Optimal Discriminatory Disclosure*

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Abstract

A seller of an indivisible good designs a selling mechanism for a buyer who knows the distribution of his valuation for the good but not the realization of his valuation. The seller can choose how much additional private information about his valuation that the buyer may access. If the buyer's valuation distributions are ranked by likelihood ratio dominance, then the seller's optimal disclosure policy has an interval structure. When price discrimination is not feasible, discriminatory disclosure—releasing different signals to buyer types with different initial information—cannot improve upon the maximal revenue achieved under non-discriminatory disclosure. When price discrimination is feasible, however, the optimal disclosure policy is in general discriminatory.

*Very preliminary. Please do not circulate. Comments are welcome.

1 Introduction

The allocation efficiency of the consumer market—matching products with consumers who value them most—critically depends on the information flow: how much firms know about the preferences of their consumers and how much consumers know about features of products available in the market. The information flow affects the distribution of market power and thus how the gain from trade is divided between firms and consumers. Hence, in order to manipute the market outcome to their favor, market participants may actively exert influence on the information structure. For example, consumers can selectively disclose their personal information, and decide whether to try out new products and leave feedback. Firms can either disclose (some) product features to a broad audience or target disclosure only for specific consumer groups.

This paper studies the role of information structure in a bilateral trade environment with one-sided incomplete information. The informed party (say the buyer) is endowed with some private information about the underlying state, but his initial private information is often incomplete and he can learn about the state over time. The uninformed party (say the seller), by controling the access to additional information, can influence the amount of additional information available for the buyer to learn subsequently. We assume that, as part of the selling mechanism, the seller can commit to disclosing additional private information to the buyer without observing its realization, and she can charge the buyer for the access to such information based on the latter's report of his private type. How should the seller design the information policy to maximize her revenue?

The buyer has two pieces of private information: his ex ante private information and the additional private information released by the seller. The more information the seller releases to the buyer, the more efficient is the final allocation, but the buyer may also possess more private information. If the seller cannot charge fee for information, she faces a trade-off between allocation efficiency and rent extraction (Ganuza, 2004; Bergemann and Pesendorfer, 2007). If the seller can charge fees for additional private information as is assumed here, Eso and Szentes (2007) use an indirect approach to show that under some regularity conditions the trade-off disappears and the seller gives up no information rent for the additional private information.[1] Li and Shi (2017) take the direct mechanism design approach to show that the absence of the above trade-off does not imply the optimality of full disclosure, because the seller can

[1]Krahmer and Strausz (2015) show that if the buyer's ex ante type is discrete so that the regularity condition is violated then the seller may also have to give rent to the released additional information.

use discriminatory disclosure to further reduce the information rent generated by the private information that the buyer already has at the time of contracting. Specifically, they show that binary partitions of the true valuation dominate full disclosure, by limiting the buyer's additional private information to only whether his true valuation is above or below some partition threshold, instead of allowing him to learn the exact valuation as under the perfect signal structure.

The key insight in Li and Shi (2017) can be understood as follows. Under full disclosure, the strike price of the option contracts offered by the seller simultaneously determines the allocation and defines the terms of trade. In contrast, under binary partitions, the allocation is determined by the partition threshold while the strike price only affects the terms of trade. Since the strike price can differ from the partition threshold, the seller can control the allocation and terms of trade separately. Therefore, with binary partitions, the seller acquires another instrument in additional to price to discriminate among buyer types to improve her revenue.

Although binary partitions can be effective in both creating trade surplus and extracting information rent, a monotone partition for the low type can be too informative for the deviating high type, generating a large information rent. Therefore, non-monotone partitioning may be needed for revenue maximizing especially when the likelihood ratios are large for the highest values. Our goal is to characterize optimal partitions and to answer the following important questions. When is monotone partition for the low type optimal? When and why is non-monotone partitioning needed to maximize revenue? When are optimal partitions discriminatory?

The assumption of first-order stochastic dominance is sufficient to prove the suboptimality of full disclosure, but it is too weak to for us to characterize optimal disclosure policy. In this paper we make the stronger assumption of likelihood ratio dominance and focus on the case of binary ex ante types. We obtain two main results. First, we show that the optimal disclosure policy has an interval structure, which nests the aforementioned binary partitions as a special case. Second, we show that the optimal disclosure policy is generally discriminatory, that is, the seller releases different amount/types of information to different ex ante buyer types. We also provide sufficient conditions for binary partitions to dominate all other (direct) disclosure policies. Our sufficient conditions impose suitable bounds on the level of the likelihood ratio for the highest value.

We are also interested the role of price discrimination in determining the optimal disclosure policy. We find that, when price discrimination is not feasible, discriminatory disclosure—releasing different signals to buyer types with different initial information—

cannot improve upon the maximal revenue achieved under non-discriminatory disclosure. When price discrimination is feasible, however, the optimal disclosure policy is in general discriminatory.

It is interesting to contrast this result to what is obtained in the literature on Bayesian persuasion of ex ante privately informed receiver. An important question there is whether and when discriminatory disclosure (also known as private disclosure) is equivalent to non-discriminatory disclosure (also known as public disclosure). Kolotilin, Mylovanov, Zapechelnyuk and Li (2017) show that if the type of the receiver (the buyer here) is independent of the sender's (the seller here) signal, then they are equivalent. This is connected to the irrelevance theorem established by Eso and Szentes (2007, 2017) in the pricing setting. Their theorem implies that if the buyer signal and the seller signal are independent then full (and thus non-discriminatory) disclosure is optimal.

If the type of the receiver is correlated with the sender's signal (as is assumed here) and the sender has state-independent preferences, Guo and Shmaya (2019) show that non-discriminatory disclosure can attain the sender's maximal payoff under discriminatory disclosure but the equivalence fails in general for non-optimal disclosure policies. The equivalence fails in general in our setup because of the interaction between price discrimination and information discrimination, which is absent in their model. On the other hand, if price discrimination is not allowed, then non-discriminatory disclosure also attains the sender's maximal payoff under discriminatory disclosure. Although the latter result is similar to Guo and Shmaya (2019), the underlying economic logic is different. In particular, here the seller's preference is not state-independent: she cares about not only the likelihood for the buyer to buy, but also at what price the transaction takes place.

This paper belongs to the literature on information disclosure in environments with ex ante private information, and the literature on dynamic mechanism design more generally. Recent contributions include Smolin (2019) who studies optimal disclosure policy for product attributes, and Krahmer (2019) who allows the seller to randomize information structures and to condition the selling mechanism on the realization. Bergemann, Bonatti and Smolin (2018) study how to design and sell information to a buyer with private information. Different from the setup here, they assume that the buyer's action choice does not have direct impact on the seller's payoff and the pricing rule cannot be contingent on the action taken by the buyer.

For the literature on information design and Bayesian persuasion initiated by Kamenica and Gentzkow (2011) (see also Aumann and Maschler, 1995), see the recent surveys by Bergemann and Morris (2019) and Kamenica (2019). Bergemann and Valimaki (2006) survey earlier literature on information in mechanism design, and Bergemann and Valimaki (2019) provides a recent surveys on on dynamic mechanism design.

(more papers to be added ....)

The rest of the paper is organized as follows. We first introduce the model in Section 2 by formally defining signal structures and disclosure policies and setting up the seller's optimization problem. In Section 3, we characterize the optimal disclosure policy. In particular, we prove that the optimal disclosure policy has an interval structure as in Guo and Shmaya (2019). Section 4 studies the equivalence of the optimal non-discriminatory disclosure policy and the optimal discriminatory disclosure policy. We show that the equivalence fails in general if the optimal disclosure policy has a strict interval structure. If price discrimination is not feasible, however, then the equivalence is restored. In Section 5, we generalize the analysis by allowing the seller to choose information structure that is dependent on the true state. In this case, we show that monotone partition is optimal. Section 6 concludes. All formal proofs are relegated to the appendix.

2 The Model

We study a two-period sequential screening model. A seller has one object for sale to a buyer. The seller and the buyer are risk-neutral, and do not discount. The buyer's valuation $\omega \in \Omega \equiv [\underline{\omega}, \overline{\omega}]$ for the good is initially unknown to both the buyer and the seller. The seller's reservation valuation is known to be c, with $c \in (\underline{\omega}, \overline{\omega})$.

At the beginning of period one, the buyer privately observes a signal $\theta \in \Theta$ about ω, which we refer to as his ex ante type. In this paper, we assume $\Theta = \{H, L\}$, with probability ϕ_H and $\phi_L = 1 - \phi_H$ respectively. For each $\theta = H, L$, let $F_\theta(\cdot)$ be the conditional distribution function over Ω, and we assume that $F_\theta(\cdot)$ has a positive and finite density $f_\theta(\cdot)$. Throughout the paper, we assume that type H is higher than type L in likelihood ratio order, i.e., $f_H(\omega)/f_L(\omega)$ is weakly increasing in ω. The seller controls information sources for ω and can release, *without observing*, a signal s about the buyer's true valuation ω at the beginning of period two for any reported type in period one. There is no disclosure cost to the seller.

The timing of our game is as follows. In period one, the seller announces and commits to a disclosure policy together with a selling mechanism, which we will describe in detail below. The buyer decides whether to participate and, if he does, he reports his ex ante type to the seller. In period two, the buyer privately receives a signal released according to the seller's disclosure policy, and reports the signal realization to the seller. The announced selling mechanism is then implemented.

A disclosure policy is modeled as a menu of signal structures, each associated with a reported type by the buyer. Following Li and Shi (2017), we focus on "direct disclosure" policies (see Section 5 for a discussion of more general disclosure policies). Formally, a (direct) signal structure $\langle S, \rho \rangle$ consists of a signal space S and a mapping $\rho : \Omega \to \Delta S$ that takes the true valuation ω to a distribution $\rho(\cdot|\omega)$ over S; a direct disclosure policy is then a menu σ that assigns a direct signal structure $\sigma(\theta)$ to each reported type θ.

A signal structure is *binary* if the signal space S contains only two elements, say s_- and s_+. An example of binary signal structures is *interval structure*: for any interval $[\underline{k}, \overline{k}] \subset [\underline{\omega}, \overline{\omega}]$, let $S = \{s_-, s_+\}$ and let the mapping $\rho(\cdot|\omega)$ be

$$\rho(s|\omega) = \begin{cases} 1 & \text{if} \quad s = s_- \text{ and } \omega \notin [\underline{k}, \overline{k}], \\ 1 & \text{if} \quad s = s_+ \text{ and } \omega \in [\underline{k}, \overline{k}], \\ 0 & \qquad\qquad \text{otherwise.} \end{cases} \tag{1}$$

If $\overline{k} = \overline{\omega}$, then the interval structure is reduced to the information structure which we call *binary partition*.

We restrict our attention to deterministic selling mechanisms (a^θ, p^θ), $\theta = H, L$. Under this restriction, there is no loss in focusing on binary signal structures. To see this note, note that for any signal structure that has more than two possible signal realizations, we can always separate the realizations by the strike price p^θ into two sets, those implying a posterior estimate of the true valuation greater than or equal to p^θ and the rest. This allows us to define a binary signal structure by merging all the realizations in the first set into a single "buy" signal and the realizations in the second set into a single "don't-buy" signal. With this new binary signal structure, type θ gets the same allocation and the same payoff under the original option contract (a^θ, p^θ) by truth-telling. Since the new binary signal structure is a garbling of original one, by Blackwell's sufficiency theorem, the type $\tilde{\theta}$ buyer, $j \neq i = H, L$, who deviates and mimics θ is weakly worse off with the new signal structure than with the original one under (a^θ, p^θ), and thus the incentive condition for type $\tilde{\theta}$ not to mimic θ remains satisfied. The seller's profit is unaffected with this change.

Since we can restrict to binary signal structures, a direct disclosure policy can be represented by a pair of probability mappings from the true valuation ω to a buy signal. Let $\sigma^\theta(\omega) \in [0, 1]$ be the probability of mapping ω to the buy signal for reported type θ, $\theta = H, L$.

For all $\theta, \tilde{\theta} = H, L$, denote the posterior estimate of a type θ buyer who reports $\tilde{\theta}$ and then observes the buy signal as

$$v_\theta^{\tilde{\theta}} = \frac{\int_{\underline{\omega}}^{\overline{\omega}} \omega \sigma^{\tilde{\theta}}(\omega) f_\theta(\omega) d\omega}{\int_{\underline{\omega}}^{\overline{\omega}} \sigma^{\tilde{\theta}}(\omega) f_\theta(\omega) d\omega}.$$

Similarly, denote the posterior estimate of a type θ_j buyer who reports θ_i and then observes the don't-buy signal as

$$u_\theta^{\tilde{\theta}} = \frac{\int_{\underline{\omega}}^{\overline{\omega}} \omega \left(1 - \sigma^{\tilde{\theta}}(\omega)\right) f_\theta(\omega) d\omega}{\int_{\underline{\omega}}^{\overline{\omega}} \left(1 - \sigma^{\tilde{\theta}}(\omega)\right) f_\theta(\omega) d\omega}.$$

It is straightforward to show that, under likelihood ratio dominance, $v_H^\theta \geq v_L^\theta$ and $u_H^\theta \geq u_L^\theta$, for each $\theta = H, L$.[2] By relabeling if necessary, we can assume $v_\theta^\theta \geq u_\theta^\theta$ for each $\theta = H, L$. Further, on the truth-telling path, without loss we can assume that both buyer types buy only upon observing the buy signal. Off the truth-telling path, a type L buyer who reports H either buys only at the buy signal or never buys, while a type H buyer who reports L may buy only at the buy signal or buy at both two signals.

2.1 Seller's optimization problem

We can thus write the seller's optimal direct disclosure problem as choosing the disclosure policy σ^θ and a selling mechanism (a^θ, p^θ), $\theta = H, L$, to maximize the profit

$$\sum_{\theta=H,L} \phi_\theta \left(a^\theta + \left(p^\theta - c\right) \int_{\underline{\omega}}^{\overline{\omega}} \sigma^\theta(\omega) f_\theta(\omega) d\omega \right), \tag{2}$$

[2] For each $\theta = H, L$, the density function $\sigma^\theta(\omega) f_H(\omega) / \int_{\underline{\omega}}^{\overline{\omega}} \sigma^\theta(w) f_H(w) dw$ dominates in likelihood ratio order the density function $\sigma^\theta(\omega) f_L(\omega) / \int_{\underline{\omega}}^{\overline{\omega}} \sigma^\theta(w) f_L(w) dw$. It then follows that $v_H^\theta \geq v_L^\theta$ because likelihood ratio dominance implies first order stochastic dominance. A similar argument shows that $u_H^\theta \geq u_L^\theta$.

subject to two IC constraints, two IR constraints, and price bounds on p^θ, $\theta = H, L$, so that truthful buyer types only buy upon observing the buy signal:

$$
\begin{aligned}
&- a^H + \int_{\underline{\omega}}^{\overline{\omega}} (\omega - p^H)\sigma^H(\omega) f_H(\omega) d\omega \\
&\quad \geq -a^L + \max\left\{ \int_{\underline{\omega}}^{\overline{\omega}} (\omega - p^L)\sigma^L(\omega) f_H(\omega) d\omega, \int_{\underline{\omega}}^{\overline{\omega}} (\omega - p^L) f_H(\omega) d\omega \right\}, \qquad (\mathrm{IC}_H) \\
&- a^L + \int_{\underline{\omega}}^{\overline{\omega}} (\omega - p^L)\sigma^L(\omega) f_L(\omega) d\omega \\
&\quad \geq -a^H + \max\left\{ \int_{\underline{\omega}}^{\overline{\omega}} (\omega - p^H)\sigma^H(\omega) f_L(\omega) d\omega, 0 \right\}, \qquad (\mathrm{IC}_L) \\
&- a^\theta + \left(v_\theta^\theta - p^\theta\right) \int_{\underline{\omega}}^{\overline{\omega}} \sigma^\theta(\omega) f_\theta(\omega) d\omega \geq 0, \ \theta = H, L, \qquad (\mathrm{IR}_\theta) \\
&u_\theta^\theta \leq p^\theta \leq v_\theta^\theta, \ \theta = H, L. \qquad (\mathrm{PB}_\theta)
\end{aligned}
$$

3 Optimal Discriminatory Disclosure

In what follows, we first identify binding constraints in the optimal solution to the (relaxed) problem.

3.1 Constraint analysis

In a dynamic mechanism design problem with exogenous full disclosure (e.g., Courty and Li, 2000), prices p^H and p^L determine cutoff rules in valuation for the purchase decision in period two, both on and off the truthful reporting path. As a result, under the weaker order of first order stochastic dominance, IR_H follows from IR_L and IC_H, and this is used to show that IR_L and IC_H bind while IC_L is satisfied. In contrast, in the present optimal direct disclosure problem, for IR_H to follow from IR_L and IC_H, we need

$$
\max\left\{ \int_{\underline{\omega}}^{\overline{\omega}} (\omega - p^L)\sigma^L(\omega) f_H(\omega) d\omega, \int_{\underline{\omega}}^{\overline{\omega}} (\omega - p^L) f_H(\omega) d\omega \right\} \geq \int_{\underline{\omega}}^{\overline{\omega}} (\omega - p^L)\sigma^L(\omega) f_L(\omega) d\omega.
$$

If $u_H^L \leq p^L$ so that in deviation type H buys only after receiving the buy signal, the above becomes

$$
\int_{\underline{\omega}}^{\overline{\omega}} (\omega - p^L)\sigma^L(\omega)(f_H(\omega) - f_L(\omega)) d\omega \geq 0, \tag{3}
$$

which does not necessarily hold even under the stronger assumption of likelihood ratio dominance. However, if

$$\int_{\underline{\omega}}^{\overline{\omega}} \sigma^L(\omega)(f_H(\omega) - f_L(\omega))d\omega \geq 0, \tag{4}$$

that is, if the signal structure σ^L for type L is such that a true type L buyer buys the good with a smaller probability than a deviating type H buyer, then (3) holds for $p^L \leq v_L^L$. This is because (3) is equivalent to

$$(v_H^L - p^L) \int_{\underline{\omega}}^{\overline{\omega}} \sigma^L(\omega) f_H(\omega) d\omega \geq (v_L^L - p^L) \int_{\underline{\omega}}^{\overline{\omega}} \sigma^L(\omega) f_L(\omega) d\omega,$$

which follows from (4) and $v_H^L \geq v_L^L$. In particular, if σ^L is given by a monotone partitioning and is therefore weakly increasing, then (4) holds, and thus IR_H is implied by IR_L, IC_H and PB_L.

Analogous to the standard relaxed problem with exogenous full disclosure, we consider a "relaxed problem" by dropping IC_L. Since here we need to choose the signal structures σ^H and σ^L, and we retain IR_H. As in the standard relaxed problem, we first establish that any solution to the relaxed problem has both IR_L and IC_H binding. The argument for why IC_H is binding is slightly complicated by the fact that we have retained IR_H in the relaxed problem.

Lemma 1 *At any solution to the relaxed problem, both IR_L and IC_H bind.*

The next hurdle in analyzing our relaxed problem is that we need to deal with the possibility of "double deviation" by type H: as already mentioned, a type H buyer who deviates and reports L may buy at both signals. This is tackled in the result below. We show that in characterizing the solution to the relaxed problem, we can restrict to no double deviation by type H.

Lemma 2 *At any solution to the relaxed problem, $u_H^L \leq p^L$.*

The idea behind Lemma 2 is simple. If double deviation by type H occurs at the solution to the relaxed problem, so that type H buys the good even after the don't-buy signal after the first deviation of misreporting as type L, the signal structure for type L must be given by a two-step function. But then double deviation by type H means that type L strictly prefers to buy after the buy signal. As a result, the seller could raise the profit by increasing the strike p^L for type L without affecting either IC_H or IR_H.

Combining Lemma 1 and Lemma 2, we can rewrite the objective function (2) in the relaxed problem as

$$\phi_H \int_{\underline{\omega}}^{\overline{\omega}} (\omega - c)\, \sigma^H(\omega) f_H(\omega) d\omega \\ + \int_{\underline{\omega}}^{\overline{\omega}} \left(\phi_L (\omega - c)\, f_L(\omega) - \phi_H \left(\omega - p^L\right) (f_H(\omega) - f_L(\omega))\right) \sigma^L(\omega) d\omega. \tag{5}$$

By Lemma 2, IR_H becomes (3). In choosing the two signal structures σ^H and σ^L and two strike prices p^H and p^L, the seller also faces the two PB_H and PB_L constraints, and the constraint of no double deviation by type H

$$u_H^L \leq p^L. \tag{ND_H}$$

Since $u_H^L \geq u_L^L$, the only part of PB_L constraints that still remains to be considered is $v_L^L \geq p^L$.

Since we have dropped IC_L in the relaxed problem, from the first integral in the the objective function (5), we have that the solution in σ^H is "efficient," given by $\sigma^H(\omega) = 1$ for all $\omega \geq c$ and 0 otherwise. The choice of the strike price p^H for type H is indeterminate as it does not appear in (5). However, it must satisfy PB_H and, together with the advance payment a^H, keep the truth-telling payoff of type H at the same level given by IC_H:

$$-a^H + \int_c^{\overline{\omega}} (\omega - p^H) f_H(\omega) d\omega = \int_{\underline{\omega}}^{\overline{\omega}} (\omega - p^L) \sigma^L(\omega) (f_H(\omega) - f_L(\omega)) d\omega.$$

Our next result establishes that there is a solution to the relaxed problem that satisfies the dropped constraint of IC_L, and is thus a solution to the original problem.[3] The intuition behind of the argument is simple. If a solution to the relaxed problem has the property that a deviating type L will buy only after receiving the buy signal, for example if $p^H = c$, and if it does not satisfy IC_L, then the rent to type H would be even higher than under the efficient and hence non-discriminatory disclosure policy for both types. This of course contradicts the assumption that we have found a solution

[3] Since p^H and a^H are indeterminate given that $\sigma^H(\omega) = 1$ for all $\omega \geq c$ and 0 otherwise, not all solutions to the relaxed problem satisfy IC_L. For example, if we set p^H to the conditional expectation of type H's valuation above c, then the solution to the relaxed problem may have $a^H < 0$, which clearly violates IC_L because IR_L binds by Lemma 1.

to the relaxed problem.

Lemma 3 *Any solution to the relaxed problem such that $p^H \leq v_H^L$ satisfies IC_L.*

We can now focus on the "residual" relaxed problem, which is choosing the signal structure σ^L and the strike price p^L for type L to maximize the second integral in (5), or

$$\int_{\underline{\omega}}^{\overline{\omega}} \left(\phi_L \left(\omega - c\right) f_L(\omega) - \phi_H \left(\omega - p^L\right) \left(f_H(\omega) - f_L(\omega)\right)\right) \sigma^L(\omega) d\omega, \tag{6}$$

subject to the constraint IR_H(equation 3) and the combined PB_L and ND_H constraints of

$$u_H^L \leq p^L \leq v_L^L. \tag{7}$$

3.2 Monotone partitioning

The binary partitions used in Li and Shi (2017) to show that full disclosure is suboptimal require σ^θ to have a threshold structure. Such partitions are monotone in that σ^θ is weakly increasing. Although they can be effective in both creating trade surplus and extracting information rent, the following example shows that non-monotone partitioning can achieve full surplus extraction. Thus, monotone partitioning is not optimal for this example.

Example 1 *Suppose that $\phi_L = \phi_H = \frac{1}{2}$, $\underline{\omega} = 0$, and $\overline{\omega} = 1$. The seller's reservation valuation $c = \frac{1}{2}$. type L has a uniform valuation distribution: $F_L(\omega) = \omega$. The valuation distribution of type H is also uniform except for an atom of size $\frac{1}{4}$ at the top:*

$$f_H(\omega) = \begin{cases} \frac{3}{4}\omega & \text{if} \quad \omega \in [0, 1) \\ 1 & \text{if} \quad \omega = 1. \end{cases}$$

Consider the following disclosure policy and selling mechanism. For type H, choose signal structure σ^H with $\sigma^H(\omega) = 1$ for any $\omega \geq c$ and $\sigma^H(\omega) = 0$ otherwise, set strike price $p^H = c$, and set advance payment $a^H = \frac{7}{32}$. For type L, choose

$$\sigma^L(\omega) = \begin{cases} 1 & \text{if} \quad \omega \in \left(\frac{1}{2}, 1\right) \\ 0 & \text{if} \quad \omega \in \left[0, \frac{1}{2}\right] \text{ or } \omega = 1, \end{cases}$$

set strike price $p^L = \frac{3}{4}$, and charge advance payment $a^L = 0$. Under these contracts and signal structures, type L will not mimic type H, and he buys only upon observing signal s_+ and receives zero expected payoff. A type H buyer will not mimic type L

because, after deviation, he buys only at signal s_+ and gets zero expected payoff since his posterior estimate when observing s_+ is $\frac{3}{4}$. This selling mechanism and disclosure policy together extract the full surplus.

In the above example, the atom in the valuation distribution of type H means that the likelihood ratio $f_H(\omega)/f_L(\omega)$ explode at the top. It captures the idea that a monotone partition for type L can be too informative for type H, generating a large information rent. Indeed, it is straightforward to show that, if the seller is restricted to binary monotone partitions for type L, the optimal partition threshold is equal to $\frac{5}{8}$, leaving an information rent of $\frac{3}{128}$ to type H. In contrast, by pooling the atom and lower realizations of ω together in the signal structure σ^L, the seller is able to extract the full surplus.[4]

Monotone partitions can only be optimal with suitable upper bounds on the likelihood ratio, as we show now. To simplifying notation, we define

$$\lambda(\omega) = \frac{f_H(\omega)}{f_L(\omega)},$$

for all $\omega \in [\underline{\omega}, \overline{\omega}]$, and

$$\Lambda(k_1, k_2) = \frac{F_H(k_2) - F_H(k_1)}{F_L(k_2) - F_L(k_1)}$$

for all $\underline{\omega} \leq k_1 < k_2 \leq \overline{\omega}$.

Proposition 1 *Suppose that $\lambda(\overline{\omega}) \leq \phi_L/\phi_H$ and $\max_\omega \lambda'(\omega) \leq 1/(\overline{\omega}-\underline{\omega})$. The optimal direct disclosure policy is a pair of binary monotone partitions.*

Although the conditions stated in Proposition 1 are restrictive, we provide an analytical example below to show how they can be satisfied.

Example 2 *For any $t \in [-1, 1]$, consider the family of density functions given by $h(\omega|t) = 1 + (2\omega - 1)t$ over $\omega \in [0, 1]$. Let $f_L(\omega) = h(\omega|t_L)$ and $f_H(\omega) = h(\omega|t_H)$, with $-1 < t_L < t_H \leq 1$. We have*

$$\lambda(\overline{\omega}) = \frac{1 + t_H}{1 + t_L}, \quad \max_{\omega \in [0,1]} \lambda'(\omega) = \frac{2(t_H - t_L)}{(\min\{1 - t_L, 1 + t_L\})^2}.$$

For any $c \in [0, 1)$, and for any $t_L \in (-1, 1)$, then so long as $\phi_L > \phi_H$, there exist values of t_H that satisfy the sufficient conditions in Proposition 1.

[4]The full-surplus extraction result exploits the fact that $F_L(\omega)$ has no atom at the top.

3.3 Regular case

Any solution to the residual relaxed problem falls in one of the two cases, depending on whether (4) holds or not.

In the "regular" case where (4) holds, at the solution IR_H must be slack. If (4) is strict, then since (6) increases with p^L, the solution must have $p^L = v^L_L$; if (4) holds with an equality, setting $p^L = v^L_L$ gives another solution to the residual relaxed problem.

In the "irregular" case where the opposite of (4) holds for the solution, if IR_H is slack the solution must have $p^L = u^L_H$ for otherwise a higher value of (6) could be obtained by decreasing p^L. If IR_H is binding, then (6) becomes

$$\int_{\underline{\omega}}^{\overline{\omega}} \phi_L (\omega - c) f_L(\omega) d\omega,$$

which is independent of p^L. If $p^L > u^L_H$, the seller can decrease p^L and increase a^L through binding IR_L to relax IC_H, which then makes it possible to increase the value of the objective in the above expression by changing σ^L, as (4) holding in the reverse direction implies that σ^L is inefficient. This contradiction means that in the irregular case, the solution satisfies $p^L = u^L_H$.

It is relatively straightforward to characterize the solution in the regular case. This is achieved in the following result. The optimal signal structure σ^L for type L turns out to take an interval form; that is, $\sigma^L(\omega) = 1$ if ω is in some interval $[\underline{k}^L, \overline{k}^L] \subset [\underline{\omega}, \overline{\omega}]$ and $\sigma^L(\omega) = 0$ otherwise. Moreover, we have $\underline{k}^L > c$. Since the optimal signal structure σ^H for type H takes a threshold form with threshold c, the optimal signal structures are represented by two "nested intervals," with $[\underline{k}^L, \overline{k}^L] \subset [c, \overline{\omega}]$.

Lemma 4 *At any regular solution, $p^L \geq c$. Further, there exist $\underline{k}^L$ and $\overline{k}^L$ satisfying $c < \underline{k}^L < \overline{k}^L \leq \overline{\omega}$ such that $\sigma^L(\omega) = 1$ if $\omega \in [\underline{k}^L, \overline{k}^L]$, and $\sigma^L(\omega) = 0$ otherwise.*

By the above result, we can represent the optimal signal structure for type L at a regular solution by two partition thresholds $\underline{k}^L$ and $\overline{k}^L$. The optimal partition may be either monotone or non-monotone. In other words, the optimal σ^L may take a threshold form, with $\overline{k}^L = \overline{\omega}$, or strict interval form, with $\overline{k}^L < \overline{\omega}$. The following result provides sufficient conditions for these two subcases.

Lemma 5 *At any regular solution, if $\phi_L/\phi_H \geq \lambda(\overline{\omega}) - \Lambda(c, \overline{\omega})$, then the optimal signal structure σ^L for type L has $\overline{k}^L = \overline{\omega}$; and if $\lambda''(\overline{\omega}) > 3\lambda'(\overline{\omega})/(\overline{\omega} - c)$, then for sufficiently small ϕ_L, the optimal σ^L has $\overline{k}^L < \overline{\omega}$.*

We are ready to present the main result in the regular case. We do so by first providing sufficient conditions for the solution to be regular. By likelihood ratio dominance there exists a unique $\omega_o \in (\underline{\omega}, \overline{\omega})$ such that $f_H(\omega_o) = f_L(\omega_o)$, or $\lambda(\omega_o) = 1$.

Proposition 2 *Suppose $\omega_o \leq c$. If there exists $\gamma > 0$ such that $\lambda(\omega) \geq 1 + \gamma(\omega - \omega_o)$ for all $\omega \in [\underline{\omega}, \overline{\omega}]$, then the optimal disclosure policy is a pair of nested intervals.*

To understand the sufficient conditions for a regular solution in Proposition 2, it is helpful to compare two increasing function $\lambda(\omega) - 1$ and $\omega - \omega_o$ for $\omega \in [\underline{\omega}, \overline{\omega}]$. Both functions pass 0 at $\omega = \omega_o$. For there to exist $\gamma > 0$ such that $\lambda(\omega) - 1 \geq \gamma(\omega - \omega_o)$, we must be able to "rotate" the function $\omega - \omega_o$ around ω_o such that it falls below $\lambda(\omega) - 1$ for $\omega \in [\underline{\omega}, \overline{\omega}]$. If $\lambda(\omega)$ is continuously differentiable at $\omega = \omega_o$, a necessary condition for this to happen is that $\lambda(\omega)$ is convex at $\omega = \omega_o$. Indeed, if $\lambda(\omega)$ is convex for all $\omega \in [\underline{\omega}, \overline{\omega}]$, we can set γ to the derivate of $\lambda(\omega)$ at ω_o to satisfy the sufficient condition. Here is an example with convex $\lambda(\omega)$:

Example 3 *Suppose $c = \omega_o \geq \frac{1}{2}$. Suppose $f_L(\omega) = 1$ and*

$$f_H(\omega) = \begin{cases} 1 + \alpha(\omega - \omega_o) & \text{if } \omega \geq \omega_o \\ 1 + \alpha\left(\frac{1-\omega_o}{\omega_o}\right)^2(\omega - \omega_o) & \text{if } \omega < \omega_o \end{cases}$$

with $\omega \in [0, 1]$, where $\alpha \in \left(0, \omega_o/(1-\omega_o)^2\right)$. The likelihood ratio $\lambda(\omega) = f_H(\omega)$ is convex for all $\omega \in [0,1]$. Moreover, the sufficient condition for $\overline{k}^L = \overline{\omega}$ in Lemma 5 is satisfied if $\alpha \leq \min\left\{2(\phi_L/\phi_H - 1), \omega_o/(1-\omega_o)^2\right\}$.

Example 2 in the previous subsection can be parameterized to satisfy the assumptions in Proposition 2.

Example 4 *Let $f_L(\omega) = 1 + (2\omega - 1)t_L$ and $f_H(\omega) = 1 + (2\omega - 1)t_H$ for $\omega \in [0, 1]$, with $-1 < t_L < t_H \leq 1$. We have*

$$\lambda(\omega) = \frac{1 + (2\omega - 1)t_H}{1 + (2\omega - 1)t_L}, \ \lambda'(\omega) = \frac{2(t_H - t_L)}{(1 + (2\omega - 1)t_L)^2}.$$

The sufficient conditions in Proposition 2 for regular solutions are therefore satisfied if $t_L \leq 0$, and $\omega_o = \frac{1}{2} \leq c$. Further, for any t_L such that $-\frac{3}{5} < t_L < 0$, there exist values of c such that $c \geq \frac{1}{2}$ and $\lambda''(1) > 3\lambda'(1)/(1-c)$. Lemma 5 then implies that the optimal $\overline{k}^L = \overline{\omega}$ for ϕ_L sufficiently close to 1 and $\overline{k}^L < \overline{\omega}$ for ϕ_L sufficiently close to 0.

The following example also satisfies the assumptions in Proposition 2.[5]

Example 5 *For any* $t > 0$*, consider the family of distribution functions given by* $H(\omega|t) = \omega^{\theta}$ *over* $\omega \in [0,1]$*. Let* $F_H(\omega) = H(\omega|t_H)$ *and* $F_L(\omega) = H(\omega|t_L)$*, with* $t_H > t_L > 0$*. We have*

$$\lambda(\omega) = \frac{t_H}{t_L}\omega^{t_H - t_L}, \;\; \lambda'(\omega) = \frac{t_H}{t_L}(t_H - t_L)\omega^{t_H - t_L - 1}.$$

The sufficient conditions in Proposition 2 for regular solutions are therefore satisfied if $t_H > t_L + 1$*, and* $\omega_o = (t_L/t_H)^{1/(t_H - t_L)} \leq c$*. Further, for any* $t_H > 4$*, there exist values of* c *such that* $c \geq \omega_o$ *and* $\lambda''(1) > 3\lambda'(1)/(1-c)$ *for sufficiently small* t_L*. Lemma 5 then implies that the optimal* $\overline{k}^L = \overline{\omega}$ *for* ϕ_L *sufficiently close to 1 and* $\overline{k}^L < \overline{\omega}$ *for* ϕ_L *sufficiently close to 0.*

The last example in this section illustrates that it is not necessary to have small ϕ_L in order for strict interval structure to be optimal for type L.

Example 6 *Suppose*

$$f_L(\omega) = \begin{cases} 1 - \frac{\omega_o}{1-\omega_o}(\omega - \omega_o) & \text{if } \omega \geq \omega_o \\ 1 - \frac{1-\omega_o}{\omega_o}(\omega - \omega_o) & \text{if } \omega < \omega_o \end{cases}$$

$$f_H(\omega) = \begin{cases} 1 + \frac{\omega_o}{1-\omega_o}(\omega - \omega_o) & \text{if } \omega \geq \omega_o \\ 1 + \frac{1-\omega_o}{\omega_o}(\omega - \omega_o) & \text{if } \omega < \omega_o \end{cases}$$

with $\omega \in [0,1]$*. Suppose* $c = \omega_o \geq \frac{1}{2}$*. The likelihood ratio* $\lambda(\omega)$ *is convex in* ω *for all* $\omega \in [0,1]$*, so the sufficient condition in Proposition 2 is satisfied. Moreover, if* $\omega_o \geq 3/5$*,*

$$\lambda''(1) = \frac{4\omega_o^2}{(1-\omega_o)^5} \geq \frac{6\omega_o}{(1-\omega_o)^4} = \frac{3\lambda'(1)}{1-\omega_o},$$

so the sufficient condition for $\overline{k}^L < \overline{\omega}$ *in Lemma 5 is satisfied. But* ϕ_L *is not necessary to be very small for strict interval to be optimal. For example, when* $c = \omega_o = 0.8$*, and* $\phi_L = 0.5$*, we have* $\underline{k}^L = 0.85$ *and* $\overline{k}^L = 0.97$*.*

[5] There is no parameterization of the example below that satisfies the sufficient conditions of Proposition 1, because $\max_\omega \lambda'(\omega) = +\infty$ if $t_H - t_L < 1$, and $\max_\omega \lambda'(\omega) = (t_H/t_L)(t_H - t_L) > 1$ if $t_H - t_L \geq 1$.

4 The Optimality of Nondiscriminatory Disclosure

frametitleThis section will investigate when nondiscriminatory disclosure can attain the revenue achieved by optimal discriminatory disclosure. Recall that the optimal information structure assigned to type H is a binary partition with threshold c.frametitle Throughout the discussion, we will hold this as given.

We first observe that equivalence holds if the optimal information structure assigned to type L is also a binary partition with threshold $\underline{k} \in (\underline{\omega}, \overline{\omega})$. To see this, consider non-discriminatory disclosure with common partition refined from binary partition $\{[\underline{\omega}, c], [c, \overline{\omega}]\}$ assigned to type H and binary partition $\{[\underline{\omega}, \underline{k}], [\underline{k}, \overline{\omega}]\}$ assigned to type L under optimal discriminatory disclosure:

$$\{[\underline{\omega}, c], [c, \underline{k}], [\underline{k}, \overline{\omega}]\},$$

and set $p^H = c$ and $p^L = \mathbb{E}_L[\omega|\omega \in [\underline{k}, \overline{\omega}]]$. Under this common partition, the on-path behavior of the two buyer types are the same as under optimal discriminatory disclosure: type H will buy if and only if $\omega \in [c, \underline{k}] \cup [\underline{k}, \overline{\omega}]$ and type L will buy if and only if $\omega \in [\underline{k}, \overline{\omega}]$. For off-path behavior, suppose type H deviates and pretends to be type L. By definition of p^L, $p^L > \underline{k}$ and thus the deviating type H buys if and only if $\omega \in [\underline{k}, \overline{\omega}]$, which is the same as under optimal discriminatory disclosure. Finally, a deviating type L will buy off-path if and only if $\omega \in [c, \underline{k}] \cup [\underline{k}, \overline{\omega}]$, which also coincides with their behavior under optimal discriminatory disclosure. Therefore, non-discriminatory disclosure with common refined partition can replicate both on- and off-path behavior for both buyer types, and thus attain the same revenue as the optimal discriminatory disclosure.

Equivalence may fail, however, if the optimal signal structure assigned to type L is a strict interval structure $[\underline{k}, \overline{k}] \subset [\underline{\omega}, \overline{\omega}]$ with $\overline{k} < \overline{\omega}$. The reason for the failure is as follows. Consider the following non-discriminatory disclosure with common partition refined from $\{[\underline{\omega}, c], [c, \overline{\omega}]\}$ and $\left\{[\underline{\omega}, \underline{k}] \cup [\overline{k}, \overline{\omega}], [\underline{k}, \overline{k}]\right\}$:

$$\left\{[\underline{\omega}, c], [c, \underline{k}] \cup [\overline{k}, \overline{\omega}], [\underline{k}, \overline{k}]\right\}.$$

Type H follows recommendation off path only if

$$\mathbb{E}_H\left[\omega|\omega \in [c, \underline{k}] \cup [\overline{k}, \overline{\omega}]\right] \leq p_L.$$

In contrast, under discriminatory disclosure, type H follows recommendation off path only if

$$\mathbb{E}_H\left[\omega|\omega\in[\underline{\omega},\underline{k}]\cup[\overline{k},\overline{\omega}]\right]\leq p_L.$$

Since

$$\mathbb{E}_H\left[\omega|\omega\in[c,\underline{k}]\cup[\overline{k},\overline{\omega}]\right]>\mathbb{E}_H\left[\omega|\omega\in[\underline{\omega},\underline{k}]\cup[\overline{k},\overline{\omega}]\right],$$

it is easier under discriminatory disclosure to provide type H incentives to follow recommendation off path. Note that if

$$\mathbb{E}_H\left[\omega|\omega\in[c,\underline{k}]\cup[\overline{k},\overline{\omega}]\right]>p^L\geq\mathbb{E}_H\left[\omega|\omega\in[\underline{\omega},\underline{k}]\cup[\overline{k},\overline{\omega}]\right],$$

the deviating type H buyer will buy more often off path and have higher deviating payoff under non-discriminatory disclosure. Therefore, the information rent for type H will be higher under non-discriminatory disclosure, leading to a lower revenue for the seller.

Example 7 *Let $c=\omega_o=0.8$ and $\phi_L=0.5$. Suppose the two distributions $F_L(\omega)$ and $F_H(\omega)$ have a common support $[0,1]$ with*

$$\begin{aligned} f_L(\omega) &= \begin{cases} 1-4(\omega-0.8) & \text{if } \omega\geq 0.8 \\ 1-\frac{1}{4}(\omega-0.8) & \text{if } \omega<0.8 \end{cases} \\ f_H(\omega) &= \begin{cases} 1+4(\omega-0.8) & \text{if } \omega\geq 0.8 \\ 1+\frac{1}{4}(\omega-0.8) & \text{if } \omega<0.8 \end{cases} \end{aligned}$$

Optimal signal structure for type L is an interval structure with $[\underline{k},\overline{k}]$, where $\underline{k}\approx 0.85$ and $\overline{k}\approx 0.97$. The optimal price p^L is given by $p^L=\mathbb{E}_H\left[\omega|\omega\in[\underline{k},\overline{k}]\right]$. It is straightforward to verify that

$$\mathbb{E}_H\left[\omega|\omega\in[0,\underline{k}]\cup[\overline{k},1]\right]<p^L \quad \text{but} \quad \mathbb{E}_H\left[\omega|\omega\in[c,\underline{k}]\cup[\overline{k},1]\right]>p^L.$$

Therefore, with nondiscriminatory disclosure $\left\{[0,c],[c,\underline{k}]\cup[\overline{k},1],[\underline{k},\overline{k}]\right\}$, the deviating type H will buy at $\omega\in[c,\underline{k}]\cup[\overline{k},1]$, in contrast to the case with discriminatory disclosure.

We conclude this section by discussing an interesting interaction between price discrimination and information discrimination. Suppose that the seller must offer the same contract (a,p) to both types, but she can still discriminate by offering different information structures $\rho_i(\omega)$ to different buyer types θ_i, $i\in\{L,H\}$. Without price discrimination, is discrimination through information disclosure effective in increasing the seller's revenue?

The answer is no. To see this, suppose the optimal signal structure assigned to type L is an interval structure $[\underline{k}, \overline{k}] \subset [\underline{\omega}, \overline{\omega}]$ with $c < \underline{k} < \overline{k} < \overline{\omega}$, and the optimal signal structure for type H is a binary partition with threshold c. Suppose further that the incentive constraint for type H is binding. The binding IC_H constraint implies that type H is indifferent between receiving binary partition

$$\{[\underline{\omega}, c], [c, \overline{\omega}]\}$$

and receiving interval structure

$$\left\{[\underline{\omega}, \underline{k}] \cup [\overline{k}, \overline{\omega}], [\underline{k}, \overline{k}]\right\}.$$

Since there is no price discrimination, the terms of trade is the same for both types. It follows that when deviating to report type L, type H gets zero expected payoff by buying at $\omega \in [c, \underline{k}] \cup [\overline{k}, \overline{\omega}]$. Therefore, if we replace the optimal discriminatory disclosure by non-discriminatory disclosure with the common refined partition

$$\left\{[\underline{\omega}, c], [c, \underline{k}] \cup [\overline{k}, \overline{\omega}], [\underline{k}, \overline{k}]\right\},$$

the off-path behavior for type H will be essentially the same in that both disclosure policies will yield the same information rent for type H. Therefore, the revenue achieved by optimal discriminatory disclosure is also attainable by non-discriminatory disclosure.

The above discussion assumes that IC_H and IR_L bind in the optimal solution. This seemly intuitive feature of the optimal solution turns out nontrivial to establish. Formally, we prove the following proposition in the Appendix:

Proposition 3 *Suppose that the seller must offer the same contract to both buyer types, but she can offer different information structures to different buyer types. If* $c \geq \max\{\omega_o, \mathbb{E}_H[\omega]\}$ *where* $\omega_o \in (\underline{\omega}, \overline{\omega})$ *is the rotation point such that* $f_H(\omega_o) = f_L(\omega_o)$*, then the optimal solution is a pair of nested intervals, and both* IR_L *and* IC_H *constraints bind. Moreover, non-discriminatory disclosure is optimal.*

The sufficient condition of $c \geq \max\{\omega_o, \mathbb{E}_H[\omega]\}$ in the above proposition is to ensure that both IR_L and IC_H constraints bind in the optimal solution. As long as these two constraints are binding and the seller cannot offer different contract to different buyer types, then it follows from the discussion preceding this proposition that non-discriminatory disclosure is optimal.

5 General Disclosure

By definition, a direct disclosure policy is a mapping $\Theta \times \Omega \to \Delta S$ from reported ex ante type $\tilde{\theta} \in \Theta$ and true valuation $\omega \in \Omega$ to a distribution over the signal space S. Because true valuation ω is correlated with the ex ante type θ, each signal structure in a direct disclosure policy implicitly depends on the ex ante type θ. In this section, we will consider a more general disclosure policy $\Theta \times \Theta \times \Omega \to \Delta S$, which is a mapping from reported type $\tilde{\theta} \in \Theta$, true ex ante type $\theta \in \Theta$ and true valuation $\omega \in \Omega$ to a signal distribution over S. That is, we allow the signal structure to explicitly depend on θ.

As in Section 2, we can focus on binary signal structures with signal space $\{s_+, s_-\}$, and use $v_i^j(s_+)$ and $v_i^j(s_-)$, with $i, j = H, L$, to denote the posterior estimates of type θ_i who observes realizations s_+ and s_-, respectively, after reporting θ_j. Let $\Lambda_i^j(s_+) \in [0, 1]$ denote the probability that a type-θ_i buyer observes signal realization s_+ when he reports θ_j. Consistency requires that for each $i, j = H, L$,

$$\Lambda_i^j(s_+)v_i^j(s_+) + (1 - \Lambda_i^j(s_+))v_i^j(s_-) = \mu_i. \tag{8}$$

Regardless of whether a buyer lies or not, the (two-point) distribution of posterior estimates must preserve the true mean. Furthermore, the true valuation distribution $F(\cdot|\theta_i)$ must be dominated by any feasible two-point distribution $(v_i^j(s_+), v_i^j(s_-), \Lambda_i^j(s_+))$ in terms of second-order stochastic dominance. That is, for $i, j = H, L$,

$$\int_{\underline{\omega}}^{v} F(\omega|\theta_i)\, d\omega \geq \begin{cases} 0 & \text{if} \quad v \in [\underline{\omega}, v_i^j(s_-)), \\ (1 - \Lambda_i^j(s_+))(v - v_i^j(s_-)) & \text{if} \quad v \in [v_i^j(s_-), v_i^j(s_+)), \\ (1 - \Lambda_i^j(s_+))(v_i^j(s_+) - v_i^j(s_-)) + (v - v_i^j(s_+)) & \text{if} \quad v \in [v_i^j(s_+), \overline{\omega}]. \end{cases} \tag{9}$$

A general disclosure policy can be then written as

$$\left\{\sigma^j = (v_i^j(s_+), v_i^j(s_-), \Lambda_i^j(s_+))_{i,j=H,L} : \sigma^j \text{ satisfies (8) and (9)}\right\}.$$

Since the disclosure policy is allowed to depend on the buyer's true ex ante type θ, in characterizing the optimal policy it is without loss to assume that a deviating buyer type learns nothing about his true valuation ω. That is, without loss we write the degenerate signal distributions for the deviating types as $v_i^j(s_+) = v_i^j(s_-) = \mu_i$ with $\Lambda_i^j(s_+) = 1$ for $i \neq j = H, L$. As before, we focus on deterministic contracts. The seller's optimal general disclosure problem can now be written as choosing a disclosure policy $(v_i^i(s_+), v_i^i(s_-), \Lambda_i^i(s_+))$ and a selling mechanism (a^i, p^i), $i = H, L$, to maximize

$$\sum_{i=H,L} \phi_i \left(a^i + \left(p^i - c\right) \Lambda_i^i(s_+)\right)$$

subject to (8) and (9), two IC constraints, two IR constraints, and price bounds:

$$- a^i + (v_i^i(s_+) - p^i)\Lambda_i^i(s_+) \geq -a^j + \max\left\{\mu_i - p^j, 0\right\}, \; i \neq j = H, L, \tag{IC$_i$}$$
$$- a^i + (v_i^i(s_+) - p^i)\Lambda_i^i(s_+) \geq 0, \; i = H, L, \tag{IR$_i$}$$
$$v_i^i(s_-) \leq p^i \leq v_i^i(s_+), \; i = H, L \tag{PB$_i$}$$

We say a binary signal structure σ^i for reported type $i = H, L$ is a "generalized" monotone partition if there is some threshold $k^i \in [\underline{\omega}, \overline{\omega}]$ such that $v_i^i(s_+) = \mu_i^+(k^i)$ and

$$v_i^i(s_-) = \mu_i^-(k^i) \equiv \frac{\int_{\underline{\omega}}^{k^i} \omega f\left(\omega|\theta_i\right) d\omega}{F(k^i|\theta_i)},$$

and $\Lambda_i^i(s_+) = 1 - F(k^i|\theta_i)$ and $\Lambda_j^i(s_+) = 1$ for $j \neq i = H, L$. That is, like a monotone partition analyzed in Section 2, σ^i allows the truthful type i to privately learn whether his true valuation ω is above some threshold k^i or not, but gives no information to the deviating type j.

The following result shows that a generalized monotone partition is the most informative to a truthful type among all binary signal structures that satisfy conditions (8) and (9). That is, for any $(v_i^i(s_+), v_i^i(s_-), \Lambda_i^i(s_+))$, there is a generalized monotone partition with some threshold k^i such that $(\mu_i^+(k^i), \mu_i^-(k^i), 1 - F(k^i|\theta_i))$ is a mean-preserving spread of $(v_i^i(s_+), v_i^i(s_-), \Lambda_i^i(s_+))$. Since no information is disclosed to the deviating type, the seller can use the generalized partition instead to increase the trade surplus with type θ_i. Thus, it is without loss to restrict to generalized monotone partitions in solving the optimal general disclosure policy. Unlike the results in the previous sections, we only need that $F(\cdot|H)$ first-order stochastic dominates $F(\cdot|\theta_L)$, which implies $\mu_H > \mu_L$.

Lemma 6 *There is a pair of generalized monotone partitions that solves the optimal general disclosure problem.*

Suppose first $\mu_H \leq \mu_L^+(c)$. Consider generalized monotone partitions with $k^H = k^L = c$, and option contracts with $a^H = (\mu_H^+(c) - c)(1 - F(c|\theta_H))$ and $p^H = c$, and $a^L = 0$ and $p^L = \mu_L^+(c)$. Since $\mu_H \leq \mu_L^+(c)$, a deviating type θ_H would not buy at

$p^L = \mu_L^+(c)$ and thus get zero information rent. In mimicking type θ_H, type θ_L gets the payoff of

$$-(\mu_H^+(c) - c)(1 - F(c|\theta_H)) + \max\{\mu_L - c, 0\} < 0,$$

where the inequality follows from either $\mu_L \leq c$ or $c < \mu_L < \mu_H$.

For the remainder of this section, we assume that $\mu_H > \mu_L^+(c)$. The following result is a counterpart of Lemma 2 and gives a characterization of the binding constraints with generalized monotone partitions.

Lemma 7 *Suppose that $\mu_H > \mu_L^+(c)$. At any solution to the optimal general disclosure problem with generalized monotone partitions, IC_H and IR_L bind.*

Since IR_L is binding, the seller's profit from type θ_L is equal to the trade surplus with this type, given by

$$T_L(k^L) \equiv \left(\mu_L^+(k^L) - c\right)(1 - F(k^L|\theta_L)).$$

Since IC_H is binding, the information rent for type θ_H is

$$R_H(k^L, p^L) \equiv \max\left\{\mu_H - p^L, 0\right\} - \left(\mu_L^+(k^L) - p^L\right)(1 - F(k^L|\theta_L)).$$

Our next result uses Lemma 7 to reduce optimal general disclosure to a constrained maximization problem.

Lemma 8 *Suppose that $\mu_H > \mu_L^+(c)$. At any solution to the optimal general disclosure problem with generalized monotone partitions, $k^H = c$, and k^L and p^L maximize $\phi_L T_L\left(k^L\right) - \phi_H R_H\left(k^L, p^L\right)$ subject to $R_H\left(k^L, p^L\right) \geq 0$ and $p^L \in [\mu_L^-(k^L), \mu_L^+(k^L)]$.*

We can now reformulate the seller's optimal general disclosure problem as choosing a threshold k^i in the generalized monotone partition and the option contract (a^i, p^i) for each reported type $i = H, L$, subject to IC_i, IR_i, and PB_i. The following proposition characterizes the solution.

Proposition 4 *The optimal generalized monotone partition for type θ_H has threshold $k_*^H = c$, with strike price $p_*^H = c$; if $\mu_H \leq \mu_L^+(c)$, the optimal generalized monotone partition for type θ_L has $k_*^L = c$, with strike price $p_*^L = \mu_L^+(c)$, and if $\mu_H > \mu_L^+(c)$, it has threshold k_*^L that maximizes*

$$\phi_L\left(\mu_L^+(k^L) - c\right)(1 - F(k^L|\theta_L)) - \phi_H\left(\mu_H - \mu_L^+(k^L)\right) \tag{10}$$

subject to $\mu_L^+(k^L) \leq \mu_H$*, with* $k_*^L > c$ *and strike price* $p_*^L = \mu_L^+(k_*^L)$*.*

6 Concluding Remarks

In Li and Shi (2017), we have established the optimality of discriminatory disclosure when there are any finite number of ex ante types or there is a continuum of them. In this paper, we characterize two important qualitative features of the optimal disclosure policy. First, it admits an interval structure. Second, it is generally discriminatory. The interaction with price discrimination is crucial for information discrimination to be effective in extracting information rent and improving revenue.

Appendix: Omitted Proofs

6.1 Proofs for Section 3

Proof of Lemma 1. First, IR_L binds; otherwise raising a^L slightly would not affect any constraint in the relaxed problem and increase the profit given in (2). Second, IC_H binds. Suppose not. Since IR_L binds, the profit from type L in (2) can be rewritten as

$$\int_{\underline{\omega}}^{\overline{\omega}} (\omega - c)\sigma^L(\omega) f_L(\omega) d\omega.$$

Since IC_H is slack, the solution to the relaxed problem must have $\sigma^L(\omega) = 1$ for all $\omega \geq c$ and 0 otherwise. Given that IR_L binds, the above implies that the deviation payoff for type H is

$$\int_{c}^{\overline{\omega}} (\omega - p^L)(f_H(\omega) - f_L(\omega)) d\omega,$$

which is strictly positive because $F_H(\omega)$ first-order stochastic dominates $F_L(\omega)$. Thus, IR_H is also slack. But then the seller's profit can be increased by raising a^H, a contradiction. ■

Proof of Lemma 2. By way of contradiction, suppose instead $u_H^L > p^L$. First, we claim that in this case, there exists $k^L \in [\underline{\omega}, \overline{\omega}]$ such that $\sigma^L(\omega) = 1$ for all $\omega \geq k^L$ and 0 for $\omega < k^L$. Suppose this is not the case. Clearly, neither $\sigma^L(\omega) = 1$ for all ω nor $\sigma^L(\omega) = 0$ for all ω can be solution to the relaxed problem. Then, since σ^L is not a two-step function, the seller could modify it by keeping $\int_{\underline{\omega}}^{\overline{\omega}} \sigma^L(\omega) f_L(\omega) d\omega$ unchanged while marginally increasing v_L^L and decreasing u_L^L. By keeping p^L unchanged, and hence PB_L still satisfied, the seller can thus increase a^L without violating IR_L. Since by assumption $u_H^L > p^L$ and thus type H strictly prefers to buy regardless of the signal after the deviation, IC_H is unaffected by the modification in σ^L, but the seller's profit from type L in (2) would increase. This is a contradiction to optimality. Thus, σ^L is given by a two-step function with some threshold k^L.

Now, using $u_H^L > p^L$ and the binding IR_L and IC_H, we can write the seller's profit as

$$\phi_H \int_{\underline{\omega}}^{\overline{\omega}} (\omega - c)\,\sigma^H(\omega) f_H(\omega) d\omega + \phi_L \int_{k^L}^{\overline{\omega}} (\omega - c)\, f_L(\omega) d\omega$$
$$- \phi_H \left(\int_{\underline{\omega}}^{\overline{\omega}} (\omega - p^L) f_H(\omega) d\omega - \int_{k^L}^{\overline{\omega}} \left(\omega - p^L\right) f_L(\omega) d\omega \right).$$

Since σ^L is a cutoff rule with threshold k^L, we have $v_L^L \geq k^L \geq u_H^L > p^L$. By slightly increasing p^L, and correspondingly decreasing a^L to keep IR_L binding and increasing a^H to keep IC_H binding, the seller can increase the profit in the relaxed problem. These changes do not affect IR_H because type H's deviating payoff is at least the left-hand side of (3), by buying only after receiving the buy signal after misreporting as type L, which is strictly positive because σ^L is weakly increasing. This is a contradiction to optimality. ■

Proof of Lemma 3. Consider any solution to the relaxed problem with p^H such that $p^H \leq v_H^L$. Since $p^H \leq v_H^L$, using Lemma 2, we can write IC_L as

$$\int_{\underline{\omega}}^{\overline{\omega}} (\omega - p^L)(f_H(\omega) - f_L(\omega))\sigma^L(\omega) d\omega \leq \int_{c}^{\overline{\omega}} (\omega - p^H)(f_H(\omega) - f_L(\omega)) d\omega.$$

Suppose the above is violated. Then, consider the alternative of setting $\hat{\sigma}^L(\omega) = 1$ for $\omega \geq c$ and 0 otherwise, and setting $\hat{p}^L = p^H$. Together with $\hat{a}^L$ that binds IR_L, and then $\hat{a}^H$ that binds IC_H, this alternative satisfies (7), as well as (3) because $\hat{\sigma}^L$ is weakly increasing. However, given that $\sigma^L(\cdot)$ and p^L violate IC_L, we have

$$\int_{\underline{\omega}}^{\overline{\omega}} (\omega - p^L)\sigma^L(\omega)(f_H(\omega) - f_L(\omega)) d\omega > \int_{\underline{\omega}}^{\overline{\omega}} \left(\omega - \hat{p}^L\right) \hat{\sigma}^L(\omega)(f_H(\omega) - f_L(\omega)) d\omega.$$

From the second integral of (5), the seller's profit under $\hat{\sigma}^L(\cdot)$ and $\hat{p}^L$ is higher than under $\sigma^L(\cdot)$ and p^L. This contradicts the assumption that $\sigma^L(\cdot)$ and p^L solve the relaxed problem. ■

Proof of Proposition 1. We show that under the conditions stated in the proposition, the solution in $\sigma^L(\cdot)$ to the relaxed problem is a two-step function, with $\sigma^L(\omega) = 1$ for all $\omega \geq \underline{k}^L$ and 0 otherwise for some $\underline{k}^L$. The objective is (6). We relax the problem further by dropping (3) and the constraint $u_H^L \leq p^L$. The remaining constraint $p^L \leq v_L^L$ can be written as

$$\int_{\underline{\omega}}^{\overline{\omega}} \left(\omega - p^L\right) \sigma^L(\omega) f_L(\omega) d\omega \geq 0.$$

Let μ be the non-negative Lagrangian multiplier associated with the above constraint. Since the objective and the constraint are linear in $\sigma^L(\omega)$, the solution is $\sigma^L(\omega) = 1$ for all ω such that $\Upsilon(\omega) \geq 0$ and 0 otherwise, where

$$\Upsilon(\omega) = \phi_L(\omega - c) + (\omega - p^L)\left(\phi_H\left(1 - \lambda(\omega)\right) + \mu\right). \tag{11}$$

From (11), for any fixed p^L, using the two assumptions in the propositions and $\mu \geq 0$, we have $\Upsilon'(\omega) \geq 0$ for all $\omega \in [\underline{\omega}, \overline{\omega}]$. It follows that there exists some $\underline{k}^L$ such that $\sigma^L(\omega) = 1$ for all $\omega \geq \underline{k}^L$ and 0 otherwise.

Given that $\sigma(\cdot)$ is a monotone partition with a threshold $\underline{k}^L$, (6) is increasing in p^L for any $\underline{k}^L$. Thus, we have $p^L = v_L^L$. The dropped constraint of $u_H^L \leq p^L$ is also satisfied, as $u_H^L < v_L^L$. Finally, the solution to the relaxed problem satisfies (3) because $\sigma^L(\cdot)$ is weakly increasing. The proposition follows immediately from Lemma 3. ■

Proof of Lemma 4. We claim that $p^L \geq c$ in any regular solution. This is because if $p^L = v_L^L < c$, then the value of the objective function given by (6) is necessarily negative, as the trade surplus from type L is negative while the rent to type H is non-negative.

Next, since $p^L = v_L^L$ in the regular case, in the residual relaxed problem we must have (4), implying that IR_H is satisfied. It follows that the residual relaxed problem becomes choosing σ^L and p^L to maximize (6), subject to a single constraint $p^L \leq v_L^L$. Let μ be the non-negative Lagrangian multiplier associated with the above constraint. As in the proof of Proposition 1, the solution is $\sigma^L(\omega) = 1$ for all ω such that $\Upsilon(\omega) \geq 0$ and 0 otherwise, where $\Upsilon(\omega)$ is given in (11). Given that $p^L \geq c$, we have

$$\Upsilon(p^L) = \phi_L(p^L - c) \geq 0.$$

Further, $\Upsilon(\omega)$ can cross 0 only once for all $\omega > p^L$. To see the latter claim, note that for $\omega > p^L$, $\Upsilon(\omega)$ has the same sign as

$$\frac{\Upsilon(\omega)}{\omega - p^L} = \phi_L \frac{\omega - c}{\omega - p^L} + \phi_H (1 - \lambda(\omega)) + \mu.$$

The second term on the right-hand side of the above expression is decreasing in ω by likelihood ratio dominance, while the first term is non-decreasing because $p^L \geq c$. Therefore, $\Upsilon(\omega)$ can cross 0 only once and only from above for all $\omega > p^L$. Similarly, $\Upsilon(\omega)$ can cross 0 only once and only from below for all $\omega < p^L$. It follows that there exists an interval of valuations $[\underline{k}^L, \overline{k}^L] \subset [\underline{\omega}, \overline{\omega}]$ such that $\sigma^L(\omega) = 1$ if and only if $\omega \in [\underline{k}^L, \overline{k}^L]$.

Finally, to show that $\underline{k}^L > c$ by contradiction, suppose instead $\underline{k}^L \leq c$. Consider increasing $\underline{k}^L$ marginally and at the same time increase p^L so as to keep it equal to v_L^L. This weakly increases the trade surplus with type L, because the effect on the first term in (6) is

$$-\phi_L(\underline{k}^L - c) f_L(\underline{k}^L) \geq 0.$$

The effect on the second term in (6) without the negative sign is

$$-\phi_H \left(v_L^L - \underline{k}^L\right) \left(\Lambda(\underline{k}^L, \overline{k}^L) - \lambda(\underline{k}^L)\right) f_L(\underline{k}^L).$$

The above expression is negative, because $v_L^L > \underline{k}^L$, and because likelihood ratio dominance implies that the difference in the last bracket is positive, implying that the rent to type H is decreased. Therefore, the seller's profit increases, which contradicts optimality. ■

Proof of Lemma 5. To establish the sufficient condition for $\overline{k}^L = \overline{\omega}$, suppose that $\overline{k}^L < \overline{\omega}$ and consider increasing $\overline{k}^L$ marginally and at the same time increase p^L so as to keep it equal to v_L^L. The effect on the first term in (6) is

$$\phi_L(\overline{k}^L - c) f_L(\overline{k}^L).$$

The effect on the second term in (6) without the negative sign is

$$\phi_H \left(\overline{k}^L - v_L^L\right) \left(\lambda(\overline{k}^L) - \Lambda(\underline{k}^L, \overline{k}^L)\right) f_L(\overline{k}^L).$$

By likelihood ratio dominance, the difference in the last bracket is positive. Further, $\Lambda(\underline{k}^L, \overline{k}^L)$ is increasing in $\underline{k}^L$ for any fixed $\overline{k}^L > \underline{k}^L$. Since $v_L^L = p^L \geq c$ and $\underline{k}^L > c$ by Lemma 4, the overall effect is positive at $\overline{k}^L = \overline{\omega}$, and hence $\overline{k}^L = \overline{\omega}$, if the condition stated in the lemma is satisfied.

To establish the sufficient condition for $\overline{k}^L < \overline{\omega}$ when ϕ_L is close to 0, suppose that for all sufficiently small ϕ_L, we have $\overline{k}^L = \overline{\omega}$. Note that in the limit of $\phi_L = 0$, we have $\underline{k}^L = \overline{k}^L$; otherwise, the first term in the objective function (6) is 0 in the limit, but the second term is strictly positive, which would be a contradiction. Then, from the proof of Lemma 4, the first order condition with respect to $\underline{k}^L$ can be written as

$$\frac{\phi_L}{1-\phi_L}(\underline{k}^L - c) - \left(v_L^L - \underline{k}^L\right)\left(\Lambda(\underline{k}^L, \overline{\omega}) - \lambda(\underline{k}^L)\right) = 0.$$

The above first order condition holds with equality for ϕ_L sufficiently close to 0; otherwise, if $\underline{k}^L = \overline{k}^L = \overline{\omega}$, then the derivative of the objective function (6) with respect to $\underline{k}^L$, evaluated at $\underline{k}^L = \overline{k}^L = \overline{\omega}$ is linear in ϕ_L and hence strictly negative when ϕ_L is sufficiently close to 0, contradicting the assumption that $\underline{k}^L = \overline{k}^L = \overline{\omega}$.

Since the above first order condition holds for all ϕ_L sufficiently small and strictly positive, we can take derivatives with respect to ϕ_L. This yields

$$\frac{\underline{k}^L - c}{(1-\phi_L)^2} + \left(\frac{\phi_L}{1-\phi_L} + (v_L^L - \underline{k}^L)\lambda'(\underline{k}^L)\right)\frac{d\underline{k}^L}{d\phi_L}$$
$$- \left(2(v_L^L - \underline{k}^L)\eta_L(\underline{k}^L) - 1\right)\left(\Lambda(\underline{k}^L, \overline{\omega}) - \lambda(\underline{k}^L)\right)\frac{d\underline{k}^L}{d\phi_L} = 0,$$

where, to save notation, we have denoted the hazard rate of $F_L(\omega)$ as

$$\eta_L(\omega) = \frac{f_L(\omega)}{1 - F_L(\omega)}.$$

Evaluating at the limit of $\phi_L = 0$, and using $\lim_{\phi_L \to 0} \underline{k}^L = \overline{\omega}$ and

$$\lim_{\underline{k}^L \to \overline{\omega}} 2(v_L^L - \underline{k}^L)\eta_L(\underline{k}^L) = 1,$$

we have

$$\lim_{\phi_L \to 0} \frac{1}{2\eta_L(\underline{k})}\frac{d\underline{k}^L}{d\phi_L} = -\frac{\overline{\omega} - c}{\lambda'(\overline{\omega})}.$$

The derivative of the objective function given by (6) with respect to $\overline{k}^L$, when evaluated at $\overline{k}^L = \overline{\omega}$, has the same sign as

$$Y(\phi_L) \equiv \frac{\phi_L}{1-\phi_L}(\overline{\omega} - c) - \left(\overline{\omega} - v_L^L\right)\left(\lambda(\overline{\omega}) - \Lambda(\underline{k}^L, \overline{\omega})\right).$$

We have $Y(0) = 0$. Taking derivative with respect to ϕ_L, we have

$$Y'(\phi_L) = \frac{\overline{\omega} - c}{(1-\phi_L)^2} + \big((v_L^L - \underline{k}^L)\lambda(\overline{\omega}) - (\overline{\omega} - v_L^L)\lambda(\underline{k}^L) + (\overline{\omega} + \underline{k}^L - 2v_L^L)\Lambda(\underline{k}^L, \overline{\omega})\big)\frac{d\underline{k}^L}{d\phi_L}.$$

Using the expression of $d\underline{k}^L/d\phi_L$, we have that $Y'(\phi_L)$ has the same sign as

$$(\overline{\omega} - c)\left(\frac{\phi_L}{1-\phi_L} + (v_L^L - \underline{k}^L)\lambda'(\underline{k}^L) - \big(2(v_L^L - \underline{k}^L)\eta_L(\underline{k}^L) - 1\big)\big(\Lambda(\underline{k}^L, \overline{\omega}) - \lambda(\underline{k}^L)\big)\right)$$
$$+ (\underline{k}^L - c)\big((v_L^L - \underline{k}^L)\lambda(\overline{\omega}) - (\overline{\omega} - v_L^L)\lambda(\underline{k}^L) + (\overline{\omega} + \underline{k}^L - 2v_L^L)\Lambda(\underline{k}^L, \overline{\omega})\big). \qquad (12)$$

The above expression is 0 in the limit of $\phi_L = 0$, and hence $Y'(0) = 0$, because $\lim_{\phi_L \to 0} v_L^L = \lim_{\phi_L \to 0} \underline{k}^L = \overline{\omega}$, and

$$\lim_{\underline{k}^L \to \overline{\omega}} 2(\overline{\omega} - v_L^L)\eta_L(\underline{k}^L) = 1.$$

The derivative of (12) with respect to ϕ_L, after dropping the terms that go to 0 at higher orders, is given by

$$(\overline{\omega} - c) + \big((\overline{\omega} - c)\lambda''(\overline{\omega}) - 2(\overline{\omega} - \underline{k}^L)\eta_L(\underline{k}^L)\lambda'(\overline{\omega})\big)\frac{1}{2\eta_L(\underline{k}^L)}\frac{d\underline{k}^L}{d\phi_L}.$$

By the limit expression of $d\underline{k}^L/d\phi_L$, the above is strictly negative at $\phi_L = 0$, and hence $Y'(\phi_L) < 0$ for sufficiently small ϕ_L, if the condition stated in the lemma is satisfied. Thus, for ϕ_L sufficiently small, $Y(\phi_L)$ is strictly negative, contradicting the assumption that $\overline{k}^L = \overline{\omega}$. ■

Proof of Proposition 2. From (6), the irregular case can be ruled out in any solution to the residual relaxed problem if, for all σ^L such that the reverse of (4) holds, the first term in (6) is non-positive. This is because having a solution in the irregular case would be worse for the seller than excluding type L altogether, as the second term in (6) is non-negative by IR$_H$. Fix any σ^L that violates (4). Since $\omega_o \leq c$, it suffices if

$$\int_{\underline{\omega}}^{\overline{\omega}} (\omega - \omega_o)\sigma^L(\omega) f_L(\omega) d\omega \leq 0.$$

By assumption, $\gamma(\omega - \omega_o) f_L(\omega) \leq f_H(\omega) - f_L(\omega)$ for all $\omega \in [\underline{\omega}, \overline{\omega}]$. Thus,

$$\gamma \int_{\underline{\omega}}^{\overline{\omega}} (\omega - \omega_o)\sigma^L(\omega) f_L(\omega) d\omega \leq \gamma \int_{\underline{\omega}}^{\overline{\omega}} \sigma^L(\omega)(f_H(\omega) - f_L(\omega)) d\omega.$$

It follows immediately that the solution cannot be irregular, and Lemma 4 applies. ■

6.2 Proofs for Section 4

This section is devoted to prove Proposition 3. Suppose that the seller must offer the same contract (a, p) to both types, but she can still discriminate by offering different information structures $\rho_i(\omega)$ to different buyer types θ_i, $i \in \{L, H\}$. The seller's maximization problem is then

$$\max_{(\rho_L(\omega),\rho_H(\omega),a,p)} a + (p-c)\left[\phi_L \int_{\underline{\omega}}^{\overline{\omega}} \rho_L(\omega)\, dF_L(\omega) + \phi_H \int_{\underline{\omega}}^{\overline{\omega}} \rho_H(\omega)\, dF_H(\omega)\right],$$

subject to two IC constraints:

$$\int_{\underline{\omega}}^{\overline{\omega}} (\omega - p)\, \rho_H(\omega)\, dF_H(\omega) \quad \geq \quad \max\left\{\int_{\underline{\omega}}^{\overline{\omega}} (\omega - p)\, \rho_L(\omega)\, dF_H(\omega), \mathbb{E}_H[\omega] - p\right\} IC_H$$

(13)

$$\int_{\underline{\omega}}^{\overline{\omega}} (\omega - p)\, \rho_L(\omega)\, dF_L(\omega) \geq \max\left\{\int_{\underline{\omega}}^{\overline{\omega}} (\omega - p)\, \rho_H(\omega)\, dF_L(\omega), \mathbb{E}_L[\omega] - p\right\} IC_L \quad (14)$$

two IR constraints:

$$-a + \int_{\underline{\omega}}^{\overline{\omega}} (\omega - p)\, \rho_H(\omega)\, dF_H(\omega) \quad \geq \quad 0 IR_H$$

(15)

-a+$\int_{\underline{\omega}}^{\overline{\omega}} (\omega - p)\, \rho_L(\omega)\, dF_L(\omega) \geq 0 IR_L$ (16)

and bounds on p so truthful types buy iff upon observing the "buy" signal:

$$\mathbb{E}_i[\omega | \text{“not buy”},\ \rho_i] \leq p \leq \mathbb{E}_i[\omega | \text{“buy”},\ \rho_i]. \qquad (\text{PB}_i)$$

Our strategy is to show that in the optimal solution to this problem, the disclosure policy is a pair of nested intervals, and IC$_H$ constraint binds. The claim of the proposition then follows from the discussion preceding the proposition.

Consider the relaxed problem with IC_H, IR_L constraints and price bounds only (i.e., dropping both IC_L and IR_H). Then IR_L constraint must bind (otherwise we can raise a to increase revenue). The objective can be rewritten as

$$
\begin{aligned}
& a+(p-c)\left[\phi_L \int_{\underline{\omega}}^{\overline{\omega}} \rho_L(\omega)\, dF_L(\omega)+\phi_H \int_{\underline{\omega}}^{\overline{\omega}} \rho_H(\omega)\, dF_H(\omega)\right] \\
= & \int_{\underline{\omega}}^{\overline{\omega}}(\omega-p)\, \rho_L(\omega)\, dF_L(\omega)+(p-c)\left[\phi_L \int_{\underline{\omega}}^{\overline{\omega}} \rho_L(\omega)\, dF_L(\omega)+\phi_H \int_{\underline{\omega}}^{\overline{\omega}} \rho_H(\omega)\, dF_H(\omega)\right] \\
= & \int_{\underline{\omega}}^{\overline{\omega}}(\omega-c)\, \rho_L(\omega)\, dF_L(\omega)-(p-c) \int_{\underline{\omega}}^{\overline{\omega}} \rho_L(\omega)\, dF_L(\omega) \\
& +(p-c)\left[\phi_L \int_{\underline{\omega}}^{\overline{\omega}} \rho_L(\omega)\, dF_L(\omega)+\phi_H \int_{\underline{\omega}}^{\overline{\omega}} \rho_H(\omega)\, dF_H(\omega)\right] \\
= & \int_{\underline{\omega}}^{\overline{\omega}}(\omega-c)\, \rho_L(\omega)\, dF_L(\omega)+\phi_H(p-c)\left[\int_{\underline{\omega}}^{\overline{\omega}} \rho_H(\omega)\, dF_H(\omega)-\int_{\underline{\omega}}^{\overline{\omega}} \rho_L(\omega)\, dF_L(\omega)\right]
\end{aligned}
$$

Therefore, we can write the seller's relaxed problem as

$$
\max_{(\rho_L(\omega),\rho_H(\omega),p)} \int_{\underline{\omega}}^{\overline{\omega}}(\omega-c)\, \rho_L(\omega)\, dF_L(\omega)+\phi_H(p-c)\left[\int_{\underline{\omega}}^{\overline{\omega}} \rho_H(\omega)\, dF_H(\omega)-\int_{\underline{\omega}}^{\overline{\omega}} \rho_L(\omega)\, dF_L(\omega)\right]
$$

subject to IC_H

$$
\int_{\underline{\omega}}^{\overline{\omega}}(\omega-p)\, \rho_H(\omega)\, dF_H(\omega) \geq \max\left\{\int_{\underline{\omega}}^{\overline{\omega}}(\omega-p)\, \rho_L(\omega)\, dF_H(\omega), \mathbb{E}_H[\omega]-p\right\}.
$$

and the price bounds

$$
\mathbb{E}_i[\omega \mid \text{“not buy”},\, \rho_i] \leq p \leq \mathbb{E}_i[\omega \mid \text{“buy”},\, \rho_i].
$$

The proof proceeds with a sequence of claims.

Claim 1 *The value of the relaxed (and in fact also the original) problem is strictly higher than* $\int_c^{\overline{\omega}}(\omega-c)\, dF_L(\omega)$.

Proof. Consider the following contract and disclosure policy: $p=c$ and $a=\int_c^{\overline{\omega}}(\omega-c)\, dF_L(\omega)$, and $\rho_L(\omega)=\rho_H(\omega)=1$ iff $\omega \geq c$. It is easy to verify that all constraints are satisfied. It yields a revenue of $\int_c^{\overline{\omega}}(\omega-c)\, dF_L(\omega)$. Now suppose we raise the price from p to $\hat{p}=\mathbb{E}_L[\omega \mid \omega \geq c]$ and reduce a to 0, but retain the disclosure policy. Then the seller can still collect $\phi_L \int_c^{\overline{\omega}}(\omega-c)\, dF_L(\omega)$ from type θ_L, but strictly more than $\phi_H \int_c^{\overline{\omega}}(\omega-c)\, dF_L(\omega)$ from type θ_H by likelihood ratio dominance. ■

The strict lower bound we derive above will be repeatedly used in the subsequent proof.

Claim 2 *In the optimal solution to the relaxed problem, $p \geq c$.*

Proof. Suppose by contradiction that the solution (ρ_H, ρ_L, p) to the relaxed problem satisfies $p < c$. Then the value of the relaxed program must satisfy

$$\begin{aligned}
& \int_{\underline{\omega}}^{\overline{\omega}} (\omega - c)\, \rho_L(\omega)\, dF_L(\omega) + \phi_H (p - c) \left[\int_{\underline{\omega}}^{\overline{\omega}} \rho_H(\omega)\, dF_H(\omega) - \int_{\underline{\omega}}^{\overline{\omega}} \rho_L(\omega)\, dF_L(\omega) \right] \\
\leq \; & \int_{\underline{\omega}}^{\overline{\omega}} (\omega - c)\, \rho_L(\omega)\, dF_L(\omega) - \phi_H (p - c) \int_{\underline{\omega}}^{\overline{\omega}} \rho_L(\omega)\, dF_L(\omega) \\
= \; & \phi_L \int_{\underline{\omega}}^{\overline{\omega}} (\omega - c)\, \rho_L(\omega)\, dF_L(\omega) - \phi_H \int_{\underline{\omega}}^{\overline{\omega}} (\omega - p)\, \rho_L(\omega)\, dF_L(\omega)
\end{aligned}$$

Since $p \leq \mathbb{E}_L [\omega | \text{“buy”},\ \rho_L(\omega)]$, we must have

$$\int_{\underline{\omega}}^{\overline{\omega}} (\omega - p)\, \rho_L(\omega)\, dF_L(\omega) \geq 0.$$

Therefore, the value of the relaxed program is bounded above by

$$\phi_L \int_{\underline{\omega}}^{\overline{\omega}} (\omega - c)\, \rho_L(\omega)\, dF_L(\omega) < \int_{\underline{\omega}}^{\overline{\omega}} (\omega - c)\, \rho_L(\omega)\, dF_L(\omega) \leq \int_{c}^{\overline{\omega}} (\omega - c)\, dF_L(\omega).$$

This contradicts to Claim 1. ■

We now assume $c \geq \mathbb{E}_H[\omega]$. Since $p \geq c$, we must have $\mathbb{E}_H[\omega] - p \leq 0$. It follows that we can simplify IC_H constraint as

$$\int_{\underline{\omega}}^{\overline{\omega}} (\omega - p)\, \rho_H(\omega)\, dF_H(\omega) \geq \int_{\underline{\omega}}^{\overline{\omega}} (\omega - p)\, \rho_L(\omega)\, dF_H(\omega)$$

Moreover, since for $i = H, L$

$$\mathbb{E}_i [\omega | \text{“not buy”},\ \rho_i] \leq \mathbb{E}_i [\omega | \text{“buy”},\ \rho_i],$$

we must have

$$\mathbb{E}_i [\omega | \text{“not buy”},\ \rho_i] \leq \mathbb{E}_i [\omega] \leq \mathbb{E}_i [\omega | \text{“buy”},\ \rho_i].$$

Therefore, the lower price bounds are not binding. The price bounds can be rewritten as

$$c \leq p \leq \min \{ \mathbb{E}_L [\omega | \text{“buy”},\ \rho_L], \mathbb{E}_H [\omega | \text{“buy”},\ \rho_H] \}$$

or equivalently

$$c \leq p \leq \min \left\{ \frac{\int_{\underline{\omega}}^{\overline{\omega}} \omega \rho_L(\omega) \, dF_L(\omega)}{\int_{\underline{\omega}}^{\overline{\omega}} \rho_L(\omega) \, dF_L(\omega)}, \frac{\int_{\underline{\omega}}^{\overline{\omega}} \omega \rho_H(\omega) \, dF_H(\omega)}{\int_{\underline{\omega}}^{\overline{\omega}} \rho_H(\omega) \, dF_H(\omega)} \right\}.$$

Claim 3 *In the optimal solution to the relaxed problem, IC_H binds.*

Proof. Suppose not. Then we can drop IC_H constraint, and the relaxed problem becomes

$$\max_{(\rho_L(\omega),\rho_H(\omega),p)} \int_{\underline{\omega}}^{\overline{\omega}} (\omega - c) \rho_L(\omega) \, dF_L(\omega) + \phi_H (p - c) \left[\int_{\underline{\omega}}^{\overline{\omega}} \rho_H(\omega) \, dF_H(\omega) - \int_{\underline{\omega}}^{\overline{\omega}} \rho_L(\omega) \, dF_L(\omega) \right]$$

subject to the price bounds and constraint $p \geq c$. Note that for any fixed p within the price bounds and with $p \geq c$, the objective is maximized by maximizing separately $\int_{\underline{\omega}}^{\overline{\omega}} \rho_H(\omega) \, dF_H(\omega)$ and

$$\int_{\underline{\omega}}^{\overline{\omega}} [\omega - c - \phi_H (p - c)] \rho_L(\omega) \, dF_L(\omega).$$

Therefore, for fixed p, the trading probability $\int_{\underline{\omega}}^{\overline{\omega}} \rho_H(\omega) \, dF_H(\omega)$ is maximized by setting

$$\rho_H(\omega) = 1 \text{ iff } \omega \geq \omega^* \text{ where } \omega^* \text{ solves } \frac{\int_{\omega^*}^{\overline{\omega}} \omega dF_H(\omega)}{1 - F_H(\omega^*)} = p$$

and the optimal $\rho_L(\omega)$ is given by

$$\rho_L(\omega) = 1 \text{ iff } \omega \geq \omega^{**} \text{ with } \omega^{**} = c + \phi_H (p - c).$$

But IC_H constraint is violated: by IR_L,

$$-a + \int_{\omega^{**}}^{\overline{\omega}} (\omega - p) \, dF_L(\omega) = 0 \Rightarrow a = \int_{\omega^{**}}^{\overline{\omega}} (\omega - p) \, dF_L(\omega) \geq 0,$$

by likelihood ratio dominance, $\mathbb{E}_H [\omega | \omega \in [\omega^{**}, \overline{\omega}]] > \mathbb{E}_L [\omega | \omega \in [\omega^{**}, \overline{\omega}]]$, and hence

$$\int_{\omega^{**}}^{\overline{\omega}} (\omega - p) \, dF_H(\omega) > 0 = \int_{\omega^*}^{\overline{\omega}} (\omega - p) \, dF_H(\omega).$$

Therefore, IC_H must be binding in the optimal solution to the relaxed problem. ■

Now we argue the optimality of interval structure.

Claim 4 *The optimal solution to the relaxed problem is a pair of intervals, $[\hat{\omega}, \overline{\omega}]$ for*

θ_H *and* $[\underline{k}, \bar{k}]$ *for* θ_L.

Proof. We first argue that PB_H must be slack: $p < \mathbb{E}_H\left[\omega | \text{“buy”}, \rho_H\right]$. Suppose by contradiction that $p = \mathbb{E}_H\left[\omega | \text{“buy”}, \rho_H\right]$. Then we must have

$$
\begin{aligned}
0 &= \int_{\underline{\omega}}^{\overline{\omega}} (\omega - p)\, \rho_H(\omega)\, dF_H(\omega) \\
&\geq \int_{\underline{\omega}}^{\overline{\omega}} (\omega - p)\, \rho_L(\omega)\, dF_H(\omega) \quad \text{by IC}_H \\
&= \int_{\underline{\omega}}^{\overline{\omega}} \rho_L(\omega)\, dF_H(\omega) \left[\frac{\int_{\underline{\omega}}^{\overline{\omega}} \omega \rho_L(\omega)\, dF_H(\omega)}{\int_{\underline{\omega}}^{\overline{\omega}} \rho_L(\omega)\, dF_H(\omega)} - p \right] \\
&> \int_{\underline{\omega}}^{\overline{\omega}} \rho_L(\omega)\, dF_H(\omega) \left[\frac{\int_{\underline{\omega}}^{\overline{\omega}} \omega \rho_L(\omega)\, dF_L(\omega)}{\int_{\underline{\omega}}^{\overline{\omega}} \rho_L(\omega)\, dF_L(\omega)} - p \right] \text{ by likelihood ratio dominance} \\
&= \frac{\int_{\underline{\omega}}^{\overline{\omega}} \rho_L(\omega)\, dF_H(\omega)}{\int_{\underline{\omega}}^{\overline{\omega}} \rho_L(\omega)\, dF_L(\omega)} \int_{\underline{\omega}}^{\overline{\omega}} (\omega - p)\, \rho_L(\omega)\, dF_L(\omega).
\end{aligned}
$$

It follows that

$$\int_{\underline{\omega}}^{\overline{\omega}} (\omega - p)\, \rho_L(\omega)\, dF_L(\omega) < 0.$$

Since the advance payment $a \geq 0$, IR_L constraint is violated. Therefore, PB_H must be slack in the optimum.

Therefore, the only relevant constraints for the relaxed problem are IC_H and PB_L. The price bound PB_L for θ_L can be rewritten as

$$\int_{\underline{\omega}}^{\overline{\omega}} (\omega - p)\, \rho_L(\omega)\, dF_L(\omega) \geq 0.$$

Fix a price $p \geq c$. Let λ_1 and λ_2 denote the multipliers attached to IC_H and PB_L. We can write the Lagrangian as

$$
\begin{aligned}
L &= \int_{\underline{\omega}}^{\overline{\omega}} (\omega - c)\,\rho_L(\omega)\,dF_L(\omega) + \phi_H (p-c)\left[\int_{\underline{\omega}}^{\overline{\omega}} \rho_H(\omega)\,dF_H(\omega) - \int_{\underline{\omega}}^{\overline{\omega}} \rho_L(\omega)\,dF_L(\omega)\right] \\
&\quad + \lambda_1 \left[\int_{\underline{\omega}}^{\overline{\omega}} (\omega - p)\,\rho_H(\omega)\,dF_H(\omega) - \int_{\underline{\omega}}^{\overline{\omega}} (\omega - p)\,\rho_L(\omega)\,dF_H(\omega)\right] + \lambda_2 \int_{\underline{\omega}}^{\overline{\omega}} (\omega - p)\,\rho_L(\omega)\,dF_L(\omega) \\
&= \int_{\underline{\omega}}^{\overline{\omega}} \left[\underbrace{(\omega - c) - \phi_H(p-c) - \lambda_1(\omega - p)\frac{f_H(\omega)}{f_L(\omega)} + \lambda_2(\omega - p)}_{\Delta_L(\omega)}\right] \rho_L(\omega)\,f_L(\omega)\,d\omega \\
&\quad + \int_{\underline{\omega}}^{\overline{\omega}} \left[\underbrace{\phi_H(p-c) + \lambda_1(\omega - p)}_{\Delta_H(\omega)}\right] \rho_H(\omega)\,f_H(\omega)\,d\omega
\end{aligned}
$$

First note that $\Delta_H(\omega)$ is increasing in ω and the optimal $\rho_H(\omega)$ is given by

$$
\rho_H(\omega) = 1 \text{ if } \omega \geq \hat{\omega} \text{ and } \rho_H(\omega) = 0 \text{ otherwise,}
$$

where $\hat{\omega}$ solves $\phi_H(p-c) + \lambda_1(\hat{\omega} - p) = 0$. Next note that $\Delta_L(\underline{\omega}) < 0$, and $\Delta_L(p) = \phi_L(p-c) \geq 0$. Moreover,

$$
\frac{\Delta_L(\omega)}{\omega - p} = \frac{\omega - c}{\omega - p} - \phi_H \frac{p-c}{\omega - p} - \lambda_1 \frac{f_H(\omega)}{f_L(\omega)} + \lambda_2 = 1 + \phi_L \frac{p-c}{\omega - p} - \lambda_1 \frac{f_H(\omega)}{f_L(\omega)} + \lambda_2
$$

is decreasing in ω for all ω. Therefore, $\Delta_L(\omega)$ can cross 0 from below only once for $\omega \leq p$, and cross 0 from above at most once for $\omega > p$. It follows that there exists an interval $[\underline{k}, \bar{k}] \subset [\underline{\omega}, \overline{\omega}]$ such that $\rho_L(\omega) = 1$ if $\omega \in [\underline{k}, \overline{k}]$ and $\rho_L(\omega) = 0$ otherwise. Since the interval structure is optimal for any p within the price bounds, the interval structure must also be optimal for the optimal p. This completes the proof. ■

It remains to show that the dropped constraints (IR_H and IC_L) are satisfied with the optimal interval structure. To do this, we first use the interval structure to rewrite the seller's objective function as

$$
\int_{\underline{k}}^{\overline{k}} (\omega - p)\,dF_L(\omega) + (p-c)\left[\phi_L\left(F_L(\overline{k}) - F_L(\underline{k})\right) + \phi_H(1 - F_H(\hat{\omega}))\right]
$$

subject to binding IC_H:

$$
\int_{\hat{\omega}}^{\overline{\omega}} (\omega - p)\,dF_H(\omega) = \int_{\underline{k}}^{\overline{k}} (\omega - p)\,dF_H(\omega)
$$

and the price bound (PB_L) for type θ_L:

$$p \leq \mathbb{E}_L[\omega|\text{"buy"}, \rho_L] = \frac{\int_{\underline{k}}^{\overline{k}} \omega dF_L(\omega)}{F_L(\overline{k}) - F_L(\underline{k})}.$$

Claim 5 *Suppose PB_L is slack in the optimal solution to the relaxed problem. Then $p \in (\underline{k}, \overline{k})$ and $p > \hat{\omega}$.*

Proof. The fact that $p < \overline{k}$ follows from PB_L constraint. Hence, we only need to argue $p > \underline{k}$ and $p > \hat{\omega}$. If $1 - F_H(\hat{\omega}) - \left(F_H(\overline{k}) - F_H(\underline{k})\right) \leq 0$, then raising p would increase the seller's revenue

$$\int_{\underline{k}}^{\overline{k}} \omega dF_L(\omega) + p\phi_H\left[(1 - F_H(\hat{\omega})) - \left(F_L(\overline{k}) - F_L(\underline{k})\right)\right] - c\left[\phi_L\left(F_L(\overline{k}) - F_L(\underline{k})\right) + \phi_H(1 - F_H(\hat{\omega}))\right],$$

while relaxing the IC_H constraint

$$\begin{aligned} \int_{\hat{\omega}}^{\overline{\omega}} (\omega - p) dF_H(\omega) &\geq \int_{\underline{k}}^{\overline{k}} (\omega - p) dF_H(\omega) \\ \Leftrightarrow \int_{\hat{\omega}}^{\overline{\omega}} \omega dF_H(\omega) &- \int_{\underline{k}}^{\overline{k}} \omega dF_H(\omega) \geq p\left[1 - F_H(\hat{\omega}) - \left(F_H(\overline{k}) - F_H(\underline{k})\right)\right]. \end{aligned}$$

The IR_L constraint can be satisfied by adjusting a accordingly. This would imply PB_L is binding, a contradiction. Therefore, we must have

$$1 - F_H(\hat{\omega}) - \left(F_H(\overline{k}) - F_H(\underline{k})\right) > 0. \tag{17}$$

In the optimal solution to the relaxed problem, IC_H constraint binds, so we have

$$p = \frac{\int_{\hat{\omega}}^{\overline{\omega}} \omega dF_H(\omega) - \int_{\underline{k}}^{\overline{k}} \omega dF_H(\omega)}{1 - F_H(\hat{\omega}) - \left(F_H(\overline{k}) - F_H(\underline{k})\right)}.$$

It follows that

$$\begin{aligned} \frac{\partial p}{\partial \hat{\omega}} &= \frac{-\hat{\omega} f_H(\hat{\omega})\left[1 - F_H(\hat{\omega}) - \left(F_H(\overline{k}) - F_H(\underline{k})\right)\right] + f_H(\hat{\omega})\left[\int_{\hat{\omega}}^{\overline{\omega}} \omega dF_H(\omega) - \int_{\underline{k}}^{\overline{k}} \omega dF_H(\omega)\right]}{\left[1 - F_H(\hat{\omega}) - \left(F_H(\overline{k}) - F_H(\underline{k})\right)\right]^2} \\ &= \frac{(p - \hat{\omega}) f_H(\hat{\omega})}{1 - F_H(\hat{\omega}) - \left(F_H(\overline{k}) - F_H(\underline{k})\right)} \end{aligned}$$

and

$$\begin{aligned}
\frac{\partial p}{\partial \underline{k}} &= \frac{\underline{k} f_H(\underline{k})\left[1-F_H(\hat{\omega})-\left(F_H(\overline{k})-F_H(\underline{k})\right)\right]-f_H(\underline{k})\left[\int_{\hat{\omega}}^{\overline{\omega}}\omega dF_H(\omega)-\int_{\underline{k}}^{\overline{k}}\omega dF_H(\omega)\right]}{\left[1-F_H(\hat{\omega})-\left(F_H(\overline{k})-F_H(\underline{k})\right)\right]^2} \\
&= \frac{(\underline{k}-p) f_H(\underline{k})}{1-F_H(\hat{\omega})-\left(F_H(\overline{k})-F_H(\underline{k})\right)}
\end{aligned}$$

The seller's objective function can be rewritten as

$$\begin{aligned}
\Pi &= \int_{\underline{k}}^{\overline{k}}(\omega-p)\,dF_L(\omega)+(p-c)\left[\phi_L\left(F_L(\overline{k})-F_L(\underline{k})\right)+\phi_H(1-F_H(\hat{\omega}))\right] \\
&= \int_{\underline{k}}^{\overline{k}}\omega dF_L(\omega)+p\phi_H\left[(1-F_H(\hat{\omega}))-\left(F_L(\overline{k})-F_L(\underline{k})\right)\right] \\
&\quad -c\left[\phi_L\left(F_L(\overline{k})-F_L(\underline{k})\right)+\phi_H(1-F_H(\hat{\omega}))\right]
\end{aligned}$$

The first-order condition for $\hat{\omega}$ is

$$\begin{aligned}
0 &= \frac{\partial p}{\partial \hat{\omega}}\phi_H\left[(1-F_H(\hat{\omega}))-\left(F_L(\overline{k})-F_L(\underline{k})\right)\right]-p\phi_H f_H(\hat{\omega})+c\phi_H f_H(\hat{\omega}) \\
&= \frac{1-F_H(\hat{\omega})-\left(F_L(\overline{k})-F_L(\underline{k})\right)}{1-F_H(\hat{\omega})-\left(F_H(\overline{k})-F_H(\underline{k})\right)}\phi_H(p-\hat{\omega})f_H(\hat{\omega})-p\phi_H f_H(\hat{\omega})+c\phi_H f_H(\hat{\omega}) \\
&= \frac{1-F_H(\hat{\omega})-\left(F_L(\overline{k})-F_L(\underline{k})\right)}{1-F_H(\hat{\omega})-\left(F_H(\overline{k})-F_H(\underline{k})\right)}\phi_H f_H(\hat{\omega})(p-\hat{\omega})-\phi_H f_H(\hat{\omega})(p-c)
\end{aligned}$$

It follows from $p>c$ that $p>\hat{\omega}$. The first-order condition for $\underline{k}$ is:

$$\begin{aligned}
0 &= -\underline{k} f_L(\underline{k})+\frac{\partial p}{\partial \underline{k}}\phi_H\left[(1-F_H(\hat{\omega}))-\left(F_L(\overline{k})-F_L(\underline{k})\right)\right]+p\phi_H f_L(\underline{k})+c\phi_L f_L(\underline{k}) \\
&= -\underline{k} f_L(\underline{k})+\frac{1-F_H(\hat{\omega})-\left(F_L(\overline{k})-F_L(\underline{k})\right)}{1-F_H(\hat{\omega})-\left(F_H(\overline{k})-F_H(\underline{k})\right)}(\underline{k}-p)\phi_H f_H(\underline{k})+p\phi_H f_L(\underline{k})+c\phi_L f_L(\underline{k}) \\
&= (p\phi_H+c\phi_L-\underline{k}) f_L(\underline{k})+\frac{1-F_H(\hat{\omega})-\left(F_L(\overline{k})-F_L(\underline{k})\right)}{1-F_H(\hat{\omega})-\left(F_H(\overline{k})-F_H(\underline{k})\right)}(\underline{k}-p)\phi_H f_H(\underline{k})
\end{aligned}$$

This condition, together with condition (17), implies that $p>\underline{k}$. ∎

Claim 6 *In the optimal solution to the relaxed problem,* $\underline{k}\geq\hat{\omega}$.

Proof. Note that if PB_L is binding, then $p\in(\underline{k},\overline{k})$. If PB_L is not binding, $p\in(\underline{k},\overline{k})$

by Claim 5. But given $p \in (\underline{k}, \overline{k})$, we have

$$\int_{\overline{k}}^{\overline{\omega}} (\omega - p)\, dF_H(\omega) \geq 0$$

which, together with binding IC_H constraint, implies that

$$\int_{\underline{k}}^{\overline{\omega}} (\omega - p)\, dF_H(\omega) \geq \int_{\underline{k}}^{\overline{k}} (\omega - p)\, dF_H(\omega) = \int_{\hat{\omega}}^{\overline{\omega}} (\omega - p)\, dF_H(\omega)\,.$$

As a result,

$$\int_{\hat{\omega}}^{\underline{k}} (\omega - p)\, dF_H(\omega) \geq 0.$$

Since $p > \underline{k}$, this is possible only if $\underline{k} \geq \hat{\omega}$. ■

Claim 7 *The optimal solution to the relaxed problem also satisfies the dropped IC_L constraint.*

Proof. Suppose by contradiction, IC_L is violated, that is,

$$\int_{\hat{\omega}}^{\overline{\omega}} (\omega - p)\, dF_L(\omega) > \int_{\underline{k}}^{\overline{k}} (\omega - p)\, dF_L(\omega) = 0.$$

Imagine that the seller replaces the interval $[\underline{k}, \overline{k}]$ for type θ_L by $[\hat{\omega}, \overline{\omega}]$, and raise price to $\hat{p} > p$ such that

$$\int_{\hat{\omega}}^{\overline{\omega}} (\omega - \hat{p})\, dF_L(\omega) = 0.$$

Since $\underline{k} \geq \hat{\omega}$, the new alternative policy weakly increases the low type's trading probability. By assmption, IR_H is strictly slack under the new policy. Moreover, both types are now offered the same information structure, IC_H and IR_H constraints are trivially satisfied by likelihood ratio dominance. Therefore, the seller's revenue is strictly higher with a weakly higher trading probability for type θ_L and a strictly higher trading price $\hat{p}$. A contradiction. ■

Claim 8 *Suppose $F_H\left(\overline{k}\right) - F_H\left(\underline{k}\right) \geq F_L\left(\overline{k}\right) - F_L\left(\underline{k}\right)$. Then the optimal solution to the relaxed problem also satisfies the dropped IR_H constraint.*

Proof. We can rewrite the payoff of type θ_H as

$$
\begin{aligned}
& -a+\int_{\hat{\omega}}^{\overline{\omega}}(\omega-p)\,dF_H(\omega) \\
= & -\int_{\underline{k}}^{\overline{k}}(\omega-p)\,dF_L(\omega)+\int_{\hat{\omega}}^{\overline{\omega}}(\omega-p)\,dF_H(\omega) \quad \text{by binding IR}_L \\
= & -\int_{\underline{k}}^{\overline{k}}(\omega-p)\,dF_L(\omega)+\int_{\underline{k}}^{\overline{k}}(\omega-p)\,dF_H(\omega) \quad \text{by IC}_H \\
= & \int_{\underline{k}}^{\overline{k}}(\omega-p)\left(f_H(\omega)-f_L(\omega)\right)d\omega \\
= & \int_{\underline{k}}^{\overline{k}}(\omega-p)\left(\frac{f_H(\omega)}{f_L(\omega)}-1\right)f_L(\omega)\,d\omega.
\end{aligned}
$$

To show it is nonnegative, we recall the following Chebychev's sum inequality (or order inequality): If $l(x)$ and $h(x)$ are real-valued, integrable functions on $[0,1]$, both increasing, then

$$\left(\int_0^1 l(x)\,dx\right)\left(\int_0^1 h(x)\,dx\right)\leq\int_0^1 l(x)\,h(x)\,dx.$$

Let's define

$$x=\frac{F_L(\omega)-F_L(\underline{k})}{F_L(\overline{k})-F_L(\underline{k})}\in[0,1]\Leftrightarrow dx=\frac{f_L(\omega)}{F_L(\overline{k})-F_L(\underline{k})}d\omega$$

and define the inverse function $\Gamma(x)$ with $\omega=\Gamma(x)$. Then $\Gamma(x)$ is increasing in x. Let

$$l(x)=\omega-p=\Gamma(x)-p \quad \text{and} \quad h(x)=\frac{f_H(\omega)}{f_L(\omega)}-1=\frac{f_H(\Gamma(x))}{f_L(\Gamma(x))}-1$$

Hence, both $l(x)$ and $h(x)$ are real-valued, increasing functions on $[0,1]$. Moreover,

$$\int_0^1 l(x)\,dx=\int_{\underline{k}}^{\overline{k}}(\omega-p)\frac{f_L(\omega)}{F_L(\overline{k})-F_L(\underline{k})}d\omega=\frac{1}{F_L(\overline{k})-F_L(\underline{k})}\int_{\hat{\omega}}^{\overline{\omega}}(\omega-p)\,f_L(\omega)\,d\omega\geq 0 \text{ by IR}_L$$

and

$$
\begin{aligned}
\int_0^1 h(x)\,dx &= \int_{\underline{k}}^{\overline{k}}\left(\frac{f_H(\omega)}{f_L(\omega)}-1\right)\frac{f_L(\omega)}{F_L(\overline{k})-F_L(\underline{k})}d\omega \\
&= \frac{\left(F_H(\overline{k})-F_H(\underline{k})\right)-\left(F_L(\overline{k})-F_L(\underline{k})\right)}{F_L(\overline{k})-F_L(\underline{k})}\geq 0* \qquad (18)
\end{aligned}
$$

where inequality follows by assumption. Therefore,

$$
\begin{aligned}
& \int_{\underline{k}}^{\overline{k}} (\omega - p) \left(\frac{f_H(\omega)}{f_L(\omega)} - 1 \right) f_L(\omega)\, d\omega \\
= & \left(F_H\left(\overline{k}\right) - F_H\left(\underline{k}\right)\right) \int_0^1 l(x)\, h(x)\, dx \\
\geq & \left(F_H\left(\overline{k}\right) - F_H\left(\underline{k}\right)\right) \left(\int_0^1 l(x)\, dx \right) \left(\int_0^1 h(x)\, dx \right) \\
\geq & \ 0
\end{aligned}
$$

This completes the proof. ■

Claim 9 (Sufficient Condition) *Let ω_o denote the crossing point of $f_L(\omega)$ and $f_H(\omega)$. Suppose $c \geq \omega_o$. Then the optimal solution to the relaxed problem also satisfies the dropped IR_H constraint.*

Proof. We consider two cases: (i) PB_L is binding and (ii) PB_L is not binding in the optimal solution to the relaxed problem. First suppose that PB_L is binding. Then $a = 0$. Hence, the payoff for type θ_H is given by

$$
\begin{aligned}
\int_{\hat{\omega}}^{\overline{\omega}} (\omega - p)\, dF_H(\omega) &= \int_{\underline{k}}^{\overline{k}} (\omega - p)\, dF_H(\omega) \\
&= \int_{\underline{k}}^{\overline{k}} \omega dF_H(\omega) - p\left[F_H\left(\overline{k}\right) - F_H\left(\underline{k}\right)\right] \\
&= \left(F_H\left(\overline{k}\right) - F_H\left(\underline{k}\right)\right) \left[\frac{\int_{\underline{k}}^{\overline{k}} \omega dF_H(\omega)}{F_H\left(\overline{k}\right) - F_H\left(\underline{k}\right)} - p \right] \\
&> \left(F_H\left(\overline{k}\right) - F_H\left(\underline{k}\right)\right) \left[\frac{\int_{\underline{k}}^{\overline{k}} \omega dF_L(\omega)}{F_L\left(\overline{k}\right) - F_L\left(\underline{k}\right)} - p \right] \\
&= 0
\end{aligned}
$$

where the first equality follows from binding IC_H, the inequality follows from likelihood ratio dominance, and the last equality follows from binding IR_L.

Next suppose PB_L is not binding. We argue that $f_H(\underline{k}) / f_L(\underline{k}) \geq 1$. If $\underline{k} \geq c$, then this immediately follows from likelihood ratio dominance. Hence, in what follows, we only consider $\underline{k} < c$. Recall that the first-order conditions for first-order conditions for

$\hat{\omega}$ and $\underline{k}$ are given by

$$\begin{aligned}\frac{1-F_H(\hat{\omega})-\left(F_L(\overline{k})-F_L(\underline{k})\right)}{1-F_H(\hat{\omega})-\left(F_H(\overline{k})-F_H(\underline{k})\right)}\phi_H f_H(\hat{\omega})(p-\hat{\omega})-\phi_H f_H(\hat{\omega})(p-c) &= 0\\ (p\phi_H+c\phi_L-\underline{k})f_L(\underline{k})+\frac{1-F_H(\hat{\omega})-\left(F_L(\overline{k})-F_L(\underline{k})\right)}{1-F_H(\hat{\omega})-\left(F_H(\overline{k})-F_H(\underline{k})\right)}(\underline{k}-p)\phi_H f_H(\underline{k}) &= 0\end{aligned}$$

which imply

$$\begin{aligned}(p\phi_H+c\phi_L-\underline{k})f_L(\underline{k}) &= \frac{p-c}{p-\hat{\omega}}(p-\underline{k})\phi_H f_H(\underline{k})\\ &\Leftrightarrow\ 1+\frac{c-\underline{k}}{(p-c)\phi_H}=\frac{p-\underline{k}}{p-\hat{\omega}}\frac{f_H(\underline{k})}{f_L(\underline{k})}\end{aligned}$$

Since by assumption $\underline{k}<c$ and by Claim 6 $p>\underline{k}\geq\hat{\omega}$, we have

$$\frac{f_H(\underline{k})}{f_L(\underline{k})}\geq\frac{p-\underline{k}}{p-\hat{\omega}}\frac{f_H(\underline{k})}{f_L(\underline{k})}>1.$$

This, together with likelihood ratio dominance, implies that

$$\left(F_H(\overline{k})-F_H(\underline{k})\right)-\left(F_L(\overline{k})-F_L(\underline{k})\right)=\int_{\underline{k}}^{\overline{k}}\left(\frac{f_H(\omega)}{f_L(\omega)}-1\right)f_L(\omega)\,d\omega>0.$$

It follows from Claim 8 that IR_H holds. ■

Therefore, if $c\geq\max\{\omega_o,\mathbb{E}_H[\omega]\}$, the solution to the relaxed problem (where both IR_L and IC_H bind) is also a solution to the original problem. Hence, in the optimal solution, both IR_L and IC_H bind, and the proof is complete.

6.3 Proofs for Section 5

Proof of Lemma 6. Fix any $(v_i^i(s_+),v_i^i(s_-),\Lambda_i^i(s_+))$ that satisfies (8) and (9) with $j=i$. Define k^i such that $F(k^i|\theta_i)=1-\Lambda_i^i(s_+)$. We claim that $\mu_i^+(k^i)\geq v_i^i(s_+)$. To see this, note that both functions on the two sides of (9) are continuous and convex in v. They take on the same valuation of 0 at $v=\underline{\omega}$, and the same valuation of $\overline{\omega}-\mu_i$ at $v=\overline{\omega}$ by (8) and by integration by parts. Furthermore, the function on the right-hand side has slope of 0 for $v\in[\underline{\omega},v_i^i(s_-))$ and slope of 1 for $v\in(v_i^i(s_+),\overline{\omega}]$. Thus, condition (9) is satisfied if and only if it holds at $v=k^i$, where the slopes of the two sides of (8) are equated, or

$$\int_{\underline{\omega}}^{k^i}F(\omega|\theta_i)\,d\omega\geq(1-\Lambda_i^i(s_+))\left(k^i-v_i^i(s_-)\right).$$

Using integration by parts and the definition of k^i, we can rewrite the above inequality as $v_i^i(s_-) \geq \mu_i^-(k^i)$. By condition (8), this is equivalent to $v_i^i(s_+) \leq \mu_i^+(k^i)$.

Now, suppose that disclosure policy $(v_i^i(s_+), v_i^i(s_-), \Lambda_i^i(s_+))$ and selling mechanism (a^i, p^i), $i = H, L$ solve the optimal general disclosure problem. The payoff of type θ_i is

$$U_i = -a^i + \left(v_i^i(s_+) - p^i\right) \Lambda_i^i(s_+).$$

The seller's profit from a type-θ_i buyer is

$$a^i + (p^i - c)\Lambda_i^i(s_+) = \left(v_i^i(s_+) - c\right) \Lambda_i^i(s_+) - U_i.$$

Consider replacing $(v_i^i(s_+), v_i^i(s_-), \Lambda_i^i(s_+))$ with a generalized monotone partition, with threshold k^i such that $F(k^i|\theta_i) = 1 - \Lambda_i^i(s_+)$. Since $\mu_i^+(k^i) \geq v_i^i(s_+)$, there is a generalized monotone partition σ^i such that, with the same strike price p^i, the seller can weakly increase the advance payment to keep U_i unchanged. Then, IR_i and IC_i are unaffected, IC_j for $j \neq i$ is weakly relaxed, but the seller's profit from type θ_i is weakly increased. ■

Proof of Lemma 7. Suppose IC_H is slack at some solution. Then IR_H must be binding, for otherwise the seller could increase a^H to raise the profit. As a result, the seller's profit from type θ_H is maximized by setting threshold k^H to c in the generalized monotone partition. This can be implemented with $a^H = \left(\mu_H^+(c) - c\right)(1 - F(c|\theta_H))$ and $p^H = c$. A deviating type-θ_L buyer gets a negative payoff, and thus IC_L is also slack at any solution such that IC_H is slack. This implies that IR_L binds, for otherwise the seller can raise a^L and increase profit. Then, by Lemma 6, there is a threshold k^L in the generalized monotone partition for θ_L such that

$$-a^L + \left(\mu_L^+(k^L) - p^L\right)\left(1 - F\left(k^L|\theta_L\right)\right) = 0.$$

We claim that $k^L > c$; otherwise, since $\mu_H > \mu_L^+(c)$ and since $\mu_L^+(k^L) \geq p^L$ by PB_L, a deviating type-θ_H buyer who always buys at p^L gets the payoff of

$$-a^L + \mu_H - p^L > -a^L + \mu_L^+(k^L) - p^L \geq -a^L + \left(\mu_L^+(k^L) - p^L\right)\left(1 - F\left(k^L|\theta_L\right)\right) = 0,$$

violating IC_H given that IR_H binds. However, given that IC_H and IC_L are both slack, since $k^L > c$, the seller could increase the profit from type θ_L, given by $(\mu_L^+(k^L) - c)(1 - F(k^L|\theta_L))$, by decreasing k^L and p^L simultaneously. This contradiction establishes that

IC_H binds at any solution.

Next, suppose that IR_L is slack at some solution. Then IR_H binds; otherwise the seller could raise a^H and a^L by the same amount and increase the profit. By Lemma 6, there is a threshold k^H in the generalized monotone partition for type θ_H, such that

$$-a^H + \left(\mu_H^+(k^H) - p^H\right)\left(1 - F\left(k^H|\theta_H\right)\right) = 0.$$

Since $\mu_H^-(k^H) < p^H \leq \mu_H^+(k^H)$ by PB_H, the above implies that $a^H \geq 0$ and

$$-a^H + \mu_H - p^H < 0.$$

From $\mu_L < \mu_H$, by deviation type θ_L gets the payoff of

$$-a^H + \max\{\mu_L - p^H, 0\} \leq -a^H + \max\{\mu_H - p^H, 0\} \leq 0.$$

Since IR_L is slack, IC_L is also slack. But then the seller could raise a^L and increase the profit. This contradiction establishes that IR_L binds at any solution. ■

Proof of Lemma 8. We can use the binding constraints IR_L and IC_H to rewrite the seller's optimal general disclosure problem as choosing k^i and p^i, $i = H, L$, to maximize

$$\phi_H\left(\mu_H^+(k^H) - c\right)(1 - F(k^H|\theta_H)) + \phi_L T_L(k^L) - \phi_H R_H\left(k^L, p^L\right)$$

subject to IR_H, IC_L and the two PB constraints. The constraint IR_H is $R_H(k^L, p^L) \geq 0$, while IC_L can be written as

$$R_H\left(k^L, p^L\right) - \left(\mu_H^+(k^H) - p^H\right)(1 - F(k^H|\theta_H)) + \max\left\{\mu_L - p^H, 0\right\} \leq 0.$$

Define the seller's relaxed problem by dropping IC_L. Since k^H and p^H do not appear in IR_H, we must have $k^H = c$, and without loss we can set $p^H = c$, which satisfies PB_H. For type θ_L, we have that k^L and p^L jointly maximize $\phi_L T_L\left(k^L\right) - \phi_H R_H\left(k^L, p^L\right)$ subject to $R_H\left(k^L, p^L\right) \geq 0$ and $p^L \in [\mu_L^-(k^L), \mu_L^+(k^L)]$. The lemma follows immediately, if we show that this solution satisfies the dropped constraint IC_L.

Suppose that k^L and p^L solve the relaxed problem, but violate IC_L:

$$R_H\left(k^L, p^L\right) - \left(\mu_H^+(c) - c\right)(1 - F(c|\theta_H)) + \max\left\{\mu_L - c, 0\right\} > 0.$$

Consider the alternative of setting both the partition threshold and the strike price to

c; note that this satisfies PB_L. We have

$$\begin{aligned} R_H(c,c) &= \max\{\mu_H - c, 0\} - \left(\mu_L^+(c) - c\right)(1 - F(c|\theta_L)) \\ &\le \left(\mu_H^+(c) - c\right)(1 - F(c|\theta_H)) - \max\{\mu_L - c, 0\}, \end{aligned}$$

which is strictly less than $R_H(k^L, p^L)$ by assumption. This implies that

$$\phi_L T_L(c) - \phi_H R_H(c,c) > \phi_L T_L(c) - \phi_H R_H(k^L, p^L) \ge \phi_L T_L(k^L) - \phi_H R(k^L, p^L).$$

However, since $\mu_H > \mu_L^+(c) > c$,

$$R_H(c,c) = \mu_H - c - \left(\mu_L^+(c) - c\right)(1 - F(c|\theta_L)) > 0,$$

This is a contradiction because k^L and p^L solve the relaxed problem. ■

Proof of Proposition 4. Now, we are ready to complete the proof of proposition. First, we argue that at any solution to the general disclosure problem, $p^L = \mu_L^+(k^L)$ so that PB_L is binding. Suppose instead $p^L < \mu_L^+(k^L)$. Since $R_H(k^L, p^L) \ge 0$, we have $p^L < \mu_H$, and so

$$R_H(k^L, p^L) = (\mu_H - p^L) - \left(\mu_L^+(k^L) - p^L\right)(1 - F(k^L|\theta_L)).$$

Furthermore, $R_H(k^L, p^L) = 0$; otherwise the seller could decrease $R_H(k^L, p^L)$ by increasing p^L without affecting $T_L(k^L)$, a contradiction. Since $\mu_H > \mu_L^+(c)$ implies that for all $k \le c$ and $p \le \mu_L^+(k)$, we have

$$\mu_H - p > (\mu_L^+(k) - p)(1 - F(k|\theta_L)),$$

from $R_H(k^L, p^L) = 0$ we have $k^L > c$. But then the seller could increase $T_L(k^L)$ by decreasing k^L, while keeping $R_H(k^L, p^L) = 0$ by changing p^L, which is a contradiction.

Next, we argue that at any solution to the general disclosure problem, $k^L > c$. Suppose instead $k^L \le c$. Then, since $\mu_H > \mu_L^+(c)$ by assumption, and since we just shown that $p^L = \mu_L^+(k^L)$, we have $\mu_H > p^L$, implying that

$$R_H\left(k^L, p^L\right) = \mu_H - p^L > 0.$$

Consider increasing k^L marginally, while at the same time increasing p^L such that $\mu_L^+(k^L) - p^L$ remains unchanged. This either increases $T_L(k^L)$ when $k^L < c$ or keeps it unchanged when $k^L = c$, but reduces $R_H(k^L, p^L)$, again a contradiction.

Finally, we show that at any solution, $\mu_L^+(k^L) \leq \mu_H$. If not, then since $p^L = \mu_L^+(k^L)$ as argued above, we have $\mu_H < p^L$ and so $R_H(k^L, p^L) = 0$. Further, we have shown above that $k^L > c$. As a result, the seller could increase $T_L(k^L)$ by decreasing k^L, while keeping $R_H(k^L, p^L) = 0$ by decreasing p^L so that $p^L = \mu_L^+(k^L)$, a contradiction.

The proposition follows immediately Lemma 8. ■

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