

# Identification in Nonparametric Models for Dynamic Treatment Effects\*

Sukjin Han

Department of Economics  
University of Texas at Austin  
[sukjin.han@austin.utexas.edu](mailto:sukjin.han@austin.utexas.edu)

First Draft: August 12, 2017

This Draft: May 13, 2018

## Abstract

This paper develops a nonparametric model that represents how sequences of outcomes and treatment choices influence one another in a dynamic manner. In this setting, we are interested in identifying the average outcome for individuals in each period, had a particular treatment sequence been assigned. The identification of this quantity allows us to identify the average treatment effects (ATE's) and the ATE's on transitions, as well as the optimal treatment regimes, namely, the regimes that maximize the (weighted) sum of the average potential outcomes, possibly less the cost of the treatments. The main contribution of this paper is to relax the sequential randomization assumption widely used in the biostatistics literature by introducing a flexible choice-theoretic framework for a sequence of endogenous treatments. We show that the parameters of interest are identified under each period's two-way exclusion restriction, i.e., with instruments excluded from the outcome-determining process and other exogenous variables excluded from the treatment-selection process. We also consider partial identification in the case where the latter variables are not available. Lastly, we extend our results to a setting where treatments do not appear in every period.

*JEL Numbers:* C14, C32, C33, C36

*Keywords:* Dynamic treatment effect, endogenous treatment, average treatment effect, optimal treatment regime, instrumental variable.

## 1 Introduction

This paper develops a nonparametric model that represents how sequences of outcomes and treatment choices influence one another in a dynamic manner. Often, treatments are endogenously chosen multiple times over a horizon, affecting a series of outcomes. Examples are medical interventions that affect health outcomes, educational interventions that affect

---

\*The author is grateful to Dan Akerberg, Xiaohong Chen, Pedro Sant'Anna, Ed Vytlacil and Nese Yildiz for their thoughtful comments and discussions.

academic achievements, job training programs that affect employment status, or online advertisements that affect consumers' preferences or purchase decisions. The relationship of interest is dynamic in the sense that the current outcome is determined by past outcomes as well as current and past treatments, and the current treatment is determined by past outcomes as well as past treatments. Such dynamic relationships are clearly present in the aforementioned examples. A static model misrepresents the nature of the problem (e.g., non-stationarity, state dependence, learning) and fails to capture important policy questions (e.g., optimal timing and schedule of interventions).

In this setting, we are interested in identifying the dynamic causal effect of a sequence of treatments on a sequence of outcomes or on a terminal outcome that may or may not be of the same kind as the transition outcomes. We are interested in learning about the average of the outcome in each period, had a particular treatment sequence been assigned *up to that period*, which defines the potential outcome in the dynamic setting. We are also interested in the average treatment effects (ATE's) and the transition-specific ATE's defined based on the average potential outcome, unconditional and conditional on the previous outcomes, respectively. For example, one may be interested in whether the success rate of a particular outcome (or the transition probability) is larger with a sequence of treatments assigned in relatively later periods rather than earlier, or with a sequence of alternating treatments rather than consistent treatments. The treatment effect is said to be dynamic, partly because the effect can vary depending upon the period of measurement, even if the same set of treatments is assigned. Lastly, we are interested in the optimal treatment regimes, namely, sequences of treatments that maximize the (weighted) sum of the average potential outcomes, possibly less the cost of the treatments. For example, a firm may be interested in the optimal timing of advertisements that maximizes its aggregate sales probabilities over time, or a sequence of educational programs may be aimed to maximize the college attendance rate. We show that the optimal regime is a natural extension of a static object commonly sought in the literature, namely, the sign of the ATE. Analogous to the static environment, knowledge about the optimal treatment regime may have useful policy implications. For example, a social planner may wish to at least exclude specific sequences of treatments that are on average suboptimal.

Dynamic treatment effects have been extensively studied in the biostatistics literature for decades under the counterfactual framework with a sequence of treatments (Robins (1986, 1987, 1997), Murphy et al. (2001), Murphy (2003), among others). In this literature, the crucial condition used to identify the average potential outcome is a dynamic version of a random assignment assumption, called the *sequential randomization*. This condition assumes that the treatment is randomized in every period within those individuals who have the same history of outcomes and treatments.<sup>1</sup> This assumption is suitable in experimental studies (e.g., clinical trials, field experiments) with the perfect compliance of subjects, but is less plausible in studies with partial compliance and in many observational contexts as in the examples described above.

The main contribution of this paper is to relax the assumption of sequential randomization widely used in the literature by introducing a flexible choice-theoretic framework for a

---

<sup>1</sup>This assumption is also called sequential conditional independence or sequential ignorability. In the econometrics literature, Vikström et al. (2017) consider treatment effects on a transition to a destination state, and carefully analyze what the sequential randomization assumption can identify in the presence of dynamic selection.

sequence of endogenous treatments. To this end, we consider a simple nonparametric structural model for a dynamic endogenous selection process and dynamic outcome formation. In this model, individuals are allowed not to fully comply with each period’s assignment in experimental settings, or are allowed to make an endogenous choice in each period as in observational settings. The heterogeneity in each period’s potential outcome is given by recursively applying a switching-regression type of models with a sequential version of rank similarity. The joint distribution of the full history of unobservable variables in the outcome and treatment equations is still flexible, allowing for arbitrary forms of treatment endogeneity as well as serial correlation. Relative to the counterfactual framework, the dynamic mechanism is clearly formulated using this structural model, which in turn facilitates our identification analysis.

We show that the average potential outcome, or equivalently, the average recursive structural function (ARSF) given the structural model we introduce, is identified under a two-way exclusion restriction (Vytlacil and Yildiz (2007), Shaikh and Vytlacil (2011), Balat and Han (2018)). That is, we assume there exist instruments excluded from the outcome-determining process and exogenous variables excluded from the treatment-selection process. Examples of the former include a sequence of randomized treatments or policy shocks. Examples of the latter include factors that agents cannot anticipate when making treatment decisions but that determine the outcome, the timing of which can be justified in this dynamic context. Using the exclusion restriction, we gain certain knowledge of each period’s structural function, which is then iteratively incorporated across periods for identification, obeying the recursive structure of the potential outcome. The proof is constructive and provides a closed form expression for the ARSF. The identification of each period’s ARSF allows us to identify the ATE’s and the optimal treatment regimes. In this paper, we also consider cases where the two-way exclusion restriction is violated in the sense that only a standard exclusion restriction holds or that the variation of the exogenous variables is limited. In these cases, we can calculate the bounds on the parameters. As an extension of our results, we consider another empirically relevant situation where treatments do not appear in every period, while outcomes are constantly observed. We show that the parameters of interest and the identification analysis can be easily modified to incorporate this situation.

This paper contributes to recent research on the identification of the effects of dynamic endogenous treatments that allows for treatment heterogeneity. Abbring and Van den Berg (2003) analyze the effect of a continuous endogenous treatment (i.e., a treatment duration) on a outcome duration and take a parametric approach by considering a mixed proportional hazard model. Cunha et al. (2007) and Heckman and Navarro (2007) consider a semiparametric discrete-time duration model for the choice of the treatment timing and associated outcomes. Building on these works, Heckman et al. (2016) consider not only ordered choice models but also unordered choice models for up-or-out treatment choices.<sup>2</sup> An interesting feature of their results is that dynamic treatment effects are decomposed into direct effects and continuation values. Abraham and Sun (2018) and Callaway and Sant’Anna (2018) extend a difference-in-differences approach to dynamic settings without specifying fixed-effect panel data models. They consider the effects of treatment timing on the treated, where

---

<sup>2</sup>As related works, the settings of Angrist and Imbens (1995) and Lee and Salanié (2017) for multiple (or multi-valued) treatment effects may be applied to a dynamic setting. Also, see Abbring and Heckman (2007) for a survey on dynamic treatment effects.

the treatment process is irreversible. We consider nonparametric dynamic models for treatment and outcome processes with a general form of evolution, where the processes can freely change states. These models can include an irreversible process as a special case. Moreover, we consider different identifying assumptions than those in the previous works and focus on the identification of the ATE’s and related parameters.

This paper’s structural approach is only relative to the counterfactual framework of Robins. A fully structural model of dynamic programming is considered in the seminal work by Rust (1987) and more recently by, e.g., Blevins (2014) and Buchholz et al. (2016). This literature typically considers a single rational agent’s optimal decision, whereas we consider multiple heterogeneous agents with no assumptions on agents’ rationality or strong parametric assumptions. Most importantly, our focus is on the identification of the effects of treatments formed as agents’ decisions. The robust approach of this paper is, in spirit, similar to Heckman and Navarro (2007) and Heckman et al. (2016). Unlike these papers, however, we do not necessarily invoke infinite supports of exogenous variables, while remaining flexible for the economic and non-economic components of the model. Lastly, Torgovitsky (2016) extends the literature on dynamic binary response models (with no treatment) by considering a counterfactual framework without imposing parametric assumptions. In his framework, the lagged outcome plays the role of a treatment for the current outcome, and the “treatment effect” captures the state dependence. Here, we consider the effects of the treatments on the outcomes, and introduce a selection equation for each treatment as an important component of the model. As an extension of our analysis, we identify the transition-specific ATE, which is related to the effect of a treatment on the state dependence.

In terms of notation, let  $\mathbf{W}^t \equiv (W_1, \dots, W_t)$  denote a row vector that collects r.v.’s  $W_t$  across time up to  $t$ , and let  $\mathbf{w}^t$  be its realization. We sometimes write  $\mathbf{W} \equiv \mathbf{W}^T$  for convenience. For a vector  $\mathbf{W}$  without the  $t$ -th element, we write  $\mathbf{W}_{-t} \equiv (W_1, \dots, W_{t-1}, W_{t+1}, \dots, W_T)$  with realization  $\mathbf{w}_{-t}$ . More generally, let  $\mathbf{W}_-$  with realization  $\mathbf{w}_-$  denote some subvector of  $\mathbf{W}$ . Lastly, for r.v.’s  $Y$  and  $W$ , we sometimes abbreviate  $\Pr[Y = y|W = w]$  and  $\Pr[Y = y|W \in \mathcal{W}]$  to  $\Pr[Y = y|w]$  (or  $P[y|w]$ ) and  $\Pr[Y = y|\mathcal{W}]$ , respectively.

## 2 Robins’s Framework

We first introduce Robins’s counterfactual framework and state the assumption of sequential randomization commonly used in the biostatistics literature (Robins (1986, 1987), Murphy et al. (2001), Murphy (2003)). For a finite horizon  $t = 1, \dots, T$  with fixed  $T$ , let  $Y_t$  be the outcome at  $t$  with realization  $y_t$  and let  $D_t$  be the binary treatment at  $t$  with realization  $d_t$ . The underlying data structure is panel data with a large number of cross-sectional observations over a short period of time (and the cross-sectional index  $i$  suppressed throughout, unless necessary). We call  $Y_T$  a *terminal outcome* and  $Y_t$  for  $t \leq T - 1$  a *transition outcome*. Let  $\mathcal{Y}$  and  $\mathcal{D} \subseteq \{0, 1\}^T$  be the supports of  $\mathbf{Y} \equiv (Y_1, \dots, Y_T)$  and  $\mathbf{D} \equiv (D_1, \dots, D_T)$ , respectively.

Consider a treatment regime  $\mathbf{d} \equiv (d_1, \dots, d_T) \in \mathcal{D}$ , which is defined as a predetermined hypothetical sequence of interventions over time, i.e., a sequence of each period’s assignment decisions on whether to treat or not, or whether to assign treatment  $A$  or treatment  $B$ .<sup>3</sup> Then,

---

<sup>3</sup>This is called a nondynamic regime in the biostatistics literature. A dynamic regime is a sequence of treatment assignments, each of which is a predetermined function of past outcomes. A nondynamic regime can be viewed as its special case, where this function is constant. See, e.g., Murphy et al. (2001); Murphy

a potential outcome at  $t$  can be written as  $Y_t(\mathbf{d})$ . This can be understood as an outcome for an individual, had a particular treatment sequence been assigned. Although the genesis of  $Y_t(\mathbf{d})$  can be very general under this counterfactual framework, the mechanism under which the sequence of treatments interacts with the sequence of outcomes is opaque. The definition of  $Y_t(\mathbf{d})$  becomes more transparent later with the structural model introduced in this paper.

Given these definitions, we state the assumption of sequential randomization by Robins: For each  $\mathbf{d} \in \mathcal{D}$ ,

$$(Y_1(\mathbf{d}), \dots, Y_T(\mathbf{d})) \perp D_t | \mathbf{Y}^{t-1}, \mathbf{D}^{t-1} \quad (2.1)$$

for  $t = 1, \dots, T$ . This assumption asserts that, holding the history of outcomes and treatments fixed, the current treatment is fully randomized. In the next section, we relax this assumption and specify dynamic selection equations for a sequence of treatments that are allowed to be endogenous. Apart from this assumption, we maintain the same preliminaries introduced in this section.

**Remark 2.1 (Irreversibility).** *As a special case of our setting, the process of  $D_t$  may be irreversible in that the process only moves from an initial state to a destination state, i.e., the destination state is an absorbing state. The up-or-out treatment decision (or the treatment timing) can be an example where the treatment process satisfies  $D_t = 1$  once  $D_{t-1} = 1$  is reached, as in Heckman and Navarro (2007), Heckman et al. (2016), Abraham and Sun (2018) and Callaway and Sant’Anna (2018). Although it is not the main focus of this paper, the process of  $Y_t$  may as well be irreversible. This case, however, requires caution due to dynamic selection; see discussions later in this paper. The survival of patients ( $Y_t = 0$ ) in discrete time duration models can be an example where the transition of the outcome satisfies  $Y_t = 1$  once  $Y_{t-1} = 1$ . In this case, it may be that  $D_t$  is missing when  $Y_{t-1} = 1$ , which can be dealt by conventionally assuming  $D_t = 0$  if  $Y_{t-1} = 1$ . When processes are irreversible, the supports  $\mathcal{D}$  and  $\mathcal{Y}$  are strict subsets of  $\{0, 1\}^T$ .*

**Remark 2.2 (Terminal outcome of a different kind).** *As in Murphy et al. (2001) and Murphy (2003), we may be interested in a terminal outcome that is of a different kind than that of the transition outcomes. For example, the terminal outcome can be college attendance, while the transition outcomes are secondary school performances. In this case, we replace  $Y_T$  with a random variable  $R_T$  to represent the terminal outcome, while maintaining  $Y_t$  for  $t \leq T - 1$  to represent the transition outcomes. Analogously,  $R_T(\mathbf{d})$  denotes the potential terminal outcome. Then, the analysis in this paper can be readily followed with the change of notation.<sup>4</sup>*

### 3 A Dynamic Structural Model and Objects of Interest

We now introduce the main framework of this paper. Consider a *dynamic structural function* for the outcomes  $Y_t$ ’s that has the form of switching regression models: For  $t = 1, \dots, T$ ,

$$Y_t = \mu_t(Y_{t-1}, D_t, X_t, U_t(D_t)),$$

---

(2003) for related discussions.

<sup>4</sup>Extending this framework to incorporate the irreversibility of the outcome variables discussed in Remark 2.1 is not straightforward. We leave this for future research.

where  $\mu_t(\cdot)$  is an unknown scalar-valued function,  $X_t$  is a set of exogenous variables, which we discuss in detail later, and  $Y_0$  is assumed to be exogenously determined, with  $Y_0 = 0$  for convenience.<sup>5</sup> The unobservable variable satisfies  $U_t(D_t) = D_t U_t(1) + (1 - D_t) U_t(0)$ , where  $U_t(d_t)$  is the “rank variable” that captures the unobserved characteristics or rank, specific to treatment state  $d_t$  (Chernozhukov and Hansen (2005)). We allow  $U_{it}(d_t)$  to contain a permanent component (i.e., individual effects) and a transitory component.<sup>6</sup> Given this structural equation, we can express the potential outcome  $Y_t(\mathbf{d})$  using a recursive structure:

$$\begin{aligned} Y_t(\mathbf{d}) &= Y_t(\mathbf{d}^t) = \mu_t(Y_{t-1}(\mathbf{d}^{t-1}), d_t, X_t, U_t(d_t)), \\ &\vdots \\ Y_2(\mathbf{d}) &= Y_2(\mathbf{d}^2) = \mu_2(Y_1(d_1), d_2, X_2, U_2(d_2)), \\ Y_1(\mathbf{d}) &= Y_1(d_1) = \mu_1(Y_0, d_1, X_1, U_1(d_1)), \end{aligned}$$

where each potential outcome at time  $t$  is only a function of  $\mathbf{d}^t$  (not the full  $\mathbf{d}$ ). This is related to the “no-anticipation” condition (Abbring and Heckman (2007)), which is implied from the structure of the model in our setting. The recursive structure provides us with a useful interpretation of the potential outcome  $Y_t(\mathbf{d})$  in a dynamic setting, and thus facilitates our identification analysis. Note that, conditional on  $\mathbf{X}^t \equiv (X_1, \dots, X_t)$ , the heterogeneity in  $Y_t(\mathbf{d})$  comes from the full vector  $\mathbf{U}^t(\mathbf{d}^t) \equiv (U_1(d_1), \dots, U_t(d_t))$ . By an iterative argument, we can readily show that the potential outcome is equal to the observed outcome when the observed treatments are consistent with the assigned regime:  $Y_t(\mathbf{d}) = Y_t$  when  $\mathbf{D} = \mathbf{d}$ , or equivalently,  $Y_t = \sum_{\mathbf{d} \in \mathcal{D}} 1\{\mathbf{D} = \mathbf{d}\} Y_t(\mathbf{d})$ .

In this paper, we consider the *average potential terminal outcome*, conditional on  $\mathbf{X} = \mathbf{x}$ , as the fundamental parameter of interest:

$$E[Y_T(\mathbf{d}) | \mathbf{X} = \mathbf{x}]. \tag{3.1}$$

We also call this parameter the *average recursive structural function* (ARSF) in the terminal period, named after the recursive structure in the model for  $Y_T(\mathbf{d})$ . Generally, in defining this parameter and all others below, we can consider the potential outcome in any time period of interest, e.g.,  $E[Y_t(\mathbf{d}) | \mathbf{X}^t = \mathbf{x}^t]$  for any given  $t$ . We focus on the terminal potential outcome only for concreteness. The knowledge of the ARSF is useful in recovering other related parameters.

First, we are interested in the conditional ATE:

$$E[Y_T(\mathbf{d}) - Y_T(\tilde{\mathbf{d}}) | \mathbf{X} = \mathbf{x}] \tag{3.2}$$

for two different regimes,  $\mathbf{d}$  and  $\tilde{\mathbf{d}}$ . For example, one may be interested in comparing more versus less consistent treatment sequences, or earlier versus later treatments.

<sup>5</sup>This assumption of an exogenous initial outcome is *not* necessary but only introduced to simplify our analysis. Instead, for example, we can allow  $Y_0$  to be endogenous and assume it is independent of the exogenous variables of the model.

<sup>6</sup>In this case, it may make sense that the permanent component does not depend on each  $d_t$ , but that the transitory component does.

Second, we consider the *optimal treatment regime*:

$$\mathbf{d}^*(\mathbf{x}) = \arg \max_{\mathbf{d} \in \mathcal{D}} E[Y_T(\mathbf{d}) | \mathbf{X} = \mathbf{x}] \quad (3.3)$$

with  $|\mathcal{D}| \leq 2^T$ . That is, we are interested in a treatment regime that delivers the maximum expected potential outcome, conditional on  $\mathbf{X} = \mathbf{x}$ . Notice that, in a static model, the identification of  $\mathbf{d}^*$  is equivalent to the identification of the sign of the static ATE, which is the information typically sought from a policy point of view. One can view  $\mathbf{d}^*$  as a natural extension of this information to a dynamic setting, which is identified by establishing the signs of *all* possible ATE's defined as in (3.2), or equivalently, by ordering all the possible ARSF's. The optimal regime may serve as a guideline in developing future policies. Moreover, it may be a realistic goal for a social planner to identify this kind of scheme that maximizes the average benefit, because it may be too costly to find a customized treatment scheme for every individual. Yet, the optimal regime is customized up to observed characteristics, as it is a function of the covariates values  $\mathbf{x}$ . More ambitious than the identification of  $\mathbf{d}^*(\mathbf{x})$  may be recovering an optimal regime based on a cost–benefit analysis, granting that each  $d_t$  can be costly:

$$\mathbf{d}^\dagger(\mathbf{x}) = \arg \max_{\mathbf{d} \in \mathcal{D}} \Pi(\mathbf{d}; \mathbf{x}), \quad (3.4)$$

where

$$\Pi(\mathbf{d}; \mathbf{x}) \equiv wE[Y_T(\mathbf{d}) | \mathbf{X} = \mathbf{x}] - \tilde{w} \sum_{t=1}^T d_t \quad \text{or} \quad \Pi(\mathbf{d}; \mathbf{x}) \equiv \sum_{t=1}^T w_t E[Y_t(\mathbf{d}) | \mathbf{X} = \mathbf{x}] - \sum_{t=1}^T \tilde{w}_t d_t$$

with  $(w, \tilde{w})$  and  $(\mathbf{w}, \tilde{\mathbf{w}})$  being predetermined weights. The latter objective function concerns the weighted sum of the average potential outcomes throughout the entire period, less the cost of treatments. Note that establishing the signs of ATE's will not identify  $\mathbf{d}^\dagger$ , and a stronger identification result becomes important, i.e., the point identification of  $E[Y_T(\mathbf{d}) | \mathbf{X} = \mathbf{x}]$  for all  $\mathbf{d}$  (or  $E[Y_t(\mathbf{d}) | \mathbf{X} = \mathbf{x}]$  for all  $t$  and  $\mathbf{d}$ ).

Lastly, we are interested in the *transition-specific ATE*:

$$E[Y_T(\mathbf{d}) | Y_{T-1}(\mathbf{d}) = y_{T-1}, \mathbf{X} = \mathbf{x}] - E[Y_T(\tilde{\mathbf{d}}) | Y_{T-1}(\tilde{\mathbf{d}}) = y_{T-1}, \mathbf{X} = \mathbf{x}] \quad (3.5)$$

for two different  $\mathbf{d}$  and  $\tilde{\mathbf{d}}$ . The knowledge of the ARSF does not directly recover this parameter, but the identification of it (and its more general form introduced later) can be paralleled by the analysis for the ARSF and ATE.

In order to facilitate identification of the parameters of interest without assuming sequential randomization, we introduce a sequence of selection equations for the binary endogenous treatments  $D_t$ 's: For  $t = 1, \dots, T$ ,

$$D_t = 1\{\pi_t(Y_{t-1}, D_{t-1}, Z_t) \geq V_t\},$$

where  $\pi_t(\cdot)$  is an unknown scalar-valued function,  $Z_t$  is the period-specific instruments,  $V_t$  is the unobservable variable that may contain permanent and transitory components, and  $D_0$  is assumed to be exogenously given as  $D_0 = 0$ . This dynamic selection process represents the

agent’s endogenous choices over time, e.g., as a result of learning or other optimal behaviors. However, the nonparametric threshold-crossing structure posits a minimal notion of optimality for the agent. We take an agnostic and robust approach by avoiding strong assumptions of the standard dynamic economic models pioneered by Rust (1987), such as forward looking behaviors and being able to compute a present value discounted flow of utilities. If we are to maintain the assumption of rational agents, the selection model can be viewed as a reduced-form approximation of a solution to a dynamic programming problem.

To simplify the exposition, we consider binary  $Y_t$  and impose weak separability in the outcome equation as in the treatment equation. The binary outcome is *not* necessary for the result of this paper, and the analysis can be easily extended to the case of continuous or censored  $Y_t$ , maintaining weak separability; see Remark 3.3. Then, the full model can be summarized as

$$Y_t = 1\{\mu_t(Y_{t-1}, D_t, X_t) \geq U_t(D_t)\}, \quad (3.6)$$

$$D_t = 1\{\pi_t(Y_{t-1}, D_{t-1}, Z_t) \geq V_t\}. \quad (3.7)$$

In this model, the observable variables are  $(\mathbf{Y}, \mathbf{D}, \mathbf{X}, \mathbf{Z})$ . All other covariates are suppressed in the equations for simplicity of exposition. Importantly, in this model, the joint distribution of the unobservable variables  $(\mathbf{U}(\mathbf{d}), \mathbf{V})$  for given  $\mathbf{d}$  is not specified, in that  $U_t(d_t)$  and  $V_{t'}$  for any  $t, t'$  are allowed to be arbitrarily correlated to each other (allowing endogeneity) as well as within themselves across time (allowing serial correlation, e.g., via time-invariant individual effects). Note that, because we allow an arbitrary form of persistence in the unobservables,  $(Y_t, D_t)$  is *not* a Markov process even after conditioning on the observables. This is in contrast to the standard dynamic economic models, where conditional independence assumptions or Markovian unobservables are commonly introduced. By considering the nonparametric index functions that depend on  $t$ , we also avoid other strong assumptions on parametric functional forms or time homogeneity.

**Remark 3.1 (Irreversibility—continued).** *A process that satisfies  $D_t = 1$  if  $D_{t-1} = 1$  is consistent with having a structural function that satisfies  $\pi_t(y_{t-1}, d_{t-1}, z_t) = +\infty$  if  $d_{t-1} = 1$ . Similarly, processes that satisfy  $Y_t = 1$  and  $D_t = 0$  if  $Y_{t-1} = 1$  are consistent with  $\mu_t(y_{t-1}, d_t, x_t) = +\infty$  and  $\pi_t(y_{t-1}, d_{t-1}, z_t) = -\infty$  if  $y_{t-1} = 1$ . This implies that  $Y_t(\mathbf{d}^t) = 1$  for any  $d_t$  if  $Y_{t-1}(\mathbf{d}^{t-1}) = 1$ . When  $Y_t$  is irreversible, the ARSF  $E[Y_T(\mathbf{d})|X]$  can be interpreted as (one minus) a potential survival rate. An important caveat is that, with irreversible  $Y_t$ , the ATE we define contains not only the treatment effect (the intensive margin) but also the effect on dynamic selection (the extensive margin), and the parameter may or may not be of interest depending on the application.*

**Remark 3.2 (Terminal outcome of a different kind—continued).** *When we replace  $Y_T$  with  $R_T$  to represent a terminal outcome of a different kind, we assume that the model (3.6) is only satisfied for  $t \leq T - 1$  and introduce  $R_T = 1\{\mu_T(Y_{T-1}, D_T, X_T) \geq U_T(D_T)\}$  as the terminal structural function. The potential terminal outcome  $R_T(\mathbf{d})$  can accordingly be expressed using the structural functions for  $(Y_1, \dots, Y_{T-1}, R_T)$ . The ARSF is written as  $E[R_T(\mathbf{d})|X]$ , and the other parameters can be defined accordingly.*

**Remark 3.3 (Non-binary  $Y_t$ ).** *Even though we focus on binary  $Y_t$  in this paper, we can obtain similar identification results with continuous  $Y_t$  or limited dependent variable  $Y_t$ , by*



maintaining a general weak separability structure:  $Y_t = m_t(\mu_t(Y_{t-1}, D_t, X_t), U_t(D_t))$ . As in the static settings of [Vytlacil and Yildiz \(2007\)](#) and [Balat and Han \(2018\)](#), we impose an assumption that guarantees certain monotonicity of each period’s average structural function with respect to the index  $\mu_t$ : For each  $t$ ,  $E[m_t(\mu_t, U_t(d_t)) | \mathbf{V}^t, \mathbf{U}^{t-1}]$  is strictly monotonic in  $\mu_t$ . Examples of the nonparametric model  $m_t(\mu_t(y_{t-1}, d_t, x_t), u_t)$  that satisfies this assumption are additively separable models or their transformation models, censored regression models, and threshold crossing models as in (3.6); see [Vytlacil and Yildiz \(2007\)](#) for more discussions.

## 4 Main Identification Analysis

We first identify the ARSF’s, i.e.,  $E[Y_t(\mathbf{d}) | \mathbf{X}^t]$  for every  $\mathbf{d}$  and  $t$ , which will then be used to identify the ATE’s and the optimal regimes  $\mathbf{d}^*$  and  $\mathbf{d}^\dagger$ . We maintain the following assumptions on  $(\mathbf{Z}, \mathbf{X})$  and  $(\mathbf{U}(\mathbf{d}), \mathbf{V})$  for every  $\mathbf{d}$ . These assumptions are written for the identification of  $E[Y_T(\mathbf{d}) | \mathbf{X}]$ , and are sufficient but not necessary for the identification of  $E[Y_t(\mathbf{d}) | \mathbf{X}^t]$  for  $t \leq T - 1$ .

**Assumption C.** *The distribution of  $(\mathbf{U}(\mathbf{d}), \mathbf{V})$  has strictly positive density with respect to Lebesgue measure on  $\mathbb{R}^{2T}$ .*

**Assumption SX.**  *$(\mathbf{Z}, \mathbf{X})$  and  $(\mathbf{U}(\mathbf{d}), \mathbf{V})$  are independent.*

Assumption C is a regularity condition to ensure the smoothness of relevant conditional probabilities. Assumption SX imposes strict exogeneity. It is implicit that the independence is conditional on the covariates suppressed in the model. The variable  $Z_t$  denotes the standard excluded instruments. Examples include a sequence of randomized treatments or policy shocks. In addition to  $Z_t$ , we introduce exogenous variables  $X_t$  in the outcome equation (3.6), that are excluded from the selection equation (3.7). We make a behavioral/information assumption that there are outcome-determining factors that the agent cannot anticipate when making a treatment decision. For example, when  $D_t$  is a medical treatment that a patient receives at every doctor visit,  $Y_{t-1}$  may be the health outcome measured by the doctor prior to the treatment during the same visit. Then  $Y_t$  is the outcome measured upon the next visit, which may create enough time gap to prevent the doctor or the patient from predicting  $X_t$ .<sup>7</sup> Note that  $(Z_t, X_t)$  are assumed to be excluded from the outcome and treatment equations of all other periods as well. Next, we introduce a sequential version of the rank similarity assumption ([Chernozhukov and Hansen \(2005\)](#)):

**Assumption RS.** *For each  $t$  and  $\mathbf{d}_{-t}$ ,  $\{\mathbf{U}(d_t, \mathbf{d}_{-t})\}_{d_t}$  are identically distributed, conditional on  $(\mathbf{U}^{t-1}(\mathbf{d}^{t-1}), \mathbf{V}^t)$ .*

Rank invariance (i.e.,  $\{\mathbf{U}(\mathbf{d})\}_{\mathbf{d}}$  being equal to each other) is particularly restrictive in the multi-period context, because it requires that the same rank be realized across different treatment states for all  $t$ . Instead, Assumption RS, which we call *sequential rank similarity*, states that  $\mathbf{U}(1, \mathbf{d}_{-t})$  and  $\mathbf{U}(0, \mathbf{d}_{-t})$  are identically distributed conditional on  $(\mathbf{U}^{t-1}(\mathbf{d}^{t-1}), \mathbf{V}^t)$ . That is, holding the history of ranks (and treatment unobservables) fixed, this requires that

<sup>7</sup>In a static scenario, [Balat and Han \(2018\)](#) motivate this reverse exclusion restriction using the notion of externalities. In their setting where multiple treatments are strategically chosen (e.g., firms’ entry decisions), factors that determine the outcome (e.g., pollution) are assumed not to appear in the firms’ payoff functions.

the joint distributions of the ranks are identical between two states that differ by  $d_t = 1$  and 0. Therefore, this allows an individual to have different realized ranks across different  $\mathbf{d}$ 's. Assumption RS can be stated in terms of the marginal distributions and conditional distributions of the ranks: Conditional on  $(\mathbf{U}^{t-1}(\mathbf{d}^{t-1}), \mathbf{V}^t)$ , (i)  $U_t(1)$  and  $U_t(0)$  are identically distributed, and (ii)  $(U_{t+1}(d_{t+1}), \dots, U_T(d_T))|U_t(1)$  and  $(U_{t+1}(d_{t+1}), \dots, U_T(d_T))|U_t(0)$  are identically distributed.

Now, we are ready to derive a period-specific result. Define the following period-specific quantity directly identified from the data, i.e., from the distribution of  $(\mathbf{Y}, \mathbf{D}, \mathbf{X}, \mathbf{Z})$ :

$$\begin{aligned} & h_t(z_t, \tilde{z}_t, x_t, \tilde{x}_t; \mathbf{z}^{t-1}, \mathbf{x}^{t-1}, \mathbf{d}^{t-1}, y_{t-1}) \\ \equiv & \Pr[Y_t = 1, D_t = 1 | \mathbf{z}^t, \mathbf{x}^t, \mathbf{d}^{t-1}, y_{t-1}] + \Pr[Y_t = 1, D_t = 0 | \mathbf{z}^t, \tilde{x}_t, \mathbf{x}^{t-1}, \mathbf{d}^{t-1}, y_{t-1}] \\ & - \Pr[Y_t = 1, D_t = 1 | \tilde{z}_t, \mathbf{z}^{t-1}, \mathbf{x}^t, \mathbf{d}^{t-1}, y_{t-1}] - \Pr[Y_t = 1, D_t = 0 | \tilde{z}_t, \mathbf{z}^{t-1}, \tilde{x}_t, \mathbf{x}^{t-1}, \mathbf{d}^{t-1}, y_{t-1}] \end{aligned}$$

for  $t \geq 1$ , where  $(\mathbf{Z}^0, \mathbf{X}^0, \mathbf{D}^0)$  is understood to mean that there is no conditioning.

**Lemma 4.1.** *Suppose Assumptions C, SX and RS hold. For each  $t$  and  $(\mathbf{z}^{t-1}, \mathbf{x}^{t-1}, \mathbf{d}^{t-1}, y_{t-1})$ , suppose  $z_t$  and  $\tilde{z}_t$  are such that*

$$\Pr[D_t = 1 | \mathbf{z}^t, \mathbf{x}^{t-1}, \mathbf{d}^{t-1}, y_{t-1}] \neq \Pr[D_t = 1 | \tilde{z}_t, \mathbf{z}^{t-1}, \mathbf{x}^{t-1}, \mathbf{d}^{t-1}, y_{t-1}]. \quad (4.1)$$

Then, for given  $(x_t, \tilde{x}_t)$ , the sign of  $h_t(z_t, \tilde{z}_t, x_t, \tilde{x}_t; \mathbf{z}^{t-1}, \mathbf{x}^{t-1}, \mathbf{d}^{t-1}, y_{t-1})$  is equal to the sign of  $\mu_t(y_{t-1}, 1, x_t) - \mu_t(y_{t-1}, 0, \tilde{x}_t)$ .

Without relying on further assumptions, the sign of  $\mu_t(y_{t-1}, 1, x_t) - \mu_t(y_{t-1}, 0, \tilde{x}_t)$  itself is already useful for calculating bounds on the ARSF's and thus on the ATE's; we discuss the partial identification in Section 6.

Proof of Lemma 4.1: For the analysis of this paper which deals with a dynamic model, it is convenient to define the  $\mathbf{U}$ -set and  $\mathbf{V}$ -set, namely the sets of histories of the unobservable variables that determine the outcomes and treatments, respectively. To focus our attention on the dependence of the potential outcomes on the unobservables, we iteratively define the potential outcome given  $(\mathbf{d}, \mathbf{x})$  as

$$Y_t(\mathbf{d}^t, \mathbf{x}^t) \equiv 1\{\mu_t(Y_{t-1}(\mathbf{d}^{t-1}, \mathbf{x}^{t-1}), d_t, x_t) \geq U_t(d_t)\}$$

for  $t \geq 2$ , with  $Y_1(d_1, x_1) = 1\{\mu_1(0, d_1, x_1) \geq U_1(d_1)\}$ . Now, define the set of  $\mathbf{U}^t(\mathbf{d}^t)$  as

$$\mathcal{U}^t(\mathbf{d}^t, y_t) \equiv \mathcal{U}^t(\mathbf{d}^t, y_t; \mathbf{x}^t) \equiv \{\mathbf{U}^t(\mathbf{d}^t) : y_t = Y_t(\mathbf{d}^t, \mathbf{x}^t)\}$$

for  $t \geq 1$ . Then,  $Y_t = y_t$  if and only if  $\mathbf{U}^t(\mathbf{d}^t) \in \mathcal{U}^t(\mathbf{d}^t, y_t; \mathbf{x}^t)$ , conditional on  $(\mathbf{D}^t, \mathbf{X}^t) = (\mathbf{d}^t, \mathbf{x}^t)$ . Realizing the dependence of  $Y_{s-1}(\mathbf{d}^{s-1}, \mathbf{x}^{s-1})$  on  $(\mathbf{U}^{s-1}(\mathbf{d}^{s-1}), \mathbf{x}^{s-1}, \mathbf{d}^{s-1})$ , let

$$\pi_s^*(\mathbf{U}^{s-1}(\mathbf{d}^{s-1}), \mathbf{x}^{s-1}, \mathbf{d}^{s-1}, z_s) \equiv \pi_s(Y_{s-1}(\mathbf{d}^{s-1}, \mathbf{x}^{s-1}), d_{s-1}, z_s),$$

and define the set of  $\mathbf{V}^t$  as

$$\mathcal{V}^t(\mathbf{d}^t, \mathbf{u}^{t-1}) \equiv \mathcal{V}^t(\mathbf{d}^t, \mathbf{u}^{t-1}; \mathbf{z}^t, \mathbf{x}^{t-1}) \equiv \{\mathbf{V}^t : d_s = 1\{V_s \leq \pi_s^*(\mathbf{u}^{s-1}, \mathbf{x}^{s-1}, \mathbf{d}^{s-1}, z_s)\} \text{ for all } s \leq t\}$$

for  $t \geq 2$ . Then,  $\mathbf{D}^t = \mathbf{d}^t$  if and only if  $\mathbf{V}^t \in \mathcal{V}^t(\mathbf{d}^t, \mathbf{U}^{t-1}(\mathbf{d}^{t-1}))$ , conditional on  $(\mathbf{Z}^t, \mathbf{X}^{t-1}) = (\mathbf{z}^t, \mathbf{x}^{t-1})$ . Fix  $t \geq 3$ . Given (4.1), consider the case  $\Pr[D_t = 1 | \mathbf{z}^t, \mathbf{x}^{t-1}, \mathbf{d}^{t-1}, y_{t-1}] > \Pr[D_t = 1 | \tilde{z}_t, \mathbf{z}^{t-1}, \mathbf{x}^{t-1}, \mathbf{d}^{t-1}, y_{t-1}]$ ; the opposite case is symmetric. Using the definitions of the sets above, we have

$$\begin{aligned} & \Pr[D_t = 1 | \mathbf{z}^t, \mathbf{x}^{t-1}, \mathbf{d}^{t-1}, y_{t-1}] \\ &= \Pr[V_t \leq \pi_t(y_{t-1}, d_{t-1}, z_t) | \mathbf{z}^t, \mathbf{x}^{t-1}, \mathcal{V}^{t-1}(\mathbf{d}^{t-1}, \mathbf{U}^{t-2}(\mathbf{d}^{t-2})), \mathcal{U}^{t-1}(\mathbf{d}^{t-1}, y_{t-1})] \\ &= \Pr[V_t \leq \pi_t(y_{t-1}, d_{t-1}, z_t) | \mathcal{V}^{t-1}(\mathbf{d}^{t-1}, \mathbf{U}^{t-2}(\mathbf{d}^{t-2})), \mathcal{U}^{t-1}(\mathbf{d}^{t-1}, y_{t-1})], \end{aligned}$$

where the last equality is given by Assumption SX. Note that the sets  $\mathcal{V}^{t-1}(\mathbf{d}^{t-1}, \mathbf{U}^{t-2}(\mathbf{d}^{t-2}))$  and  $\mathcal{U}^{t-1}(\mathbf{d}^{t-1}, y_{t-1})$  do not change with the change in  $z_t$ . Therefore, a parallel expression can be derived for  $\Pr[D_t = 1 | \tilde{z}_t, \mathbf{z}^{t-1}, \mathbf{x}^{t-1}, \mathbf{d}^{t-1}, y_{t-1}]$ . Let  $\pi_t \equiv (y_{t-1}, d_{t-1}, z_t)$  and  $\tilde{\pi}_t \equiv (y_{t-1}, d_{t-1}, \tilde{z}_t)$  for abbreviation. Then, under Assumption C,

$$\begin{aligned} 0 &< \Pr[D_t = 1 | \mathbf{z}^t, \mathbf{x}^{t-1}, \mathbf{d}^{t-1}, y_{t-1}] - \Pr[D_t = 1 | \tilde{z}_t, \mathbf{z}^{t-1}, \mathbf{x}^{t-1}, \mathbf{d}^{t-1}, y_{t-1}] \\ &= \Pr[V_t \leq \pi_t | \mathcal{V}^{t-1}(\mathbf{d}^{t-1}, \mathbf{U}^{t-2}(\mathbf{d}^{t-2})), \mathcal{U}^{t-1}(\mathbf{d}^{t-1}, y_{t-1})] \\ &\quad - \Pr[V_t \leq \tilde{\pi}_t | \mathcal{V}^{t-1}(\mathbf{d}^{t-1}, \mathbf{U}^{t-2}(\mathbf{d}^{t-2})), \mathcal{U}^{t-1}(\mathbf{d}^{t-1}, y_{t-1})], \end{aligned}$$

which implies  $\pi_t > \tilde{\pi}_t$ . Next, we have

$$\begin{aligned} & \Pr[Y_t = 1, D_t = 1 | \mathbf{z}^t, \mathbf{x}^t, \mathbf{d}^{t-1}, y_{t-1}] \\ &= \Pr[U_t(1) \leq \mu_t(y_{t-1}, 1, x_t), V_t \leq \pi_t | \mathcal{V}^{t-1}(\mathbf{d}^{t-1}, \mathbf{U}^{t-2}(\mathbf{d}^{t-2})), \mathcal{U}^{t-1}(\mathbf{d}^{t-1}, y_{t-1})] \end{aligned}$$

by Assumption SX. Again, note that  $\mathcal{V}^{t-1}(\mathbf{d}^{t-1}, \mathbf{U}^{t-2}(\mathbf{d}^{t-2}))$  and  $\mathcal{U}^{t-1}(\mathbf{d}^{t-1}, y_{t-1})$  do not change with the change in  $(z_t, x_t)$ , which is key. Therefore, similar expressions can be derived for the other terms involved in  $h_t$ , and we have

$$\begin{aligned} & h_t(z_t, \tilde{z}_t, x_t, \tilde{x}_t; \mathbf{z}^{t-1}, \mathbf{x}^{t-1}, \mathbf{d}^{t-1}, y_{t-1}) \\ &= \Pr[U_t(1) \leq \mu_t(y_{t-1}, 1, x_t), \tilde{\pi}_t \leq V_t \leq \pi_t | \mathcal{V}^{t-1}(\mathbf{d}^{t-1}, \mathbf{U}^{t-2}(\mathbf{d}^{t-2})), \mathcal{U}^{t-1}(\mathbf{d}^{t-1}, y_{t-1})] \\ &\quad - \Pr[U_t(0) \leq \mu_t(y_{t-1}, 0, \tilde{x}_t), \tilde{\pi}_t \leq V_t \leq \pi_t | \mathcal{V}^{t-1}(\mathbf{d}^{t-1}, \mathbf{U}^{t-2}(\mathbf{d}^{t-2})), \mathcal{U}^{t-1}(\mathbf{d}^{t-1}, y_{t-1})], \end{aligned}$$

the sign of which identifies the sign of  $\mu_t(y_{t-1}, 1, x_t) - \mu_t(y_{t-1}, 0, \tilde{x}_t)$  by Assumption RS. For example, when this quantity is zero, then  $\mu_t(y_{t-1}, 1, x_t) - \mu_t(y_{t-1}, 0, \tilde{x}_t) = 0$ . The case  $t \leq 2$  can be shown analogously with  $\mathcal{V}^1(d_1) \equiv \mathcal{V}^1(d_1; z_1) \equiv \{V_1 : d_1 = 1\{V_1 \leq \pi_1(0, 0, z_1)\}\}$ .  $\square$

For the point identification of the ARSF's, the final assumption we introduce concerns the variation of the exogenous variables  $(\mathbf{Z}, \mathbf{X})$ . Define the following sets:

$$\mathcal{S}_t(d_t, y_{t-1}) \equiv \left\{ (x_t, \tilde{x}_t) : \mu_t(y_{t-1}, d_t, x_t) = \mu_t(y_{t-1}, \tilde{d}_t, \tilde{x}_t) \text{ for } \tilde{d}_t \neq d_t \right\}, \quad (4.2)$$

$$\begin{aligned} \mathcal{T}_t(\mathbf{x}_{-t}, \mathbf{z}_{-t}) &\equiv \left\{ (x_t, \tilde{x}_t) : \exists (z_t, \tilde{z}_t) \text{ such that (4.1) holds and} \right. \\ &\quad \left. (x_t, z_t), (\tilde{x}_t, z_t), (x_t, \tilde{z}_t), (\tilde{x}_t, \tilde{z}_t) \in \text{Supp}(X_t, Z_t | \mathbf{x}_{-t}, \mathbf{z}_{-t}) \right\}, \quad (4.3) \end{aligned}$$

$$\mathcal{X}_t(d_t, y_{t-1}; \mathbf{x}_{-t}, \mathbf{z}_{-t}) \equiv \{x_t : \exists \tilde{x}_t \text{ with } (x_t, \tilde{x}_t) \in \mathcal{S}_t(d_t, y_{t-1}) \cap \mathcal{T}_t(\mathbf{x}_{-t}, \mathbf{z}_{-t})\}, \quad (4.4)$$

$$\mathcal{X}_t(d_t; \mathbf{x}_{-t}, \mathbf{z}_{-t}) \equiv \mathcal{X}_t(d_t, 0; \mathbf{x}_{-t}, \mathbf{z}_{-t}) \cap \mathcal{X}_t(d_t, 1; \mathbf{x}_{-t}, \mathbf{z}_{-t}), \quad (4.5)$$

where (4.2) is related to the sufficient variation of  $X_t$  and (4.3) is related to the rectangular variation of  $(X_t, Z_t)$ .

**Assumption SP.** For each  $t$  and  $d_t$ ,  $\Pr[X_t \in \mathcal{X}_t(d_t; \mathbf{x}_{-t}, \mathbf{z}_{-t}) | \mathbf{x}_{-t}, \mathbf{z}_{-t}] > 0$  almost everywhere.

This assumption requires that  $X_t$  varies sufficiently to achieve  $\mu_t(y_{t-1}, d_t, x_t) = \mu_t(y_{t-1}, \tilde{d}_t, \tilde{x}_t)$ , while holding  $Z_t$  to be  $z_t$  and  $\tilde{z}_t$ , respectively, conditional on  $(\mathbf{X}_{-t}, \mathbf{Z}_{-t})$ . This is a dynamic version of the support assumption found in Vytlacil and Yildiz (2007).<sup>8</sup> Note that even though this assumption seems to be written in terms of the unknown object  $\mu_t(\cdot)$ , it is testable because the sets defined above have empirical analogs, according to Lemma 4.1. Let  $\mathcal{X}_t(d_t; \mathbf{x}_{-t}) \equiv \{x_t : x_t \in \mathcal{X}_t(d_t; \mathbf{x}_{-t}, \mathbf{z}_{-t}) \text{ for some } \mathbf{z}_{-t} \in \text{Supp}(\mathbf{Z}_{-t} | \mathbf{x}_{-t})\}$  and  $\mathcal{X}(\mathbf{d}) \equiv \{\mathbf{x} : x_t \in \mathcal{X}_t(d_t; \mathbf{x}_{-t}) \text{ for some } (x_{t+1}, \dots, x_T), \text{ for } t \geq 1\}$ , which sequentially collect  $x_t \in \mathcal{X}_t(d_t; \mathbf{x}_{-t}, \mathbf{z}_{-t})$  for all  $t$ . We are now ready to state the main identification result.

**Theorem 4.1.** Under Assumptions C, SX, RS and SP,  $E[Y_T(\mathbf{d}) | \mathbf{x}]$  is identified for  $\mathbf{d} \in \mathcal{D}$  and  $\mathbf{x} \in \mathcal{X}(\mathbf{d})$ .

Based on Theorem 4.1, we can identify the ATE's. Since the identification of all  $E[Y_t(\mathbf{d}) | \mathbf{x}^t]$ 's can be shown analogously to Theorem 4.1, we can identify the optimal treatment regimes  $\mathbf{d}^*(\mathbf{x})$  and  $\mathbf{d}^\dagger(\mathbf{x})$  as well.

**Corollary 4.1.** Under Assumptions C, SX, RS and SP,  $E[Y_T(\mathbf{d}) - Y_T(\tilde{\mathbf{d}}) | \mathbf{x}]$  is identified for  $\mathbf{d}, \tilde{\mathbf{d}} \in \mathcal{D}$  and  $\mathbf{x} \in \mathcal{X}(\mathbf{d}) \cap \mathcal{X}(\tilde{\mathbf{d}})$ , and  $\mathbf{d}^*(\mathbf{x})$  and  $\mathbf{d}^\dagger(\mathbf{x})$  are identified for  $\mathbf{x} \in \bigcap_{\mathbf{d} \in \mathcal{D}} \mathcal{X}(\mathbf{d})$ .

Proof of Theorem 4.1: To begin with, note that  $E[Y_T(\mathbf{d}) | \mathbf{x}] = \Pr[Y_T(\mathbf{d}) = 1 | \mathbf{x}] = \Pr[\mathbf{U}(\mathbf{d}) \in \mathcal{U}^T(\mathbf{d}, 1; \mathbf{x}) | \mathbf{x}] = \Pr[\mathbf{U}(\mathbf{d}) \in \mathcal{U}^T(\mathbf{d}, 1; \mathbf{x}) | \mathbf{x}, \mathbf{z}] = E[Y_T(\mathbf{d}) | \mathbf{x}, \mathbf{z}]$  by Assumption SX.<sup>9</sup> As the first step of identifying  $E[Y_T(\mathbf{d}) | \mathbf{x}, \mathbf{z}]$  for given  $\mathbf{d} = (d_1, \dots, d_T)$ ,  $\mathbf{x} = (x_1, \dots, x_T)$  and  $\mathbf{z} = (z_1, \dots, z_T)$ , we apply the result of Lemma 4.1. Fix  $t \geq 2$  and  $y_{t-1} \in \{0, 1\}$ . Suppose  $x'_t$  is such that  $\mu_t(y_{t-1}, d_t, x_t) = \mu_t(y_{t-1}, d'_t, x'_t)$  with  $d'_t \neq d_t$  by applying Lemma 4.1. The existence of  $x'_t$  is guaranteed by Assumption SP, as  $x_t \in \mathcal{X}_t(d_t, y_{t-1}; \mathbf{x}_{-t}, \mathbf{z}_{-t}) \subset \mathcal{X}_t(d_t; \mathbf{x}_{-t}, \mathbf{z}_{-t})$ . The implication of  $\mu_t(y_{t-1}, d_t, x_t) = \mu_t(y_{t-1}, d'_t, x'_t)$  for relevant  $\mathbf{U}$ -sets is as follows: By the definition of the  $\mathbf{U}$ -set,  $\mathbf{U} \in \mathcal{U}(\mathbf{d}, y_T; \mathbf{x})$  is equivalent to  $\mathbf{U} \in \mathcal{U}^T(d'_t, \mathbf{d}_{-t}, y_T; x'_t, \mathbf{x}_{-t})$  conditional on  $Y_{t-1}(\mathbf{d}^{t-1}, \mathbf{x}^{t-1}) = y_{t-1}$  for all  $\mathbf{x}_{-t}$  and  $\mathbf{d}_{-t}$ .<sup>10</sup> Based on this result, in what follows we equate the unobserved quantity  $E[Y_T(\mathbf{d}) | \mathbf{x}, \mathbf{z}, \mathbf{d}^{t-1}, d'_t, y_{t-1}]$  with a quantity that partly matches the assigned treatment and the observed treatment. Analogous to the  $\mathbf{U}$ -set defined earlier, define

$$\mathcal{U}^t(\mathbf{d}^t, \mathbf{y}^t) \equiv \mathcal{U}^t(\mathbf{d}^t, \mathbf{y}^t; \mathbf{x}^t) \equiv \{\mathbf{U}^t(\mathbf{d}^t) : y_s = Y_s(\mathbf{d}^s, \mathbf{x}^s) \text{ for all } s \leq t\}.$$

<sup>8</sup>Although Assumption SP requires sufficient rectangular variation in  $(X_t, Z_t)$ , it clearly differs from the large variation assumptions in Heckman and Navarro (2007) and Heckman et al. (2016), which are employed for identification at infinity arguments. In our setting, it is possible that  $\mathcal{X}_t(d_t; \mathbf{x}_{-t}, \mathbf{z}_{-t})$  is nonempty even when  $Z_t$  is discrete, as long as  $X_t$  contains continuous elements with sufficient support (Vytlacil and Yildiz (2007)). In all these works, including the present one, the support requirement is conditional on the exogenous variables in other periods; see also Cameron and Heckman (1998).

<sup>9</sup>When we are to identify the average potential outcome at  $t$  instead, the conditioning variables we use are the vectors of exogenous variables up to  $t$ , i.e.,  $E[Y_t(\mathbf{d}^t) | \mathbf{x}^t, \mathbf{z}^t]$ . Then the entire proof can be easily modified based on this expression.

<sup>10</sup>The subsequent analysis is substantially simplified when  $\mu_t(y_{t-1}, d_t, x_t) = \mu_t(y_{t-1}, d'_t, x'_t)$  is satisfied for all  $y_{t-1}$ , but this situation is unlikely to occur. Therefore, it is important to condition on  $Y_{t-1}(\mathbf{d}^{t-1}, \mathbf{x}^{t-1}) = y_{t-1}$  in the analysis.

For  $t \geq 2$ ,

$$\begin{aligned}
& E[Y_T(\mathbf{d})|\mathbf{x}, \mathbf{z}, \mathbf{y}^{t-1}, \mathbf{d}^{t-1}, d'_t] \\
&= \Pr \left[ \mathbf{U}(\mathbf{d}) \in \mathcal{U}^T(\mathbf{d}, 1; \mathbf{x}) \left| \begin{array}{l} \mathbf{x}, \mathbf{z}, \\ \mathbf{U}^{t-1}(\mathbf{d}^{t-1}) \in \mathcal{U}^{t-1}(\mathbf{d}^{t-1}, \mathbf{y}^{t-1}), \\ \mathbf{V}^t \in \mathcal{V}^t(\mathbf{d}^{t-1}, d'_t, \mathbf{U}^{t-1}(\mathbf{d}^{t-1})) \end{array} \right. \right] \\
&= \Pr \left[ \mathbf{U}(\mathbf{d}) \in \mathcal{U}^T(\mathbf{d}, 1; \mathbf{x}) \left| \begin{array}{l} \mathbf{U}^{t-1}(\mathbf{d}^{t-1}) \in \mathcal{U}^{t-1}(\mathbf{d}^{t-1}, \mathbf{y}^{t-1}), \\ \mathbf{V}^t \in \mathcal{V}^t(\mathbf{d}^{t-1}, d'_t, \mathbf{U}^{t-1}(\mathbf{d}^{t-1})) \end{array} \right. \right], \tag{4.6}
\end{aligned}$$

where the last equality follows from Assumption SX. Then, by Assumption RS and the discussion above, (4.6) is equal to

$$\begin{aligned}
& \Pr \left[ \mathbf{U}(d'_t, \mathbf{d}_{-t}) \in \mathcal{U}^T(d'_t, \mathbf{d}_{-t}, 1; x'_t, \mathbf{x}_{-t}) \left| \begin{array}{l} \mathbf{U}^{t-1}(\mathbf{d}^{t-1}) \in \mathcal{U}^{t-1}(\mathbf{d}^{t-1}, \mathbf{y}^{t-1}), \\ \mathbf{V}^t \in \mathcal{V}^t(\mathbf{d}^{t-1}, d'_t, \mathbf{U}^{t-1}(\mathbf{d}^{t-1})) \end{array} \right. \right] \\
&= \Pr \left[ \mathbf{U}(d'_t, \mathbf{d}_{-t}) \in \mathcal{U}^T(d'_t, \mathbf{d}_{-t}, 1; x'_t, \mathbf{x}_{-t}) \left| \begin{array}{l} x'_t, \mathbf{x}_{-t}, \mathbf{z}, \\ \mathbf{U}^{t-1}(\mathbf{d}^{t-1}) \in \mathcal{U}^{t-1}(\mathbf{d}^{t-1}, \mathbf{y}^{t-1}), \\ \mathbf{V}^t \in \mathcal{V}^t(\mathbf{d}^{t-1}, d'_t, \mathbf{U}^{t-1}(\mathbf{d}^{t-1})) \end{array} \right. \right] \\
&= E[Y_T(d'_t, \mathbf{d}_{-t})|x'_t, \mathbf{x}_{-t}, \mathbf{z}, \mathbf{y}^{t-1}, \mathbf{d}^{t-1}, d'_t], \tag{4.7}
\end{aligned}$$

where the first equality is by Assumption SX. Note that the last quantity in (4.7) is still unobserved, since  $d_s$  for  $s \geq t+1$  are not realized treatments; e.g., when  $T=3$  and  $t=2$ ,

$$E[Y_3(\mathbf{d})|\mathbf{x}, \mathbf{z}, y_1, d_1, d'_2] = E[Y_3(d_1, d'_2, d_3)|x_1, x'_2, x_3, \mathbf{z}, y_1, d_1, d'_2].$$

The quantity, however, will be useful in the iterative argument below. Recall the abbreviations  $\mathcal{V}^t(\mathbf{d}^{t-1}, d'_t, \mathbf{U}^{t-1}(\mathbf{d}^{t-1})) \equiv \mathcal{V}^t(\mathbf{d}^{t-1}, d'_t, \mathbf{U}^{t-1}(\mathbf{d}^{t-1}); \mathbf{z}^t, \mathbf{x}^{t-1})$  and  $\mathcal{U}^{t-1}(\mathbf{d}^{t-1}, \mathbf{y}^{t-1}) \equiv \mathcal{U}^{t-1}(\mathbf{d}^{t-1}, \mathbf{y}^{t-1}; \mathbf{x}^{t-1})$ . That is, in the derivation of (4.7), the key is to consider the average potential outcome for a group of individuals that is defined by the treatments at time  $t$  or earlier and the lagged outcome, for which  $x_t$  is excluded. We use the result (4.7) in the next step.

First, note that  $E[Y_T(\mathbf{d})|\mathbf{x}, \mathbf{z}, \mathbf{y}^{T-1}, \mathbf{d}^T] = E[Y_T|\mathbf{x}, \mathbf{z}, \mathbf{y}^{T-1}, \mathbf{d}^T]$  is trivially identified for any generic values  $(\mathbf{d}, \mathbf{x}, \mathbf{z}, \mathbf{y}^{T-1})$ . We prove by means of mathematical induction. For given  $2 \leq t \leq T-1$ , suppose  $E[Y_T(\mathbf{d})|\mathbf{x}, \mathbf{z}, \mathbf{y}^{t-1}, \mathbf{d}^t]$  is identified for any generic values  $(\mathbf{d}, \mathbf{x}, \mathbf{z}, \mathbf{y}^{t-1})$ , and consider the identification of

$$\begin{aligned}
E[Y_T(\mathbf{d})|\mathbf{x}, \mathbf{z}, \mathbf{y}^{t-2}, \mathbf{d}^{t-1}] &= \Pr[D_t = d_t|\mathbf{x}, \mathbf{z}, \mathbf{y}^{t-2}, \mathbf{d}^{t-1}]E[Y_T(\mathbf{d})|\mathbf{x}, \mathbf{z}, \mathbf{y}^{t-2}, \mathbf{d}^{t-1}, d_t] \\
&\quad + \Pr[D_t = d'_t|\mathbf{x}, \mathbf{z}, \mathbf{y}^{t-2}, \mathbf{d}^{t-1}]E[Y_T(\mathbf{d})|\mathbf{x}, \mathbf{z}, \mathbf{y}^{t-2}, \mathbf{d}^{t-1}, d'_t]. \tag{4.8}
\end{aligned}$$

The first main term  $E[Y_T(\mathbf{d})|\mathbf{x}, \mathbf{z}, \mathbf{y}^{t-2}, \mathbf{d}^{t-1}, d_t]$  in (4.8) is identified, by integrating over  $y_{t-1} \in \{0, 1\}$  the quantity  $E[Y_T(\mathbf{d})|\mathbf{x}, \mathbf{z}, \mathbf{y}^{t-1}, \mathbf{d}^t]$ , which is assumed to be identified in the

previous iteration. The remaining unknown term in (4.8) satisfies

$$E[Y_T(\mathbf{d})|\mathbf{x}, \mathbf{z}, \mathbf{y}^{t-2}, \mathbf{d}^{t-1}, d'_t] = \Pr[Y_{t-1} = 1|\mathbf{x}, \mathbf{z}, \mathbf{y}^{t-2}, \mathbf{d}^{t-1}, d'_t]E[Y_T(\mathbf{d})|\mathbf{x}, \mathbf{z}, (\mathbf{y}^{t-2}, 1), \mathbf{d}^{t-1}, d'_t] \\ + \Pr[Y_{t-1} = 0|\mathbf{x}, \mathbf{z}, \mathbf{y}^{t-2}, \mathbf{d}^{t-1}, d'_t]E[Y_T(\mathbf{d})|\mathbf{x}, \mathbf{z}, (\mathbf{y}^{t-2}, 0), \mathbf{d}^{t-1}, d'_t].$$

By applying (4.7) to the unknown terms in this expression, we have

$$E[Y_T(\mathbf{d})|\mathbf{x}, \mathbf{z}, \tilde{\mathbf{y}}^{t-1}, \mathbf{d}^{t-1}, d'_t] = E[Y_T(d'_t, \mathbf{d}_{-t})|x'_t, \mathbf{x}_{-t}, \mathbf{z}, \tilde{\mathbf{y}}^{t-1}, \mathbf{d}^{t-1}, d'_t] \quad (4.9)$$

for each  $\tilde{\mathbf{y}}^{t-1}$ , which is identified from the previous iteration. Therefore,  $E[Y_T(\mathbf{d})|\mathbf{x}, \mathbf{z}, \mathbf{y}^{t-2}, \mathbf{d}^{t-1}]$  is identified. Note that when  $t = 2$ ,  $\mathbf{Y}^0$  is understood to mean there is no conditioning. Lastly, when  $t = 1$ ,

$$E[Y_T(\mathbf{d})|\mathbf{x}, \mathbf{z}] = \Pr[D_1 = d_1|\mathbf{x}, \mathbf{z}]E[Y_T(\mathbf{d})|\mathbf{x}, \mathbf{z}, d_1] + \Pr[D_1 = d'_1|\mathbf{x}, \mathbf{z}]E[Y_T(\mathbf{d})|\mathbf{x}, \mathbf{z}, d'_1].$$

Noting that  $Y_0 = 0$ , suppose  $x'_1$  is such that  $\mu_1(0, d_1, x_1) = \mu_1(0, d'_1, x'_1)$  with  $d'_1 \neq d_1$  by applying Lemma 4.1. Then,

$$E[Y_T(\mathbf{d})|\mathbf{x}, \mathbf{z}, d'_1] = \Pr[\mathbf{U}(\mathbf{d}) \in \mathcal{U}^T(\mathbf{d}, 1; \mathbf{x})|V_1 \in \mathcal{V}^1(d'_1)] \\ = \Pr[\mathbf{U}(d'_1, \mathbf{d}_{-1}) \in \mathcal{U}^T(d'_1, \mathbf{d}_{-1}, 1; x'_1, \mathbf{x}_{-1})|V_1 \in \mathcal{V}^1(d'_1)] \\ = E[Y_T(d'_1, \mathbf{d}_{-1})|x'_1, \mathbf{x}_{-1}, \mathbf{z}, d'_1],$$

which is identified from the previous iteration for  $t = 2$ . Therefore,  $E[Y_T(\mathbf{d})|\mathbf{x}, \mathbf{z}]$  is identified, which completes the proof of Theorem 4.1.  $\square$

Note that this proof provides a closed-form expression for  $E[Y_T(\mathbf{d})|\mathbf{x}]$  in an iterative manner, which can immediately be used for estimation. For concreteness, we provide an expression for  $E[Y_T(\mathbf{d})|\mathbf{x}]$  when  $T = 2$ :

$$E[Y_2(\mathbf{d})|\mathbf{x}] = P[\mathbf{d}|\mathbf{x}, \mathbf{z}]E[Y_2|\mathbf{x}, \mathbf{z}, \mathbf{d}] + P[d_1, d'_2|\mathbf{x}, \mathbf{z}]\mu_{2,d_1,d'_2} \\ + P[d'_1, d_2|\mathbf{x}, \mathbf{z}]E[Y_2|x'_1, x_2, \mathbf{z}, d'_1, d_2] + P[d'_1, d'_2|\mathbf{x}, \mathbf{z}]\mu_{2,d'_1,d'_2}, \quad (4.10)$$

where

$$\mu_{2,d_1,d'_2} \equiv P[y_1|\mathbf{x}, \mathbf{z}, d_1, d'_2]E[Y_2|x_1, x'_2, \mathbf{z}, d_1, d'_2, y_1] \\ + P[y'_1|\mathbf{x}, \mathbf{z}, d_1, d'_2]E[Y_2|x_1, x''_2, \mathbf{z}, d_1, d'_2, y'_1], \\ \mu_{2,d'_1,d'_2} \equiv P[y_1|x'_1, x_2, \mathbf{z}, d'_1, d'_2]E[Y_2|x'_1, x'_2, \mathbf{z}, d'_1, d'_2, y_1] \\ + P[y'_1|x'_1, x_2, \mathbf{z}, d'_1, d'_2]E[Y_2|x'_1, x''_2, \mathbf{z}, d'_1, d'_2, y'_1]$$

for  $(x'_1, x'_2, x''_2)$  such that  $\mu_1(0, d_1, x_1) = \mu_1(0, d'_1, x'_1)$ ,  $\mu_2(y_1, d_2, x_2) = \mu_2(y_1, d'_2, x'_2)$ , and  $\mu_2(y'_1, d_2, x_2) = \mu_2(y'_1, d'_2, x''_2)$ .

**Remark 4.1.** *In estimating the parameters identified in this section, one can improve efficiency by aggregating the conditional expectations (4.9) with respect to  $X_t = x'_t$  over the*

following set:

$$\lambda_t(x_t; \mathbf{z}^{t-1}, \mathbf{x}^{t-1}, \mathbf{d}^{t-1}, y_{t-1}) \equiv \{\tilde{x}_t : h_t(z_t, \tilde{z}_t, x_t, \tilde{x}_t; \mathbf{z}^{t-1}, \mathbf{x}^{t-1}, \mathbf{d}^{t-1}, y_{t-1}) = 0 \text{ for some } (z_t, \tilde{z}_t)\}.$$

Similarly, one can aggregate the identifying equation for  $E[Y_T(\mathbf{d})|\mathbf{x}]$  (e.g., equation (4.10)) with respect to  $\mathbf{Z} = \mathbf{z}$  conditional on  $\mathbf{X} = \mathbf{x}$ .

## 5 Treatment Effects on Transitions

In fact, the identification strategy introduced in the previous section can tackle a more general problem. In this section, we extend the identification analysis of the ATE (Theorem 4.1 and Corollary 4.1) and show identification of the transition-specific ATE. Given the vector  $\mathbf{Y}(\mathbf{d}) \equiv (Y_1(\mathbf{d}), \dots, Y_T(\mathbf{d}))$  of potential outcomes, let  $\mathbf{Y}_-(\mathbf{d}) \equiv (Y_{t_1}(\mathbf{d}), \dots, Y_{t_L}(\mathbf{d})) \in \mathcal{Y}_- \subseteq \{0, 1\}^L$  be its  $1 \times L$  subvector, where  $t_1 < t_2 < \dots < t_L \leq T-1$  and  $L < T$ . Then, the transition-specific ATE can be defined as  $E[Y_T(\mathbf{d})|\mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-, \mathbf{X} = \mathbf{x}] - E[Y_T(\tilde{\mathbf{d}})|\mathbf{Y}_-(\tilde{\mathbf{d}}) = \mathbf{y}_-, \mathbf{X} = \mathbf{x}]$  for some sequences  $\mathbf{d}$  and  $\tilde{\mathbf{d}}$ .

**Theorem 5.1.** *Under Assumptions C, SX, RS and SP, for each  $\mathbf{y}_-$ ,  $E[Y_T(\mathbf{d})|\mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-, \mathbf{X} = \mathbf{x}] - E[Y_T(\tilde{\mathbf{d}})|\mathbf{Y}_-(\tilde{\mathbf{d}}) = \mathbf{y}_-, \mathbf{X} = \mathbf{x}]$  is identified for  $\mathbf{d}, \tilde{\mathbf{d}} \in \mathcal{D}$  and  $\mathbf{x} \in \mathcal{X}(\mathbf{d}) \cap \mathcal{X}(\tilde{\mathbf{d}})$ .*

The proof of this theorem extends that of Theorem 4.1; see the Appendix.<sup>11</sup> The transition-specific ATE defined in Theorem 5.1 concerns a transition from a state that is specified by the value of the vector of previous potential outcomes,  $\mathbf{Y}_-(\mathbf{d})$ . When  $Y_T(\mathbf{d})$  is binary,  $E[Y_T(\mathbf{d})|\mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-, \mathbf{X} = \mathbf{x}]$  can be viewed as a generalization of the transition probability. As a simple example, with  $L = T-1$ , one may be interested in a transition to one state when all previous potential outcomes have stayed in the other state until  $T-1$ . When  $L = 1$  with  $\mathbf{Y}_-(\mathbf{d}) = Y_{T-1}(\mathbf{d})$ , the transition-specific ATE becomes  $\Pr[Y_T(\mathbf{d}) = 1|Y_{T-1}(\mathbf{d}) = 0] - \Pr[Y_T(\tilde{\mathbf{d}}) = 1|Y_{T-1}(\tilde{\mathbf{d}}) = 0]$  introduced in Section 3. This is a particular example of the treatment effect on the transition probability. The treatment effects on transitions have been studied by, e.g., Abbring and Van den Berg (2003), Heckman and Navarro (2007), Fredriksson and Johansson (2008) and Vikström et al. (2017).<sup>12</sup> Let  $Y_t(d_t) \equiv \mu_t(Y_{t-1}, d_t, X_t, U_t(d_t))$  be the *period-specific potential outcome* at time  $t$ . Since  $Y_{t-1} = Y_{t-1}(\mathbf{D}^{t-1})$ , the period-specific potential outcome can be expressed as  $Y_t(d_t) = Y_t(\mathbf{D}^{t-1}, d_t)$  using the usual potential outcome. As a corollary of the result above, we also identify a related parameter that specifies the previous state by the observed outcome:  $E[Y_T(1) - Y_T(0)|Y_{T-1} = y_{T-1}]$ .

**Corollary 5.1.** *Under Assumptions C, SX, RS and SP, for each  $y_{T-1}$ ,  $E[Y_T(1)|y_{T-1}, \mathbf{x}] - E[Y_T(0)|y_{T-1}, \mathbf{x}]$  is identified for  $\mathbf{x} \in \mathcal{X}(\mathbf{d}) \cap \mathcal{X}(\tilde{\mathbf{d}})$ .*

<sup>11</sup>As before, the parameters in Theorem 5.1 and Corollary 5.1 below can be defined for any given period instead of the terminal period  $T$ . The identification analysis of such parameters is essentially the same, and thus omitted.

<sup>12</sup>The definition of the treatment effect on the transition probability in this paper differs from those defined in the literature on duration models, e.g., that in Vikström et al. (2017). Since Vikström et al. (2017)'s main focus is on  $Y_t$  that is irreversible, they define a different treatment parameter that yields a specific interpretation under dynamic selection; see their paper for details. In addition, they assume sequential randomization and that treatments are assigned earlier than the transition of interest.

The corollary is derived by observing that  $Y_T(d_T) = Y_T(\mathbf{D}^{T-1}, d_T)$ , and thus

$$\begin{aligned} & E[Y_T(d_T)|y_{T-1}, \mathbf{x}] \\ = & \sum_{\mathbf{d}^{T-1} \in \mathcal{D}^{T-1}} \Pr[\mathbf{D}^{T-1} = \mathbf{d}^{T-1} | \mathbf{x}] E[Y_T(\mathbf{d}^{T-1}, d_T) | Y_{T-1}(\mathbf{d}^{T-1}) = y_{T-1}, \mathbf{D}^{T-1} = \mathbf{d}^{T-1}, \mathbf{x}], \end{aligned}$$

where each  $E[Y_T(\mathbf{d}^{T-1}, d_T) | Y_{T-1}(\mathbf{d}^{T-1}) = y_{T-1}, \mathbf{D}^{T-1} = \mathbf{d}^{T-1}, \mathbf{x}]$  is identified from the iteration at  $t = T - 1$  in the proof of Theorem 5.1 by taking  $Y_-(\mathbf{d}) = Y_{T-1}(\mathbf{d}^{T-1})$ .

## 6 Partial Identification

Suppose Assumption SP does not hold in that  $X_t$  does not exhibit sufficient rectangular variation, or that there is no  $X_t$  that is excluded from the selection equation at time  $t$ . In this case, we partially identify the ARSF's, ATE's and  $\mathbf{d}^*(\mathbf{x})$  (or  $\mathbf{d}^\dagger(\mathbf{x})$ ).

We briefly illustrate the calculation of the bounds on the ARSF  $E[Y_T(\mathbf{d}) | \mathbf{x}]$  when the sufficient rectangular variation is not guaranteed; the case where  $X_t$  does not exist at all can be dealt in a similar manner, and so is omitted. For each  $E[Y_T(\mathbf{d}) | \mathbf{x}, \mathbf{z}, \mathbf{y}^{t-1}, \mathbf{d}^{t-1}, d'_t]$  in the proof of Theorem 4.1, we can calculate its upper and lower bounds depending on the sign of  $\mu_t(y_{t-1}, 1, x_t) - \mu_t(y_{t-1}, 0, \tilde{x}_t)$ , which is identified in Lemma 4.1. Note that, in the context of this section,  $\tilde{x}_t$  does *not* necessarily differ from  $x_t$ . For example, for the lower bound on  $E[Y_T(\mathbf{d}) | \mathbf{x}] = E[Y_T(\mathbf{d}) | \mathbf{x}, \mathbf{z}]$ , suppose  $\mu_t(y_{t-1}, d_t, x_t) - \mu_t(y_{t-1}, d'_t, x'_t) \geq 0$  for given  $y_{t-1}$ , where  $x'_t$  is allowed to equal  $x_t$ . Then, by the definition of the  $\mathcal{U}$ -set and under Assumption RS, it satisfies that  $\mathcal{U}^T(\mathbf{d}, y_T; \mathbf{x}) \supseteq \mathcal{U}^T(d'_t, \mathbf{d}_{-t}, y_T; x'_t, \mathbf{x}_{-t})$ , conditional on  $Y_{t-1}(\mathbf{d}^{t-1}, \mathbf{x}^{t-1}) = y_{t-1}$ . Therefore, we have a lower bound on as  $E[Y_T(\mathbf{d}) | \mathbf{x}, \mathbf{z}, \mathbf{y}^{t-1}, \mathbf{d}^{t-1}, d'_t]$  as

$$\begin{aligned} & E[Y_T(\mathbf{d}) | \mathbf{x}, \mathbf{z}, \mathbf{y}^{t-1}, \mathbf{d}^{t-1}, d'_t] \\ = & \Pr[\mathbf{U}(\mathbf{d}) \in \mathcal{U}^T(\mathbf{d}, 1; \mathbf{x}) | \mathbf{U}^{t-1}(\mathbf{d}^{t-1}) \in \mathcal{U}^{t-1}(\mathbf{d}^{t-1}, \mathbf{y}^{t-1}), \mathbf{V}^t \in \mathcal{V}^t(\mathbf{d}^{t-1}, d'_t, \mathbf{U}^{t-1}(\mathbf{d}^{t-1}))] \\ \geq & \Pr \left[ \begin{array}{l} \mathbf{U}(d'_t, \mathbf{d}_{-t}) \in \mathcal{U}^T(d'_t, \mathbf{d}_{-t}, 1; x'_t, \mathbf{x}_{-t}) \\ \mathbf{U}^{t-1}(\mathbf{d}^{t-1}) \in \mathcal{U}^{t-1}(\mathbf{d}^{t-1}, \mathbf{y}^{t-1}), \\ \mathbf{V}^t \in \mathcal{V}^t(\mathbf{d}^{t-1}, d'_t, \mathbf{U}^{t-1}(\mathbf{d}^{t-1})) \end{array} \right] \\ = & E[Y_T(d'_t, \mathbf{d}_{-t}) | x'_t, \mathbf{x}_{-t}, \mathbf{z}, \mathbf{y}^{t-1}, \mathbf{d}^{t-1}, d'_t]. \end{aligned} \tag{6.1}$$

Then, it is possible to calculate the lower bounds on  $E[Y_T(\mathbf{d}) | \mathbf{x}, \mathbf{z}]$  using the iterative scheme introduced in the proof of Theorem 4.1. That is, at each iteration, we take the previous iteration's lower bound as given, expand each main term in (4.8) as before, and apply (6.1) for necessary terms.

Lastly, depending on the signs of the ATE's, we can construct bounds on  $\mathbf{d}^*(\mathbf{x})$  (or  $\mathbf{d}^\dagger(\mathbf{x})$ ), which will be expressed as strict subsets of  $\mathcal{D}$ . The partial identification of the optimal regimes may not yield sufficiently narrow bounds unless there are a sufficient number of ATE's whose bounds are informative about their signs. In general, however, the informativeness of bounds truly depends on the policy questions. Note that  $\mathcal{D}$  is a discrete set. Even though the bounds may not be informative about the optimal regime, they may still be useful from the planner's perspective if they can help her exclude a few suboptimal regimes, i.e.,  $\mathbf{d}^\circ$  such that  $E[Y_T(\mathbf{d}) | \mathbf{x}] \geq E[Y_T(\mathbf{d}^\circ) | \mathbf{x}]$  for some  $\mathbf{d}$ .



## 7 Subsequences of Treatments

An important extension of the model introduced in this paper is to the case where treatments do not appear in every period, while the outcomes are constantly observed. For example, institutionally, there may only be a one-shot treatment at the beginning of time or a few treatments earlier in the horizon, or there may be evenly spaced treatment decisions with a lower frequency than outcomes. A potential outcome that corresponds to this situation can be defined as a function of a certain subsequence  $\mathbf{d}_-$  of  $\mathbf{d}$ . Let  $\mathbf{d}_- \equiv (d_{t_1}, \dots, d_{t_K}) \in \mathcal{D}_- \subseteq \{0, 1\}^K$  be a  $1 \times K$  subvector of  $\mathbf{d}$ , where  $t_1 < t_2 < \dots < t_K \leq T$  and  $K < T$ . Then, the potential outcomes  $Y_t(\mathbf{d}_-)$  and the associated structural functions are defined as follows: Let  $\mathbf{d}_-^{t_k} \equiv (d_{t_1}, \dots, d_{t_k})$ . A potential outcome in the period when a treatment exists is expressed using a switching regression model as

$$Y_{t_k}(\mathbf{d}_-) = Y_{t_k}(\mathbf{d}_-^{t_k}) = \mu_{t_k}(Y_{t_k-1}(\mathbf{d}_-^{t_{k-1}}), d_{t_k}, X_{t_k}, U_{t_k}(d_{t_k}))$$

for  $k \geq 1$  with  $Y_{t_1-1}(\mathbf{d}_-^{t_0}) = Y_{t_1-1}$ , and a potential outcome when there is no treatment is expressed as

$$Y_t(\mathbf{d}_-) = Y_t(\mathbf{d}_-^{t_k}) = \mu_t(Y_{t-1}(\mathbf{d}_-^{t_k}), U_t)$$

for  $t$  such that  $t_k < t < t_{(k+1)}$  ( $1 \leq k \leq K-1$ ). Lastly,  $Y_t(\mathbf{d}_-) = Y_t = \mu_t(Y_{t-1}, U_t)$  for  $t < t_1$  and  $Y_t(\mathbf{d}_-) = Y_t(\mathbf{d}_-^{t_K}) = \mu_t(Y_{t-1}(\mathbf{d}_-^{t_K}), U_t)$  for  $t > t_K$ . Each structural model at the time of no treatment is a plain dynamic model with a lagged dependent variable. Let  $T = 4$  and  $\mathbf{d}_- = (d_1, d_3)$  for illustration. Then the sequence of potential outcomes can be expressed as

$$\begin{aligned} Y_4(\mathbf{d}_-) &= Y_4(\mathbf{d}_-^3) = \mu_4(Y_3(\mathbf{d}_-^3), U_4), \\ Y_3(\mathbf{d}_-) &= Y_3(\mathbf{d}_-^3) = \mu_3(Y_2(d_1), d_3, X_3, U_3(d_3)), \\ Y_2(\mathbf{d}_-) &= Y_2(d_1) = \mu_2(Y_1(d_1), U_2), \\ Y_1(\mathbf{d}_-) &= Y_1(d_1) = \mu_1(Y_0, d_1, X_1, U_1(d_1)). \end{aligned}$$

The selection equations are of the following form: For  $k \geq 1$ ,

$$D_{t_k} = 1\{\pi_{t_k}(Y_{t_k-1}, D_{t_{(k-1)}}, Z_{t_k}) \geq V_{t_k}\},$$

where the lagged outcome and the *latest* treatment enter each equation. The observable variables are  $(\mathbf{Y}, \mathbf{D}_-, \mathbf{X}_-, \mathbf{Z}_-)$ .<sup>13</sup>

Now all the parameters introduced in Section 3 can be readily modified by replacing  $\mathbf{d}$  with  $\mathbf{d}_-$  for some  $\mathbf{d}_-$ ; we omit the definitions for the sake of brevity. Moreover, the identification analysis of Section 4 can be easily modified in accordance with the extended setting. Let  $\mathbf{U}_-(\mathbf{d}_-) \equiv (U_{t_1}(d_{t_1}), \dots, U_{t_K}(d_{t_K}))$  and let  $\mathbf{U}(\mathbf{d}_-)$  be the vector of all the outcome unobservables that consists of  $\mathbf{U}_-(\mathbf{d}_-)$  and  $\{U_t\}_{t \in \{1, \dots, T\} \setminus \{t_1, \dots, t_K\}}$ .

**Assumption C'**. *The distribution of  $(\mathbf{U}_-(\mathbf{d}_-), \mathbf{V}_-)$  has strictly positive density with respect to Lebesgue measure on  $\mathbb{R}^{2K}$ .*

<sup>13</sup>It may be the case that  $X_t$  is observed whenever  $Y_t$  is observed, and thus is included in the  $Y_t$ -equations for  $t \neq t_k$  as well. We ignore that case here.

**Assumption SX'.**  $(\mathbf{Z}_-, \mathbf{X}_-)$  and  $(\mathbf{U}(\mathbf{d}_-), \mathbf{V}_-)$  are independent.

Let  $\mathbf{d}_{-, -t_k}$  be  $\mathbf{d}_-$  without the  $t_k$ -th element.

**Assumption RS'.** For each  $t_k$  and  $\mathbf{d}_{-, -t_k}$ ,  $\{\mathbf{U}_-(d_{t_k}, \mathbf{d}_{-, -t_k})\}_{d_{t_k}}$  are identically distributed conditional on  $(\mathbf{U}^{t_k-1}(\mathbf{d}_-^{t(k-1)}), \mathbf{V}_-^{t_k})$ .

Under these modified assumptions, Lemma 4.1 is now only relevant for  $t = t_k$ . Restrict the definitions of  $\mathcal{X}_t(d_t; \mathbf{x}_{-t}, \mathbf{z}_{-t})$  in (4.5) and  $\mathcal{X}_t(d_t; \mathbf{x}_{-t})$  to hold only for  $t = t_k$ .

**Assumption SP'.** For each  $t_k$  and  $d_{t_k}$ ,  $\Pr[X_{t_k} \in \mathcal{X}_{t_k}(d_{t_k}; \mathbf{x}_{-, -t_k}, \mathbf{z}_{-, -t_k}) | \mathbf{x}_{-, -t_k}, \mathbf{z}_{-, -t_k}] > 0$  almost everywhere.

Let  $\mathcal{X}_-(\mathbf{d}_-) \equiv \{\mathbf{x}_- : x_{t_k} \in \mathcal{X}_{t_k}(d_{t_k}; \mathbf{x}_{-, -t_k}) \text{ for some } (x_{t_{(k+1)}}, \dots, x_{t_K}), \text{ for } k \geq 1\}$ .

**Theorem 7.1.** Under Assumptions C', SX', RS' and SP',  $E[Y_T(\mathbf{d}_-) | \mathbf{x}_-]$  is identified for  $\mathbf{d}_- \in \mathcal{D}_-$ ,  $\mathbf{x}_- \in \mathcal{X}_-(\mathbf{d}_-)$ .

**Corollary 7.1.** Under Assumptions C', SX', RS' and SP',  $E[Y_T(\mathbf{d}_-) - Y_T(\tilde{\mathbf{d}}_-) | \mathbf{x}_-]$  is identified for  $\mathbf{d}_-, \tilde{\mathbf{d}}_- \in \mathcal{D}_-$  and  $\mathbf{x}_- \in \mathcal{X}_-(\mathbf{d}_-) \cap \mathcal{X}_-(\tilde{\mathbf{d}}_-)$ , and  $\mathbf{d}_-^*(\mathbf{x}_-)$  and  $\mathbf{d}_-^\dagger(\mathbf{x}_-)$  are identified for  $\mathbf{x}_- \in \bigcap_{\mathbf{d}_- \in \mathcal{D}_-} \mathcal{X}_-(\mathbf{d}_-)$ .

## References

- Abbring, J. H., Heckman, J. J., 2007. Econometric evaluation of social programs, part iii: Distributional treatment effects, dynamic treatment effects, dynamic discrete choice, and general equilibrium policy evaluation. *Handbook of econometrics* 6, 5145–5303. [2](#), [3](#)
- Abbring, J. H., Van den Berg, G. J., 2003. The nonparametric identification of treatment effects in duration models. *Econometrica* 71 (5), 1491–1517. [1](#), [5](#)
- Abraham, S., Sun, L., 2018. Estimating dynamic treatment effects in event studies with heterogeneous treatment effects. MIT. [1](#), [2.1](#)
- Angrist, J. D., Imbens, G. W., 1995. Two-stage least squares estimation of average causal effects in models with variable treatment intensity. *Journal of the American statistical Association* 90 (430), 431–442. [2](#)
- Balat, J., Han, S., 2018. Multiple treatments with strategic interaction. UT Austin. [1](#), [3.3](#), [7](#)
- Blevins, J. R., 2014. Nonparametric identification of dynamic decision processes with discrete and continuous choices. *Quantitative Economics* 5, 531–554. [1](#)
- Buchholz, N., Shum, M., Xu, H., 2016. Semiparametric estimation of dynamic discrete choice models. Princeton, Caltech and UT Austin. [1](#)
- Callaway, B., Sant'Anna, P. H., 2018. Difference-in-differences with multiple time periods and an application on the minimum wage and employment. Temple University and Vanderbilt University. [1](#), [2.1](#)

- Cameron, S. V., Heckman, J. J., 1998. Life cycle schooling and dynamic selection bias: Models and evidence for five cohorts of american males. *Journal of Political economy* 106 (2), 262–333. [8](#)
- Chernozhukov, V., Hansen, C., 2005. An IV model of quantile treatment effects. *Econometrica* 73 (1), 245–261. [3](#), [4](#)
- Cunha, F., Heckman, J. J., Navarro, S., 2007. The identification and economic content of ordered choice models with stochastic thresholds. *International Economic Review* 48 (4), 1273–1309. [1](#)
- Fredriksson, P., Johansson, P., 2008. Dynamic treatment assignment: the consequences for evaluations using observational data. *Journal of Business & Economic Statistics* 26 (4), 435–445. [5](#)
- Heckman, J. J., Humphries, J. E., Veramendi, G., 2016. Dynamic treatment effects. *Journal of econometrics* 191 (2), 276–292. [1](#), [2.1](#), [8](#)
- Heckman, J. J., Navarro, S., 2007. Dynamic discrete choice and dynamic treatment effects. *Journal of Econometrics* 136 (2), 341–396. [1](#), [2.1](#), [8](#), [5](#)
- Lee, S., Salanié, B., 2017. Identifying effects of multivalued treatments. Columbia University. [2](#)
- Murphy, S. A., 2003. Optimal dynamic treatment regimes. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 65 (2), 331–355. [1](#), [2](#), [3](#), [2.2](#)
- Murphy, S. A., van der Laan, M. J., Robins, J. M., Group, C. P. P. R., 2001. Marginal mean models for dynamic regimes. *Journal of the American Statistical Association* 96 (456), 1410–1423. [1](#), [2](#), [3](#), [2.2](#)
- Robins, J., 1986. A new approach to causal inference in mortality studies with a sustained exposure period—application to control of the healthy worker survivor effect. *Mathematical Modelling* 7 (9-12), 1393–1512. [1](#), [2](#)
- Robins, J., 1987. A graphical approach to the identification and estimation of causal parameters in mortality studies with sustained exposure periods. *Journal of chronic diseases* 40, 139S–161S. [1](#), [2](#)
- Robins, J. M., 1997. Causal inference from complex longitudinal data. In: *Latent variable modeling and applications to causality*. Springer, pp. 69–117. [1](#)
- Rust, J., 1987. Optimal replacement of gmc bus engines: An empirical model of harold zurcher. *Econometrica: Journal of the Econometric Society*, 999–1033. [1](#), [3](#)
- Shaikh, A. M., Vytlacil, E. J., 2011. Partial identification in triangular systems of equations with binary dependent variables. *Econometrica* 79 (3), 949–955. [1](#)
- Torgovitsky, A., 2016. Nonparametric inference on state dependence with applications to employment dynamics. University of Chicago. [1](#)

Vikström, J., Ridder, G., Weidner, M., 2017. Bounds on treatment effects on transitions. Uppsala University, USC and UCL. [1](#), [3.3](#), [4](#), [12](#)

Vytlacil, E., Yildiz, N., 2007. Dummy endogenous variables in weakly separable models. *Econometrica* 75 (3), 757–779. [1](#), [3.3](#), [4](#), [8](#)

## A Additional Proof

Proof of Theorem [5.1](#): We analyze the identification of  $E[Y_T(\mathbf{d})|\mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-, \mathbf{x}, \mathbf{z}]$ . Since

$$E[Y_T(\mathbf{d})|\mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-, \mathbf{x}, \mathbf{z}] = \Pr[Y_T(\mathbf{d}) = 1, \mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-|\mathbf{x}, \mathbf{z}] / \Pr[\mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-|\mathbf{x}, \mathbf{z}],$$

we identify each term in the fraction. For each term, the proof is parallel to that of Theorem [4.1](#). Let  $\tilde{\mathbf{y}}_- \equiv (y_1, \dots, y_{t_{\tilde{L}}})$  be a subvector (not necessarily strict) of  $\mathbf{y}$ , where  $t_1 < t_2 < \dots < t_{\tilde{L}} \leq T$  and  $\tilde{L} \leq T$ ; e.g., when  $\tilde{L} = T$ ,  $\tilde{\mathbf{y}}_- = \mathbf{y}$ . Generalizing the  $\mathcal{U}$ -sets introduced in Section [4](#), define

$$\mathcal{U}^{t_{\tilde{L}}}(\mathbf{d}^{t_{\tilde{L}}}, \tilde{\mathbf{y}}_-) \equiv \mathcal{U}^{t_{\tilde{L}}}(\mathbf{d}^{t_{\tilde{L}}}, \tilde{\mathbf{y}}_-; \mathbf{x}^{t_{\tilde{L}}}) \equiv \{\mathcal{U}^{t_{\tilde{L}}}(\mathbf{d}^{t_{\tilde{L}}}) : y_s = Y_s(\mathbf{d}^s, \mathbf{x}^s) \text{ for all } s \in \{t_1, \dots, t_{\tilde{L}}\}\}.$$

In the first part of the proof, we identify  $\Pr[Y_T(\mathbf{d}) = 1, \mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-|\mathbf{x}, \mathbf{z}]$ . Take  $\tilde{\mathbf{Y}}_-(\mathbf{d}) = (\mathbf{Y}_-(\mathbf{d}), Y_T(\mathbf{d}))$  with realization  $\tilde{\mathbf{y}}_- = (\mathbf{y}_-, 1)$ . Then, for  $2 \leq t \leq T - 1$ , we have

$$\begin{aligned} & \Pr[Y_T(\mathbf{d}) = 1, \mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-, \mathbf{Y}^{t-1} = \mathbf{y}^{t-1}, \mathbf{D}^t = (\mathbf{d}^{t-1}, d'_t)|\mathbf{x}, \mathbf{z}] \\ &= \Pr[\mathcal{U}(\mathbf{d}) \in \mathcal{U}^T(\mathbf{d}, \mathbf{y}_-, 1; \mathbf{x}), \mathcal{U}^{t-1}(\mathbf{d}^{t-1}) \in \mathcal{U}^{t-1}(\mathbf{d}^{t-1}, \mathbf{y}^{t-1}), \mathcal{V}^t \in \mathcal{V}^t(\mathbf{d}^{t-1}, d'_t, \mathcal{U}^{t-1}(\mathbf{d}^{t-1}))] \\ &= \Pr \left[ \begin{array}{l} \mathcal{U}(d'_t, \mathbf{d}_{-t}) \in \mathcal{U}^T(d'_t, \mathbf{d}_{-t}, \mathbf{y}_-, 1; \mathbf{x}'_t, \mathbf{x}_{-t}), \\ \mathcal{U}^{t-1}(\mathbf{d}^{t-1}) \in \mathcal{U}^{t-1}(\mathbf{d}^{t-1}, \mathbf{y}^{t-1}), \\ \mathcal{V}^t \in \mathcal{V}^t(\mathbf{d}^{t-1}, d'_t, \mathcal{U}^{t-1}(\mathbf{d}^{t-1})) \end{array} \right] \\ &= \Pr[Y_T(d'_t, \mathbf{d}_{-t}) = 1, \mathbf{Y}_-(d'_t, \mathbf{d}_{-t}) = \mathbf{y}_-, \mathbf{Y}^{t-1} = \mathbf{y}^{t-1}, \mathbf{D}^t = (\mathbf{d}^{t-1}, d'_t)|\mathbf{x}'_t, \mathbf{x}_{-t}, \mathbf{z}], \end{aligned} \quad (\text{A.1})$$

where the second equality uses  $\mathbf{x}'_t$  such that  $\mu_t(y_{t-1}, d_t, x_t) = \mu_t(y_{t-1}, d'_t, x'_t)$  by applying Lemma [4.1](#). First,  $\Pr[Y_T(\mathbf{d}) = 1, \mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-, \mathbf{Y}^{T-1} = \mathbf{y}^{T-1}, \mathbf{D}^T = \mathbf{d}^T|\mathbf{x}, \mathbf{z}] = \Pr[Y_T = 1, \mathbf{Y}_- = \mathbf{y}_-, \mathbf{Y}^{T-1} = \mathbf{y}^{T-1}, \mathbf{D}^T = \mathbf{d}^T|\mathbf{x}, \mathbf{z}]$  is trivially identified for any generic values  $(\mathbf{d}, \mathbf{x}, \mathbf{z}, \mathbf{y}^{T-1})$ . For given  $2 \leq t \leq T - 1$ , suppose  $\Pr[Y_T(\mathbf{d}) = 1, \mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-, \mathbf{Y}^{t-1} = \mathbf{y}^{t-1}, \mathbf{D}^t = \mathbf{d}^t|\mathbf{x}, \mathbf{z}]$  is identified for any generic values  $(\mathbf{d}, \mathbf{x}, \mathbf{z}, \mathbf{y}^{t-1})$ , and consider identification of

$$\begin{aligned} & \Pr[Y_T(\mathbf{d}) = 1, \mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-, \mathbf{Y}^{t-2} = \mathbf{y}^{t-2}, \mathbf{D}^{t-1} = \mathbf{d}^{t-1}|\mathbf{x}, \mathbf{z}] \\ &= \Pr[Y_T(\mathbf{d}) = 1, \mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-, \mathbf{Y}^{t-2} = \mathbf{y}^{t-2}, \mathbf{D}^t = (\mathbf{d}^{t-1}, d_t)|\mathbf{x}, \mathbf{z}] \\ & \quad + \Pr[Y_T(\mathbf{d}) = 1, \mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-, \mathbf{Y}^{t-2} = \mathbf{y}^{t-2}, \mathbf{D}^t = (\mathbf{d}^{t-1}, d'_t)|\mathbf{x}, \mathbf{z}]. \end{aligned} \quad (\text{A.2})$$

The first term in the expression is identified, by summing over  $y_{t-1}$  the quantity  $\Pr[Y_T(\mathbf{d}) = 1, \mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-, \mathbf{Y}^{t-1} = \mathbf{y}^{t-1}, \mathbf{D}^t = \mathbf{d}^t|\mathbf{x}, \mathbf{z}]$ , which is identified from the previous iteration.

The second unknown term in (A.2) satisfies

$$\begin{aligned} & \Pr[Y_T(\mathbf{d}) = 1, \mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-, \mathbf{Y}^{t-2} = \mathbf{y}^{t-2}, \mathbf{D}^t = (\mathbf{d}^{t-1}, d'_t) | \mathbf{x}, \mathbf{z}] \\ &= \Pr[Y_T(\mathbf{d}) = 1, \mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-, \mathbf{Y}^{t-1} = (\mathbf{y}^{t-2}, 1), \mathbf{D}^t = (\mathbf{d}^{t-1}, d'_t) | \mathbf{x}, \mathbf{z}] \\ & \quad + \Pr[Y_T(\mathbf{d}) = 1, \mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-, \mathbf{Y}^{t-1} = (\mathbf{y}^{t-2}, 0), \mathbf{D}^t = (\mathbf{d}^{t-1}, d'_t) | \mathbf{x}, \mathbf{z}]. \end{aligned} \quad (\text{A.3})$$

But note that, by (A.1), each term in (A.3) satisfies

$$\begin{aligned} & \Pr[Y_T(\mathbf{d}) = 1, \mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-, \mathbf{Y}^{t-1} = \tilde{\mathbf{y}}^{t-1}, \mathbf{D}^t = (\mathbf{d}^{t-1}, d'_t) | \mathbf{x}, \mathbf{z}] \\ &= \Pr[Y_T(d'_t, \mathbf{d}_{-t}) = 1, \mathbf{Y}_-(d'_t, \mathbf{d}_{-t}) = \mathbf{y}_-, \mathbf{Y}^{t-1} = \tilde{\mathbf{y}}^{t-1}, \mathbf{D}^t = (\mathbf{d}^{t-1}, d'_t) | x'_t, \mathbf{x}_{-t}, \mathbf{z}] \end{aligned} \quad (\text{A.4})$$

for each  $\tilde{\mathbf{y}}^{t-1}$ , which is identified from the previous iteration. Therefore,  $\Pr[Y_T(\mathbf{d}) = 1, \mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-, \mathbf{Y}^{t-2} = \mathbf{y}^{t-2}, \mathbf{D}^{t-1} = \mathbf{d}^{t-1} | \mathbf{x}, \mathbf{z}]$  is identified. Lastly, when  $t = 1$ ,

$$\begin{aligned} \Pr[Y_T(\mathbf{d}) = 1, \mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_- | \mathbf{x}, \mathbf{z}] &= \Pr[Y_T(\mathbf{d}) = 1, \mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-, D_1 = d_1 | \mathbf{x}, \mathbf{z}] \\ & \quad + \Pr[Y_T(\mathbf{d}) = 1, \mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-, D_1 = d'_1 | \mathbf{x}, \mathbf{z}]. \end{aligned}$$

The first term is identified from the iteration for  $t = 2$ . Noting that  $Y_0 = 0$ , suppose  $x'_1$  is such that  $\mu_1(0, d_1, x_1) = \mu_1(0, d'_1, x'_1)$  with  $d'_1 \neq d_1$  by Lemma 4.1. Then, similarly to (A.1),

$$\begin{aligned} & \Pr[Y_T(\mathbf{d}) = 1, \mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-, D_1 = d'_1 | \mathbf{x}, \mathbf{z}] \\ &= \Pr[Y_T(d'_1, \mathbf{d}_{-1}) = 1, \mathbf{Y}_-(d'_1, \mathbf{d}_{-1}) = \mathbf{y}_-, D_1 = d'_1 | x'_1, \mathbf{x}_{-1}, \mathbf{z}], \end{aligned}$$

which is also identified from the previous iteration for  $t = 2$ . Therefore  $\Pr[Y_T(\mathbf{d}) = 1, \mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_- | \mathbf{x}, \mathbf{z}]$  is identified.

In the second part of the proof, we identify  $\Pr[\mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_- | \mathbf{x}, \mathbf{z}]$ . Take  $\tilde{\mathbf{Y}}_-(\mathbf{d}) = \mathbf{Y}_-(\mathbf{d}) \equiv (Y_{t_1}(\mathbf{d}), \dots, Y_{t_L}(\mathbf{d}))$  with realization  $\tilde{\mathbf{y}}_- = \mathbf{y}_-$ . Then, for  $2 \leq t \leq t_L - 1$ , we can show the following equivalence, analogous to (A.1):

$$\begin{aligned} & \Pr[\mathbf{Y}_-(\mathbf{d}) = \mathbf{y}_-, \mathbf{Y}^{t-1} = \mathbf{y}^{t-1}, \mathbf{D}^t = (\mathbf{d}^{t-1}, d'_t) | \mathbf{x}, \mathbf{z}] \\ &= \Pr[\mathbf{U}^{t_L}(\mathbf{d}^{t_L}) \in \mathcal{U}^{t_L}(\mathbf{d}^{t_L}, \mathbf{y}_-; \mathbf{x}^{t_L}), \mathbf{U}^{t-1}(\mathbf{d}^{t-1}) \in \mathcal{U}^{t-1}(\mathbf{d}^{t-1}, \mathbf{y}^{t-1}), \mathbf{V}^t \in \mathcal{V}^t(\mathbf{d}^{t-1}, d'_t, \mathbf{U}^{t-1}(\mathbf{d}^{t-1}))] \\ &= \Pr \left[ \begin{array}{l} \mathbf{U}^{t_L}(d'_t, \mathbf{d}_{-t}^{t_L}) \in \mathcal{U}^{t_L}(d'_t, \mathbf{d}_{-t}^{t_L}, \mathbf{y}_-; x'_t, \mathbf{x}_{-t}^{t_L}), \\ \mathbf{U}^{t-1}(\mathbf{d}^{t-1}) \in \mathcal{U}^{t-1}(\mathbf{d}^{t-1}, \mathbf{y}^{t-1}), \\ \mathbf{V}^t \in \mathcal{V}^t(\mathbf{d}^{t-1}, d'_t, \mathbf{U}^{t-1}(\mathbf{d}^{t-1})) \end{array} \right] \\ &= \Pr[\mathbf{Y}_-(d'_t, \mathbf{d}_{-t}) = \mathbf{y}_-, \mathbf{Y}^{t-1} = \mathbf{y}^{t-1}, \mathbf{D}^t = (\mathbf{d}^{t-1}, d'_t) | x'_t, \mathbf{x}_{-t}, \mathbf{z}]. \end{aligned}$$

The rest of the proof is an immediate modification of the iterative argument in the first part, and hence is omitted.  $\square$