Efficiency in Search and Matching Models: A Generalized Hosios Condition*

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Abstract

When is entry efficient in markets with search and matching frictions? This paper generalizes the well-known Hosios condition to dynamic search environments where the expected match output depends on the market tightness. Entry is efficient when buyers' surplus share equals the *matching elasticity* plus the *surplus elasticity* (i.e. the elasticity of the expected match surplus with respect to buyers). This ensures agents are paid for their contribution to both *match creation* and *surplus creation*. In search-theoretic models of the labor market, for example, vacancy entry is efficient only when firms are compensated for the effect of job creation on both employment and labor productivity.

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1 Introduction

The well-known "Hosios condition" (Hosios, 1990) characterizes efficient entry in markets with search and matching frictions.[1] When there is entry of buyers, this condition states that entry is efficient only when buyers' share of the joint match surplus equals the elasticity of the matching function with respect to buyers.[2] The condition has proven to be widely applicable across a broad range of search-and-matching models. For example, in the labor market, the Hosios condition tells us when the level of vacancy creation and therefore the unemployment rate is efficient, e.g. see Rogerson, Shimer, and Wright (2005). In monetary environments with seller entry, such as Rocheteau and Wright (2005), the Hosios condition determines when the level of seller entry and therefore the number of trades is efficient.

While the Hosios condition applies to a broad range of search models, it does not always apply in settings where the *expected match output* – i.e. the expected value of the distribution of output across matches – is a function of the market tightness or buyer-seller ratio.[3] One example where this may occur is when sellers (or buyers) face a choice among heterogeneous buyers (or sellers) in many-on-one or multilateral meetings, e.g. the competing auctions environment in Albrecht, Gautier, and Vroman (2014). Another example is when there are sequential markets and agents in the first market face potential gains from trade in the second, e.g. the labor market and goods market in Berentsen, Menzio, and Wright (2011).

In this paper, we provide a generalization of the Hosios condition that characterizes efficient entry in a wide class of dynamic search-and-matching models where the expected match output depends on the market tightness. This simple, intuitive condition provides a *unifying lens* for understanding the efficiency of entry in a range of search-and-matching models which may appear quite different on the surface.

Consider an environment with buyer entry. An equilibrium allocation is efficient only when buyers are paid their marginal contribution to the social surplus. If the expected match output is exogenous, buyers need only be paid for their effect on *match creation*, i.e. on the total number of matches. In such environments, the

[1] This condition is sometimes called the "Hosios-Mortensen" condition. Early versions of it were discussed in Mortensen (1982b), Mortensen (1982a), and Pissarides (1986).

[2] By efficiency, we mean *constrained efficiency*, i.e. the social planner is constrained by the same frictions as the decentralized market economy.

[3] We use the term "match output" because our examples focus mainly on labor markets, but the term *output* can be interpreted more broadly.

standard Hosios condition applies. If the expected match output is endogenous, however, buyers must also be compensated for their effect on *surplus creation*, i.e. on the expected value of the joint surplus created by each match. We show that entry is efficient only when buyers' surplus share equals the *matching elasticity* plus the *surplus elasticity* (i.e. the elasticity of the expected match surplus with respect to buyers). We call this the "generalized Hosios condition".

When the standard Hosios condition obtains, markets internalize the search externalities that arise through the frictional matching process. However, when the expected match output is a function of the market tightness, a novel externality arises: the *output externality*. Depending on the environment, this externality may be either positive or negative. When the standard Hosios condition holds, the output externality is not internalized and there may be either over-entry or under-entry relative to the efficient level. For example, in labor markets featuring Nash bargaining, imposing the standard Hosios condition may lead to inefficiently high unemployment because firms are not compensated for their effect on labor productivity. Only when the generalized Hosios condition obtains are both the search externalities and the output externality fully internalized by a market economy.

In the examples we consider, entry is typically efficient when prices are determined by directed or competitive search – precisely because the generalized Hosios condition holds *endogenously*. Competitive search allows agents to trade off prices against both the probability of trade and the expected match surplus if trade occurs, thus internalizing both the search externalities and the output externality.[4] In markets where prices are determined by Nash bargaining, however, the generalized Hosios condition does not hold and entry is generically inefficient. Importantly, efficiency cannot simply be restored by *imposing* the generalized Hosios condition through a particular choice of matching technology and bargaining parameter.[5] This is because the expected match surplus is endogenous and we cannot assume the surplus elasticity is constant. In this way, the generalized Hosios condition highlights the importance of competitive search as a way of decentralizing the efficient allocation.

[4]Generally, competitive search is efficient because it enables agents to trade off prices against the probability of trade. See the survey on directed and competitive search by Wright, Kircher, Julien, and Guerrieri (2017). It is not always efficient, however, as shown in Guerrieri (2008).

[5]By contrast, the standard Hosios condition is often imposed when calibrating search models with Nash bargaining by using a Cobb-Douglas matching technology and setting buyers' bargaining parameter equal to the constant matching elasticity, e.g. Shimer (2005).

Overview. In Section 2, we first provide a simple example to build intuition and motivate our analysis. Next, we formulate the planner's problem and present two versions of our key result: the generalized Hosios condition. We first use the approach of Menzio and Shi (2011) to derive the condition in a discrete-time dynamic environment. While the proof is simpler using this approach – and the condition characterizes efficiency for the entire equilibrium path, not just the steady state – it requires the assumption that match output is *bounded*. Next, we use an alternative approach to derive the result in a continuous-time environment. This approach delivers a steady state version of the generalized Hosios condition that does not require the assumption that match output is bounded. We also provide a corollary that determines when there is over- or under-entry under the standard Hosios condition.

Section 3 presents two examples. Section 3.1 considers the efficiency of job creation in an environment that is inspired by the model of the goods market and labor market in Berentsen et al. (2011). In this setting, the labor market tightness affects the expected match "output" because it influences the expected *gains from trade* in the goods market by affecting the goods market tightness and thus the probability of trade. The output externality from vacancy entry is always positive. When wages are determined by Nash bargaining, job creation is generically inefficient. However, competitive search (wage posting) can decentralize the efficient allocation by ensuring that the generalized Hosios condition holds endogenously.[6]

Section 3.2 discusses two related environments in which meetings are many-on-one or multilateral. We first consider the efficiency of vacancy creation in the labor market model of Mangin (2017), in which firms directly compete to hire workers and wages are determined by auctions. Market tightness affects match output because greater competition to hire workers allows workers to be more *selective*, thereby increasing labor productivity. The output externality from vacancy entry is always positive. Next, we discuss the efficiency of seller entry in the competing auctions environment found in Albrecht et al. (2014). Market tightness affects match "output" because it increases the expected value of the highest valuation among buyers who approach a given seller. The output externality from seller entry is always negative because greater seller entry decreases the buyer-seller ratio. In both examples, since prices are

[6] Petrosky-Nadeau, Wasmer, and Weil (2018) also study efficiency in sequential labor and goods markets. Their environment differs in a key respect: the labor market tightness does not directly affect the expected match surplus in the labor market because buyers in the goods market are not the same agents as workers.

determined by competitive search (auctions), the generalized Hosios condition holds endogenously and the equilibrium level of entry is efficient.

The Appendix provides additional examples and extensions that we describe in Section 3.3. One example, in particular, shows that competitive search does *not* always decentralize the efficient allocation in environments where the generalized Hosios condition is necessary for constrained efficiency.

2 Generalized Hosios Condition

To motivate our question and build intuition, we start with a simple example where the expected match output depends on the market tightness. In Sections 2.1 and 2.2, we then derive two general versions of our main results for dynamic economies.

Consider a static environment where firms are ex ante identical and all workers are initially unemployed. All agents are risk-neutral. The measure of vacancies or firms is v, the measure of unemployed workers is u, and the labor market tightness is $\theta \equiv v/u$. Meetings are many-on-one or multilateral (i.e. many firms can meet one worker). Specifically, the probability that n firms meet a given worker is $P_n(\theta)$ and we assume the meeting technology is Poisson, i.e. $P_n(\theta) = \frac{\theta^n e^{-\theta}}{n!}$. The matching probabilities for workers and firms are therefore $m(\theta) = 1 - e^{-\theta}$ and $m(\theta)/\theta$ respectively.

After meetings occur, each worker draws an i.i.d. match-specific productivity x for every firm he meets and then chooses to work for exactly one of them. Match output $x \in X = \{x_L, x_H\} \subseteq \mathbb{R}_+$, where $x_L < x_H$ and the probability of low productivity is $\alpha \in [0, 1]$. The value of non-market activity is $z \geq 0$ and we assume $x_L > z$.

Suppose workers are hired by the firm they meet with the highest match-specific productivity.[7] A worker only produces output x_L if *all* n firms he meets draw x_L, so the probability $f(x_L; \theta)$ that a worker produces output x_L (conditional $n \geq 1$) is

$$f(x_L; \theta) = \frac{\sum_{n=1}^{\infty} P_n(\theta)\alpha^n}{1 - P_0(\theta)} \tag{1}$$

[7] This will be true, for example, if wages are determined by either Nash bargaining or auctions.

and $f(x_H;\theta) = 1 - f(x_L;\theta)$. Since $P_n(\theta) = \frac{\theta^n e^{-\theta}}{n!}$, we have[8]

$$f(x_L;\theta) = \frac{e^{-\theta(1-\alpha)} - e^{-\theta}}{1 - e^{-\theta}}. \tag{2}$$

The expected match output, or labor productivity, is

$$y(\theta) = x_L + f(x_H;\theta)(x_H - x_L). \tag{3}$$

While the distribution of match-specific productivities is exogenous, the distribution of productivities across *realized matches* – i.e. matches actually chosen by workers – is *endogenous* and depends on the market tightness θ. In the limit as $\theta \to 0$, we have $f(x_L;\theta) \to \alpha$. However, for any $\theta > 0$, the probability $f(x_H;\theta)$ that a worker produces output x_H is strictly increasing in θ, and the expected match output $y(\theta)$ is also strictly increasing θ, i.e. $y'(\theta) > 0$. Intuitively, this is because a higher number of vacancies per unemployed worker allows workers to be more *selective*. We call this the *selection channel*.

What is the efficient level of vacancy creation in this economy? Suppose the social planner can create vacancies at cost $c > 0$. We are interested in *constrained efficiency* in the sense that the planner is constrained by both the matching technology $m(.)$ *and* the output technology $y(.)$ – or, equivalently, the distribution $f(x;\theta)$. Taking both $m(.)$ and $y(.)$ as given, the planner is restricted to simply choose a market tightness θ to maximize the social surplus per worker, which is given by

$$\Omega(\theta) = m(\theta)y(\theta) + (1 - m(\theta))z - c\theta. \tag{4}$$

Letting $s(\theta)$ denote the *expected match surplus*, we have $s(\theta) = y(\theta) - z$ and

$$\Omega(\theta) = m(\theta)s(\theta) + z - c\theta. \tag{5}$$

Any solution θ^P satisfies the first-order condition

$$m'(\theta)s(\theta) + m(\theta)s'(\theta) \leq c \tag{6}$$

and $\theta^P \geq 0$, with complementary slackness. It can be shown that there exists a unique

[8]This follows from the fact that $e^{-\theta}\sum_{n=1}^{\infty}\frac{(\alpha\theta)^n}{n!} = e^{-\theta}(e^{\theta\alpha} - 1) = e^{-\theta(1-\alpha)} - e^{-\theta}$.

solution $\theta^P > 0$ to (6) that maximizes $\Omega(\theta)$ provided that $c < x_L + (1-\alpha)(x_H - x_L) - z$.

Let $\eta_m(\theta) \equiv m'(\theta)\theta/m(\theta)$, the *matching elasticity*, and let $\eta_s(\theta) \equiv s'(\theta)\theta/s(\theta)$, the *surplus elasticity*. Multiplying (6) by θ and dividing by $m(\theta)s(\theta)$, the optimal θ^P satisfies

(7) $$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{firms' surplus share}}.$$

With free entry of vacancies, the total expected payoff for firms equals cv, and the total surplus created by all matches is $m(\theta)s(\theta)u$, therefore the term on the right of (7) is *firms' surplus share*.

We call this the "generalized Hosios condition". Like the original Hosios (1990) condition, it turns out that this simple condition characterizes efficiency across a wide range of search-and-matching environments. In general, efficiency requires that buyers' surplus share equals the matching elasticity plus the surplus elasticity because this ensures that agents are paid for their contribution to both *match creation* and *surplus creation*. In this example, efficiency requires that firms' entry decisions internalize the effects of vacancy creation on both employment and labor productivity. In the special case where $y'(\theta) = 0$, we recover the standard Hosios condition, which sets firms' surplus share equal to the matching elasticity.

2.1 Dynamic economy (bounded output)

We now present two versions of the generalized Hosios condition for dynamic economies. First, we use the approach of Menzio and Shi (2011) to derive the condition in a discrete-time dynamic environment where match output is bounded. This approach delivers a condition that characterizes efficiency across the entire equilibrium path, not just the steady state. In the next section, we use an alternative approach to derive a version of the generalized Hosios condition in a continuous-time dynamic environment. This approach delivers a steady state version of the generalized Hosios condition that does not require the assumption that match output is bounded.

Consider a discrete-time dynamic environment. In any period $t \in \{0, 1, ...\}$, there is measure one of sellers and a large number of potential buyers. All agents are risk-neutral. The measure of buyers who enter is denoted by v and the measure of unmatched sellers is denoted by u. The market tightness is $\theta \equiv v/u$. Buyer-seller matches are destroyed at an exogenous rate $\delta \in (0, 1]$.[9] The flow payoff for unmatched sellers is $z \geq 0$.

Buyers and sellers are matched according to a constant-returns-to-scale matching function. The matching probabilities for sellers and buyers are denoted respectively by $m(\theta)$ and $m(\theta)/\theta$. We call the function $m(.)$ the *matching technology*.

Assumption 1. *The function $m(.)$ satisfies: (i) $m'(\theta) > 0$ and $m''(\theta) < 0$ for all $\theta \in \mathbb{R}_+$, (ii) $\lim_{\theta \to 0} m(\theta) = 0$, (iii) $\lim_{\theta \to 0} m'(\theta) = 1$, (iv) $\lim_{\theta \to \infty} m(\theta) = 1$, (v) $\lim_{\theta \to \infty} m'(\theta) = 0$, and (vi) $m(\theta)/\theta$ is strictly decreasing for all $\theta \in \mathbb{R}_+$.*

Within each period, the timing is as follows. First, match destruction occurs. Next, buyers enter and the search and matching process takes place. Finally, production occurs.

Match output $x \in X \equiv \{x_1, x_2, ..., x_N\} \subseteq \mathbb{R}_+$ where $x_1 < x_2 < ... < x_N$. We assume that all matches have positive surplus.[10] While the distribution used here is discrete, the results also apply to continuous distributions provided that X is *bounded*, i.e. $X = [x_{\min}, x_{\max}]$ where $x_{\min}, x_{\max} \in \mathbb{R}_+$ and $x_{\min} > z$.

Given market tightness θ, the output of a *new match* is an i.i.d draw from a discrete probability distribution $f : X \to [0, 1]$ where $\sum_{x \in X} f(x; \theta) = 1$. The distribution $f(x; \theta)$ will be determined endogenously by features of the environment. The expected output of new matches is

$$y(\theta) \equiv \sum_{x \in X} x f(x; \theta). \tag{8}$$

If the distribution $f(x; \theta)$ does not depend on the market tightness, i.e. $f(x; \theta) = f(x)$, then $y(\theta) = \bar{y} \in \mathbb{R}_+$ for all $\theta \in \mathbb{R}_+$. In general, however, the expected match output $y(\theta)$ may depend directly on the market tightness. We call the function $y(.)$ the *output technology*.

Let $g : X \to [0, 1]$ be the probability distribution of match output x across *all* sellers in the economy. This includes newly matched sellers, unmatched sellers, and sellers matched in previous periods whose matches have survived. Here, $g(x)$ denotes the measure of sellers producing output x, where $x = 0$ for unmatched sellers, so

[9] In the special case $\delta = 1$, the economy features non-enduring matches, i.e. all matches are destroyed at the end of each period. All of the following results therefore apply to economies with non-enduring matches.

[10] In Appendix C, we provide an example of an environment where not all matches are accepted.

$\sum_{x \in X} g(x) = 1 - u$. The average match output across all sellers is $\sum_{x \in X} xg(x)$ and the average output of *matched* sellers is

$$y = \frac{\sum_{x \in X} xg(x)}{1 - u}. \tag{9}$$

The distribution g is given by the following law of motion:

$$\hat{g}(x) = um(\theta)f(x;\theta) + (1 - \delta)g(x) \quad \text{for all } x \in X, \tag{10}$$

and the measure of unmatched sellers u is given by the standard law of motion:

$$\hat{u} = u(1 - m(\theta)) + \delta(1 - u). \tag{11}$$

Here, $\hat{g}(x)$ is the distribution of match output across sellers, and $\hat{u}$ is the measure of unemployed workers, at the production stage, i.e. the beginning of the next period.

2.1.1 Planner's problem

Suppose the planner can create vacancies at cost $c > 0$ each period. At the start of a period, the planner observes the aggregate state of the economy, $\psi = (u, g)$, and chooses a market tightness $\theta = v/u$ where $\theta \in \mathbb{R}_+$. The planner is restricted to take both the matching technology $m(.)$ and the distribution $f(x;\theta)$ as given, and chooses the market tightness θ to maximize the sum of present and future social surplus. The discount factor is $\beta \in (0, 1)$.

Letting $\hat{\psi} = (\hat{u}, \hat{g})$ denote the next period's state, the Bellman equation for the planner's value function $W(\psi)$ can be written as:

$$W(\psi) = \max_{\theta \in \mathbb{R}_+} \left\{ F(\theta|\psi) + \beta W(\hat{\psi}) \right\} \tag{12}$$

where

$$F(\theta|\psi) = -c\theta u + z\hat{u} + \sum_{x \in X} x\hat{g}(x) \tag{13}$$

subject to the following laws of motion:

$$\hat{u} = u(1 - m(\theta)) + \delta(1 - u) \tag{14}$$

and

$$\hat{g}(x) = um(\theta)f(x;\theta) + (1-\delta)g(x) \quad \text{for all } x \in X. \tag{15}$$

Our planner's problem is a generalization of that considered in Menzio and Shi (2011) to environments where the distribution of output across new matches, given by $f(x;\theta)$, is *endogenous* and may depend directly on the market tightness in that period. If match output is bounded, i.e. X is bounded, Theorem 1 in Menzio and Shi (2011) generalizes to our environment.[11]

Lemma 1. *Assume that X is bounded. (i) The planner's value function $W(\psi)$ is the unique solution to (12). (ii) $W(\psi)$ is linear in u and g:*

$$W(\psi) = W_u u + \sum_{x \in X} W_e(x) g(x), \tag{16}$$

where W_u and $W_e(x)$ are the component value functions given by

$$W_u = \max_{\theta \in \mathbb{R}_+} \{-c\theta + (1-m(\theta))(z + \beta W_u) + m(\theta)(y(\theta) + \beta W_e(\theta))\} \tag{17}$$

and

$$W_e(x) = \delta(z + \beta W_u) + (1-\delta)(x + \beta W_e(x)) \tag{18}$$

where

$$W_e(\theta) \equiv \sum_{x \in X} W_e(x) f(x;\theta). \tag{19}$$

Proof. *Part (i).* Letting $C(\Psi)$ be the set of bounded, continuous functions $R : \Psi \to \mathbb{R}$ with the sup norm $\|\mathbb{R}\| = \sup_{\psi \in \Psi} R(\psi)$, we can define an operator T by

$$(TR)(\psi) = \max_{\theta \in \mathbb{R}_+} \; F(\theta|\psi) + \beta R(\hat{\psi}) \tag{20}$$

subject to (14) and (15), where the return function F is defined by (13).

[11] More precisely, the special case of our planner's problem where $f(x;\theta) = f(x)$ is a special case of that considered in Menzio and Shi (2011). This is because the authors incorporate additional features – such as on-the-job search, aggregate productivity shocks, and signals regarding match-specific productivity. We abstract from these features to focus attention on what is novel here.

We first prove that TR is bounded. Consider any function $R \in C(\Psi)$. Since R is bounded, there exist R_0 and $\bar{R}$ such that $R_0 \leq R(\hat{\psi}) \leq \bar{R}$ for all $\hat{\psi} \in \Psi$. Therefore, using (20) and (13), $(TR)(\psi)$ is bounded below by $\min\{z, x_1\} + \beta R_0$ and bounded above by $\max\{z, x_N\} + \beta \bar{R}$.

Next, we prove that TR is continuous in ψ. To do this, observe that since X is bounded we can replace the constraint $\theta \in \mathbb{R}_+$ with the constraint $\theta \in [0, \bar{\theta}]$, where $\bar{\theta}$ is defined as:

$$\bar{\theta} = c^{-1}u^{-1}\{[\max\{z, x_N\} - \min\{z, x_1\}] + \beta[\bar{R} - R_0]\}. \tag{21}$$

For the modified problem, the maximand is continuous in (ψ, θ) and the set of feasible choices for θ is compact, so it follows from the Theorem of the Maximum that TR is continuous in ψ (Theorem 3.6 in Stokey, Lucas, and Prescott, 1989).

Therefore, $T : C(\Psi) \to C(\Psi)$, i.e. the operator T maps the set of bounded, continuous functions into itself. It is straightforward to verify that T satisfies Blackwell's sufficient conditions for a contraction (Theorem 3.3 in Stokey et al., 1989). Therefore, T is a contraction mapping and it has exactly one fixed point $R^* \in C(\Psi)$. Since $\lim_{t\to\infty} \beta^t R^*(\psi) = 0$ for all $\psi \in \Psi$, R^* is equal to the planner's function W (Theorem 4.3 in Stokey et al., 1989).

Part (ii). Let $C'(\Psi) \subseteq C(\Psi)$ be the set of bounded, continuous functions $R : \Psi \to \mathbb{R}$ that are *linear* in the measure of unmatched sellers, u, and the measure $g(x)$ of sellers producing output x. We have $R \in C'(\Psi)$ if and only if there exist R_u and $R_e : X \to \mathbb{R}$ such that

$$R(\psi) = R_u u + \sum_{x \in X} R_e(x) g(x). \tag{22}$$

Consider any function $R \in C'(\Psi)$. Substituting (14) and (15) into (13), and then substituting into the maximand in (20) and simplifying, we obtain

$$(TR)(\psi) = \hat{R}_u u + \sum_{x \in X} \hat{R}_e(x) g(x) \tag{23}$$

where $\hat{R}_u$ is given by

$$\hat{R}_u = \max_{\theta \in \mathbb{R}_+} \left\{ -c\theta + (1 - m(\theta))(z + \beta \hat{R}_u) + m(\theta) \left(y(\theta) + \beta \sum_{x \in X} \hat{R}_e(x) f(x; \theta) \right) \right\}, \tag{24}$$

and $y(\theta)$ is given by definition (8), and $\hat{R}_e(x)$ is given by

$$\hat{R}_e(x) = \delta(z + \beta\hat{R}_u) + (1-\delta)(x + \beta\hat{R}_e(x)). \tag{25}$$

Therefore, we have $T : C'(\Psi) \to C'(\Psi)$ and, since $C'(\Psi)$ is a closed subset of $C(\Psi)$, we have $W \in C'(\Psi)$ by Corollary 1 to Theorem 3.2 in Stokey et al. (1989). ■

We now use Lemma 1 to prove our main result. Proposition 1 says that the planner chooses to set buyers' surplus share equal the *matching elasticity* plus the *surplus elasticity*. When this condition holds, the level of buyer entry is efficient because agents are compensated for their effect on both *match creation* and *surplus creation.* Importantly, this intuitive condition characterizes efficiency along the entire equilibrium path, not just in steady state. Moreover, in the absence of aggregate productivity shocks, the optimal market tightness is constant, i.e. $\theta_t = \theta$ for all t.

Assumption 2. *The function $\Lambda(.)$ defined by $\Lambda(\theta) \equiv m(\theta)s(\theta)$ satisfies: (i) $\lim_{\theta\to 0}\Lambda(\theta) = 0$; (ii) $\lim_{\theta\to 0}\Lambda'(\theta) > c$; (iii) $\lim_{\theta\to\infty}\Lambda'(\theta) < c$; and (iv) $\Lambda''(\theta) < 0$ for all $\theta \in \mathbb{R}_+$.*

Assumption 2 is sufficient for the existence and uniqueness of the efficient choice θ^P. If Assumption 2 does not hold, (26) is still a necessary condition for efficiency.

Proposition 1 (Generalized Hosios Condition). *Assume X is bounded. There exists a unique efficient allocation $(\theta_t^P)_{t=0}^{\infty}$ where $\theta_t^P = \theta^P > 0$ for all t and θ^P satisfies*

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{buyers' surplus share}}. \tag{26}$$

An equilibrium allocation $(\theta_t^*)_{t=0}^{\infty}$ is efficient if and only if θ_t^* satisfies (26) for all t.

Proof. Rearranging (17), the socially optimal market tightness θ^P is given by

$$\theta^P = \arg\max_{\theta\in\mathbb{R}_+} \left\{-c\theta + z + \beta W_u + m(\theta)(y(\theta) - z + \beta(W_e(\theta) - W_u))\right\}. \tag{27}$$

We can write the expected match surplus as

$$s(\theta) = y(\theta) - z + \beta(W_e(\theta) - W_u), \tag{28}$$

and thus (27) can be rewritten as

$$\theta^P = \arg\max_{\theta \in \mathbb{R}_+} \{-c\theta + m(\theta)s(\theta) + z + \beta W_u\}. \tag{29}$$

Taking the first-order condition for (29), the optimal market tightness θ^P satisfies

$$m'(\theta)s(\theta) + m(\theta)s'(\theta) \leq c \tag{30}$$

and $\theta^P \geq 0$ with complementary slackness. If Assumption 2 holds, there exists a unique solution $\theta^P > 0$ that satisfies $\Lambda'(\theta) = c$ where $\Lambda(\theta) \equiv m(\theta)s(\theta)$. Therefore, there exists a unique $\theta^P > 0$ that satisfies (30) with equality. Multiplying both sides of (30) by $\theta/m(\theta)s(\theta)$, we obtain (26). Since θ^P is unique, an equilibrium allocation is efficient if and only if the equilibrium θ_t^* satisfies condition (26) for all t. ■

The following provides a useful version of the generalized Hosios condition that is easier to apply in practice. It tells us that the derivative $y'(\theta)$ of the expected match output is key. If $y'(\theta) = 0$, we recover the standard Hosios condition.

Proposition 2. *Assume that X is bounded. There exists a unique efficient allocation $(\theta_t^P)_{t=0}^{\infty}$ where $\theta_t^P = \theta^P > 0$ for all t and θ^P satisfies*

$$\eta_m(\theta) + \frac{y'(\theta)\theta}{(1-\beta(1-\delta))s(\theta)} = \frac{c\theta}{m(\theta)s(\theta)}. \tag{31}$$

An equilibrium allocation $(\theta_t^*)_{t=0}^{\infty}$ is efficient if and only if θ_t^* satisfies (31) for all t.

Proof. From (28), we see that $s(\theta)$ depends on the market tightness θ only through $y(\theta)$ and $W_e(\theta)$. The component value function W_u does not depend on θ since it is a maximized value given by (17). Differentiating (28), we have

$$s'(\theta) \equiv y'(\theta) + \beta W_e'(\theta). \tag{32}$$

Using (18), we obtain

$$W_e(x) = \frac{\delta(z + \beta W_u) + (1-\delta)x}{1-\beta(1-\delta)} \tag{33}$$

and, using the fact that $W_e(\theta) \equiv \sum_{x \in X} W_e(x) f(x;\theta)$, we obtain

$$W_e(\theta) = \frac{\delta(z + \beta W_u) + (1-\delta) y(\theta)}{1 - \beta(1-\delta)}. \tag{34}$$

Differentiating (34) yields

$$W_e'(\theta) = \frac{(1-\delta) y'(\theta)}{1 - \beta(1-\delta)}. \tag{35}$$

Substituting (35) into (32) and simplifying, we obtain

$$s'(\theta) \equiv \frac{y'(\theta)}{1 - \beta(1-\delta)}. \tag{36}$$

Finally, substituting (36) into (26) yields condition (31). ■

While this approach delivers a powerful version of the generalized Hosios condition, one limitation is that Lemma 1 requires the assumption that match output is *bounded.* If match output is not bounded – for example, if the distribution $f(x;\theta)$ is continuous and has unbounded upper support, $X = [x_{\min}, \infty)$ – then Lemma 1 does not apply, and therefore Propositions 1 and 2 do not apply. Since many important applications feature unbounded upper support, we provide an alternative approach in the next section that does not require bounded match output.

2.2 Dynamic economy (unbounded output)

We now use an alternative approach to derive a version of the generalized Hosios condition in a continuous-time dynamic environment where match output is not necessarily bounded. This approach delivers steady state versions of Propositions 1 and 2 that apply to *any* distribution of match output $f(x;\theta)$ with finite mean.

Consider a continuous-time dynamic environment. At any time t, there is measure one of sellers and a large number of potential buyers. As in Section 2.1, all agents are risk neutral and the measure of buyers who enter is denoted by v_t, the measure of unmatched sellers is denoted by u_t, and market tightness is $\theta_t \equiv v_t/u_t$. Matches are destroyed at a rate $\delta \in (0,1]$ and the flow payoff for unmatched sellers is $z \geq 0$.

In continuous time, $m(\theta_t)$ and $m(\theta_t)/\theta_t$ are now arrival rates rather than matching probabilities for buyers and sellers respectively, so Assumption 1 needs to be amended.

Assumption 1a. *The function* $m(.)$ *satisfies: (i)* $m'(\theta) > 0$ *and* $m''(\theta) < 0$ *for all* $\theta \in \mathbb{R}_+$*, (ii)* $\lim_{\theta \to 0} m(\theta) = 0$*, (iii)* $\lim_{\theta \to 0} m'(\theta) = +\infty$*, (iv)* $\lim_{\theta \to +\infty} m(\theta) = +\infty$*, (v)* $\lim_{\theta \to \infty} m'(\theta) = 0$*, and (vi)* $m(\theta)/\theta$ *is strictly decreasing in* θ *for all* $\theta \in \mathbb{R}_+$*.*

Match output $x \in X = [x_{\min}, x_{\max}] \subseteq \mathbb{R}_+$ where $x_{\max} \in \mathbb{R}_+ \cup \{\infty\}$. We assume that all matches have positive surplus. At time t, the output of a *new match* is an i.i.d. draw from a probability distribution with pdf $f(x; \theta_t)$ and a finite mean. We define $y(\theta_t) \equiv \int_{x_{\min}}^{x_{\max}} x f(x; \theta_t) dx$, the expected output for *new matches* at time t. We assume a continuous distribution but the results easily extend to discrete distributions.

Let y_t denote the average match output across all matched sellers at time t. Note that y_t is not equal to $y(\theta_t)$, since y_t is a weighted average across *all* active matches, i.e. both new matches and existing matches that have survived from earlier times. In Appendix A, we derive the following law of motion for y_t:

$$\dot{y}_t = (y(\theta_t) - y_t) \frac{m(\theta_t) u_t}{1 - u_t} \tag{37}$$

and the law of motion for the measure of unmatched sellers is standard:

$$\dot{u}_t = \delta(1 - u_t) - m(\theta_t) u_t. \tag{38}$$

Before solving the planner's problem, we first derive an expression for the steady state expected match surplus, $s(\theta) \equiv V_B + V_S - U_B - U_S$, where V_S and V_B denote the steady state asset values for matched sellers and buyers respectively, and U_S and U_B denote the steady state asset values for unmatched sellers and buyers respectively.

Lemma 2. *With free entry of buyers, the steady state expected match surplus is*

$$s(\theta) = \frac{y(\theta) - z + c\theta}{r + \delta + m(\theta)}. \tag{39}$$

Proof. See Appendix A.

2.2.1 Planner's problem

Suppose the planner can create vacancies at cost $c > 0$. At any time t, the planner observes the aggregate state of the economy, $\psi_t = (u_t, y_t)$, and chooses a market tightness ratio $\theta = v/u$ where $\theta \in \mathbb{R}_+$. The planner is restricted to take both the matching technology $m(.)$ and the output technology $y(.)$ as given, and chooses the market tightness θ to maximize the sum of present and future social surplus. The discount rate is $r > 0$.

Given initial conditions u_0 and y_0, the planner chooses θ_t for all $t \in \mathbb{R}_+$ to maximize

$$\Omega = \int_0^\infty e^{-rt}((1-u_t)y_t + zu_t - c\theta_t u_t)dt \tag{40}$$

subject to the following laws of motion:

$$\dot{u}_t = \delta(1-u_t) - m(\theta_t)u_t \tag{41}$$

and

$$\dot{y}_t = (y(\theta_t) - y_t)\frac{m(\theta_t)u_t}{1-u_t}. \tag{42}$$

Propositions 3 and 4 are steady state versions of the generalized Hosios condition presented in Propositions 1 and 2 that do not require bounded match output.

If Assumption 2 does not hold, (43) is still a necessary condition for efficiency.

Proposition 3 (**Generalized Hosios Condition**). *There exists a unique steady state efficient allocation $\theta^P > 0$ and it satisfies*

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{buyers' surplus share}}. \tag{43}$$

A steady state equilibrium allocation θ^* is efficient if and only if it satisfies (43).

Proof. The current value Hamiltonian for the planner's problem is

$$H = ((1-u_t)y_t + zu_t - c\theta_t u_t) + \lambda_t(\delta(1-u_t) - m(\theta_t)u_t) + \mu_t\left(\frac{m(\theta_t)u_t\,(y(\theta_t)-y_t)}{1-u_t}\right). \tag{44}$$

The first-order necessary conditions are

$$\frac{\partial H}{\partial \theta_t} = -cu_t - \lambda_t m'(\theta_t)u_t + \mu_t\left(\frac{m'(\theta_t)u_t\,(y(\theta_t)-y_t) + m(\theta_t)u_t y'(\theta_t)}{1-u_t}\right) = 0 \tag{45}$$

$$\text{(46)} \quad \frac{dH}{du_t} = -(y_t - z + c\theta_t) - \lambda_t(\delta + m(\theta_t)) + \mu_t \left(\frac{m(\theta_t)\,(y(\theta_t) - y_t)}{(1-u_t)^2} \right) = -\dot{\lambda}_t + r\lambda_t$$

$$\text{(47)} \qquad \frac{\partial H}{\partial y_t} = 1 - u_t - \mu_t \left(\frac{m(\theta_t)u_t}{1-u_t} \right) = -\dot{\mu}_t + r\mu_t$$

$$\text{(48)} \qquad \frac{\partial H}{\partial \lambda_t} = \delta(1-u_t) - m(\theta_t)u_t = \dot{u}_t$$

$$\text{(49)} \qquad \frac{\partial H}{\partial \mu_t} = \frac{m(\theta_t)u_t\,(y(\theta_t) - y_t)}{1-u_t} = \dot{y}_t.$$

The transversality conditions are $\lim_{t\to\infty} e^{-rt}\lambda_t u_t = 0$ and $\lim_{t\to\infty} e^{-rt}\mu_t y_t = 0$.

In steady state, we have $\dot{u}_t = \dot{y}_t = \dot{\theta}_t = 0$ and therefore $y(\theta_t) = y_t = y(\theta)$. Also, $\dot{\mu}_t = 0$ and $\dot{\lambda}_t = 0$. Substituting into the first-order conditions, we obtain

$$\text{(50)} \qquad -\lambda m'(\theta)u + \mu\delta y'(\theta) = cu,$$

$$\text{(51)} \qquad \lambda = -\left(\frac{y(\theta) - z + c\theta}{r + \delta + m(\theta)} \right),$$

$$\text{(52)} \qquad \mu = \frac{1-u}{r+\delta}.$$

Substituting λ and μ into (50), then using $\delta(1-u) = m(\theta)u$ and (39), yields

$$\text{(53)} \qquad \eta_m(\theta) + \frac{y'(\theta)\theta}{(r+\delta)s(\theta)} = \frac{c\theta}{m(\theta)s(\theta)}$$

where $\eta_m(\theta) \equiv m'(\theta)\theta/m(\theta)$. Again using expression (39) for $s(\theta)$, we obtain

$$\text{(54)} \qquad \eta_m(\theta) + \left(\frac{(y'(\theta) + c)\theta}{y(\theta) - z + c\theta} - \frac{m'(\theta)\theta}{r + \delta + m(\theta)} \right) = \frac{c\theta}{m(\theta)s(\theta)}.$$

Using Lemma 2, we can write $\eta_s(\theta) \equiv s'(\theta)\theta/s(\theta)$ as the elasticity of the numerator minus the elasticity of the denominator:

$$\eta_s(\theta) = \frac{(y'(\theta)+c)\theta}{y(\theta)-z+c\theta} - \frac{m'(\theta)\theta}{r+\delta+m(\theta)}. \tag{55}$$

Substituting (55) into (54), (53) is equivalent to (43). Both are necessary conditions for any steady state solutions $\theta^P > 0$. Assumption 2 implies there exists a unique $\theta^P > 0$ that satisfies $\Lambda'(\theta) = c$ where $\Lambda(\theta) \equiv m(\theta)s(\theta)$, and therefore there exists a unique $\theta^P > 0$ that satisfies (43).

We can apply Arrow's Sufficiency Theorem to prove that θ^P is indeed a global maximum.[12] To show this, we formulate the current value Hamiltonian in terms of the state variable $\tilde{x}_t \equiv (1-u_t)y_t$. Using (41) and $\dot{x}_t = -\delta\tilde{x}_t + m(\theta_t)u_t y(\theta_t)$ (see Appendix A), the current value Hamiltonian as a function of state and control variables is

$$H(\tilde{x}, u, \theta) = \tilde{x} + zu - c\theta u + \lambda_1(\delta(1-u) - m(\theta)u) + \mu_1(-\delta\tilde{x} + m(\theta)uy(\theta)) \tag{56}$$

and the maximized Hamiltonian is $M_H(\tilde{x}, u) \equiv \max_{\theta\in\mathbb{R}_+} H(\tilde{x}, u, \theta)$. Applying Arrow's Sufficiency Theorem, the solution θ^P to (43) is a global maximum provided that $M_H(\tilde{x}, u)$ is jointly weakly concave in u and $\tilde{x}$.

To find $\theta^*(\tilde{x}, u) \equiv \arg\max_{\theta\in\mathbb{R}_+} H(\tilde{x}, u, \theta)$, we set

$$\frac{\partial H}{\partial \theta} = -cu - \lambda_1 m'(\theta)u + \mu_1 u(m'(\theta)y(\theta) + m(\theta)y'(\theta)) = 0 \tag{57}$$

and, differentiating (57), we have

$$\frac{\partial^2 H}{\partial \theta^2} = -\lambda_1 m''(\theta)u + \mu_1 u(m''(\theta)y(\theta) + 2m'(\theta)y'(\theta) + m(\theta)y''(\theta)) < 0, \tag{58}$$

provided that $m''(\theta)y(\theta)+2m'(\theta)y'(\theta)+m(\theta)y''(\theta) < 0$ and $m''(\theta) < 0$ since $\lambda_1 < 0$ and $\mu_1 > 0$. Assumption 1a states that $m''(\theta) < 0$ for all $\theta \in \mathbb{R}_+$ and Assumption 2 says that $\Lambda''(\theta) < 0$ for all $\theta \in \mathbb{R}_+$ where $\Lambda(\theta) \equiv m(\theta)s(\theta)$. In particular, in the special case where $s(\theta) = y(\theta)$, Assumption 2 implies $m''(\theta)y(\theta) + 2m'(\theta)y'(\theta) + m(\theta)y''(\theta) < 0$. Therefore, $\theta^*(\tilde{x}, u)$ is indeed a maximum and we have

$$M_H(\tilde{x}, u) = \tilde{x} + zu - c\theta^* u + \lambda_1(\delta(1-u) - m(\theta^*)u) + \mu_1(-\delta\tilde{x} + m(\theta^*)uy(\theta^*)) \tag{59}$$

[12] Arrow's Sufficiency Theorem generalizes Mangasarian's sufficiency conditions. See Kamien and Schwartz (1991), p. 221-222.

where $\theta^* \equiv \theta^*(\tilde{x}, u)$. Since u cancels out in (57) and $\tilde{x}$ does not appear in (57), θ^* does not depend directly on u or $\tilde{x}$. Also, it can be verified that neither λ_1 nor μ_1 depends on either u or $\tilde{x}$.[13] The function $M_H(\tilde{x}, u)$ is linear in both $\tilde{x}$ and u and it is therefore weakly concave, thus the solution θ^P to (43) is a global maximum. ■

Again, we obtain a useful version of the generalized Hosios condition.

Proposition 4. *There exists a unique steady state efficient allocation $\theta^P > 0$ and it satisfies*

$$\eta_m(\theta) + \frac{y'(\theta)\theta}{(r+\delta)s(\theta)} = \frac{c\theta}{m(\theta)s(\theta)}. \tag{60}$$

A steady state equilibrium allocation θ^* is efficient if and only if it satisfies (60).

Proof. Condition (60) is derived as an intermediate step in the proof of Proposition 3 as condition (53), which is shown to be equivalent to (43). ■

2.3 Discussion

In search-and-matching models with free entry of buyers, there are two standard externalities related to the frictional matching process: the congestion and thick market externalities.[14] Both of these search externalities are fully internalized by markets when the Hosios condition holds. In environments where the expected match output depends on market tightness, however, a novel externality arises. Depending on the specific environment, a higher buyer/seller ratio may either increase or decrease the expected match output and this effect may not be internalized by the market. We call this the *output externality*.

When the standard Hosios condition holds, buyers' entry decisions fail to internalize the output externality and entry may not be efficient. Applying the standard Hosios condition may therefore result in either over-entry or under-entry of buyers

[13] Note that the co-state variables λ_1 and μ_1 for the current value Hamiltonian with state variables u_t and $\tilde{x}_t$ are different to the co-state variables λ and μ for the current value Hamiltonian with state variables u_t and p_t.

[14] The congestion externality is a negative externality that arises because a higher buyer/seller ratio reduces the matching probability of each buyer. The thick market externality is a positive externality that arises because a higher buyer/seller ratio increases the matching probability for each seller.

relative to the efficient level. Corollary 1 tells us that the direction of the inefficiency depends only on the derivative of the output technology $y(.)$ at the equilibrium θ^*.

Corollary 1. *A steady state equilibrium allocation features* under-entry *(*over-entry*) of buyers under the standard Hosios condition if and only if* $y'(\theta^*) > (<) \ 0$.

Proof. Suppose the standard Hosios condition holds, i.e. $\eta_m(\theta^*) = c\theta^*/m(\theta^*)s(\theta^*)$. First, we show there is under-entry (over-entry) of buyers if and only if $s'(\theta^*) > (<) 0$. Second, we show that $s'(\theta^*) > 0$ if and only if $y'(\theta^*) > 0$. Letting $\Lambda(\theta) = m(\theta)s(\theta)$, Proposition 3 says there exists a unique efficient $\theta^P > 0$ that satisfies $\Lambda'(\theta^P) = c$. Since the standard Hosios condition holds, $m'(\theta^*)s(\theta^*) = c$ and therefore we have $\Lambda'(\theta^P) = m'(\theta^*)s(\theta^*)$. Now, $\Lambda'(\theta^*) = m'(\theta^*)s(\theta^*) + m(\theta^*)s'(\theta^*)$, and thus $\Lambda'(\theta^P) = \Lambda'(\theta^*) - m(\theta^*)s'(\theta^*)$, so if $s'(\theta^*) > 0$ then $\Lambda'(\theta^P) < \Lambda'(\theta^*)$. If Assumption 2 holds then $\Lambda''(\theta) < 0$ for all $\theta \in \mathbb{R}_+$ and therefore $\Lambda'(\theta^P) < \Lambda'(\theta^*)$ implies that $\theta^* < \theta^P$, i.e. there is *under-entry* of buyers. Similarly, if $s'(\theta^*) < 0$, there is *over-entry* of buyers, $\theta^* > \theta^P$. Using expression (55) for $\eta_s(\theta)$, and rearranging using (39), we have $s'(\theta^*) > 0$ if and only if $y'(\theta^*) > m'(\theta^*)s(\theta^*) - c$. Finally, since $m'(\theta^*)s(\theta^*) = c$ by assumption, we have $s'(\theta^*) > 0$ if and only if $y'(\theta^*) > 0$. ■

When $y'(\theta^*) > 0$, the output externality arising from buyer entry is *positive* and the standard Hosios condition results in under-entry. Alternatively, if $y'(\theta^*) < 0$, the output externality is *negative* and the standard Hosios condition results in over-entry of buyers. If $y'(\theta^*) = 0$, there is no output externality and buyer entry is efficient under the standard Hosios condition.

Returning to our motivating example, the standard Hosios condition would result in under-entry of vacancies, or inefficiently high unemployment, since $y'(\theta) > 0$ and the output externality is positive. Intuitively, this is because it does not incorporate the fact that higher job creation leads not only to lower unemployment for workers, but also higher labor productivity. In an alternative environment where workers instead apply to firms, we would have $y'(\theta) < 0$. In this case, the output externality is negative and the standard Hosios rule would result in over-entry of vacancies, or inefficiently low unemployment.[15]

[15] When workers apply to firms, a greater number of vacancies per unemployed worker implies *fewer* applicants per vacancy, which decreases the expected match output because firms can be less selective. For example, see the model of worker applications in Gavrel (2012), or the model of worker selection using interviews in Wolthoff (2017).

Seller entry. When there is *seller entry* instead of buyer entry, the direction of the effect of entry is reversed. Since the buyers' surplus share and the sellers' surplus share add to one, an efficient $\theta^P > 0$ must satisfy

$$(61) \qquad 1 - \underbrace{\eta_m(\theta)}_{\text{matching elasticity}} - \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{\pi(\theta)}{m(\theta)s(\theta)}}_{\text{sellers' surplus share}}$$

where $\pi(\theta)$ is the expected payoff for sellers. If there is free entry of sellers at cost $\kappa > 0$, substituting $\pi(\theta) = \kappa$ into (61) delivers the generalized Hosios condition for seller entry:

$$(62) \qquad 1 - \underbrace{\eta_m(\theta)}_{\text{matching elasticity}} - \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{\kappa}{m(\theta)s(\theta)}}_{\text{sellers' surplus share}} .$$

Corollary 2. *A steady state equilibrium allocation features* over-entry *(*under-entry*) of sellers under the standard Hosios condition if and only if* $y'(\theta^*) > (<) \ 0$.

Proof. With seller entry, the direction of Corollary 1 is reversed since $\theta^* < \theta^P$ implies *over-entry* of sellers because $\theta = v/u$. Similarly, $\theta^* > \theta^P$ implies *under-entry* of sellers relative to the efficient level. ■

When $y'(\theta^*) > 0$, the output externality arising from seller entry is *negative* since $\theta = v/u$ and therefore $y(.)$ is decreasing in the measure of unmatched sellers u. In this case, the standard Hosios condition results in over-entry of sellers. If $y'(\theta^*) < 0$, the output externality from seller entry is *positive* and the standard Hosios condition results in under-entry. If $y'(\theta^*) = 0$, there is no output externality and seller entry is efficient under the standard Hosios condition.

2.3.1 Applying the condition

In any decentralized market, the equilibrium surplus shares of buyers and sellers will depend on the price determination mechanism. Depending on how prices are determined, the economy may or may not decentralize the efficient allocation – or, equivalently, the generalized Hosios condition may or may not hold. Efficiency arises only when the price determination mechanism ensures that both the search externalities and the output externality are internalized by a decentralized market.

Competitive search. If prices are determined by directed or competitive search, the generalized Hosios condition typically (but not always) decentralizes the efficient allocation. For example, we show that competitive search (price posting) can decentralize the constrained efficient allocation in environments with bilateral meetings such as Example 3.1, where the expected match output depends on the market tightness (see Appendix B). We will also see in Example 3.2 that competitive search (auctions) can decentralize the efficient allocation in environments with many-on-one or multilateral meetings. In both cases, the reason why competitive search delivers efficiency is precisely because the generalized Hosios condition holds *endogenously*. This is because competitive search allows agents to trade off prices against both the probability of trade and the expected match surplus if trade occurs, thus internalizing both the search externalities and the output externality.

Nash bargaining. Returning to our motivating example, suppose that wages are determined by Nash bargaining and workers' bargaining power is $\beta \in (0,1)$. With probability $m(\theta)/\theta$, firms successfully hire a worker and receive a share $1-\beta$ of the expected match surplus $s(\theta)$. The equilibrium market tightness $\theta^* > 0$ satisfies the following free entry condition:

$$\frac{m(\theta)}{\theta}(1-\beta)s(\theta) = c, \tag{63}$$

or, equivalently, the equilibrium $\theta^* > 0$ satisfies

$$\underbrace{1-\beta}_{\text{firms' bargaining power}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{firms' surplus share}} . \tag{64}$$

Applying the generalized Hosios condition, and using (64), entry is efficient only if

$$\underbrace{\eta_m(\theta^*)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta^*)}_{\text{surplus elasticity}} = \underbrace{1-\beta}_{\text{firms' bargaining power}} . \tag{65}$$

In the special case where $x_H = x_L$, i.e. match output is constant, we have efficiency if and only if $\eta_m(\theta^*) = 1-\beta$, a well-known version of the Hosios condition. If the matching technology is Cobb-Douglas and $m(\theta)$ has constant elasticity, i.e. $\eta_m(\theta) = \eta$ for all $\theta \in \mathbb{R}_+$, we can restore efficiency by *imposing* the Hosios condition. To do so, we make the following choice of parameter values: $\eta = 1 - \beta$. This approach, used for example in Shimer (2005), ensures efficiency in search models with Nash bargaining, regardless of the value of the equilibrium θ^*. In environments where the generalized Hosios condition is necessary for efficiency, however, it will generally not be possible to impose efficiency in this manner. This is because the surplus elasticity $\eta_s(\theta^*)$ is endogenous and it will not typically be constant. Instead, competitive search is necessary for decentralizing the efficient allocation.

3 Examples

We now present two main examples of search-and-matching environments where the expected match output depends on the market tightness. The Appendix contains additional examples and extensions, as described in Section 3.3.

3.1 Effect of labor market on goods market

One way in which the expected match output can depend on the labor market tightness is when there are *sequential markets*, such as a labor market and a goods market, and the possibility of trade in the goods market depends on the matching outcomes in the labor market. A classic example is Berentsen et al. (2011), which features both a labor market and a goods market. We present a static, highly simplified version of that model in order to focus attention.[16]

Workers first sell their labor to firms in the labor market and then purchase goods from firms in the goods market. Importantly, while all workers can search in the goods market, only *active* firms (i.e. filled vacancies) can produce and trade in the goods market. In this way, the labor market tightness affects the goods market tightness by affecting the measure of firms who search in the goods market. In turn, the goods market tightness determines the probability of trade for both workers and firms. This implies that the labor market tightness affects the expected match "output" because this includes both the direct match output in the labor market *and* the expected gains from trade in the goods market.[17]

[16] In particular, we simplify the model in Berentsen et al. (2011) by eliminating the third market, the Arrow-Debreu market, since it is unnecessary in the static model considered here.

[17] The environment in Kaplan and Menzio (2016), while different to that found in Berentsen et al. (2011), shares a similar feature because sellers' expected revenue in the product market depends on

The labor market is a standard DMP style environment with bilateral meetings. The labor market tightness is $\theta = v/u$ and the matching probabilities for workers and firms respectively are $m(\theta)$ and $m(\theta)/\theta$ where $m(.)$ satisfies Assumption 1. There is free entry of vacancies at cost $\kappa > 0$ and all matches produce direct output $\bar{y} > z$, where $z \geq 0$ is the value of non-market activity for the unemployed.

In the goods market, the probabilities of trade for workers and firms respectively are $m^G(\phi)$ and $q^G(\phi) \equiv m^G(\phi)/\phi$, where ϕ is the ratio of sellers to buyers and $m^G(.)$ satisfies Assumption 1. Since all workers (including the unemployed) search but only active firms search (i.e. those that have successfully hired a worker), we have $\phi = (m(\theta)/\theta)v/u = m(\theta)$. Since the unemployment rate is $u(\theta) = 1 - m(\theta)$, we have $\phi(\theta) = 1 - u(\theta)$.

Active firms can produce a single unit of an indivisible good at a production cost $c > 0$. Unemployed workers value the good at $v_u > 0$ and employed workers value the good at $v_e > v_u \geq c$. We assume, for simplicity, that $v_u = c$ (i.e. trades in the goods market with unemployed workers do not have any surplus).

While there is no heterogeneity here, matches that are formed in the labor market face different outcomes in terms of the surplus created, depending on whether or not workers trade in the goods market. Match "output" $x \in X = [\bar{y}, \bar{y} + (v_e - c)]$ and the distribution of output across matches is

$$f(x;\theta) = \begin{cases} 1 - m^G(\phi(\theta)) & \text{if } x = \bar{y} \\ m^G(\phi(\theta)) & \text{if } x = \bar{y} + (v_e - c) \end{cases} \tag{66}$$

and 0 otherwise. In the first case, the worker does not trade in the goods market and the match "output" x is just $\bar{y}$. In the next case, the worker does trade and x equals the direct output $\bar{y}$ *plus* the total gains from trade in the goods market (for both the worker and the firm).

The expected match output in the labor market is $y(\theta) = \sum_{x \in X} x f(x;\theta)$ and the expected match surplus is $s(\theta) = y(\theta) - z$. Taking the expected value of the distribution $f(x;\theta)$, we obtain

$$y(\theta) = \underbrace{\bar{y}}_{\text{direct output}} + \underbrace{m^G(\phi(\theta))(v_e - c)}_{\text{expected gains from trade}} . \tag{67}$$

the unemployment rate and thereby on the labor market tightness.

A sufficient condition for Assumption 2 to hold is that $\bar{y} - z > \kappa$ and $\frac{-m''(\theta)m(\theta)}{(m'(\theta))^2} > 2$ for all $\theta \in \mathbb{R}_+$. Applying Proposition 1, there exists a unique efficient choice $\theta^P > 0$ and it satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{firms' surplus share}}. \tag{68}$$

Whether or not this condition holds depends on the wage determination mechanism. For example, if wages are determined by Nash bargaining with workers' bargaining power $\beta \in (0, 1)$, then, similarly to (63), we have efficiency only if and only if the equilibrium θ^* satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{1 - \beta}_{\text{firms' bargaining power}}. \tag{69}$$

With competitive search and wage posting, on the other hand, the generalized Hosios condition holds *endogenously* and we always have constrained efficiency. Appendix B shows how competitive search with wage posting can decentralize the efficient allocation in environments such as this.

In this example, the output externality is *positive*, i.e. $y'(\theta^*) > 0$ or $y'(\theta^*) < 0$. Since $\phi'(\theta) > 0$ and $\frac{dm^G(\phi)}{d\phi} > 0$, an increase in the labor market tightness θ has a positive effect on the expected gains from trade in the goods market through an increase in workers' probability of trade. This means that imposing the standard Hosios condition in the labor market would result in *under-entry* of vacancies and inefficiently high unemployment. Since Berentsen et al. (2011) use Nash bargaining to determine wages and impose the standard Hosios condition in the labor market to calibrate their model, this may be quantitatively important.

3.2 Many-on-one meetings and competing auctions

We now consider two examples of environments that feature many-on-one meetings (where many buyers may meet one seller) and auctions. In a competing auctions environment, a large number of sellers compete to attract buyers by posting auctions.[18]

[18] Following the seminal work of Peters and Severinov (1997), recent papers using competing auctions include Albrecht, Gautier, and Vroman (2012); Albrecht et al. (2014); Albrecht, Gautier, and Vroman (2016); Kim and Kircher (2015); Lester, Visschers, and Wolthoff (2015); Mangin (2017).

Such an environment features the *selection channel* because the auction mechanism enables sellers to "select" the buyer with the highest valuation. The expected match "output" $y(\theta)$ is increasing in the market tightness because more buyers per seller implies a greater expected value of the *highest* valuation.

3.2.1 Labor market: competing auctions with vacancy entry

Consider the labor market environment in Mangin (2017).[19] Workers are identical sellers who post reservation wages to attract firms and then auction their labor using second-price auctions. Firms are ex ante identical buyers who pay a cost $c > 0$ to enter and search for workers. The labor market tightness is $\theta \equiv v/u$, the ratio of vacancies or firms to unemployed workers. The meeting technology is Poisson and $P_n(\theta) = \frac{\theta^n e^{-\theta}}{n!}$ is the probability that n firms approach a given worker. The matching probability for workers is $m(\theta) = 1 - e^{-\theta}$.

Firms' valuations x of workers' labor are match-specific productivity draws that are private information. Valuations are drawn *ex post* (i.e. after meetings) independently from a distribution with cdf G with density $g = G' > 0$, finite mean, and support $X = [x_0, \infty) \subseteq \mathbb{R}_+$.

It can be shown that the distribution of output across all matches has pdf

$$f(x;\theta) = \frac{\theta g(x) e^{-\theta(1-G(x))}}{1 - e^{-\theta}} \tag{70}$$

and the expected match output is given by

$$y(\theta) = \frac{\int_{x_0}^{\infty} \theta g(x) e^{-\theta(1-G(x))} x dx}{1 - e^{-\theta}}. \tag{71}$$

Mangin (2017) proves that $y'(\theta) > 0$ for all $\theta \in \mathbb{R}_+$ if G is well-behaved, i.e. if it satisfies a mild regularity condition. Therefore, the output externality from vacancy creation is always positive. Intuitively, a higher number of vacancies per unemployed worker allows workers to be more *selective*, increasing labor productivity.

We can now apply Proposition 3. If $E_G(x) - z > c$, Assumption 2 holds and we

[19] Mangin and Sedláček (2018) extends this model to a dynamic economy with aggregate shocks.

have constrained efficiency of vacancy entry if and only if the equilibrium θ^* satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{firms' surplus share}}. \tag{72}$$

Mangin (2017) shows that when wages are determined through auctions, the equilibrium θ^* satisfies a condition that can be shown to be equivalent to (72). In the limit as $\theta \to \infty$, firms' surplus share converges to the *tail index* λ_G of the distribution G, where $\lambda_G > 0$ only if G has unbounded upper support.[20] In general, the generalized Hosios condition holds *endogenously* and we have constrained efficient vacancy entry.

In the special case where $y(\theta) = \bar{y} \in \mathbb{R}_+$, we recover the large economy version of the directed search model found in Julien, Kennes, and King (2000), which is closely related to Burdett, Shi, and Wright (2001). In this special case, the standard Hosios condition holds endogenously and we therefore have constrained efficiency.

3.2.2 Business-stealing: competing auctions with seller entry

Albrecht et al. (2014) examines the efficiency of *seller entry* in a competing auctions environment. The authors consider both ex ante and ex post buyer heterogeneity, as well as seller heterogeneity, and they prove that seller entry is always efficient. In particular, Albrecht et al. (2014) identifies a negative externality from seller entry called the *business-stealing externality.* When an additional seller enters, the seller "steals" potential buyers from existing sellers, thereby reducing the expected surplus for those sellers. Although they do not explicitly identify it, the generalized Hosios condition applies in their setting and it is the fact that this condition holds endogenously that ensures efficiency.

Consider a simple version of their model featuring identical sellers with reservation value $z = 0$. Buyers are ex ante identical but heterogeneous ex post. Sellers pay a cost κ to enter and attract buyers by posting second-price auctions with reserve prices. The buyer-seller ratio is $\theta \equiv N_B/N_S$. The meeting technology is Poisson and $P_n(\theta) = \frac{\theta^n e^{-\theta}}{n!}$ is the probability that n buyers approach a given seller. The matching probability for sellers is $m(\theta) = 1 - e^{-\theta}$.

Buyers' valuations x are private information and are drawn *ex post* (i.e. after

[20]The fact that G has unbounded upper support is thus important. See Proposition 2 in Mangin (2017). The *tail index* is a measure of fatness of the tails of the distribution G.

meetings) independently from a distribution with cdf G, density $g = G' > 0$, and support $X = [0, 1]$.

The distribution of *valuations* of successful buyers has pdf

$$f(x;\theta) = \frac{\theta g(x) e^{-\theta(1-G(x))}}{1-e^{-\theta}} \tag{73}$$

and the expected valuation of a successful buyer is given by

$$y(\theta) = \frac{\int_0^1 \theta g(x) e^{-\theta(1-G(x))} x dx}{1-e^{-\theta}}. \tag{74}$$

We can now directly apply Proposition 1. If $E_G(x) > c$, then Assumption 2 holds and we have constrained efficiency of seller entry if and only if the equilibrium θ^* satisfies the generalized Hosios condition for *seller entry*:

$$1 - \underbrace{\eta_m(\theta)}_{\text{matching elasticity}} - \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{\kappa}{m(\theta)s(\theta)}}_{\text{sellers' surplus share}}. \tag{75}$$

Albrecht et al. (2014) show that an equivalent condition holds endogenously in this environment, and we therefore have constrained efficiency of seller entry.[21]

The output externality that arises in Example 3.2.1 also appears in Albrecht et al. (2014) due to the selection channel. Through the auction mechanism, sellers choose to trade with the buyer who has the highest valuation. From Example 3.2.1, we know that $y'(\theta) > 0$ if the distribution G is well-behaved. Importantly, this is a *negative* externality with regard to seller entry since $\theta = N_B/N_S$ and thus $y(.)$ is decreasing in the number of sellers. When there is a fixed number of buyers, more seller entry implies fewer buyers for each seller, thereby reducing the power of the selection channel.

The *business-stealing externality* is closely related to this negative output externality. Consider the expected surplus per seller, $\Lambda(\theta) = m(\theta)s(\theta)$. The business-stealing externality reflects the fact that $\Lambda'(\theta) > 0$ and thus the expected surplus per seller is

[21] In Albrecht et al. (2014), the planner maximizes the total social surplus, $\Lambda(\theta)N_S - \kappa N_S$, where $\Lambda(\theta)$ is the expected surplus per seller. The social surplus per *buyer* is $\Omega_B(\theta) = \Lambda(\theta)/\theta - \kappa/\theta$ and the first-order condition for the planner's problem is $\Omega_B'(\theta) = \Lambda'(\theta)/\theta - \Lambda(\theta)/\theta^2 + \kappa/\theta^2 = 0$. Rearranging, the efficient θ^P satisfies $1 - \eta_\Lambda(\theta) = \kappa/\Lambda(\theta)$ where $\eta_\Lambda(\theta) \equiv \Lambda'(\theta)\theta/\Lambda(\theta)$. Since the surplus per seller is $\Lambda(\theta) = m(\theta)s(\theta)$, we have $\eta_\Lambda(\theta) = \eta_m(\theta) + \eta_s(\theta)$, which is equivalent to (75).

decreasing in N_S. In fact, since $\Lambda(\theta) = m(\theta)s(\theta)$, the "business-stealing" effect can be decomposed into two effects: the effect on sellers' matching probability $m(\theta)$, and the effect on the expected match surplus $s(\theta)$. Both effects are clearly reflected in the generalized Hosios condition via the matching elasticity and the surplus elasticity.

3.3 Other examples

Appendix B extends the competitive search (price posting) approach of Moen (1997) to an environment where the expected match output depends on the market tightness. Since the generalized Hosios condition holds *endogenously* in this economy, competitive search with price posting provides a way of decentralizing the constrained efficient allocation in environments where Proposition 1 applies and meetings are bilateral, such as Example 3.1. (In environments where Proposition 1 applies and meetings are many-on-one or multilateral, competitive search (auctions) decentralizes the constrained efficient allocation, as seen in Example 3.2.)

Appendix C presents two related examples. The first example is a Diamond-Mortensen-Pissarides (DMP) style model with bilateral meetings and job acceptance decisions. Since the cut-off productivity for accepting a match depends on the market tightness, labor productivity also depends on the market tightness. It is well known that the standard Hosios condition characterizes efficient entry in this environment. We show that, in fact, the generalized Hosios condition applies but it reduces to the standard Hosios condition simply because the positive effect of vacancy entry on the expected match surplus and the negative effect of vacancy entry on the job acceptance probability *exactly* offset each other.

The second example in Appendix C features firms that are ex ante heterogeneous with respect to productivity. Since firms' entry decisions are affected by the probability of hiring and therefore the market tightness, labor productivity is also affected by the market tightness and the generalized Hosios condition is necessary for constrained efficiency. This example is related to the model of labor force participation found in Albrecht, Navarro, and Vroman (2010). (In a follow-up paper to the present one, Julien and Mangin (2017) applies and extends the generalized Hosios condition to the environment in Albrecht et al. (2010) with labor force participation.)

Appendix D shows that competitive search does *not* always decentralize the constrained efficient allocation in environments that require the generalized Hosios condition for efficiency. We develop a dynamic model of the market for referrals that is inspired by the model of word-of-mouth communication in Campbell, Leister, and Zenou (2017). Consumers cannot directly observe firms' quality, but they can purchase "referrals". Proposition 1 does not apply directly in this environment since there is an additional law of motion for the distribution of traded goods' quality. However, we show that the generalized Hosios condition is still necessary for constrained efficiency. With competitive search (price posting), the market internalizes both the search externalities and the *direct* component of the output externality (i.e. the impact of referrals on the average quality of goods traded in the current period). However, there is an additional externality – a *dynamic* composition externality – that is not internalized. Consumers do not internalize the effect of their decisions on future consumers through the impact of referrals on the dynamics of the quality distribution. As a result, competitive search does not deliver efficiency.[22]

4 Conclusion

This paper generalizes the well-known Hosios (1990) condition that characterizes efficient entry in search-and-matching models. We extend this simple rule to dynamic search environments where the expected match output depends on the market tightness. Such environments give rise to a novel externality – the *output externality* – that is not captured by the standard Hosios condition. To ensure constrained efficiency, markets must internalize the effect of entry on both the number of matches created and the average value created by each match. We show that this occurs precisely when buyers' surplus share equals the *matching elasticity* plus the *surplus elasticity.* We call this intuitive condition the "generalized Hosios condition". When it holds, agents are fully compensated for the effect of entry on both *match creation* and *surplus creation.* In search-theoretic models of the labor market, for example, vacancy entry and unemployment are constrained efficient only when firms are compensated for the effect of job creation on both employment and labor productivity.

[22] This dynamic externality is similar in flavor to that found in Guerrieri (2008), which develops a dynamic competitive search model with informational asymmetries. In that paper, the inefficiency arises because firms do not internalize the effect of their decisions on the outside options of workers hired in earlier periods. In our example of the market for referrals, consumers do not internalize the effect of their decisions on consumers that trade in future periods.

Appendix A: Omitted proofs

Proof of Lemma 2. In steady state, we have the following Bellman equations:

$$rU_B = -c + \frac{m(\theta)}{\theta}(V_B - U_B), \tag{76}$$

$$rV_B = y(\theta) - w(\theta) + \delta(U_B - V_B), \tag{77}$$

$$rU_S = z + m(\theta)(V_S - U_S), \tag{78}$$

$$rV_S = w(\theta) + \delta(U_S - V_S), \tag{79}$$

where $w(\theta)$ is the expected transfer. Using $s(\theta) = V_B + V_S - U_B - U_S$, we obtain

$$r(V_B + V_S) = y(\theta) - \delta s(\theta). \tag{80}$$

Setting $U_B = 0$ and substituting back into $s(\theta) = V_B + V_S - U_S$ yields $s(\theta) = (y(\theta) - rU_S)/(r + \delta)$. Next, using (78) and (79), we find that

$$U_S = \frac{z(r + \delta) + m(\theta)w(\theta)}{r(r + \delta + m(\theta))}, \tag{81}$$

and, substituting into $s(\theta) = (y(\theta) - rU_S)/(r + \delta)$, we obtain

$$s(\theta) = \frac{y(\theta) - z + m(\theta)\left(\frac{y(\theta)-w(\theta)}{r+\delta}\right)}{r + \delta + m(\theta)}. \tag{82}$$

Now (76) implies $V_B = c\theta/m(\theta)$ when $U_B = 0$. Substituting into (77), we have

$$\frac{y(\theta) - w(\theta)}{r + \delta} = \frac{c\theta}{m(\theta)}. \tag{83}$$

Finally, substituting (83) into (82), we obtain expression (39). ■

Derivation of laws of motion in continuous time. The law of motion for the unemployment rate u_t in discrete time is

$$u_{t+dt} - u_t = \delta dt(1-u_t) - m(\theta_t)dt\, u_t \tag{84}$$

and the law of motion for average match output y_t is given by

$$y_{t+dt} = \frac{(1-\delta dt)(1-u_t)y_t + m(\theta_t)dt\, u_t y(\theta_t)}{1-u_{t+dt}}. \tag{85}$$

Defining $\tilde{x}_t \equiv (1-u_t)y_t$, we have

$$\tilde{x}_{t+dt} - \tilde{x}_t = -\delta dt \tilde{x}_t + m(\theta_t)dt\, u_t y(\theta_t). \tag{86}$$

In continuous time ($dt \to 0$), the laws of motion for u_t and $\tilde{x}_t$ are

$$\dot{u}_t \equiv \frac{du_t}{dt} = \lim_{dt\to 0}\left(\frac{u_{t+dt}-u_t}{dt}\right) = \delta(1-u_t) - m(\theta_t)u_t \tag{87}$$

and

$$\dot{x}_t \equiv \frac{d\tilde{x}_t}{dt} = \lim_{dt\to 0}\left(\frac{\tilde{x}_{t+dt}-\tilde{x}_t}{dt}\right) = -(\delta\tilde{x}_t - m(\theta_t)u_t y(\theta_t)). \tag{88}$$

Also, since $\tilde{x}_t \equiv (1-u_t)y_t$, we have

$$\dot{x}_t = -\dot{u}_t y_t + (1-u_t)\dot{y}_t \tag{89}$$

and, rearranging, we have

$$\dot{y}_t = \frac{\dot{x}_t + \dot{u}_t y_t}{1-u_t}. \tag{90}$$

Substituting in $\dot{x}_t$ and $\dot{u}_t$ from (88) and (87), and using $\tilde{x}_t \equiv (1-u_t)y_t$, leads to:

$$\dot{y}_t = \frac{m(\theta_t)u_t(y(\theta_t) - y_t)}{1-u_t}. \tag{91}$$

Appendix B: Competitive search (posting)

It is well-known that competitive search equilibrium is typically (but not always) constrained efficient in the sense that it decentralizes the planner's allocation (Shimer, 1996; Moen, 1997). In competitive search models where the expected match output is constant, agents simply trade off prices against the probability of trade. The fact that competitive search allows agents to do so is what delivers efficiency. In environments where the expected match output depends on the market tightness, agents trade off prices against both the probability of trade *and* the expected match surplus if trade occurs. Again, the fact that agents can do so is what delivers efficiency.

Consider a simple competitive search model in the spirit of Moen (1997). There is a continuum of submarkets indexed by $i \in [0, 1]$ and free entry of vacancies at cost $c > 0$. Workers in submarket i post the same wage w_i and face the same market tightness θ_i, the ratio of vacancies to workers in that submarket. Firms' search is *directed* by observing the posted wages and deciding which submarkets to enter. Within each submarket, workers and firms are matched according to a frictional meeting technology. Matching probabilities for workers and firms are $m(\theta_i)$ and $m(\theta_i)/\theta_i$ respectively, where $m(.)$ satisfies Assumption 1.

In any submarket, match output $x \in X = [x_{\min}, x_{\max}] \subseteq \mathbb{R}_+$ where $x_{\max} \in \mathbb{R}_+ \cup \{\infty\}$. In submarket i, match output is an i.i.d. draw from a probability distribution with pdf $f(x; \theta_i)$ and a finite mean. Let $y(\theta_i) \equiv \int_{x_{\min}}^{x_{\max}} x f(x; \theta_i) dx$, the expected match output. The flow payoff for unmatched sellers is $z \geq 0$ and we assume that $x_{\min} > z$. The expected match surplus in submarket i is $s(\theta_i) = y(\theta_i) - z$.

The expected payoff for firms in submarket i with wage w_i and tightness θ_i is

$$\Pi(\theta_i, w_i) = \frac{m(\theta_i)}{\theta_i}(y(\theta_i) - w_i), \tag{92}$$

and the expected payoff for workers in submarket i with market tightness θ_i is

$$V(\theta_i, w_i) = m(\theta_i)w_i + (1 - m(\theta_i))z. \tag{93}$$

Workers in submarket i choose a wage w_i^* and market tightness θ_i^* that solve

$$\max_{w_i, \theta_i \in \mathbb{R}_+} (m(\theta_i)w_i + (1 - m(\theta_i))z) \tag{94}$$

subject to $\Pi(\theta_i, w_i) \leq c$ and $\theta_i \geq 0$ with complementary slackness. To induce participation by firms in submarket i, i.e. $\theta_i > 0$, the constraint $\Pi(\theta_i, w_i) \leq c$ is binding:

$$\frac{m(\theta_i)}{\theta_i}(y(\theta_i) - w_i) = c. \tag{95}$$

Solving for w_i as a function of θ_i using (95), we obtain

$$w(\theta_i) = y(\theta_i) - \frac{c\theta_i}{m(\theta_i)}. \tag{96}$$

Choosing a wage w_i^* is thus equivalent to choosing a market tightness θ_i^* where

$$\theta_i^* = \arg\max_{\theta_i \in \mathbb{R}_+} \left(m(\theta_i)w(\theta_i) + (1 - m(\theta_i))z\right) \tag{97}$$

and using (96), this is equivalent to

$$\theta_i^* = \arg\max_{\theta_i \in \mathbb{R}_+} \left(m(\theta_i)y(\theta_i) + (1 - m(\theta_i))z - c\theta_i\right). \tag{98}$$

The equilibrium θ_i^* satisfies the first-order condition

$$m'(\theta_i)s(\theta_i) + m(\theta_i)s'(\theta_i) = c, \tag{99}$$

or, equivalently, the equilibrium θ_i^* solves

$$\underbrace{\eta_m(\theta_i)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta_i)}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta_i}{m(\theta_i)s(\theta_i)}}_{\text{firms' surplus share}}. \tag{100}$$

The generalized Hosios condition holds endogenously *within each active submarket* i. If we consider a symmetric equilibrium in which firms are indifferent across submarkets and all workers post the same wage, then $\theta_i^* = \theta^*$ for all submarkets i. If Assumption 2 holds, then Proposition 1 tells us that the equilibrium level of vacancy entry is constrained efficient. While we consider only a static model here, the same result holds in dynamic environments where Proposition 1 applies.[23]

[23] Proof for dynamic economy is available on request.

Appendix C: Constrained planner

The expected match output may depend on the market tightness when agents make a decision about whether or not to enter the market, or whether to accept or reject a match, and that decision depends on the market tightness. In the next two examples, we consider such environments. Importantly, we assume that in choosing the market tightness θ, the planner is constrained not only by the matching frictions but also by the entry or acceptance decision rules that agents would choose in the decentralized equilibrium. This is because the function $y(.)$, or equivalently, the distribution of match output $f(x;\theta)$, arises as a consequence of these entry or acceptance decisions. Since the planner is restricted to take both the matching technology $m(.)$ *and* the output technology $y(.)$ as given, the planner is constrained by these.[24]

Endogenous job acceptance

Consider the steady state of a continuous-time dynamic Diamond-Mortensen-Pissarides (DMP) style environment with a job acceptance decision.[25] Workers and firms discount future payoffs at a rate $r > 0$. The market tightness is $\theta = v/u$ and workers' arrival rate for meetings is $m(\theta)$. After workers and firms meet, a match-specific productivity x is drawn from a distribution with cdf G and density $g = G'$ where $g(x) > 0$ for all $x \in X = [0, 1]$. After observing the productivity x, workers and firms decide whether to accept the match. There is free entry of vacancies at cost $c > 0$ and matches are destroyed at an exogenous rate $\delta > 0$. The flow value of non-market activity is $z \geq 0$.

A job with match-specific productivity x is acceptable to both worker and firm if and only if the match surplus $S(x) \geq 0$.[26] There is a cut-off productivity x^* such that all jobs with productivity $x \geq x^*$ are acceptable to both workers and firms. We write $x^*(\theta)$ since the cut-off productivity will depend on the value of unemployment U_S and therefore on the market tightness. The probability a match is *acceptable* is $a(\theta) = 1 - G(x^*(\theta))$ and the probability a worker is hired is $\hat{m}(\theta) \equiv a(\theta)m(\theta)$.

[24] In these two examples, the constrained efficiency is "doubly constrained" since the planner's problem is solved subject to an additional constraint which is one of the equilibrium conditions.

[25] The classic references are Mortensen and Pissarides (1994) and Pissarides (2000).

[26] The productivity-specific match surplus $S(x)$ is defined by $S(x) \equiv V_S(x) + V_B(x) - U_S - U_B$ where the Bellman equations for V_S and V_B found in Appendix A are adjusted to be productivity-specific.

The distribution of output across all *realized* (i.e. accepted) matches has pdf

$$f(x;\theta) = \frac{g(x)}{1 - G(x^*(\theta))} \tag{101}$$

and the expected match output across all (accepted) matches is

$$y(\theta) = \int_{x^*(\theta)}^{1} \frac{xg(x)}{1 - G(x^*(\theta))} dx, \tag{102}$$

where the equilibrium cut-off productivity $x^*(\theta)$ is given by equating $S(x^*) = 0$. With free entry of firms, it can be shown that $S(x) = \frac{x - rU_S}{r+\delta}$, so we have $x^*(\theta) = rU_S$. Clearly, since the cut-off productivity $x^*(\theta)$ depends on the market tightness, the expected match output $y(\theta)$ also depends on the market tightness.

The planner chooses θ^P to maximize the total social surplus net of entry costs. Importantly, we assume the planner uses the same cut-off productivity rule as in the equilibrium, i.e. $x^*(\theta) = rU_S$. Solving the planner's problem yields the generalized Hosios condition.[27] In particular, the social optimum θ^P must satisfy

$$\eta_{\hat{m}}(\theta) + \eta_{\hat{s}}(\theta) = \frac{c\theta}{\hat{m}(\theta)\hat{s}(\theta)}. \tag{103}$$

where $\hat{m}(\theta) \equiv a(\theta)m(\theta)$ and $\hat{s}(\theta)$ is the expected match surplus for accepted matches.[28]

Now, condition (103) is equivalent to

$$\eta_m(\theta) + \eta_a(\theta) + \eta_{\hat{s}}(\theta) = \frac{c\theta}{\hat{m}(\theta)\hat{s}(\theta)} \tag{104}$$

where $\eta_a(\theta) \equiv a'(\theta)\theta/a(\theta)$. Using Proposition 3 (adjusted), we have

$$\eta_{\hat{s}}(\theta) = \frac{y'(\theta)\theta}{(r+\delta)\hat{s}(\theta)}, \tag{105}$$

and differentiating $a(\theta) = 1 - G(x^*(\theta))$ yields

$$\eta_a(\theta) = -\frac{g(x^*)\frac{dx^*}{d\theta}\theta}{1 - G(x^*)}. \tag{106}$$

[27] All of the results for this example can be easily obtained by modifying the proofs of Lemma 2, as well as Propositions 3 and 4, so that $m(\theta)$ is replaced by $\hat{m}(\theta) = a(\theta)m(\theta)$ throughout.

[28] The adjusted steady state expected match surplus is given by $\hat{s}(\theta) = \frac{y(\theta) - z + c\theta}{r+\delta+\hat{m}(\theta)}$.

Using (106) and (105), and the fact that

$$y'(\theta) = (y(\theta) - x^*(\theta)) \frac{g(x^*) \frac{dx^*}{d\theta} \theta}{1 - G(x^*)}, \tag{107}$$

we obtain the following:

$$\eta_a(\theta) + \eta_{\hat{s}}(\theta) = \left(\frac{y(\theta) - x^*(\theta)}{(r + \delta)\hat{s}(\theta)} - 1 \right) \frac{g(x^*) \frac{dx^*}{d\theta} \theta}{1 - G(x^*)}. \tag{108}$$

Combining $x^*(\theta) = rU_S$ with the fact that $(r + \delta)\hat{s}(\theta) = y(\theta) - rU_S$, we have $(r + \delta)\hat{s}(\theta) = y(\theta) - x^*(\theta)$. Substituting into (108), we obtain $\eta_a(\theta) + \eta_{\hat{s}}(\theta) = 0$, and substituting into (104), an efficient $\theta^P > 0$ must satisfy

$$\eta_m(\theta) = \frac{c\theta}{\hat{m}(\theta)\hat{s}(\theta)}. \tag{109}$$

While the generalized Hosios condition (103) does indeed apply here, the standard Hosios condition is sufficient for constrained efficiency.[29] Intuitively, this is because there are two offsetting effects of an increase in the market tightness θ. First, there is an increase in the cut-off productivity x^*, which *decreases* the job acceptance probability $a(\theta)$ since workers are more selective. Second, the increase in x^* leads to an *increase* in the expected match surplus $\hat{s}(\theta)$ for acceptable matches, since these matches have higher productivity. The fact that these two effects *exactly offset* each other is reflected in the fact that $\eta_a(\theta) + \eta_{\hat{s}}(\theta) = 0$, which implies the generalized Hosios condition reduces to the standard Hosios condition.

Ex ante heterogeneity and market composition

When there is *ex ante* heterogeneity among buyers or sellers, dependence of the expected match output on market tightness can arise naturally through market composition. If the market tightness influences the individual entry decisions of buyers or sellers that are ex ante heterogeneous with respect to characteristics that affect match output, then average output per match will depend on market tightness.[30] We

[29] Note that Corollary 1 does not directly apply here since it assumes that all matches are accepted.

[30] For example, Albrecht et al. (2010) consider an environment where workers are ex ante heterogeneous with respect to their market productivity. There is both firm entry *and* a labor force participation decision. Related literature includes Albrecht, Navarro, and Vroman (2009), Gavrel

call this the *composition channel*.

Suppose there is a measure u of unemployed workers and a fixed measure M of firms that may choose to search. Firms' productivities x are distributed according to a twice differentiable distribution with cdf G and density g where $G(0) = 0$ and $g(x) > 0$ for all $x \in X = [0, 1]$. Firms learn their own productivity before deciding whether to pay the entry cost $c > 0$ and search. Expected wages paid by a firm with productivity x is $w(x, \theta) \leq x$.

Let v be the measure of *searching* firms and define $\theta \equiv v/u$. Meetings are bilateral and the probabilities of matching for workers and firms are $m(\theta)$ and $m(\theta)/\theta$ respectively, where we assume $m(.)$ satisfies Assumption 1.

A firm with productivity x chooses to search for a worker if and only if

$$\frac{m(\theta)}{\theta}(x - w(x, \theta)) > c. \tag{110}$$

If $\partial w(x, \theta)/\partial x < 1$, there is a unique cut-off productivity $x^*(\theta)$ such that firms enter if and only if $x \geq x^*(\theta)$.[31] The distribution of output across matches has pdf

$$f(x; \theta) = \frac{g(x)}{1 - G(x^*(\theta))} \tag{111}$$

and the expected match output, or labor productivity, is given by

$$y(\theta) = \int_{x^*(\theta)}^{1} \frac{xg(x)}{1 - G(x^*(\theta))} dx. \tag{112}$$

It can be verified that x^* is strictly increasing in θ provided that $\partial w(x, \theta)/\partial x < 1$. This is intuitive: as the market tightness increases, the probability of finding a worker is lower so only high productivity firms choose to pay the cost c and search. At the same time, the average match output $y(\theta)$ is increasing in the cut-off productivity x^*. Therefore, $y'(\theta) > 0$ for all $\theta \in \mathbb{R}_+$ and the output externality is positive.

Suppose the planner chooses a market tightness θ to maximize the total social surplus minus the total entry costs. As in the previous example, we assume the planner uses the same cut-off productivity rule $x^*(\theta)$ as in the decentralized economy.

(2011), Charlot, Malherbet, and Ulus (2013), and Masters (2015). In a follow-up paper to the present one, Julien and Mangin (2017) applies and extends the generalized Hosios condition to the environment in Albrecht et al. (2010).

[31] This is true, for example, if wages are determined by Nash bargaining with $\beta < 1$.

If Assumption 2 is satisfied, there exists a unique social optimum θ^P and we have constrained efficiency if and only if θ^* satisfies the generalized Hosios condition in Proposition 1.[32] Since $y'(\theta^*) > 0$, Corollary 1 implies that there is under-entry of firms under the standard Hosios condition.

Appendix D: Endogenous quality dynamics

We now present an example that illustrates how competitive search may not always endogenize the generalized Hosios condition. In the model we present, an endogenous quality distribution arises through the possibility of "referrals". The model is closely related to – but different from – Campbell et al. (2017), which presents a dynamic model of consumer sales with word-of-mouth communication through social networks. In our setting, the key variable θ is the ratio of *referrals* to consumers and the endogenous quality distribution is the probability that a traded good is low quality, i.e. the market share of low-quality firms. We use competitive search to model the *market for referrals* (not goods) and consider whether the entry of *sellers of referrals* (not firms) is constrained efficient.

There is a fixed measure of consumers who seek to purchase one unit of a durable good. After purchasing the good, consumers exit the market and are replaced by new consumers. Goods are produced by a large number of competitive firms of two types: high quality and low quality. The share of firms that produce low quality goods is $\mu \in (0, 1)$.[33] The low-quality good has quality x_L and the high-quality good has quality $x_H > x_L$. The price of the good is p for both types of firm.

Consumers cannot directly observe firms' quality, but they can receive referrals. A single referral tells a consumer about the quality of a good purchased in the previous period. In each period $t \in \{0, 1, ...\}$, the expected number of referrals per consumer is θ_t (which is endogenous) and $P_n(\theta_t)$ is the probability a consumer receives n referrals at time t. This is a kind of "meeting technology" which matches referrals with consumers. If a consumer receives at least one referral, they pick the "best" referral and then choose whether to purchase from that firm or instead choose a firm randomly.[34]

[32] For example, if G is uniform on $[0, 1]$ and wages are determined by Nash bargaining, Assumption 2 holds provided that $c < 1/2$ and $\beta < 1/2$.

[33] Since our focus is on the market for *referrals*, we do not endogenize the entry of low and high quality firms as in Campbell et al. (2017) but instead assume that μ is exogenous.

[34] If the consumer is indifferent between two referrals, they pick one at random.

If a consumer receives no referrals, they purchase the good from a random firm, i.e. they buy a low-quality good with probability μ.

Let α_t denote the *market share* of low-quality firms, i.e. the probability that a good traded in period t is low quality. Low-quality goods are purchased only if *all* n of a consumer's referrals are to low-quality firms (which occurs with probability α_t^n) *and* the consumer picks a low-quality firm when choosing randomly (which occurs with probability μ). We therefore obtain the following law of motion for α_t:

$$\alpha_{t+1} = \mu \sum_{n=0}^{\infty} P_n(\theta_{t+1}) \alpha_t^n \tag{113}$$

where $\alpha_0 = \mu \in (0, 1)$. If $P_n(\theta)$ is Poisson, i.e., $P_n(\theta) = \frac{\theta^n e^{-\theta}}{n!}$, we have

$$\alpha_{t+1} = \mu e^{-\theta_{t+1}(1-\alpha_t)}. \tag{114}$$

Both the selection channel and an additional channel are present. The *selection channel* implies that the average quality of a traded good is increasing in the number of referrals per consumer θ_t since consumers can be more selective. An additional channel, which is a kind of dynamic *composition channel,* ensures that the quality distribution α_t itself evolves over time. This is because the market composition, i.e. the composition of the pool of referrals, depends on the previous period's θ_t since referrals are only drawn from traded goods.[35]

Suppose there is a large number of potential entrants who can pay a cost $c > 0$ to acquire information about a random good purchased in the previous period. This information can be sold to consumers as a "referral". In the market for selling referrals, consumers post *referral fees* and commit to paying a single fee for the best referral they receive. Similarly to the competitive search environment in Appendix B, consumers form a submarket i by choosing a referral fee r_i^* and a ratio of referrals

[35] In terms of our earlier notation, the law of motion for the distribution of quality across "matches", i.e. trades which occur when a consumer receives at least one referral, is given by

$$f_{t+1}(x_L; \theta_{t+1}) = \frac{\mu \sum_{n=1}^{\infty} P_n(\theta_{t+1}) \alpha_t^n}{1 - P_0(\theta_{t+1})}. \tag{115}$$

Since the distribution f_t evolves over time and does not depend only on the current period's market tightness θ_t, Propositions 1 and 2 do not directly apply in this setting. However, we will show that the generalized Hosios condition is also a necessary condition for efficiency in this environment.

to consumers θ_i^* to maximize their expected payoff:

$$m(\theta_i)(y(\theta_i,\alpha) - r_i - p) + (1 - m(\theta_i))(y_\mu - p) \tag{116}$$

subject to the following condition for sellers of referrals:

$$\frac{m(\theta_i)}{\theta_i} r_i \leq c, \tag{117}$$

and $\theta_i \geq 0$, with complementary slackness. Here, $m(\theta_i) = 1 - e^{-\theta_i}$ is the probability a consumer receives at least one referral, $m(\theta_i)/\theta_i$ is the probability a seller is paid a referral fee, $y(\theta_i, \alpha)$ is the expected quality of a good purchased *if the consumer receives at least one referral*, and $y_\mu = \mu x_L + (1-\mu)x_H$ is the expected quality of a good purchased from a random firm.

Using (117), the choice of a consumer in submarket i is equivalent to

$$\theta_i^* = \arg\max_{\theta_i \in \mathbb{R}_+} (m(\theta_i)(y(\theta_i,\alpha) - y_\mu) + y_\mu - p - c\theta_i) \tag{118}$$

and θ_i^* satisfies the first-order condition

$$m'(\theta_i)s(\theta_i) + m(\theta_i)\frac{\partial y(\theta_i,\alpha)}{\partial \theta_i} = c \tag{119}$$

where the expected match surplus is $s(\theta_i) = y(\theta_i,\alpha) - y_\mu$, i.e. the difference between the expected quality in submarket i with and without receiving at least one referral. In symmetric equilibrium, $\theta_i^* = \theta^*$ for all submarkets i and θ^* satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\frac{\frac{\partial y(\theta,\alpha)}{\partial\theta}\theta}{s(\theta)}}_{\text{direct surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{surplus share of referral sellers}} \tag{120}$$

as well as the steady state condition

$$\alpha = \mu e^{-\theta(1-\alpha)}. \tag{121}$$

If $\mu < \frac{1}{2}$, there exists a unique steady state equilibrium (θ^*, α^*) where $\alpha^* \in (0,1)$.[36]

Now consider a planner who can directly choose the number of referrals per con-

[36] A detailed derivation of the steady state equilibrium can be found below.

sumer θ. Importantly, while consumers take the distribution of quality α *as given*, the planner takes the effect of θ on α into account. In the proof below, we solve the dynamic planner's problem subject to the law of motion for α. The resulting steady state condition is identical to the one obtained when the planner maximizes the steady state social surplus per consumer,

$$\Omega(\theta) = m(\theta)(y(\theta, \alpha(\theta)) - y_\mu) + y_\mu - c\theta, \tag{122}$$

where $\alpha(\theta)$ is given by (121) and

$$y(\theta, \alpha(\theta)) = \frac{(1 - \mu e^{-\theta(1-\alpha(\theta))})x_H + \mu e^{-\theta(1-\alpha(\theta))}x_L - e^{-\theta}y_\mu}{1 - e^{-\theta}}. \tag{123}$$

Using $s(\theta) = y(\theta, \alpha(\theta)) - y_\mu$, the planner's first-order condition is equivalent to

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\eta_s(\theta)}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{surplus share of referral sellers}}, \tag{124}$$

which is just the generalized Hosios condition. Differentiating $s(\theta)$, we obtain

$$s'(\theta) = \underbrace{\frac{\partial y(\theta, \alpha(\theta))}{\partial \theta}}_{\text{direct output externality}} + \underbrace{\frac{\partial y(\theta, \alpha(\theta))}{\partial \alpha}\alpha'(\theta)}_{\text{indirect output externality}} \tag{125}$$

and thus we have efficiency only if θ^* satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\frac{\frac{\partial y(\theta,\alpha)}{\partial \theta}\theta}{s(\theta)} + \frac{\frac{\partial y(\theta,\alpha)}{\partial \alpha}\alpha'(\theta)\theta}{s(\theta)}}_{\text{surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{surplus share of referral sellers}}. \tag{126}$$

Comparing (126) with (120), it is clear that the economy is not efficient. The decentralized market internalizes both the search externalities and the *direct* "output externality", i.e. the direct effect of θ on the quality of traded goods via the selection channel. However, there is an additional externality arising from the use of referrals. This is reflected in the term $\frac{\partial y(\theta,\alpha(\theta))}{\partial \alpha}\alpha'(\theta)$, which captures the *indirect* or *dynamic* "output externality", i.e. the indirect effect of θ via the dynamic composition channel. Since $\frac{\partial y(\theta,\alpha(\theta))}{\partial \alpha} < 0$ and the market share of low-quality firms is decreasing in the number of referrals per consumer at any equilibrium θ^*, i.e. $\alpha'(\theta^*) < 0$, this is a positive externality that is not internalized by the decentralized economy. Therefore, the equilibrium number of referrals is inefficiently *low*.

Proofs for Appendix D

Equilibrium. In period t, the expected payoff for a seller of a referral in submarket i with referral fee $r_{i,t}$ and market tightness $\theta_{i,t}$ is

$$\Pi(\theta_{i,t}, r_{i,t}) = \frac{m(\theta_{i,t})}{\theta_{i,t}} r_{i,t} - c \tag{127}$$

and the expected payoff for consumers in submarket i is

$$V(\theta_{i,t}, r_{i,t}) = m(\theta_{i,t})(y(\theta_{i,t}, \alpha_{t-1}) - r_{i,t} - p) + (1 - m(\theta_{i,t}))(y_\mu - p)). \tag{128}$$

Consumers in submarket i choose a referral fee $r^*_{i,t}$ and market tightness $\theta^*_{i,t}$ that maximize $V(\theta_{i,t}, r_{i,t})$ subject to $\Pi(\theta_{i,t}, r_{i,t}) \leq c$ and $\theta_{i,t} \geq 0$, with complementary slackness. To induce participation by sellers in submarket i, i.e. $\theta_{i,t} > 0$, the constraint $\Pi(\theta_{i,t}, r_{i,t}) \leq c$ is binding:

$$\frac{m(\theta_{i,t})}{\theta_{i,t}} r_{i,t} = c. \tag{129}$$

Using (129) to replace $r_{i,t}$ in $V(\theta_{i,t}, r_{i,t})$, the choice of a consumer in submarket i is

$$\theta^*_{i,t} = \arg \max_{\theta_{i,t} \in \mathbb{R}_+} (m(\theta_{i,t})(y(\theta_{i,t}, \alpha_{t-1}) - y_\mu) + y_\mu - p - c\theta_{i,t}). \tag{130}$$

Differentiating with respect to $\theta_{i,t}$, the first-order condition of this problem is

$$m'(\theta_{i,t})(y(\theta_{i,t}, \alpha_{t-1}) - y_\mu) + m(\theta_{i,t}) \frac{\partial y(\theta_{i,t}, \alpha_{t-1})}{\partial \theta_{i,t}} - c = 0. \tag{131}$$

In symmetric equilibrium, $\theta^*_{i,t} = \theta^*_t$ for all submarkets i and θ^*_t satisfies

$$m'(\theta_t)(y(\theta_t, \alpha_{t-1}) - y_\mu) + m(\theta_t) \frac{\partial y(\theta_t, \alpha_{t-1})}{\partial \theta_t} = c. \tag{132}$$

In steady state, $\theta_t = \theta_{t-1} = \theta$ and $\alpha_t = \alpha_{t-1} = \alpha$ and any equilibrium (θ^*, α^*) satisfies

$$\underbrace{\eta_m(\theta)}_{\text{matching elasticity}} + \underbrace{\frac{\frac{\partial y(\theta,\alpha)}{\partial \theta}\theta}{s(\theta)}}_{\text{direct surplus elasticity}} = \underbrace{\frac{c\theta}{m(\theta)s(\theta)}}_{\text{surplus share of referral sellers}} \tag{133}$$

where the expected match surplus is $s(\theta) = y(\theta, \alpha) - y_\mu$.

To solve for the equilibrium, we use the fact that the average quality of a traded good in period t is given by

$$(1-\alpha_t)x_H + \alpha_t x_L = g(\theta_t, \alpha_{t-1}) + (1 - m(\theta_t))y_\mu \tag{134}$$

where $g(\theta_t, \alpha_{t-1}) \equiv m(\theta_t)y(\theta_t, \alpha_{t-1})$. Using the fact that $\alpha_t = \mu e^{-\theta_t(1-\alpha_{t-1})}$,

$$g(\theta_t, \alpha_{t-1}) = x_H - \mu \Delta x e^{-\theta_t(1-\alpha_{t-1})} - e^{-\theta_t} y_\mu \tag{135}$$

where $\Delta x = x_H - x_L$. The first-order condition (132) is equivalent to

$$\frac{\partial g(\theta_t, \alpha_{t-1})}{\partial \theta_t} - m'(\theta_t)y_\mu - c = 0. \tag{136}$$

Differentiating (135) with respect to θ_t, this is equivalent to

$$(1-\alpha_{t-1})\mu \Delta x e^{-\theta_t(1-\alpha_{t-1})} - c = 0 \tag{137}$$

and the second-order condition is

$$-(1-\alpha_{t-1})^2 \mu \Delta x e^{-\theta_t(1-\alpha_{t-1})} < 0. \tag{138}$$

Using the fact that $\alpha_t = \mu e^{-\theta_t(1-\alpha_{t-1})}$, this is equivalent to

$$(1-\alpha_{t-1})\alpha_t = \frac{c}{\Delta x}. \tag{139}$$

In steady state, $\theta_t = \theta_{t-1} = \theta$ and $\alpha_t = \alpha_{t-1} = \alpha$ and any equilibrium α satisfies

$$-\alpha^2 + \alpha - \frac{c}{\Delta x} = 0 \tag{140}$$

as well as $\alpha = \mu e^{-\theta(1-\alpha)}$. Since $\mu \in (0,1)$, there are two solutions $\alpha \in (0,1)$ provided that $\frac{c}{\Delta x} < \frac{1}{4}$ and one solution if $\frac{c}{\Delta x} = \frac{1}{4}$. The two solutions are $\alpha = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{c}{\Delta x}}$. Since $\alpha < \mu$ for $\theta > 0$, if $\mu < \frac{1}{2}$ we obtain a unique steady state equilibrium:

$$\alpha^* = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{c}{\Delta x}} \tag{141}$$

and

$$\theta^* = \frac{1}{1-\alpha} \ln\left(\frac{\mu}{\alpha}\right). \tag{142}$$

Planner. Given $\alpha_0 = \mu \in (0,1)$, the planner chooses $\{\theta_t\}_{t=1}^{\infty}$ to maximize the total discounted social surplus per consumer:

$$\Omega(\{\theta_t\}_{t=1}^{\infty}) = \sum_{t=1}^{\infty} \beta^t \left(m(\theta_t) y(\theta_t, \alpha_{t-1}) + (1 - m(\theta_t)) y_\mu - c\theta_t\right) \tag{143}$$

subject to $\theta_t \geq 0$ and the law of motion for α_t :

$$\alpha_t = \mu e^{-\theta_t(1-\alpha_{t-1})} \tag{144}$$

The Lagrangian for this problem is

$$\mathcal{L} = \sum_{t=1}^{\infty} \beta^t \left(m(\theta_t)(y(\theta_t, \alpha_{t-1}) - y_\mu) + y_\mu - c\theta_t\right) + \lambda_t(\alpha_t - \mu e^{-\theta_t(1-\alpha_{t-1})}). \tag{145}$$

The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial \theta_t} = \beta^t (m'(\theta_t)(y(\theta_t, \alpha_{t-1}) - y_\mu) + m(\theta_t) \frac{\partial y(\theta_t, \alpha_{t-1})}{\partial \theta_t} - c) + \lambda_t (1 - \alpha_{t-1}) \mu e^{-\theta_t(1-\alpha_{t-1})} = 0 \tag{146}$$

$$\frac{\partial \mathcal{L}}{\partial \alpha_{t-1}} = \lambda_{t-1} + \beta^t m(\theta_t) \frac{\partial y(\theta_t, \alpha_{t-1})}{\partial \alpha_{t-1}} - \lambda_t \theta_t \mu e^{-\theta_t(1-\alpha_{t-1})} = 0 \tag{147}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = \alpha_t - \mu e^{-\theta_t(1-\alpha_{t-1})} = 0 \tag{148}$$

In steady state, $\theta_{t+1} = \theta_t = \theta$ and $\alpha_{t+1} = \alpha_t = \alpha$, so we have

$$\beta^t(m'(\theta)(y(\theta,\alpha) - y_\mu) + m(\theta)\frac{\partial y(\theta,\alpha)}{\partial \theta} - c) = -\lambda(1-\alpha)\mu e^{-\theta(1-\alpha)} \tag{149}$$

$$\lambda + \beta^t m(\theta)\frac{\partial y(\theta,\alpha)}{\partial \alpha} = \lambda\theta\mu e^{-\theta(1-\alpha)} \tag{150}$$

$$\alpha = \mu e^{-\theta(1-\alpha)} \tag{151}$$

Rearranging (149), we obtain

$$\lambda = \frac{-\beta^t(m'(\theta)(y(\theta,\alpha) - y_\mu) + m(\theta)\frac{\partial y(\theta,\alpha)}{\partial \theta} - c)}{(1-\alpha)\mu e^{-\theta(1-\alpha)}}, \tag{152}$$

and rearranging (150) delivers

$$\lambda = \frac{-\beta^t m(\theta)\frac{\partial y(\theta,\alpha)}{\partial \alpha}}{1-\theta\mu e^{-\theta(1-\alpha)}}. \tag{153}$$

Equating (152) and (153) yields

$$\frac{m'(\theta)(y(\theta,\alpha) - y_\mu) + m(\theta)\frac{\partial y(\theta,\alpha)}{\partial \theta} - c}{(1-\alpha)\mu e^{-\theta(1-\alpha)}} = \frac{m(\theta)\frac{\partial y(\theta,\alpha)}{\partial \alpha}}{1-\theta\mu e^{-\theta(1-\alpha)}}. \tag{154}$$

Rearranging, and substituting in (151), we obtain

$$m'(\theta)(y(\theta,\alpha) - y_\mu) + m(\theta)\left(\frac{\partial y(\theta,\alpha)}{\partial \theta} - \frac{\partial y(\theta,\alpha)}{\partial \alpha}\frac{(1-\alpha)\alpha}{1-\theta\alpha}\right) = c. \tag{155}$$

Implicitly differentiating $\alpha = \mu e^{-\theta(1-\alpha)}$, we have $\alpha'(\theta) = -(1-\alpha)\alpha/(1-\theta\alpha)$, and substituting $\alpha'(\theta)$ into (155) yields

$$m'(\theta)s(\theta) + m(\theta)\left(\frac{\partial y(\theta,\alpha)}{\partial \theta} + \frac{\partial y(\theta,\alpha)}{\partial \alpha}\alpha'(\theta)\right) = c. \tag{156}$$

Rearranging (156), we obtain (126). This is a necessary condition for efficiency.

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