m(\theta_{j-1}^{CI})u_j(a_{j-1}^{CI}) - m(\theta_j^{CI})u_j(a_j^{CI}) \right. \\
&\quad \left. - m(\theta_{j-1}^{CI})u_k(a_{j-1}^{CI}) + m(\theta_j^{CI})u_k(a_j^{CI}) \right] \\
&\geq \sum_{j=i+1}^k \left[m(\theta_{j-1}^{CI})u_j(a_{j-1}^{CI}) - m(\theta_j^{CI})u_j(a_j^{CI}) \right. \\
&\quad \left. - m(\theta_{j-1}^{CI})u_k(a_{j-1}^{CI}) + m(\theta_j^{CI})u_k(a_j^{CI}) \right] \\
&\geq \sum_{j=k}^{i-1} \left[m(\theta_{j-1}^{CI})(u_j(a_{j-1}^{CI}) - u_k(a_{j-1}^{CI}) - u_j(a_j^{CI}) + u_k(a_j^{CI})) \right] \geq 0.
\end{aligned}$$

The first equality is again derived by using telescoping technique. The first inequality follows construction of t_i . The second inequality uses $\theta_{j+1}^{CI} \geq \theta_j^{CI}$ and also the fact that u_i is increasing in i for every a . The last inequality is the implication of sorting assumption on u and the fact that $a_{j+1}^{CI} \geq a_j^{CI}$.

To show $U_i \geq 0$ for all i , consider equation 19. The first term in the right hand side is zero following the definition of t_1 . The summation is positive following the assumption that u_i is increasing in i for every a .

Step 3:

From construction of $t_{m,i}$, buyers' zero profit condition is satisfied.

Step 4:

Consider the budget constraint:

$$\begin{aligned}
B &= \int_{YP} [q(\theta(y))t_m + t_u] d\lambda(\{y\}) \\
&= \sum \pi_i \left[m(\theta_i)(v_i(a_i) - p_i) - k\theta_i \right] \\
&= \sum_{i=1}^I \pi_i \left[m(\theta_i)(v_i(a_i) + u_i(a_i)) - k\theta_i \right] - \sum_{i=1}^I \pi_i \left[m(\theta_i)(p_i + u_i(a_i)) \right] \\
&= \sum_{i=1}^I \pi_i \left[m(\theta_i)(v_i(a_i) + u_i(a_i)) - k\theta_i \right] - \sum_{i=1}^I \pi_i \left[m(\theta_1)(p_1 + u_1(a_1)) + \sum_{j=2}^i [K_j(\theta_{j-1}, a_{j-1}) - K_{j-1}(\theta_{j-1}, a_{j-1})] \right] \\
&= \sum_{i=1}^I \pi_i \left[m(\theta_i)(v_i(a_i) + u_i(a_i)) - k\theta_i \right] - \sum_{i=1}^I \pi_i \left[\sum_{j=2}^i [K_j(\theta_{j-1}, a_{j-1}) - K_{j-1}(\theta_{j-1}, a_{j-1})] \right] - m(\theta_1)(p_1 + u_1(a_1))
\end{aligned} \tag{20}$$

where I used definition of p_i for the fourth equality.

Now, suppose assumption part 5(a) holds. I want to show that $B \geq 0$. To do that, I prove that the elements in the summation (equation 20) for each i are positive. I proceed with induction on i . If $I = 1$, then $B = 0$ trivially by the choice of p_1 . For $I = 2$:

$$\begin{aligned}
m(\theta_2)(v_2(a_2) + u_2(a_2)) - k\theta_2 &\geq m(\theta_1)(v_2(a_1) + u_2(a_1)) - k\theta_1 \\
&\geq m(\theta_1)(u_2(a_1) - u_1(a_1)) + m(\theta_1)(v_2(a_1) + u_1(a_1)) - k\theta_1 \\
&\geq m(\theta_1)(u_2(a_1) - u_1(a_1)) + m(\theta_1)(v_1(a_1) + u_1(a_1)) - k\theta_1 + \\
&\geq m(\theta_1)(u_2(a_1) - u_1(a_1)) + m(\theta_1)(u_2(a_1) + p_1)
\end{aligned}$$

The first inequality is due to the fact that θ_1 and a_1 are feasible for the second type maximization problem ($\max_{\theta,a} \{m(\theta)(v_2(a) + u_2(a)) - k\theta\}$). The second inequality is because v is increasing in i . The last inequality holds due to the construction of p_1 .

Now assume that the induction hypothesis for type $i - 1$ is correct. Then, I show that the hypothesis will be correct for type i as well:

$$\begin{aligned}
m(\theta_i)(v_i(a_i) + u_i(a_i)) - k\theta_i &\geq m(\theta_{i-1})(v_i(a_{i-1}) + u_i(a_{i-1})) - k\theta_{i-1} \\
&= m(\theta_{i-1})(v_i(a_{i-1}) + u_{i-1}(a_{i-1})) - k\theta_{i-1} + m(\theta_{i-1})(u_i(a_{i-1}) - u_{i-1}(a_{i-1})) \\
&\geq m(\theta_{i-1})(v_{i-1}(a_{i-1}) + u_{i-1}(a_{i-1})) - k\theta_{i-1} + m(\theta_{i-1})(u_i(a_{i-1}) - u_{i-1}(a_{i-1}))
\end{aligned}$$

$$\begin{aligned}
&\geq m(\theta_1)(p_1+u_1(a_1))+\sum_{j=2}^{i-1} [K_j(\theta_{j-1}, a_{j-1})-K_{j-1}(\theta_{j-1}, a_{j-1})]+K_i(\theta_{i-1}, a_{i-1})-K_{i-1}(\theta_{i-1}, a_{i-1}) \\
&= m(\theta_1)(p_1+u_1(a_1))+\sum_{j=2}^i [K_j(\theta_{j-1}, a_{j-1})-K_{j-1}(\theta_{j-1}, a_{j-1})]
\end{aligned}$$

Similar to the case for $I = 2$, the first inequality is because θ_{i-1} and a_{i-1} are feasible for type i maximization problem ($\max_{\theta,a}\{m(\theta)(v_i(a) + u_i(a)) - k\theta\}$). The second inequality is because v is increasing in i . The last inequality holds due to the induction hypothesis.

Now, suppose assumption part 5(b) and 5(c) hold. Again, I want to show that $B \geq 0$. The budget balance condition can be written in the following form (following equation 20):

$$\begin{aligned}
B &= \sum_{i=1}^I \pi_i \left[m(\theta_i)(v_i(a_i)+u_i(a_i))-k\theta_i \right] - \sum_{j=2}^I \left[[K_j(\theta_{j-1}, a_{j-1})-K_{j-1}(\theta_{j-1}, a_{j-1})] \sum_{i=j}^I \pi_i \right] - m(\theta_1)(p_1+u_1(a_1)) \\
&= \sum_{i=1}^I \pi_i \left[m(\theta_i)(v_i(a_i)+u_i(a_i))-k\theta_i \right] - \sum_{j=2}^I \left[\pi_j [K_j(\theta_{j-1}, a_{j-1})-K_{j-1}(\theta_{j-1}, a_{j-1})] \frac{\Delta_j}{\pi_j} \right] - m(\theta_1)(p_1+u_1(a_1)) \\
&= \sum_{i=2}^I \pi_i \left[m(\theta_i)(v_i(a_i)+u_i(a_i))-k\theta_i - [K_j(\theta_{j-1}, a_{j-1})-K_{j-1}(\theta_{j-1}, a_{j-1})] \frac{\Delta_i}{\pi_i} \right] + \pi_1 U_1^{CI} - m(\theta_1)(p_1+u_1(a_1)) \geq 0
\end{aligned}$$

For the second equality, we changed the order of integrals for the double sigma term. For the third equality, we defined a new variable Δ_i as follows

$$\Delta_i = \sum_{k=i}^I \pi_k$$

to simplify the expression. We used assumption 3 and choice of p_1 to establish the inequality in the last line.

Step 5:

Finally, for all i , let $p'_i = p_i + \frac{B}{m(\theta_i^{CI})}$ and $t'_{m,i} = t_{m,i} - \frac{B}{m(\theta_i^{CI})}$. Now consider the following allocation:

$$Y^P = \{y'_i\}_{1 \leq i \leq I}, \theta(y_i) = \theta_i^{CI}, \gamma_i(y_i) = 1, \lambda(\{y_i\}) = \theta_i^{CI} \pi_i. \quad (21)$$

where y'_i is defined as follows:

$$y'_i = (a_i^{CI}, p'_i, t'_{m,i}, 0). \quad (22)$$

This allocation is feasible due to the following reasons: It's easy to check that now budget constraint is satisfied with equality (similar to the proof of the first lemma). Since $p'_i + t'_{m,i} = p_i + t_{m,i}$ for all i , buyers' zero profit is also satisfied. Sellers' maximization is still

satisfied, because in the new allocation, all sellers get an expected increase of B equally over all sub-markets, so their maximization decisions are not affected.

This allocation is constraint feasible because the level of θ and a assigned to every type is exactly equal to that under complete information allocation, so it is not possible to increase the objective any more. The proof is complete. \square

8.4 Proofs of Asset market with lemons

Asset market with lemons if $\pi_1 b_1 + \pi_2 b_2 \geq c_2$. Here the complete information allocation is achievable through a pooling allocation. Seller's maximization problem is satisfied, because there is just one sub-market. Also, both types get positive payoff. Buyers' zero profit condition is satisfied by construction of $t_{m,i}$. Budget balance is also trivially satisfied because the planner does not make any transfers. The objective is maximized because the θ and α allocated to both types is the same as what they get under complete information. The proof is complete. \square

Asset market with lemons when $\pi_1 b_1 + \pi_2 b_2 < c_2$. Here the complete information allocation is not achievable through a pooling allocation, because type 2 gets strictly negative payoff, therefore, pooling allocation is not feasible. If $b_2 - c_2$ is greater than $b_1 - c_1$, part 5(b) of assumption 2 is violated. If $b_2 - c_2$ is less than or equal to $b_1 - c_1$, then it's easy to check that although part 5(b) is satisfied, part 5 is violated. Therefore, I cannot use result 2. Hence, I need to solve the planner's problem completely, taking all constraints into account. We proceed in 6 steps. In the first step, we use lemmas 3 and 1 to simplify the problem. Specifically, I show that we can assume without loss of generality that there are only two sub-markets. In the second step, we show that the market tightness in both sub-markets must be strictly positive. In the third step, we show that the market tightness in both sub-markets must be less than or equal to 1. In the fourth step, we show that α (probability that the seller gives the asset to the buyer) in both sub-markets must be equal to 1. In the fifth step, we show that market tightness in the submarket that type 1 applies to must be equal to 1. In the last step, I calculate the market tightness in the other sub-market and conclude the characterization of the constrained efficient allocation.

Step 1: Simplifying the problem

According to lemma 1, we can assume without loss of generality that $t_u = 0$ for all sub-markets. Also, the planner's budget constraint must be binding. Moreover, according to lemma 3, we can assume without loss of generality that different types are allocated to

different sub-markets. Let $y_i \equiv (\alpha_i, p_i, t_{m,i})$ denote the sub-market that type i applies to. (As noted earlier, $t_u = 0$ for all sub-markets, so we eliminate that to reduce the notation.). Also, let $\theta_i \equiv \theta(y_i)$. Now, we can write planner's problem as follows:

Problem 2 (Asset market with lemons, 1).

$$\max_{\{(\theta_i, \alpha_i, p_i, t_{m,i})\}_{i=1,2}} \sum_1^2 \pi_i (\min\{\theta_i, 1\} \alpha_i (h_i - c_i) - k\theta_i),$$

subject to

$$\begin{aligned} \min\{\theta_i^{-1}, 1\} (\alpha_i h_i - p_i - t_{m,i}) &= k, \text{ for all } i \\ \min\{\theta_1, 1\} (p_1 - \alpha_1 c_1) &\geq \min\{\theta_2, 1\} (p_2 - \alpha_2 c_1) \text{ (IC-12) }, \\ \min\{\theta_2, 1\} (p_2 - \alpha_2 c_2) &\geq \min\{\theta_1, 1\} (p_1 - \alpha_1 c_2) \text{ (IC-21) }, \\ \min\{\theta_1, 1\} (p_1 - \alpha_1 c_1) &\geq 0 \text{ (IR-1)} \\ \min\{\theta_2, 1\} (p_2 - \alpha_2 c_2) &\geq 0 \text{ (IR-2) and} \\ \sum_1^2 \pi_i \min\{\theta_i, 1\} t_{m,i} &= 0 \text{ (BB)}. \end{aligned}$$

To write the objective, we take into account that the planner's budget constraint is bonding. Therefore, no transfers appear in the objective. The first constraint is buyers' zero profit condition. I break sellers' maximization condition into two parts. The first part of sellers' maximization condition includes sellers' Incentive compatibility constraints: In the second (the third) line, we ensure that type 1 (2) does not want to apply to the sub-market designed for the other type. I call this constraint IC-12 (IC-21). The second part of sellers' maximization condition includes basically sellers' participation or individual rationality constraints: In the fourth (fifth) line, we ensure that type 1 (2) gets strictly positive payoff. I call this constraint IR-1 (IR-2). The last line is planner's budget constraint.

Step 2: $\theta_1 > 0$ and $\theta_2 > 0$

If both θ_1 and θ_2 are 0 then the welfare is 0. But this is not possible because we know at least that equilibrium allocation is feasible and delivers strictly positive utility. To rule out the case that even one of them cannot be 0, first note that IC-12 and IC-21 together imply that:

$$(m(\theta_1)\alpha_1 - m(\theta_2)\alpha_2)c_1 \leq m(\theta_1)p_1 - m(\theta_2)p_2 \leq (m(\theta_1)\alpha_1 - m(\theta_2)\alpha_2)c_2. \quad (23)$$

But $c_1 < c_2$, therefore,

$$m(\theta_1)\alpha_1 \geq m(\theta_2)\alpha_2. \quad (24)$$

If $\theta_1 = 0$, then θ_2 must be 0 as well, which leads to 0 welfare. Nevertheless, this cannot be part of a planner's allocation, given the fact that the equilibrium allocation is feasible and delivers strictly positive welfare. Thus $\theta_1 > 0$. If $\theta_2 = 0$, then it's easy to check that the maximum possible welfare in this case (even if $\theta_1 = 1$) is less than the welfare under the proposed planner's allocation.

Let $r_i \equiv \min\{\theta_i, 1\}p_i$ for all i . For any θ_i and $r_i \in \mathbb{R}$, we can find a unique $p_i \in \mathbb{R}$ which solves the maximization problem. From now on, we work with r_i instead because it simplifies the analysis. Also, we solve buyers' zero profit condition for $t_{m,i}$ in terms of other variables to remove $t_{m,i}$ from the problem completely and then rewrite the problem as follows:

Problem 3 (Asset market with lemons, 2).

$$\max_{\{\theta_i, \alpha_i, r_i\}_{i=1,2}} \sum_1^2 \pi_i (\min\{\theta_i, 1\} \alpha_i (h_i - c_i) - k\theta_i),$$

subject to

$$r_1 - \min\{\theta_1, 1\} \alpha_1 c_1 \geq r_2 - \min\{\theta_2, 1\} \alpha_2 c_1 \quad (IC-12),$$

$$r_2 - \min\{\theta_2, 1\} \alpha_2 c_2 \geq r_1 - \min\{\theta_1, 1\} \alpha_1 c_2 \quad (IC-21),$$

$$r_1 - \min\{\theta_1, 1\} \alpha_1 c_1 \geq 0 \quad (IR-1)$$

$$r_2 - \min\{\theta_2, 1\} \alpha_2 c_2 \geq 0 \quad (IR-2) \text{ and}$$

$$\sum_1^2 \pi_i (\min\{\theta_i, 1\} \alpha_i h_i - k\theta_i - r_i) = 0 \quad (BB).$$

Step 3: $\theta_1 \leq 1$ and $\theta_2 \leq 1$

Suppose $\theta_i > 1$ for some i , now I propose the following: $\theta'_i = 1$, $r'_i = r_i + k(\theta_i - 1)\pi_i$ and $r'_j = r_j + k(\theta_i - 1)\pi_i$ where $j \neq i$. Therefore, if we replace θ_i , r_1 and r_2 by $\theta'_i = 1$, r'_1 and r'_2 , we can increase the objective by $k(\theta_i - 1)$. Also, the new solution satisfies all the constraints because of the following: Obviously, IC-12 and IC-21 are still satisfied, because the change in r_1 is the same as the change in r_2 and also $\min\{\theta_1, 1\}$ and $\min\{\theta_2, 1\}$ have not changed. IR-1 and IR-2 are satisfied because $r'_1 > r_1$ and $r'_2 > r_2$. BB is also satisfied by construction of r'_1 and r'_2 . A contradiction. Therefore, for all $i \in \{1, 2\}$, $\theta_i \leq 1$.

Step 4: $\alpha_1 = \alpha_2 = 1$

Suppose $\alpha_i < 1$ for some i . Let α'_i be defined such that $\alpha'(\theta_i - \epsilon)$ equals $\alpha\theta_i$, where $0 < \epsilon < \theta_i(1 - \alpha_i)$. Now consider the following: $\theta'_i = \theta_i - \epsilon$, $r'_i = r_i + k\epsilon\pi_i$ and $r'_j = r_j + k\epsilon\pi_i$ where $j \neq i$.

Now, if we replace α_i , θ_i , r_1 and r_2 by α'_i , θ'_i , r'_1 and r'_2 , we can increase the objective by $k\epsilon$. But it's a contradiction if we show that the new solution satisfies all the constraints.

But the new solution indeed satisfies all the constraints because of the following: Obviously, IC-12 and IC-21 are still satisfied, because $\min\{\theta_i, 1\}\alpha_i = \min\{\theta'_i, 1\}\alpha'_i$. IR-1 and IR-2 are satisfied because $r'_1 > r_1$ and $r'_2 > r_2$. BB is also satisfied by construction of r'_1 and r'_2 . A contradiction. Therefore, for all $i \in \{1, 2\}$, $\alpha_i = 1$.

Step 5: $\theta_1 = 1$

For simplicity, we write the planner's problem again incorporating the results so far:

Problem 4 (Asset market with lemons, 2).

$$\max_{\{(\theta_i, r_i)\}_{i=1,2}} \sum_1^2 \pi_i(\theta_i(h_i - c_i) - k\theta_i),$$

subject to

$$r_1 - \theta_1 c_1 \geq r_2 - \theta_2 c_1 \quad (IC-12) ,$$

$$r_2 - \theta_2 c_2 \geq r_1 - \theta_1 c_2 \quad (IC-21) ,$$

$$r_1 - \theta_1 c_1 \geq 0 \quad (IR-1)$$

$$r_2 - \theta_2 c_2 \geq 0 \quad (IR-2) \text{ and}$$

$$\sum_1^2 \pi_i(\theta_i h_i - k\theta_i - r_i) = 0 \quad (BB).$$

First note that $\theta_1 \geq \theta_2$ following equation 24 and because $\alpha_1 = \alpha_2 = 1$ according to step 4. By way of contradiction, assume that $\theta_1 < 1$ at a solution. We consider two cases. First, assume that IR-2 is not binding. I propose the following: $\theta'_i = \theta_i + \epsilon$ for all i where $\epsilon < \min\{1 - \theta_1, \frac{r_2 - \theta_2 c_2}{c_2 - \pi_1 b_1 - \pi_2 b_2}\}$, and $r'_i = r_i + (\pi_1 b_1 + \pi_2 b_2)\epsilon$ for all i . It's easy to check that all constraints are satisfied, but the objective now has increased by $(\pi_1(b_1 - c_1) + \pi_2(b_2 - c_2))\epsilon$. A contradiction. Note that we used equation 24 to ensure that $\theta_2 + \epsilon < 1$.

Therefore, IR-2 is binding. I propose the following: $\theta'_1 = \theta_1 + \epsilon$ where $\epsilon < 1 - \theta_1$, $r'_1 = r_1 + b_1 \epsilon$. It's again easy to check that all constraints are satisfied. The only tricky thing here is to check that IC-21 is satisfied. But The LHS in IC-21 is fixed. The RHS increases by $\epsilon(b_1 - c_2)$ which is a negative number, so IC-21 is not violated. (Note that $b_1 - c_2 < 0$, otherwise $\pi_1 b_1 + \pi_2 b_2 > \pi_1 c_2 + \pi_2 c_2 = c_2$ which contradicts the initial assumption that $\pi_1 b_1 + \pi_2 b_2 < c_2$). But the objective now has increased by $\pi_1 b_1 \epsilon$. A contradiction.

Step 6: Calculating θ_2 and the rest of unknowns

We write r_1 from the budget constraint in terms of other variables, specially r_2 :

$$r_1 = b_1 + \frac{\pi_2}{\pi_1} \theta_2 b_2 - \frac{\pi_2}{\pi_1} r_2 \quad (25)$$

Now, one can write equation 23 as follows after replacing r_1 from the above equation:

$$(1 - \theta_2)c_1 \leq b_1 + \frac{\pi_2}{\pi_1}\theta_2b_2 - \frac{r_2}{\pi_1} \leq (1 - \theta_2)c_2. \quad (26)$$

First, note that IR-1 is implied by IC-12 and IR-2. Second, I argue that IR-2 must be binding at the solution. By way of contradiction, suppose not. Then only equation 26 is sufficient to determine θ_2 . But in order to maximize the objective, we need to choose the highest possible θ_2 consistent with equation 26, which is $\theta_2 = 1$. But according to equation 26, $r_2 = \pi_1b_1 + \pi_2b_2$ and $r_2 > c_2$ from IR-2, which is a contradiction with $\pi_1b_1 + \pi_2b_2 < c_2$. Therefore, IR-2 is binding. Third, since IR-2 is binding, we replace r_2 by θ_2c_2 and rewrite equation 26 again:

$$(1 - \theta_2)c_1 \leq b_1 + \frac{\theta_2}{\pi_1}(\pi_2b_2 - c_2) \leq (1 - \theta_2)c_2. \quad (27)$$

Now, it's easy to see that the right inequality in 27 is satisfied for any $\theta_2 \in [0, 1]$, because $b_1 < c_2$. In order to maximize the objective, we need to find the maximum value for θ_2 under which the left inequality in 27 is satisfied ($(1 - \theta_2)c_1 \leq b_1 + \frac{\theta_2}{\pi_1}(\pi_2b_2 - c_2)$). This implies that $\theta_2 = \frac{\pi_1(b_1 - c_1)}{c_2 - \pi_2b_2 - \pi_1c_1}$.

Now the proof is complete, because we have found the values for α_i , θ_i and r_i . We can calculate values of p_i from θ_i and r_i . We can calculate values $t_{m,i}$ from buyers' zero profit conditions and check that they are the same as them in table 1.

□

8.5 Proof of the rat race

Proof. This proposition is basically a special case of result 2. It's straight forward to check that all conditions are satisfied. Specially note that, part 5(a) of assumption 2 is satisfied, therefore, we do not need any assumption on the distribution.

□

8.6 Asset market with continuous type space

8.6.1 Preliminaries of the proofs

First of all, note that the ideas used here are the same those used in the simple two-type model (asset market with lemons). However, mathematical tools that we use here are different, because the state space is continuous.

One way of proving proposition 5 is to take a Guess-And-Verify approach. We guess that the complete information allocation is achievable. Then we check whether conditions for feasibility are satisfied. One problem is that if complete information allocation is not

achievable (like proposition 6), this approach does not work, because checking for feasibility is not sufficient, since there might be other feasible allocations which deliver a higher value of objective for the planner. Therefore, in order to be able to use a general solution method, we first characterize incentive compatible schemes, as is common in the mechanism design literature. Then, we work with a modified problem in which sellers' maximization condition has been replaced by some other constraints (monotonicity and Envelope condition). See below for the details.

TB CORRECTED: We proceed in 6 steps. In the first step, we use lemmas NO POOLING and BB BINDING to simplify the problem. Specifically, we show that we can assume without loss of generality that there are only two sub-markets. In the second step, we show that the market tightness in both sub-markets must strictly positive. In the third step, we show that the market tightness in both sub-markets must be less than or equal to 1. In the fourth step, we show that α (probability that the seller gives the asset to the buyer) in both sub-markets must be equal to 1. In the fifth step, we show that market tightness in the submarket that type 1 applies to must be equal to 1. In the last step, I calculate the market tightness in the other sub-market and introduce the constrained efficient allocation.

Step 1: Simplifying the problem

Similar to the discrete type case, we can assume without loss of generality that $t_u(\phi) = 0$ for all sub-markets. Suppose for an allocation and for some ϕ , $t_u(\phi) \neq 0$. Consider another allocation with $t'_m(\phi) = t_m(\phi) + \frac{t_u}{q(\theta(y))}$ and $t'_u(\phi) = 0$ for any ϕ such that $t_u(\phi) \neq 0$. In this new allocation, the maximization problem of sellers and zero profit of buyers do not change. The planner also makes the same expected amount of money over all sub-market. Therefore, we can always make another allocation with $t_u = 0$. The only caveat is that if $q(\theta(\phi))$ is equal to 0, or $\theta(\phi) = \infty$, this arguments fails. But the point is that $\theta(\phi)$ can be ∞ only for a zero measure of sub-markets on the support of G . Otherwise, the cost of entry for those sub-markets goes to ∞ although the gains from trade for all types of sellers are finite, which leads to $-\infty$ value for the planner's objective. This contradicts with the allocation to be constrained efficient because we know that at least the planner can get 0 value for the objective by shutting down all sub-markets ($\theta(\phi) = 0$ for all ϕ).

The planner's budget constraint must be binding. Suppose for the constrained efficient allocation, the budget constraint is not binding, that is,

$$B \equiv \int_{\text{supp } G} [q(\theta_{GH}(\phi))t_m(\phi) + t_u(\phi)] dG > 0.$$

For any ϕ , define new t'_s and t'_m as follows:

$$t'_s(\phi) = t_s(\phi) + \frac{B}{m(\theta(\phi))} \left(\int_{\text{supp}G} \frac{dG}{\theta(\phi)} \right)^{-1},$$

$$t'_m(\phi) = t_m(\phi) - \frac{B}{m(\theta(\phi))} \left(\int_{\text{supp}G} \frac{dG}{\theta(\phi)} \right)^{-1}.$$

All G , H and t_u remain the same, so the new allocation is just different from the old original one in transfers (t_s and t_m). In this new allocation, the maximization problem of sellers and zero profit of buyers do not change. It's similar to the discrete type case that the planner's budget constraint now holds with equality, but the objective has increased by amount B . A contradiction with the original allocation to be constrained efficient.

Also, similar to discrete type case, we can assume without loss of generality that different types are allocated to different sub-markets. To show that ignoring pooling allocations is without loss of generality, suppose just two types are allocated to one sub-market ϕ with strictly positive probability. Then, we can construct another allocation and allocate those two types to two different sub-markets and adjust t_m for those sub-markets accordingly. The proof is exactly the same as in lemma 3.

So far we have shown that any constrained efficient allocation can be assumed to be separating without loss of generality. Now we want to show that any allocation can be implemented without any lotteries. Assume by way of contradiction that the constrained efficient allocation involves lotteries, that is, there exists some type z which is assigned to ϕ_1 with probability $\alpha > 0$ and to ϕ_2 with probability $\beta > 0$. This seller gets $\alpha m(\theta_1)(p_1 - c(y)) + \beta m(\theta_2)(p_2 - c(y))$. Replace these two sub-markets with ϕ_3 with market tightness $m(\theta_3) = \alpha m(\theta_1) + \beta m(\theta_2)$ and transfers $t_{s,3} = \frac{\alpha m(\theta_1)p_1 + \beta m(\theta_2)p_2}{(\alpha + \beta)m(\theta_3)}$ and $t_{m,3} = h(z) - t_{s,3} - \frac{k}{3}$. The sellers get exactly the same expected payoff. Buyers also get zero payoff from this new sub-market. The planner's budget constraint is changed in the following way. Originally the planner get the expected payoff of $(\alpha m(\theta_1) + \beta m(\theta_2))h(y) - (\alpha m(\theta_1)p_1 + \beta m(\theta_2)p_1) - k(\alpha\theta_1 + \beta\theta_2)$ from sub-markets ϕ_1 and ϕ_2 . By creating a new sub-market, the planner gets $(\alpha + \beta)m(\theta_3)h(z) - (\alpha + \beta)m(\theta_3)t_{s,3} - k(\alpha + \beta)\theta_3$. The first two terms are exactly the same as in the original case by construction of θ_3 and $t_{s,3}$. But $-k(\alpha\theta_1 + \beta\theta_2)$ is weakly less than $-k(\alpha + \beta)\theta_3$ due to concavity of $m(\cdot)$. Therefore, we can always find another feasible allocation which does not involve pooling or lotteries and creates weakly higher surplus while delivering the planner weakly higher surplus.

I summarize the results that we have discussed so far in the following lemma. In particular, since each type only applies to one sub-market, I index sub-markets by types of sellers, z .

Lemma 4. *Suppose $(\theta(z), p(z))$ solves problem 5 and $t_m(z) = h(z) - p(z) - \frac{k}{q(\theta(z))}$, $t_u(z) = 0$, $G(z) = \int^z f(\tilde{z})\theta(\tilde{z})d\tilde{z}$ and $H(\phi, z) = \int^{\min\{\phi, z\}} f(\tilde{z})d\tilde{z}$. Then (G, H, p, t_m, t_u) is a constrained efficient allocation.*

Problem 5.

$$\begin{aligned} & \max_{\theta(z), p(z)} \int \left[m(\theta(z))(h(z) - c(z)) - k\theta(z) \right] dF \\ & \text{s.t. } z \in \arg \max_{\hat{z}} U(z, \hat{z}), \quad (IC) \\ & \quad U(z, z) \geq 0 \quad (IR), \\ & \text{and } \int \left[m(\theta(z))(h(z) - t_S(z)) - k\theta(z) \right] dF = 0 \quad (BB), \\ & \text{in which } V(z, \hat{z}) \equiv m(\theta(\hat{z}))(p(\hat{z}) - c(z)). \end{aligned}$$

To understand this problem, I break down the sellers' maximization problem into two constraints. The first one is Incentive compatibility (IC) constraint, stating that type z should get weakly lower payoff by applying to sub-market \hat{z} . Formally, let $U(z, \hat{z})$ denote the payoff of type z from applying to sub-market \hat{z} . IC implies that $U(z, \hat{z}) \leq U(z, z)$. The second one is participation or individual rationality (IR) constraint, stating that all types need to get at least 0 payoff to participate. The last constraint in the problem is planner's budget constraint. I denote it by BB which stands for Budget Balance condition.

In the lemma, definitions of $G(\cdot)$ and $H(\cdot, \cdot)$ require explanation. Regarding G distribution, since each type z applies only to sub-market z , the measure of buyers at each sub-market z is $f(z)\theta(z)$. To calculate $G(z)$, we just need to integrate over all sub-markets with indices less than z . Regarding H distribution, first assume that $\phi \geq z$. Since each type z applies only to sub-market z , the measure of types who apply to sub-markets less than ϕ is the measure of all types below z . Now assume that $\phi < z$. The measure of types who apply to sub-markets less than ϕ is the measure of all types below ϕ .

Step 2: Characterizing Incentive Compatible Schemes

Suppose $c(z)$ is strictly monotone in z . The first two parts of lemma state that $c'(z) \frac{d\theta(y)}{dy} \leq 0$ is necessary and sufficient for $\theta(z)$ to be implementable, that is, there exists some transfers that together with $\theta(z)$ satisfies IC. The third part gives us necessary condition for $U(z)$, where $U(z)$ denotes the payoff that type z gets from some allocation. (I suppress the functionality of $U(\cdot)$ to allocation to reduce the notation.)

Lemma 5 (Necessary and sufficient condition for $\theta(z)$ to be implementable). *Assume that $c(z)$ is strictly monotone in y .*

1. A piecewise C^1 function $\theta(z)$ is implementable only if $c'(z)\frac{d\theta(z)}{dz} \leq 0$ wherever $\theta(z)$ is differentiable at z .
2. Any piecewise C^1 function $\theta(z)$ satisfying $c'(z)\frac{d\theta(z)}{dz} \leq 0$ is implementable.
3. If $(a(\cdot), r(\cdot))$ satisfies I.C. constraint, then $U(z)$ must satisfy

$$U(z) = U(z_H) + \int_z^{z_H} m(\theta(z_0))c'(z_0)dz_0.$$

Proof. $U(z, \hat{z})$ can be written as $U(z, \hat{z}) = x(\hat{z})c(z) + r(\hat{z})$ where $x(\hat{y}) = -m(\theta(\hat{y}))$ and $r(\hat{y}) = m(\theta(\hat{y}))p(\hat{y})$. Following Fudenberg and Tirole (1990) ([7] p. 257), we say, formally, that $\theta(z)$ is implementable if there exists a function $r(z)$ such that the allocation $\theta(z), r(z)$ satisfy IC constraint in problem 5. According to [7] theorem 7.1, the **necessary** condition for $x(\cdot)$ to satisfy IC is

$$\frac{\partial}{\partial z} \left(\frac{\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial r}} \right) \frac{dx}{dz} \geq 0.$$

But $\frac{\partial}{\partial z} \left(\frac{\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial r}} \right) \frac{dx}{dz} = \frac{\partial}{\partial z} \left(\frac{c(z)}{1} \right) (-m'(\theta(z))) \frac{d\theta(z)}{dz}$. But $c'(\cdot) > 0$ and $m'(\cdot) \geq 0$, therefore, the necessary condition is equivalent to

$$\frac{d\theta(z)}{dz} \leq 0. \quad (28)$$

According to [7] theorem 7.3, the **sufficient** condition for $x(\cdot)$ to be implementable is that $\frac{dx(z)}{dz} \geq 0$, or equivalently, $\frac{d\theta(z)}{dz} \leq 0$.¹²

For the third part of the lemma, we use corollary 1 from Milgrom and Segal (2002) ([12]), if $\theta(y)$ satisfies IC, then $U(\cdot)$ can be written as follows:

$$U(z) = U(z_H) - \int_z^{z_H} \left[\frac{\partial U(z_0, z_0)}{\partial z} \right] dz_0 = U(z_H) - \int_z^{z_H} [-m(\theta(z_0))c'(z_0)] dz_0. \quad (29)$$

This equation is derived from Envelop theorem and is standard in mechanism design literature. The requirements that we need to check are as follows:

1. $U(z, \hat{z})$ is differentiable and absolutely continuous in z . This is satisfied because c is assumed to be twice differentiable.
2. $\sup_{\hat{z}} \left| \frac{\partial U(z, \hat{z})}{\partial z} \right|$ is integrable. This is satisfied because $\sup_{\hat{z}} \left| \frac{\partial U(z, \hat{z})}{\partial z} \right| \leq |c'(z)| < M$ for some $M \in \mathbb{R}$, because $c'(\cdot)$ is continuous and is defined over a compact set $[z_L, z_H]$.
3. $\theta(z)$ is obviously non-empty. □

¹²Briefly, the idea of the proof for necessity is that the second order condition for IC maximization problem ($\max_{\hat{z}} V(z, \hat{z})$) should hold. For sufficiency, the proof goes by contradiction. The proof of this lemma is standard in mechanism design literature thus omitted from here.

We know that under IC $U(z) = m(\theta(z))(p(z) - c(z))$, for all z . We can substitute $U(\cdot)$ from equation 29 to derive the price which satisfies IC:

$$p(z) = c(z) + \frac{U(z_H) + \int_z^{z_H} m(\theta(z_0))c'(z_0)dz_0}{m(\theta(z))}. \quad (30)$$

So far we have reduced IC constraint in the planner's problem to two conditions 28 and 30. Therefore, thanks to lemma 5, we can rewrite the planner's problem as follows:

Problem 6. *Planner's problem*

$$\begin{aligned} & \max_{\theta(z), p(z)} \int \left[m(\theta(z))[h(z) - c(z)] - k\theta(z) \right] f(z) dz \\ \text{s.t. } & \frac{d\theta(z)}{dz} \leq 0, \quad p(z) = c(z) + \frac{U(z_H) + \int_z^{z_H} m(\theta(z_0))c'(z_0)dz_0}{m(\theta(z))}, \\ & U(z) \geq 0, \quad \text{and} \quad \int \left[m(\theta(z))[h(z) - t_S(z)] - k\theta(z) \right] f(z) dz = 0. \end{aligned}$$

From now on, we work with this problem and characterize the solution to this problem.

8.6.2 Proof of Proposition 5 under part 1 of assumption 5

We prove proposition 5 under the first assumption ($h'(\cdot) \leq 0$). In order to solve the planner's problem, we use somewhat a backward approach. We first guess that the planner can achieve the complete information level of welfare. That is, the planner can maximize his objective function point-wise. What we need to do then is to check that the monotonicity constraint is satisfied. Finally, we need to check that $U(z)$ for all z is non-negative while planner's budget constraint is satisfied.

Proposition 8. *Proposition 5 under part 1 of assumption 5*

Assume that $c'(\cdot) > 0$ and $h'(\cdot) \leq 0$. Then $\theta(z) = \theta^{CI}(z)$ for all z and

$$U(z_H) = \int \int_z^{z_H} m(\theta(z_0))h'(z_0)f(z)dz_0dz + U^{CI}(z_H).$$

Also, $p(\cdot)$ is pinned down by equation 30.

Proof. Point-wise maximization yields the following FOC:

$$m'(\theta(z))(h(z) - c(z)) - k = 0. \quad (31)$$

By differentiating 31 with respect to z , one yields

$$\frac{d\theta(z)}{dz} = -\frac{k(h'(z) - c'(z))}{m''(\theta(z))(h(z) - c(z))^2}. \quad (32)$$

By assumption, $h'(z) - c'(z) \leq 0$ and $m''(\theta(z)) \leq 0$, therefore $\frac{d\theta(z)}{dz}$ is negative. Hence, $c'(z) \frac{d\theta(z)}{dz} \leq 0$ constraint is satisfied in problem 6. Now, we calculate that budget constraint in terms of $U(z_H)$ and other variables and equate it to 0. Then we derive $U(z_H)$ and ensure that $U(z_H)$ is positive.

$$\begin{aligned}
& \int [m(\theta(z))[h(z) - p(z)] - k\theta(z)] f(z) dz \\
&= \int [m(\theta(z))[h(z) - c(z)] - k\theta(z) - m(\theta(z))(p(z) - c(z))] f(z) dz \\
&= \int \left[\left[- \int_z^{z_H} m(\theta(z_1))[h'(z_1) - c'(z_1)] dz_1 + U^{CI}(z_H) \right] - \int_z^{z_H} m(\theta(z_0))c'(z_0) dz_0 - U(z_H) \right] f(z) dz \\
&= \int \int_z^{z_H} [-m(\theta(z_0))[h'(z_0) - c'(z_0)] - m(\theta(z_0))c'(z_0)] f(z) dz_0 dz + U^{CI}(z_H) - U(z_H) \\
&= - \int \int_z^{z_H} m(\theta(z_0))h'(z_0) f(z) dz_0 dz + U^{CI}(z_H) - U(z_H) = 0 \tag{33}
\end{aligned}$$

Therefore, if $U(z_H)$ is chosen as follows, the budget constraint holds:

$$U(z_H) = - \int \int_z^{z_H} m(\theta(z_0))h'(z_0) f(z) dz_0 dz + U^{CI}(z_H).$$

Since $U^{CI}(z_H) \geq 0$ and $h' \leq 0$, so $U(z) \geq 0$. Moreover since $c'(z) > 0$, the integral in 30 is positive and consequently, $p(z) \geq c(z)$ is also satisfied, implying that all types get positive payoff. \square

The proposition and its proof can be written in the exactly same fashion if instead $c(\cdot)$ is strictly decreasing and $h(\cdot) - c(\cdot)$ is increasing. The result are not reported to save space. Note that in order to show that IC is satisfied, it is sufficient to have $h' - c' \leq 0$ (see equation 32). In the next part of the proposition, we replace the assumption $h' \leq 0$ with this weaker assumption $h' - c' \leq 0$. To satisfy planner's budget constraint, we need another assumption summarized in the second part of assumption 5.

8.6.3 Proof of Proposition 5 under part 2 of assumption 5

Proposition 9. *Suppose the second part of assumption 5 holds. Then for all z , $\theta(z) = \theta^{CI}(z)$. Also,*

$$U(z_H) = \int \left[m(\theta(z))[h(z) - c(z)] - k\theta(z) - m(\theta(z))c'(z) \left(\frac{F(z)}{f(z)} \right) \right] g(z) dz.$$

Prices are pinned down by equation 30.

Proof. The proof is similar to proof of the first part discussed above. Note that $h'(z) - c'(z)$ is negative. According to equation 32, monotonicity constraint (that $\theta(z)$ is decreasing in θ) is satisfied. Also $U(z_H)$ is positive as will be shown below. The integral in 30 is positive because $c'(z) > 0$ and $U(z_H) \geq 0$, therefore all types (not only z_H) get a positive payoff. To check that planner's budget constraint holds, we write the planner's budget in terms of $U(z_H)$ and other variables and equate it to 0. Then we ensure that $U(z_H) \geq 0$. Now we calculate planner's budget:

$$\begin{aligned}
& \int [m(\theta(z))[h(z) - p(z)] - k\theta(z)] f(z) dz \\
&= \int [m(\theta(z))[h(z) - c(z)] - k\theta(z) - m(\theta(z))(p(z) - c(z))] f(z) dz \\
&= \int \left[m(\theta(z))[h(z) - c(z)] - k\theta(z) - \int_z^{z_H} m(\theta(z_0))c'(z_0) dz_0 \right] f(z) dz - U(z_H) \\
&= \int \left[m(\theta(z))[h(z) - c(z)] - k\theta(z) - m(\theta(z))c'(z) \frac{F(z)}{f(z)} \right] f(z) dz - U(z_H) = 0. \quad (34)
\end{aligned}$$

The second equality follows equation 30. The third equality is established using integration by parts¹³. From equation 34, we can write

$$U(z_H) = \int \left[m(\theta(z))[h(z) - c(z)] - k\theta(z) - m(\theta(z))c'(z) \frac{F(z)}{f(z)} \right] f(z) dz.$$

A sufficient condition for the integral to be positive is that the sum of terms in the brackets is always positive, that is, for all z : $m(\theta(z))[h(z) - c(z)] - k\theta(z) - m(\theta(z))c'(z) \frac{F(z)}{f(z)} \geq 0$. But at the solution $m'(\theta(z))[h(z) - c(z)] = k$, therefore

$$\begin{aligned}
& \frac{m(\theta(z))[h(z) - c(z)] - k\theta(z)}{m(\theta(z))} \\
&= \frac{m(\theta(z))[h(z) - c(z)] - m'(\theta(z))[h(z) - c(z)]\theta(z)}{m(\theta(z))} = -\frac{\theta(z)q'(\theta(z))}{q(\theta(z))}(h(z) - c(z)).
\end{aligned}$$

Hence, for the $U(z_H)$ to be positive, it is sufficient to have:

$$\eta(\theta(z)) \frac{h(z) - c(z)}{c'(z)} \geq \frac{F(z)}{f(z)}.$$

From $m'(\theta(z))[h(z) - c(z)] = k$, we can write $\theta(z)$ as the following $\theta(z) = m'^{-1}\left(\frac{k}{h(z) - c(z)}\right)$. Replacing $\theta(\cdot)$ in the sufficient condition yields $\psi\left(\frac{k}{h(z) - c(z)}\right) \frac{h(z) - c(z)}{c'(z)} \geq \frac{F(z)}{f(z)}$ which is the same as the left hand side of the second part of assumption 5. This concludes the proof. \square

¹³For any differentiable functions F and G, if $G(z_H) = 1$, $G(z_L) = 0$, $F(z_H) = 1$ and $F(z_L) = 0$ we will have: $\int_z^{z_H} F(z)g(z)dz = \int_{z_L}^{z_H} F'(z)(1 - G(z))dz$ using integration by parts.

When $c'(\cdot) < 0$ and $h'(\cdot) - c'(\cdot) \geq 0$, a similar result can be obtained. We want to understand the condition in assumption ?? better. It's easy to show that if $1/q(\theta)$ is convex, $\eta(\theta)$ is increasing in θ (WHY?). Also, m' is decreasing in θ by assumption. Hence, $\psi(\cdot)$ is a decreasing function. The second part of assumption 5 states that for a given distribution, for a given z and a given value for $c'(z)$, $h(z) - c(z)$ should be sufficiently high or k should be sufficiently low. The intuition is that the surplus generated by type z should be sufficiently high (or the entry cost sufficiently low) to provide enough resources for the planner to implement the full information allocation. This assumption is exactly the counterpart of the third part of assumption 2 in the discrete type case.

8.6.4 Proof of proposition 6

Proof. By assumption, the complete information case is not achievable. Therefore, the guess-and-verify approach does not work. To solve for planner's problem, consider problem 6. We first ignore monotonicity constraint. Then we form the Lagrangian and derive first order condition (FOC). Then we verify that the monotonicity constraint (and consequently IC) is also satisfied. Denote the Lagrangian by \mathcal{L} and the Lagrangian multiplier by μ :

$$\begin{aligned} \mathcal{L} = & \int \left[m(\theta(z))(h(z) - c(z)) - k\theta(z) \right. \\ & \left. + \mu \left[m(\theta(z))(h(z) - c(z)) - k\theta(z) - m(\theta(z))c'(z)\left(\frac{F(z)}{f(z)}\right) - U(z_H) \right] \right] f(z) dz. \end{aligned} \quad (35)$$

To write the Lagrangian, we substituted the price from equation 30 to the BB constraint (see the derivation in the proof of proposition 9, equation 33). The FOC with respect to $\theta(z)$ for all z is given by:

$$m'(\theta(z))(h(z) - c(z)) - k + \mu(m'(\theta(z))(h(z) - c(z)) - k) - \mu m'(\theta(z))c'(z)\frac{F(z)}{f(z)} = 0. \quad (36)$$

It can be simplified to conform to equation 3 exactly.

According to assumptions of the proposition, $h - c$ and $h - c + c'\frac{F(z)}{f(z)}$ are decreasing in z . Also, μ is non-negative, so $h(z) - c(z) + \frac{\mu}{1+\mu}c'(z)\frac{F(z)}{f(z)}$ is also decreasing in z . Therefore, FOC implies that $\theta(z)$ is decreasing in z as well. As a result, the monotonicity constraint ($c'(z)\frac{d\theta(z)}{dz} \leq 0$) is satisfied. The complete information allocation is not achievable, so if the planner allocates all types the market tightness $\theta^{CI}(z)$ (and the corresponding transfers from equation 30), then the derived value for $U(z_H)$ becomes negative. (otherwise the previous proposition 9 implies that the complete information allocation can be achieved).

Assume there exists a $\mu > 0$ that the FOC and BB both hold. Since the objective function is strictly concave in $\theta(z)$ for all y , and the objective is just sum of some concave functions, the objective is also concave in $\theta(\cdot)$ ¹⁴. Because of concavity of the objective function, the FOC is sufficient for the solution. Hence, it only remains to show that such a $\mu > 0$ exists.

Note that $\theta(\cdot)$ obtained from equation 36 is continuous in μ . Accordingly, the LHS of the equation is continuous in μ as well. We need to show that the LHS is negative at $\mu = 0$ and is positive when $\mu \rightarrow \infty$.

If $\mu = 0$, then $\theta(z) = \theta^{CI}(z)$. Because complete information allocation is not achievable, $\int \left[m(\theta^{CI}(z)) \left[h(z) - c(z) - c'(z) \frac{F(y)}{f(y)} \right] - k\theta(z) \right] g(z) dz < 0$. (Otherwise, according to proposition 5, we can find a set of transfers such that the complete information allocation is achievable.)

If $\mu \rightarrow \infty$, then $m'(\theta(y)) \left[h(z) - c(z) - c'(z) \frac{F(z)}{f(z)} \right] = k$. Therefore,

$$\begin{aligned} & \int \left[m(\theta(z)) \left[h(z) - c(z) - c'(z) \frac{F(z)}{f(z)} \right] - k\theta(z) \right] g(z) dz \\ &= \int \left[m(\theta(z)) \frac{k}{m'(\theta(z))} - k\theta(z) \right] g(z) dz = k \int \frac{m(\theta(z)) - \theta m'(\theta(z))}{m'(\theta(z))} g(z) dz \\ &= -k \int \frac{\theta(z)^2 q'(\theta(z))}{m'(\theta(z))} g(z) dz > 0, \end{aligned}$$

where the first equality is derived from the FOC when $\mu \rightarrow \infty$ and the last inequality follows from $q' < 0$ and $m' > 0$. According to intermediate value theorem, there exists a strictly positive μ which satisfy equation ???. \square

Example 1. Model parameters:

$m(\theta) = 1 - e^{-\theta}$, $Z = [z_L, z_H] \subset R_{++}$, $c(z) = \beta z$, $h(z) = \beta z + s$ where $\beta > 0$. $f(\cdot)$ is log-concave (like uniform, truncated normal and many popular distributions), so $\frac{1-F}{f}$ is decreasing in y . Then BB is satisfied if $\frac{s-k(1-\ln(\frac{k}{s}))}{s-k} \frac{s}{\beta} > \frac{1}{f(z_L)}$. $p^{CI}(z) = \eta(m^{-1}(\frac{k}{s}))h(z) + (1 - \eta(m^{-1}(\frac{k}{s})))c(z) = \psi(\frac{k}{s})s + c(z)$.

Constrained efficient allocation:

¹⁴To show this point, consider a simpler version where the objective is a function of two variables, that is, $g(x_1, x_2) = f(x_1) + h(x_2)$. Also assume $f(\cdot)$ and $h(\cdot)$ are concave in x_1 and x_2 respectively. We want to show that g is concave in (x_1, x_2) . TO show that, we form the Hessian as follows:

$$\begin{bmatrix} f'' & 0 \\ 0 & h'' \end{bmatrix}$$

Since f'' and h'' are both negative, the determinant of Hessian is negative. Therefore g is concave.

- $\theta(z) = \theta^{CI}(z) = m^{-1}\left(\frac{k}{s}\right) = \ln\left(\frac{s}{k}\right)$ and $p(z) = c(z_L)$.

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