Identification in Nonparametric Models for Dynamic Treatment Effects*

Sukjin Han
Department of Economics
University of Texas at Austin
sukjin.han@austin.utexas.edu

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Abstract

This paper develops a nonparametric model that represents how sequences of outcomes and treatment choices influence one another in a dynamic manner. In this setting, we are interested in identifying the average outcome for individuals in each period, had a particular treatment sequence been assigned. The identification of this quantity allows us to identify the average treatment effects (ATE's) and the ATE's on transitions, as well as the optimal treatment regimes, namely, the regimes that maximize the (weighted) sum of the average potential outcomes, possibly less the cost of the treatments. The main contribution of this paper is to relax the sequential randomization assumption widely used in the biostatistics literature by introducing a flexible choice-theoretic framework for a sequence of endogenous treatments. This framework allows non-compliance of subjects in experimental studies or endogenous treatment decisions in observational settings. We show that the parameters of interest are identified under each period's two-way exclusion restriction, i.e., with instruments excluded from the outcome-determining process and other exogenous variables excluded from the treatment-selection process. We also consider partial identification in the case where the latter variables are not available. Lastly, we extend our results to a setting where treatments do not appear in every period.

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1 Introduction

This paper develops a nonparametric model that represents how sequences of outcomes and treatment choices influence one another in a dynamic manner. Often, treatments are chosen

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multiple times over a horizon, affecting a series of outcomes. Examples are medical interventions that affect health outcomes, educational interventions that affect academic achievements, job training programs that affect employment status, or online advertisements that affect consumers' preferences or purchase decisions. Agents endogenously make decisions of receiving treatments, e.g., whether to comply with random assignments. The relationship of interest is dynamic in the sense that the current outcome is determined by past outcomes as well as current and past treatments, and the current treatment is determined by past outcomes as well as past treatments. Such dynamic relationships are clearly present in the aforementioned examples. A static model misrepresents the nature of the problem (e.g., nonstationarity, state dependence, learning) and fails to capture important policy questions (e.g., optimal timing and schedule of interventions).

In this setting, we are interested in identifying the dynamic causal effect of a sequence of treatments on a sequence of outcomes or on a terminal outcome that may or may not be of the same kind as the intermediate outcomes. We are interested in learning about the average of the outcome in each period, had a particular treatment sequence been assigned *up to that period*, which defines the potential outcome in this dynamic setting. We are also interested in the average treatment effects (ATE's) and the transition-specific ATE's defined based on the average potential outcome, unconditional and conditional on the previous outcomes, respectively. For example, one may be interested in whether the success rate of a particular outcome (or the transition probability) is larger with a sequence of treatments assigned in relatively later periods rather than earlier, or with a sequence of alternating treatments rather than consistent treatments. The treatment effect is said to be dynamic, partly because the effect can vary depending upon the period of measurement, even if the same set of treatments is assigned. Lastly, we are interested in the optimal treatment regimes, namely, sequences of treatments that maximize the (weighted) sum of the average potential outcomes, possibly less the cost of the treatments. For example, a firm may be interested in the optimal timing of advertisements that maximizes its aggregate sales probabilities over time, or a sequence of educational programs may be aimed to maximize the college attendance rate. We show that the optimal regime is a natural extension of a static object commonly sought in the literature, namely, the sign of the ATE. Analogous to the static environment, knowledge about the optimal treatment regime may have useful policy implications. For example, a social planner may wish to at least exclude specific sequences of treatments that are on average suboptimal.

Dynamic treatment effects have been extensively studied in the biostatistics literature for decades under the counterfactual framework with a sequence of treatments (Robins (1986, 1987, 1997), Murphy et al. (2001), Murphy (2003), among others). In this literature, the crucial condition used to identify the average potential outcome is a dynamic version of a random assignment assumption, called the *sequential randomization*. This condition assumes that the treatment is randomized in every period within those individuals who have the same history of outcomes and treatments.[1] This assumption is only suitable in experimental studies with the perfect compliance of subjects, which is often not easy to achieve (Robins (1994); Robins and Rotnitzky (2004)). When interventions continue for multiple periods as

[1]This assumption is also called sequential conditional independence or sequential ignorability. In the econometrics literature, Vikström et al. (2018) consider treatment effects on a transition to a destination state, and carefully analyze what the sequential randomization assumption can identify in the presence of dynamic selection.

in the examples described above, the issue of non-compliance may become more prevalent than in one-time experiments, e.g., due to the cost of enforcement or the subjects' learning. In addition to partial compliance in experimental settings, sequential randomization is invalid in many observational contexts as well.

The main contribution of this paper is to relax the assumption of sequential randomization widely used in the literature by introducing a flexible choice-theoretic framework for a sequence of endogenous treatments. To this end, we consider a simple nonparametric structural model for a dynamic endogenous selection process and dynamic outcome formation. In this model, individuals are allowed not to fully comply with each period's assignment in experimental settings, or are allowed to make an endogenous choice in each period as in observational settings. The heterogeneity in each period's potential outcome is given by recursively applying a switching-regression type of models with a sequential version of rank similarity. The joint distribution of the full history of unobservable variables in the outcome and treatment equations is still flexible, allowing for arbitrary forms of treatment endogeneity as well as serial correlation. Relative to the counterfactual framework, the dynamic mechanism is clearly formulated using this structural model, which in turn facilitates our identification analysis.

We show that the average potential outcome, or equivalently, the average recursive structural function (ARSF) given the structural model we introduce, is identified under a two-way exclusion restriction (Vytlacil and Yildiz (2007), Shaikh and Vytlacil (2011), Balat and Han (2018)). That is, we assume there exist instruments excluded from the outcome-determining process and exogenous variables excluded from the treatment-selection process. A leading example of the former is a sequence of randomized treatment assignments from, e.g., clinical trials, field experiments, and A/B testings, and other examples include sequential policy shocks. Examples of the latter include factors that agents cannot anticipate when making treatment or compliance choices but that determine the outcome. We show that such timing can be justified in this dynamic context, and some covariates in the outcome process may be valid candidates. Using the exclusion restriction, we gain certain knowledge of each period's structural function, which is then iteratively incorporated across periods for identification, obeying the recursive structure of the potential outcome. The proof is constructive and provides a closed form expression for the ARSF. The identification of each period's ARSF allows us to identify the ATE's and the optimal treatment regimes. In this paper, we also consider cases where the two-way exclusion restriction is violated in the sense that only a standard exclusion restriction holds or that the variation of the exogenous variables is limited. In these cases, we can calculate the bounds on the parameters. As an extension of our results, we consider another empirically relevant situation where treatments do not appear in every period, while outcomes are constantly observed. We show that the parameters of interest and the identification analysis can be easily modified to incorporate this situation.

This paper contributes to recent research on the identification of the effects of dynamic endogenous treatments that allows for treatment heterogeneity. Cunha et al. (2007) and Heckman and Navarro (2007) consider a semiparametric discrete-time duration model for the choice of the treatment timing and associated outcomes. Building on these works, Heckman et al. (2016) consider not only ordered choice models but also unordered choice models for up-or-out treatment choices.[2] An interesting feature of their results is that dynamic treatment

[2]As related works, the settings of Angrist and Imbens (1995), Jun et al. (2016), and Lee and Salanié (2017)

effects are decomposed into direct effects and continuation values. As an important feature, these papers consider attrition based on the irreversible treatment decisions; see also Sasaki (2015). Similar to our approach, Heckman and Navarro (2007) and Heckman et al. (2016) utilize exclusion restrictions. Unlike these papers, however, we do not necessarily invoke infinite supports of exogenous variables but instead use the two-way exclusion restriction. Abraham and Sun (2018), Athey and Imbens (2018), and Callaway and Sant'Anna (2018) extend a difference-in-differences approach to dynamic settings without specifying fixed-effect panel data models. They consider the effects of treatment timing on the treated, where the treatment process is irreversible as in the previous works. We consider nonparametric dynamic models for treatment and outcome processes with a general form of evolution, where the processes can freely change states. These models can include an irreversible process as a special case. Moreover, we consider different identifying assumptions than those in the previous works and focus on the identification of the ATE's and related parameters.

This paper's structural approach is only relative to the counterfactual framework of Robins. A fully structural model of dynamic programming is considered in the seminal work by Rust (1987) and more recently by, e.g., Blevins (2014) and Buchholz et al. (2016). This literature typically considers a single rational agent's optimal decision, whereas we consider a large group of heterogenous agents with no assumptions on agents' rationality or strong parametric assumptions. Most importantly, our focus is on the identification of the effects of treatments formed as agents' decisions. The robust approach of this paper is, in spirit, similar to Heckman and Navarro (2007) and Heckman et al. (2016), in that we remain flexible for the economic and non-economic components of the model. Lastly, Torgovitsky (2016) extends the literature on dynamic binary response models (with no treatment) by considering a counterfactual framework without imposing parametric assumptions. In his framework, the lagged outcome plays the role of a treatment for the current outcome, and the "treatment effect" captures the state dependence. Here, we consider the effects of the treatments on the outcomes, and introduce a selection equation for each treatment as an important component of the model. As an extension of our analysis, we identify the transition-specific ATE, which is related to the effect of a treatment on the state dependence.

In terms of notation, let $\boldsymbol{W}^t \equiv (W_1, .., W_t)$ denote a row vector that collects r.v.'s W_t across time up to t, and let $\boldsymbol{w}^t$ be its realization. We sometimes write $\boldsymbol{W} \equiv \boldsymbol{W}^T$ for convenience. For a vector $\boldsymbol{W}$ without the t-th element, we write $\boldsymbol{W}_{-t} \equiv (W_1, ..., W_{t-1}, W_{t+1}, ..., W_T)$ with realization $\boldsymbol{w}_{-t}$. More generally, let $\boldsymbol{W}_{-}$ with realization $\boldsymbol{w}_{-}$ denote some subvector of $\boldsymbol{W}$. Lastly, for r.v.'s Y and W, we sometimes abbreviate $\Pr[Y = y|W = w]$ and $\Pr[Y = y|W \in \mathcal{W}]$ to $\Pr[Y = y|w]$ (or $P[y|w]$) and $\Pr[Y = y|\mathcal{W}]$, respectively.

2 Robins's Framework

We first introduce Robins's counterfactual framework and state the assumption of sequential randomization commonly used in the biostatistics literature (Robins (1986, 1987), Murphy et al. (2001), Murphy (2003)). For a finite horizon $t = 1, ..., T$ with fixed T, let Y_t be the outcome at t with realization y_t and let D_t be the binary treatment at t with realization d_t. The underlying data structure is panel data with a large number of cross-sectional observations

for multiple (or multi-valued) treatment effects may be applied to a dynamic setting. Also, see Abbring and Heckman (2007) for a survey on dynamic treatment effects.

over a short period of time (and the cross-sectional index i suppressed throughout, unless necessary). We call Y_T a *terminal outcome* and Y_t for $t \leq T-1$ a *intermediate outcome*.[3] Let $\mathcal{Y}$ and $\mathcal{D} \subseteq \{0,1\}^T$ be the supports of $\boldsymbol{Y} \equiv (Y_1, ..., Y_T)$ and $\boldsymbol{D} \equiv (D_1, ..., D_T)$, respectively.

Consider a treatment regime $\boldsymbol{d} \equiv (d_1, ..., d_T) \in \mathcal{D}$, which is defined as a predetermined hypothetical sequence of interventions over time, i.e., a sequence of each period's decisions on whether to treat or not, or whether to choose treatment A or treatment B.[4] Then, a potential outcome at t can be written as $Y_t(\boldsymbol{d})$. This can be understood as an outcome for an individual, had a particular treatment sequence been assigned. Although the genesis of $Y_t(\boldsymbol{d})$ can be very general under this counterfactual framework, the mechanism under which the sequence of treatments interacts with the sequence of outcomes is opaque. The definition of $Y_t(\boldsymbol{d})$ becomes more transparent later with the structural model introduced in this paper.

Given these definitions, we state the assumption of sequential randomization by Robins: For each $\boldsymbol{d} \in \mathcal{D}$,

$$(Y_1(\boldsymbol{d}), ..., Y_T(\boldsymbol{d})) \perp D_t | \boldsymbol{Y}^{t-1}, \boldsymbol{D}^{t-1} \tag{2.1}$$

for $t = 1, ..., T$. This assumption asserts that, holding the history of outcomes and treatments (and potentially other covariates) fixed, the current treatment is fully randomized. Sequential randomization can be violated if agents make decisions D_t based on time-varying or time-invariant factors, unobserved to the analyst. In the next section, we relax this assumption and specify dynamic selection equations for a sequence of treatments that are allowed to be endogenous, i.e., to be dependent on unobservable factors. Apart from this assumption, we maintain the same preliminaries introduced in this section.

Remark 2.1 (Irreversibility). *As a special case of our setting, the process of D_t may be irreversible in that the process only moves from an initial state to a destination state, i.e., the destination state is an absorbing state. The up-or-out treatment decision (or the treatment timing) can be an example where the treatment process satisfies $D_t = 1$ once $D_{t-1} = 1$ is reached, as in Heckman and Navarro (2007), Heckman et al. (2016), Abraham and Sun (2018) and Callaway and Sant'Anna (2018). Although it is not the main focus of this paper, the process of Y_t may as well be irreversible. This case, however, requires caution due to dynamic selection; see discussions later in this paper. The survival of patients ($Y_t = 0$) in discrete time duration models can be an example where the transition of the outcome satisfies $Y_t = 1$ once $Y_{t-1} = 1$. In this case, it may be that D_t is missing when $Y_{t-1} = 1$, which can be dealt by conventionally assuming $D_t = 0$ if $Y_{t-1} = 1$. When processes are irreversible, the supports $\mathcal{D}$ and $\mathcal{Y}$ are strict subsets of $\{0,1\}^T$.*

Remark 2.2 (Terminal outcome of a different kind). *As in Murphy et al. (2001) and Murphy (2003), we may be interested in a terminal outcome that is of a different kind than that of the intermediate outcomes. For example, the terminal outcome can be college attendance, while the intermediate outcomes are secondary school performances. In this case, we replace Y_T with a random variable R_T to represent the terminal outcome, while maintaining*

[3] The terminal period T may be an administrative end of follow-up time.

[4] This is called a nondynamic regime in the biostatistics literature. A dynamic regime is a sequence of treatment assignments, each of which is a predetermined function of past outcomes. A nondynamic regime can be viewed as its special case, where this function is constant. See, e.g., Murphy et al. (2001); Murphy (2003) for related discussions.

Y_t for $t \leq T-1$ to represent the intermediate outcomes. Analogously, $R_T(\boldsymbol{d})$ denotes the potential terminal outcome. Then, the analysis in this paper can be readily followed with the change of notation.[5]

3 A Dynamic Structural Model and Objects of Interest

We now introduce the main framework of this paper. Consider a *dynamic structural function* for the outcomes Y_t's that has the form of switching regression models: For $t = 1, ..., T$,

$$Y_t = \mu_t(Y_{t-1}, D_t, X_t, U_t(D_t)),$$

where $\mu_t(\cdot)$ is an unknown scalar-valued function, X_t is a set of exogenous variables, which we discuss in detail later, and Y_0 is assumed to be exogenously determined, with $Y_0 = 0$ for convenience.[6] The unobservable variable satisfies $U_t(D_t) = D_t U_t(1) + (1 - D_t) U_t(0)$, where $U_t(d_t)$ is the "rank variable" that captures the unobserved characteristics or rank, specific to treatment state d_t (Chernozhukov and Hansen (2005)). We allow $U_{it}(d_t)$ to contain a permanent component (i.e., individual effects) and a transitory component.[7] Given this structural equation, we can express the potential outcome $Y_t(\boldsymbol{d})$ using a recursive structure:

$$\begin{aligned} Y_t(\boldsymbol{d}) &= Y_t(\boldsymbol{d}^t) = \mu_t(Y_{t-1}(\boldsymbol{d}^{t-1}), d_t, X_t, U_t(d_t)), \\ &\vdots \\ Y_2(\boldsymbol{d}) &= Y_2(\boldsymbol{d}^2) = \mu_2(Y_1(d_1), d_2, X_2, U_2(d_2)), \\ Y_1(\boldsymbol{d}) &= Y_1(d_1) = \mu_1(Y_0, d_1, X_1, U_1(d_1)), \end{aligned}$$

where each potential outcome at time t is only a function of $\boldsymbol{d}^t$ (not the full $\boldsymbol{d}$). This is related to the "no-anticipation" condition (Abbring and Heckman (2007)) or the "consistency" condition (Robins (2000)), which is implied from the structure of the model in our setting. The recursive structure provides us with a useful interpretation of the potential outcome $Y_t(\boldsymbol{d})$ in a dynamic setting, and thus facilitates our identification analysis. Note that, conditional on $\boldsymbol{X}^t \equiv (X_1, ..., X_t)$, the heterogeneity in $Y_t(\boldsymbol{d})$ comes from the full vector $\boldsymbol{U}^t(\boldsymbol{d}^t) \equiv (U_1(d_1), ..., U_t(d_t))$. By an iterative argument, we can readily show that the potential outcome is equal to the observed outcome when the observed treatments are consistent with the assigned regime: $Y_t(\boldsymbol{d}) = Y_t$ when $\boldsymbol{D} = \boldsymbol{d}$, or equivalently, $Y_t = \sum_{\boldsymbol{d} \in \mathcal{D}} 1\{\boldsymbol{D} = \boldsymbol{d}\} Y_t(\boldsymbol{d})$.

In this paper, we consider the *average potential terminal outcome*, conditional on $\boldsymbol{X} = \boldsymbol{x}$, as the fundamental parameter of interest:

$$E[Y_T(\boldsymbol{d}) | \boldsymbol{X} = \boldsymbol{x}]. \tag{3.1}$$

We also call this parameter the *average recursive structural function* (ARSF) in the terminal

[5] Extending this framework to incorporate the irreversibility of the outcome variables discussed in Remark 2.1 is not straightforward. We leave this for future research.

[6] This assumption of an exogenous initial outcome is *not* necessary but only introduced to simplify our analysis; see Remark 4.2 for alternative assumptions.

[7] In this case, it may make sense that the permanent component does not depend on each d_t, but that the transitory component does.

period, named after the recursive structure in the model for $Y_T(\boldsymbol{d})$. Generally, in defining this parameter and all others below, we can consider the potential outcome in any time period of interest, e.g., $E[Y_t(\boldsymbol{d})|\boldsymbol{X}^t = \boldsymbol{x}^t]$ for any given t. We focus on the terminal potential outcome only for concreteness. The knowledge of the ARSF is useful in recovering other related parameters.

First, we are interested in the conditional ATE:

$$ATE(\boldsymbol{d}, \tilde{\boldsymbol{d}}) \equiv E[Y_T(\boldsymbol{d}) - Y_T(\tilde{\boldsymbol{d}})|\boldsymbol{X} = \boldsymbol{x}] \tag{3.2}$$

for two different regimes, $\boldsymbol{d}$ and $\tilde{\boldsymbol{d}}$. For example, one may be interested in comparing more versus less consistent treatment sequences, or earlier versus later treatments.

Second, we consider the *optimal treatment regime*:

$$\boldsymbol{d}^*(\boldsymbol{x}) = \arg\max_{\boldsymbol{d} \in \mathcal{D}} E[Y_T(\boldsymbol{d})|\boldsymbol{X} = \boldsymbol{x}] \tag{3.3}$$

with $|\mathcal{D}| \leq 2^T$. That is, we are interested in a treatment regime that delivers the maximum expected potential outcome, conditional on $\boldsymbol{X} = \boldsymbol{x}$. Notice that, in a static model, the identification of $\boldsymbol{d}^*$ is equivalent to the identification of the sign of the static ATE, which is the information typically sought from a policy point of view. One can view $\boldsymbol{d}^*$ as a natural extension of this information to a dynamic setting, which is identified by establishing the signs of *all* possible ATE's defined as in (3.2), or equivalently, by ordering all the possible ARSF's. The optimal regime may serve as a guideline in developing future policies. Moreover, it may be a realistic goal for a social planner to identify this kind of scheme that maximizes the average benefit, because it may be too costly to find a customized treatment scheme for every individual. Yet, the optimal regime is customized up to observed characteristics, as it is a function of the covariates values $\boldsymbol{x}$. More ambitious than the identification of $\boldsymbol{d}^*(\boldsymbol{x})$ may be recovering an optimal regime based on a cost–benefit analysis, granting than each d_t can be costly:

$$\boldsymbol{d}^\dagger(\boldsymbol{x}) = \arg\max_{\boldsymbol{d} \in \mathcal{D}} \Pi(\boldsymbol{d}; \boldsymbol{x}), \tag{3.4}$$

where

$$\Pi(\boldsymbol{d}; \boldsymbol{x}) \equiv wE[Y_T(\boldsymbol{d})|\boldsymbol{X} = \boldsymbol{x}] - \tilde{w} \sum_{t=1}^{T} d_t \quad \text{or} \quad \Pi(\boldsymbol{d}; \boldsymbol{x}) \equiv \sum_{t=1}^{T} w_t E[Y_t(\boldsymbol{d})|\boldsymbol{X} = \boldsymbol{x}] - \sum_{t=1}^{T} \tilde{w}_t d_t$$

with $(w, \tilde{w})$ and $(\boldsymbol{w}, \tilde{\boldsymbol{w}})$ being predetermined weights. The latter objective function concerns the weighted sum of the average potential outcomes throughout the entire period, less the cost of treatments. Note that establishing the signs of ATE's will not identify $\boldsymbol{d}^\dagger$, and a stronger identification result becomes important, i.e., the point identification of $E[Y_T(\boldsymbol{d})|\boldsymbol{X} = \boldsymbol{x}]$ for all $\boldsymbol{d}$ (or $E[Y_t(\boldsymbol{d})|\boldsymbol{X} = \boldsymbol{x}]$ for all t and $\boldsymbol{d}$).

Lastly, we are interested in the *transition-specific ATE*:

$$E[Y_T(\boldsymbol{d})|Y_{T-1}(\boldsymbol{d}) = y_{T-1}, \boldsymbol{X} = \boldsymbol{x}] - E[Y_T(\tilde{\boldsymbol{d}})|Y_{T-1}(\tilde{\boldsymbol{d}}) = y_{T-1}, \boldsymbol{X} = \boldsymbol{x}] \tag{3.5}$$

for two different $\boldsymbol{d}$ and $\tilde{\boldsymbol{d}}$. The knowledge of the ARSF does not directly recover this parameter, but the identification of it (and its more general form introduced later) can be paralleled by the analysis for the ARSF and ATE.

In order to facilitate identification of the parameters of interest without assuming sequential randomization, we introduce a sequence of selection equations for the binary endogenous treatments D_t's: For $t = 1, ..., T$,

$$D_t = 1\{\pi_t(Y_{t-1}, D_{t-1}, Z_t) \geq V_t\},$$

where $\pi_t(\cdot)$ is an unknown scalar-valued function, Z_t is the period-specific instruments, V_t is the unobservable variable that may contain permanent and transitory components, and D_0 is assumed to be exogenously given as $D_0 = 0$.[8] This dynamic selection process represents the agent's endogenous choices over time, e.g., as a result of learning or other optimal behaviors. However, the nonparametric threshold-crossing structure posits a minimal notion of optimality for the agent. We take an agnostic approach by avoiding strong assumptions of the standard dynamic economic models pioneered by Rust (1987), such as forward looking behaviors and being able to compute a present value discounted flow of utilities. If we are to maintain the assumption of rational agents, the selection model can be viewed as a reduced-form approximation of a solution to a dynamic programming problem.

To simplify the exposition, we consider binary Y_t and impose weak separability in the outcome equation as in the treatment equation. The binary outcome is *not* necessary for the result of this paper, and the analysis can be easily extended to the case of continuous or censored Y_t, maintaining weak separability; see Remark 3.3. Then, the full model can be summarized as

$$Y_t = 1\{\mu_t(Y_{t-1}, D_t, X_t) \geq U_t(D_t)\}, \tag{3.6}$$

$$D_t = 1\{\pi_t(Y_{t-1}, D_{t-1}, Z_t) \geq V_t\}. \tag{3.7}$$

In this model, the observable variables are $(\boldsymbol{Y}, \boldsymbol{D}, \boldsymbol{X}, \boldsymbol{Z})$. All other covariates are suppressed in the equations for simplicity of exposition. Importantly, in this model, the joint distribution of the unobservable variables $(\boldsymbol{U}(\boldsymbol{d}), \boldsymbol{V})$ for given $\boldsymbol{d}$ is not specified, in that $U_t(d_t)$ and $V_{t'}$ for any t, t' are allowed to be arbitrarily correlated to each other (allowing endogeneity) as well as within themselves across time (allowing serial correlation, e.g., via time-invariant individual effects). Note that, because we allow an arbitrary form of persistence in the unobservables, (Y_t, D_t) is *not* a Markov process even after conditioning on the observables. This is in contrast to the standard dynamic economic models, where conditional independence assumptions or Markovian unobservables are commonly introduced. By considering the nonparametric index functions that depend on t, we also avoid other strong assumptions on parametric functional forms or time homogeneity.

Example 1. *As a concrete example of our setting, consider a multi-period experiment, which is common in clinical trials, such as the Fast Track Prevention Program in Conduct Problems Prevention Research Group (1992); also see the biostatistics literature referenced in the introduction. A clinical research organization is interested in improving patients' symptoms (Y_t), and runs an experiment of randomly assigning treatments at each t (Z_t). Based on*

[8] This is an alternative to simply assuming there is no treatment at $t = 0$. We maintain the current assumption to avoid additional definitions for $\pi_1(\cdot)$ and other relevant objects.

what is assigned, each patient decides whether or not to receive the treatment (D_t) by being a complier, defier, always-taker or never-taker. In doing so, the patient has a habit in her decision (D_{t-1}) and takes into account her last symptom (Y_{t-1}). The current symptom (Y_t) is formed based on the previous symptom (Y_{t-1}), the current treatment (D_t), and other symptom-influencing factors (X_t) occurring at time t.

Remark 3.1 (**Irreversibility**—continued)**.** *A process that satisfies $D_t = 1$ if $D_{t-1} = 1$ is consistent with having a structural function that satisfies $\pi_t(y_{t-1}, d_{t-1}, z_t) = +\infty$ if $d_{t-1} = 1$. Similarly, processes that satisfy $Y_t = 1$ and $D_t = 0$ if $Y_{t-1} = 1$ are consistent with $\mu_t(y_{t-1}, d_t, x_t) = +\infty$ and $\pi_t(y_{t-1}, d_{t-1}, z_t) = -\infty$ if $y_{t-1} = 1$. This implies that $Y_t(\boldsymbol{d}^t) = 1$ for any d_t if $Y_{t-1}(\boldsymbol{d}^{t-1}) = 1$. When Y_t is irreversible, the ARSF $E[Y_T(\boldsymbol{d})|X]$ can be interpreted as (one minus) a potential survival rate. An important caveat is that, with irreversible Y_t, the ATE we define contains not only the treatment effect (the intensive margin) but also the effect on dynamic selection (the extensive margin), and the parameter may or may not be of interest depending on the application.*

Remark 3.2 (**Terminal outcome of a different kind**—continued)**.** *When we replace Y_T with R_T to represent a terminal outcome of a different kind, we assume that the model (3.6) is only satisfied for $t \leq T-1$ and introduce $R_T = 1\{\mu_T(Y_{T-1}, D_T, X_T) \geq U_T(D_T)\}$ as the terminal structural function. The potential terminal outcome $R_T(\boldsymbol{d})$ can accordingly be expressed using the structural functions for $(Y_1, ..., Y_{T-1}, R_T)$. The ARSF is written as $E[R_T(\boldsymbol{d})|X]$, and the other parameters can be defined accordingly.*

Remark 3.3 (**Non-binary** Y_t)**.** *Even though we focus on binary Y_t in this paper, we can obtain similar identification results with continuous Y_t or limited dependent variable Y_t, by maintaining a general weak separability structure: $Y_t = m_t(\mu_t(Y_{t-1}, D_t, X_t), U_t(D_t))$. As in the static settings of Vytlacil and Yildiz (2007) and Balat and Han (2018), we impose an assumption that guarantees certain monotonicity of each period's average structural function with respect to the index μ_t: For each t, $E[m_t(\mu_t, U_t(d_t))|\boldsymbol{V}^t, \boldsymbol{U}^{t-1}]$ is strictly monotonic in μ_t. Examples of the nonparametric model $m_t(\mu_t(y_{t-1}, d_t, x_t), u_t)$ that satisfies this assumption are additively separable models or their transformation models, censored regression models, and threshold crossing models as in (3.6); see Vytlacil and Yildiz (2007) for more discussions.*

4 Main Identification Analysis

We first identify the ARSF's, i.e., $E[Y_t(\boldsymbol{d})|\boldsymbol{X}^t]$ for every $\boldsymbol{d}$ and t, which will then be used to identify the ATE's and the optimal regimes $\boldsymbol{d}^*$ and $\boldsymbol{d}^\dagger$. We maintain the following assumptions on $(\boldsymbol{Z}, \boldsymbol{X})$ and $(\boldsymbol{U}(\boldsymbol{d}), \boldsymbol{V})$ for every $\boldsymbol{d}$. These assumptions are written for the identification of $E[Y_T(\boldsymbol{d})|\boldsymbol{X}]$, and are sufficient but not necessary for the identification of $E[Y_t(\boldsymbol{d})|\boldsymbol{X}^t]$ for $t \leq T-1$.

Assumption C. *The distribution of $(\boldsymbol{U}(\boldsymbol{d}), \boldsymbol{V})$ has strictly positive density with respect to Lebesgue measure on $\mathbb{R}^{2T}$.*

Assumption SX. *$(\boldsymbol{Z}, \boldsymbol{X})$ and $(\boldsymbol{U}(\boldsymbol{d}), \boldsymbol{V})$ are independent.*

Assumption C is a regularity condition to ensure the smoothness of relevant conditional probabilities. Assumption SX imposes strict exogeneity, which is a simple sufficient condition for necessary requirements we need for identification; see Remark 4.3. It is implicit that the independence is conditional on the covariates suppressed in the model. Just as the treatments $\boldsymbol{D}$, these covariates may be correlated with the individual effects contained in $(\boldsymbol{U}(\boldsymbol{d}), \boldsymbol{V})$. The variable Z_t denotes the standard excluded instruments. A leading example is a sequence of randomized treatment assignments. Other examples include sequential policy shocks. In addition to Z_t, we introduce exogenous variables X_t in the outcome equation (3.6), that are excluded from the selection equation (3.7). We make a behavioral/information assumption that there are outcome-determining factors that the agent cannot anticipate when making a treatment decision. Continuing with Example 1, when D_t is a compliance choice that a patient makes at the t-th visit, Y_{t-1} may be the symptom measured prior to the decision during the *same* visit. Then Y_t is the symptom measured upon the next visit, which may create enough time gap to prevent the patient from predicting X_t.[9] Note that (Z_t, X_t) are assumed to be excluded from the outcome and treatment equations of all other periods as well. Next, we introduce a sequential version of the rank similarity assumption (Chernozhukov and Hansen (2005)):

Assumption RS. *For each t and $\boldsymbol{d}_{-t}$, $\boldsymbol{U}(1, \boldsymbol{d}_{-t})$ and $\boldsymbol{U}(0, \boldsymbol{d}_{-t})$ are identically distributed, conditional on $\boldsymbol{V}^t$ and $(\boldsymbol{Z}, \boldsymbol{X})$.*

Rank invariance (i.e., $\{\boldsymbol{U}(\boldsymbol{d})\}_{\boldsymbol{d}}$ being equal to each other) is particularly restrictive in the multi-period context, because it requires that the same rank be realized across 2^T different treatment states. Significantly weaker than the rank invariance would be a joint rank similarity assumption that $\boldsymbol{U}(\boldsymbol{d})$'s are identically distributed across 2^T states (conditional on the observables and treatment unobservables). This allows an individual to have different realized ranks across different $\boldsymbol{d}$'s. Assumption RS, which we call *sequential rank similarity*, relaxes this even further by only requiring that $\boldsymbol{U}(1, \boldsymbol{d}_{-t})$ and $\boldsymbol{U}(0, \boldsymbol{d}_{-t})$ are identically distributed instead. That is, the assumption requires that, within individuals with the same observed characteristics and history of the treatment unobservables, the joint distributions of the ranks are identical between just two states that differ by $d_t = 1$ and 0.[10]

Now, we are ready to derive a period-specific result. Define the following period-specific quantity directly identified from the data, i.e., from the distribution of $(\boldsymbol{Y}, \boldsymbol{D}, \boldsymbol{X}, \boldsymbol{Z})$:

$$\begin{aligned}
& h_t(z_t, \tilde{z}_t, x_t, \tilde{x}_t; \boldsymbol{z}^{t-1}, \boldsymbol{x}^{t-1}, \boldsymbol{d}^{t-1}, y_{t-1}) \\
\equiv & \Pr[Y_t = 1, D_t = 1 | \boldsymbol{z}^t, \boldsymbol{x}^t, \boldsymbol{d}^{t-1}, y_{t-1}] + \Pr[Y_t = 1, D_t = 0 | \boldsymbol{z}^t, \tilde{x}_t, \boldsymbol{x}^{t-1}, \boldsymbol{d}^{t-1}, y_{t-1}] \\
& - \Pr[Y_t = 1, D_t = 1 | \tilde{z}_t, \boldsymbol{z}^{t-1}, \boldsymbol{x}^t, \boldsymbol{d}^{t-1}, y_{t-1}] - \Pr[Y_t = 1, D_t = 0 | \tilde{z}_t, \boldsymbol{z}^{t-1}, \tilde{x}_t, \boldsymbol{x}^{t-1}, \boldsymbol{d}^{t-1}, y_{t-1}]
\end{aligned}$$

for $t \geq 1$, where $(\boldsymbol{Z}^0, \boldsymbol{X}^0, \boldsymbol{D}^0, Y_0)$ is understood to mean that there is no conditioning.

Lemma 4.1. *Suppose Assumptions C, SX and RS hold. For each t and $(\boldsymbol{z}^{t-1}, \boldsymbol{x}^{t-1}, \boldsymbol{d}^{t-1}, y_{t-1})$, suppose z_t and $\tilde{z}_t$ are such that*

$$\Pr[D_t = 1 | \boldsymbol{z}^t, \boldsymbol{x}^{t-1}, \boldsymbol{d}^{t-1}, y_{t-1}] \neq \Pr[D_t = 1 | \tilde{z}_t, \boldsymbol{z}^{t-1}, \boldsymbol{x}^{t-1}, \boldsymbol{d}^{t-1}, y_{t-1}]. \quad (4.1)$$

[9] In a static scenario, Balat and Han (2018) motivate this reverse exclusion restriction using the notion of externalities. In their setting where multiple treatments are strategically chosen (e.g., firms' entry decisions), factors that determine the outcome (e.g., pollution) are assumed not to appear in the firms' payoff functions.

[10] In fact, we can further relax Assumption RS by allowing $U_t(d_t)$ to be a function of x_t from the outset; see Remark 4.4.

Then, for given $(x_t, \tilde{x}_t)$*, the sign of* $h_t(z_t, \tilde{z}_t, x_t, \tilde{x}_t; \boldsymbol{z}^{t-1}, \boldsymbol{x}^{t-1}, \boldsymbol{d}^{t-1}, y_{t-1})$ *is equal to the sign of* $\mu_t(y_{t-1}, 1, x_t) - \mu_t(y_{t-1}, 0, \tilde{x}_t)$*.*

Without relying on further assumptions, the sign of $\mu_t(y_{t-1}, 1, x_t) - \mu_t(y_{t-1}, 0, \tilde{x}_t)$ itself is already useful for calculating bounds on the ARSF's and thus on the ATE's; we discuss the partial identification in Section 6.

For the analysis of this paper which deals with a dynamic model, it is convenient to define the $\boldsymbol{U}$-set and $\boldsymbol{V}$-set, namely the sets of histories of the unobservable variables that determine the outcomes and treatments, respectively. To focus our attention on the dependence of the potential outcomes on the unobservables, we iteratively define the potential outcome given $(\boldsymbol{d}, \boldsymbol{x})$ as

$$Y_t(\boldsymbol{d}^t, \boldsymbol{x}^t) \equiv 1\{\mu_t(Y_{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{x}^{t-1}), d_t, x_t) \geq U_t(d_t)\}$$

for $t \geq 2$, with $Y_1(d_1, x_1) = 1\{\mu_1(0, d_1, x_1) \geq U_1(d_1)\}$. Now, define the set of $\boldsymbol{U}^t(\boldsymbol{d}^t)$ as

$$\mathcal{U}^t(\boldsymbol{d}^t, y_t) \equiv \mathcal{U}^t(\boldsymbol{d}^t, y_t; \boldsymbol{x}^t) \equiv \{\boldsymbol{U}^t(\boldsymbol{d}^t) : y_t = Y_t(\boldsymbol{d}^t, \boldsymbol{x}^t)\}$$

for $t \geq 1$. Then, $Y_t = y_t$ if and only if $\boldsymbol{U}^t(\boldsymbol{d}^t) \in \mathcal{U}^t(\boldsymbol{d}^t, y_t; \boldsymbol{x}^t)$, conditional on $(\boldsymbol{D}^t, \boldsymbol{X}^t) = (\boldsymbol{d}^t, \boldsymbol{x}^t)$. The $\boldsymbol{V}$-set $\mathcal{V}^t(\boldsymbol{d}^t, \boldsymbol{u}^{t-1}) \equiv \mathcal{V}^t(\boldsymbol{d}^t, \boldsymbol{u}^{t-1}; \boldsymbol{z}^t, \boldsymbol{x}^{t-1})$ is similarly defined within the proof of Lemma 4.1 in the Appendix. Then, $\boldsymbol{D}^t = \boldsymbol{d}^t$ if and only if $\boldsymbol{V}^t \in \mathcal{V}^t(\boldsymbol{d}^t, \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1}))$, conditional on $(\boldsymbol{Z}^t, \boldsymbol{X}^{t-1}) = (\boldsymbol{z}^t, \boldsymbol{x}^{t-1})$. Given these sets, what we show in the proof of this lemma is that, under Assumptions C and SX,

$$\begin{aligned}
&h_t(z_t, \tilde{z}_t, x_t, \tilde{x}_t; \boldsymbol{z}^{t-1}, \boldsymbol{x}^{t-1}, \boldsymbol{d}^{t-1}, y_{t-1}) \\
=& \Pr[U_t(1) \leq \mu_t(y_{t-1}, 1, x_t), \tilde{\pi}_t \leq V_t \leq \pi_t | \mathcal{V}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{U}^{t-2}(\boldsymbol{d}^{t-2})), \mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, y_{t-1})] \\
&- \Pr[U_t(0) \leq \mu_t(y_{t-1}, 0, \tilde{x}_t), \tilde{\pi}_t \leq V_t \leq \pi_t | \mathcal{V}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{U}^{t-2}(\boldsymbol{d}^{t-2})), \mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, y_{t-1})],
\end{aligned}$$

the sign of which identifies the sign of $\mu_t(y_{t-1}, 1, x_t) - \mu_t(y_{t-1}, 0, \tilde{x}_t)$ by Assumption RS. For example, when this quantity is zero, then $\mu_t(y_{t-1}, 1, x_t) - \mu_t(y_{t-1}, 0, \tilde{x}_t) = 0$.

For the point identification of the ARSF's, the final assumption we introduce concerns the variation of the exogenous variables $(\boldsymbol{Z}, \boldsymbol{X})$. Define the following sets:

$$\mathcal{S}_t(d_t, y_{t-1}) \equiv \left\{(x_t, \tilde{x}_t) : \mu_t(y_{t-1}, d_t, x_t) = \mu_t(y_{t-1}, \tilde{d}_t, \tilde{x}_t) \text{ for } \tilde{d}_t \neq d_t\right\}, \tag{4.2}$$

$$\begin{aligned}
\mathcal{T}_t(\boldsymbol{x}_{-t}, \boldsymbol{z}_{-t}) \equiv \{&(x_t, \tilde{x}_t) : \exists (z_t, \tilde{z}_t) \text{ such that (4.1) holds and} \\
&(x_t, z_t), (\tilde{x}_t, z_t), (x_t, \tilde{z}_t), (\tilde{x}_t, \tilde{z}_t) \in \mathrm{Supp}(X_t, Z_t | \boldsymbol{x}_{-t}, \boldsymbol{z}_{-t})\},
\end{aligned} \tag{4.3}$$

$$\mathcal{X}_t(d_t, y_{t-1}; \boldsymbol{x}_{-t}, \boldsymbol{z}_{-t}) \equiv \{x_t : \exists \tilde{x}_t \text{ with } (x_t, \tilde{x}_t) \in \mathcal{S}_t(d_t, y_{t-1}) \cap \mathcal{T}_t(\boldsymbol{x}_{-t}, \boldsymbol{z}_{-t})\}, \tag{4.4}$$

$$\mathcal{X}_t(d_t; \boldsymbol{x}_{-t}, \boldsymbol{z}_{-t}) \equiv \mathcal{X}_t(d_t, 0; \boldsymbol{x}_{-t}, \boldsymbol{z}_{-t}) \cap \mathcal{X}_t(d_t, 1; \boldsymbol{x}_{-t}, \boldsymbol{z}_{-t}), \tag{4.5}$$

where (4.2) is related to the sufficient variation of X_t and (4.3) is related to the rectangular variation of (X_t, Z_t).

Assumption SP. *For each* t *and* d_t*,* $\Pr[X_t \in \mathcal{X}_t(d_t; \boldsymbol{x}_{-t}, \boldsymbol{z}_{-t}) | \boldsymbol{x}_{-t}, \boldsymbol{z}_{-t}] > 0$ *almost everywhere.*

This assumption requires that X_t varies sufficiently to achieve $\mu_t(y_{t-1}, d_t, x_t) = \mu_t(y_{t-1}, \tilde{d}_t, \tilde{x}_t)$, while holding Z_t to be z_t and $\tilde{z}_t$, respectively, conditional on $(\boldsymbol{X}_{-t}, \boldsymbol{Z}_{-t})$. This is a dynamic version of the support assumption found in Vytlacil and Yildiz (2007).[11] Note that even though this assumption seems to be written in terms of the unknown object $\mu_t(\cdot)$, it is testable because the sets defined above have empirical analogs, according to Lemma 4.1. Let $\mathcal{X}_t(d_t; \boldsymbol{x}_{-t}) \equiv \{x_t : x_t \in \mathcal{X}_t(d_t; \boldsymbol{x}_{-t}, \boldsymbol{z}_{-t}) \text{ for some } \boldsymbol{z}_{-t} \in \text{Supp}(\boldsymbol{Z}_{-t}|\boldsymbol{x}_{-t})\}$ and $\mathcal{X}(\boldsymbol{d}) \equiv \{\boldsymbol{x} : x_t \in \mathcal{X}_t(d_t; \boldsymbol{x}_{-t}) \text{ for some } (x_{t+1}, ..., x_T), \text{ for } t \geq 1\}$, which sequentially collect $x_t \in \mathcal{X}_t(d_t; \boldsymbol{x}_{-t}, \boldsymbol{z}_{-t})$ for all t. We are now ready to state the main identification result.

Theorem 4.1. *Under Assumptions C, SX, RS and SP, $E[Y_T(\boldsymbol{d})|\boldsymbol{x}]$ is identified for $\boldsymbol{d} \in \mathcal{D}$ and $\boldsymbol{x} \in \mathcal{X}(\boldsymbol{d})$.*

Based on Theorem 4.1, we can identify the ATE's. Since the identification of all $E[Y_t(\boldsymbol{d})|\boldsymbol{x}^t]$'s can be shown analogously to Theorem 4.1, we can identify the optimal treatment regimes $\boldsymbol{d}^*(\boldsymbol{x})$ and $\boldsymbol{d}^\dagger(\boldsymbol{x})$ as well.

Corollary 4.1. *Under Assumptions C, SX, RS and SP, $ATE(\boldsymbol{d}, \tilde{\boldsymbol{d}})$ is identified for $\boldsymbol{d}, \tilde{\boldsymbol{d}} \in \mathcal{D}$ and $\boldsymbol{x} \in \mathcal{X}(\boldsymbol{d}) \cap \mathcal{X}(\tilde{\boldsymbol{d}})$, and $\boldsymbol{d}^*(\boldsymbol{x})$ and $\boldsymbol{d}^\dagger(\boldsymbol{x})$ are identified for $\boldsymbol{x} \in \bigcap_{\boldsymbol{d} \in \mathcal{D}} \mathcal{X}(\boldsymbol{d})$.*

We sketch the identification analysis here; the full proof of Theorem 4.1 is found in the Appendix. We consider the identification of $E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}]$, since $E[Y_T(\boldsymbol{d})|\boldsymbol{x}] = E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}]$ by Assumption SX.[12] As the first step of identifying $E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}]$ for given $\boldsymbol{d} = (d_1, ..., d_T)$, $\boldsymbol{x} = (x_1, ..., x_T)$ and $\boldsymbol{z} = (z_1, ..., z_T)$, we apply the result of Lemma 4.1. Fix $t \geq 2$ and $y_{t-1} \in \{0, 1\}$. Suppose x'_t is such that $\mu_t(y_{t-1}, d_t, x_t) = \mu_t(y_{t-1}, d'_t, x'_t)$ with $d'_t \neq d_t$ by applying Lemma 4.1. The existence of x'_t is guaranteed by Assumption SP, as $x_t \in \mathcal{X}_t(d_t, y_{t-1}; \boldsymbol{x}_{-t}, \boldsymbol{z}_{-t}) \subset \mathcal{X}_t(d_t; \boldsymbol{x}_{-t}, \boldsymbol{z}_{-t})$. The implication of $\mu_t(y_{t-1}, d_t, x_t) = \mu_t(y_{t-1}, d'_t, x'_t)$ for relevant $\boldsymbol{U}$-sets is as follows: By the definition of the $\boldsymbol{U}$-set, $\boldsymbol{U} \in \mathcal{U}(\boldsymbol{d}, y_T; \boldsymbol{x})$ is equivalent to $\boldsymbol{U} \in \mathcal{U}^T(d'_t, \boldsymbol{d}_{-t}, y_T; x'_t, \boldsymbol{x}_{-t})$ *conditional on* $Y_{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{x}^{t-1}) = y_{t-1}$ for all $\boldsymbol{x}_{-t}$ and $\boldsymbol{d}_{-t}$.[13] Based on this result, we equate the unobserved quantity $E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{y}^{t-1}, \boldsymbol{d}^{t-1}, d'_t]$ with a quantity that partly matches the assigned treatment and the observed treatment. Once we define $\mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{y}^{t-1}) \equiv \mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{y}^{t-1}; \boldsymbol{x}^{t-1})$ analogous to the $\boldsymbol{U}$-set defined earlier, we can show that

$$\begin{aligned} &E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{y}^{t-1}, \boldsymbol{d}^{t-1}, d'_t] \\ &= \Pr\left[\boldsymbol{U}(\boldsymbol{d}) \in \mathcal{U}^T(\boldsymbol{d}, 1; \boldsymbol{x}) \;\middle|\; \begin{array}{c} \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1}) \in \mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{y}^{t-1}), \\ \boldsymbol{V}^t \in \mathcal{V}^t(\boldsymbol{d}^{t-1}, d'_t, \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1})) \end{array} \right] \end{aligned}$$

[11]Although Assumption SP requires sufficient rectangular variation in (X_t, Z_t), it clearly differs from the large variation assumptions in Heckman and Navarro (2007) and Heckman et al. (2016), which are employed for identification at infinity arguments. In our setting, it is possible that $\mathcal{X}_t(d_t; \boldsymbol{x}_{-t}, \boldsymbol{z}_{-t})$ is nonempty even when Z_t is discrete, as long as X_t contains continuous elements with sufficient support (Vytlacil and Yildiz (2007)). In all these works, including the present one, the support requirement is conditional on the exogenous variables in other periods; see also Cameron and Heckman (1998).

[12]When we are to identify the average potential outcome at t instead, the conditioning variables we use are the vectors of exogenous variables up to t, i.e., $E[Y_t(\boldsymbol{d}^t)|\boldsymbol{x}^t, \boldsymbol{z}^t]$. Then the entire proof can be easily modified based on this expression.

[13]The subsequent analysis is substantially simplified when $\mu_t(y_{t-1}, d_t, x_t) = \mu_t(y_{t-1}, d'_t, x'_t)$ is satisfied *for all* y_{t-1}, but this situation is unlikely to occur. Therefore, it is important to condition on $Y_{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{x}^{t-1}) = y_{t-1}$ in the analysis.

for $t \geq 2$, by Assumption SX. Then, by Assumption RS and the discussion above, this quantity is shown to be equal to

$$
\begin{aligned}
&\Pr\left[\boldsymbol{U}(d_t', \boldsymbol{d}_{-t}) \in \mathcal{U}^T(d_t', \boldsymbol{d}_{-t}, 1; x_t', \boldsymbol{x}_{-t}) \;\middle|\; \begin{array}{c} \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1}) \in \mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{y}^{t-1}), \\ \boldsymbol{V}^t \in \mathcal{V}^t(\boldsymbol{d}^{t-1}, d_t', \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1})) \end{array} \right] \\
=&E[Y_T(d_t', \boldsymbol{d}_{-t})|x_t', \boldsymbol{x}_{-t}, \boldsymbol{z}, \boldsymbol{y}^{t-1}, \boldsymbol{d}^{t-1}, d_t'], \qquad (4.6)
\end{aligned}
$$

by Assumption SX. Note that this last quantity is still unobserved, since d_s for $s \geq t+1$ are not realized treatments; e.g., when $T = 3$ and $t = 2$,

$$E[Y_3(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}, y_1, d_1, d_2'] = E[Y_3(d_1, d_2', d_3)|x_1, x_2', x_3, \boldsymbol{z}, y_1, d_1, d_2'].$$

The quantity, however, will be useful in the remaining proof where we use mathematical induction to recover $E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}]$; see the Appendix. Recall the abbreviations $\mathcal{V}^t(\boldsymbol{d}^{t-1}, d_t', \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1})) \equiv \mathcal{V}^t(\boldsymbol{d}^{t-1}, d_t', \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1}); \boldsymbol{z}^t, \boldsymbol{x}^{t-1})$ and $\mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{y}^{t-1}) \equiv \mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{y}^{t-1}; \boldsymbol{x}^{t-1})$. That is, in the derivation of (4.6), the key is to consider the average potential outcome for a group of individuals that is defined by the treatments at time t or earlier and the lagged outcome, for which x_t is excluded.

The proof of Theorem 4.1 is constructive in that it provides a closed-form expression for $E[Y_T(\boldsymbol{d})|\boldsymbol{x}]$ in an iterative manner, which can immediately be used for estimation. For concreteness, we provide an expression for $E[Y_T(\boldsymbol{d})|\boldsymbol{x}]$ when $T = 2$:

$$
\begin{aligned}
E[Y_2(\boldsymbol{d})|\boldsymbol{x}] =&P[\boldsymbol{d}|\boldsymbol{x}, \boldsymbol{z}]E[Y_2|\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{d}] + P[d_1, d_2'|\boldsymbol{x}, \boldsymbol{z}]\mu_{2,d_1,d_2'} \\
&+ P[d_1', d_2|\boldsymbol{x}, \boldsymbol{z}]E[Y_2|x_1', x_2, \boldsymbol{z}, d_1', d_2] + P[d_1', d_2'|\boldsymbol{x}, \boldsymbol{z}]\mu_{2,d_1',d_2'}, \qquad (4.7)
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_{2,d_1,d_2'} \equiv&P[y_1|\boldsymbol{x}, \boldsymbol{z}, d_1, d_2']E[Y_2|x_1, x_2', \boldsymbol{z}, d_1, d_2', y_1] \\
&+ P[y_1'|\boldsymbol{x}, \boldsymbol{z}, d_1, d_2']E[Y_2|x_1, x_2'', \boldsymbol{z}, d_1, d_2', y_1'], \\
\mu_{2,d_1',d_2'} \equiv&P[y_1|x_1', x_2, \boldsymbol{z}, d_1', d_2']E[Y_2|x_1', x_2', \boldsymbol{z}, d_1', d_2', y_1] \\
&+ P[y_1'|x_1', x_2, \boldsymbol{z}, d_1', d_2']E[Y_2|x_1', x_2'', \boldsymbol{z}, d_1', d_2', y_1']
\end{aligned}
$$

for (x_1', x_2', x_2'') such that $\mu_1(0, d_1, x_1) = \mu_1(0, d_1', x_1')$, $\mu_2(y_1, d_2, x_2) = \mu_2(y_1, d_2', x_2')$, and $\mu_2(y_1', d_2, x_2) = \mu_2(y_1', d_2', x_2'')$.

Remark 4.1. *In estimating the parameters identified in this section, one can improve efficiency by aggregating the conditional expectations (A.4) with respect to $X_t = x_t'$ over the following set:*

$$\lambda_t(x_t; \boldsymbol{z}^{t-1}, \boldsymbol{x}^{t-1}, \boldsymbol{d}^{t-1}, y_{t-1}) \equiv \{\tilde{x}_t : h_t(z_t, \tilde{z}_t, x_t, \tilde{x}_t; \boldsymbol{z}^{t-1}, \boldsymbol{x}^{t-1}, \boldsymbol{d}^{t-1}, y_{t-1}) = 0 \text{ for some } (z_t, \tilde{z}_t)\}.$$

Similarly, one can aggregate the identifying equation for $E[Y_T(\boldsymbol{d})|\boldsymbol{x}]$ (e.g., equation (4.7)) with respect to $\boldsymbol{Z} = \boldsymbol{z}$ conditional on $\boldsymbol{X} = \boldsymbol{x}$.

Remark 4.2. *The assumption that the initial condition Y_0 is exogenously determined is not necessary but imposed for convenience. Such an assumption appears in, e.g., Heckman and*

Navarro (2007). In an alternative setting where Y_0 is endogenously determined in the model, a similar identification analysis as in this section can be followed by modifying Assumption SX. We may consider two alternatives depending upon whether Y_0 is observable or not: (a) $(\boldsymbol{U}(\boldsymbol{d}), \boldsymbol{V})$ and $(\boldsymbol{Z}, \boldsymbol{X})$ are independent conditional on Y_0; or (b) $(\boldsymbol{U}(\boldsymbol{d}), \boldsymbol{V}, Y_0)$ and $(\boldsymbol{Z}, \boldsymbol{X})$ are independent. First, recall that each of these statements is "conditional on other covariates." The assumption (a) can be imposed when Y_0 is observable, maybe because $t = 1$ is not the start of sample period. The assumption (b) can be imposed when Y_0 is unobservable, maybe because $t = 1$ is the start of sample period and the logical start of the process. The analysis in these alternative scenarios is omitted as it is a straightforward extension of the current one. In this analysis, there is no need to assume the distribution of initial conditions, unlike in the literature on dynamic models with random effects. Still, we recover certain treatment effects, unlike in the literature on nonseparable models with unobservable individual effects where, in general, partial effects are hard to recover. The trade-off is that we require variables that are independent of the individual effects, even though other covariates are allowed not to be.

Remark 4.3. *The strict exogeneity of Assumption SX is a simple sufficient condition for what we actually need for the identification analysis. As described in Lemma A.1 of the Appendix, the conditions we need to show Lemma 4.1 and Theorem 4.1, respectively, are the following: For each t, (i) $(Z_t, X_t) \perp (U_t(d_t), V_t) | \boldsymbol{Z}^{t-1}, \boldsymbol{X}^{t-1}$; (ii) $Z_t \perp (\boldsymbol{U}(\boldsymbol{d}), \boldsymbol{V}^t) | \boldsymbol{Z}^{t-1}, \boldsymbol{X}_{-t}$ and $X_t \perp (\boldsymbol{U}(\boldsymbol{d}), \boldsymbol{V}^t) | \boldsymbol{Z}^{t-1}, \boldsymbol{X}_{-t}$. In these high-level conditions, the condition for Z_t is reminiscent of the sequential randomization assumption. In fact, this is consistent with our leading example of experimental studies with partial compliance. Here, it is apparently the treatment assignment Z_t rather than the received treatment D_t for which the sequential randomization assumption should be imposed.*

Remark 4.4. *In order to define the $\boldsymbol{U}$-set, recall that we use an alternative potential outcome $Y_t(\boldsymbol{d}^t, \boldsymbol{x}^t) = \mu_t(Y_{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{x}^{t-1}), d_t, x_t, U_t(d_t))$. Motivated from this, we may consider a structural model that adds another dimension for heterogeneity by allowing $U_t(d_t)$ to be a function of x_t as well:*

$$Y_t(\boldsymbol{d}^t, \boldsymbol{x}^t) = \mu_t(Y_{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{x}^{t-1}), d_t, x_t, U_t(d_t, x_t)).$$

Given this extension, we can relax Assumption RS and impose that $\{\boldsymbol{U}(d_t, \boldsymbol{d}_{-t}, x_t, \boldsymbol{x}_{-t})\}_{d_t, x_t}$ are identically distributed conditional on $\boldsymbol{V}^t$ and $(\boldsymbol{Z}, \boldsymbol{X})$. The current Assumption RS can be viewed as requiring rank invariance in terms of x_t, while it allows rank similarity in d_t.

5 Treatment Effects on Transitions

In fact, the identification strategy introduced in the previous section can tackle a more general problem. In this section, we extend the identification analysis of the ATE (Theorem 4.1 and Corollary 4.1) and show identification of the transition-specific ATE. Given the vector $\boldsymbol{Y}(\boldsymbol{d}) \equiv (Y_1(\boldsymbol{d}), ..., Y_T(\boldsymbol{d}))$ of potential outcomes, let $\boldsymbol{Y}_-(\boldsymbol{d}) \equiv (Y_{t_1}(\boldsymbol{d}), ..., Y_{t_L}(\boldsymbol{d})) \in \mathcal{Y}_- \subseteq \{0,1\}^L$ be its $1 \times L$ subvector, where $t_1 < t_2 < \cdots < t_L \leq T-1$ and $L < T$. Then, the transition-specific ATE can be defined as $E[Y_T(\boldsymbol{d})|\boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-, \boldsymbol{X} = \boldsymbol{x}] - E[Y_T(\tilde{\boldsymbol{d}})|\boldsymbol{Y}_-(\tilde{\boldsymbol{d}}) = \boldsymbol{y}_-, \boldsymbol{X} = \boldsymbol{x}]$ for some sequences $\boldsymbol{d}$ and $\tilde{\boldsymbol{d}}$.

Theorem 5.1. *Under Assumptions C, SX, RS and SP, for each $\boldsymbol{y}_-$, $E[Y_T(\boldsymbol{d})|\boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-, \boldsymbol{X} = \boldsymbol{x}] - E[Y_T(\tilde{\boldsymbol{d}})|\boldsymbol{Y}_-(\tilde{\boldsymbol{d}}) = \boldsymbol{y}_-, \boldsymbol{X} = \boldsymbol{x}]$ is identified for $\boldsymbol{d}, \tilde{\boldsymbol{d}} \in \mathcal{D}$ and $\boldsymbol{x} \in \mathcal{X}(\boldsymbol{d}) \cap \mathcal{X}(\tilde{\boldsymbol{d}})$.*

The proof of this theorem extends that of Theorem 4.1; see the Appendix.[14] The transition-specific ATE defined in Theorem 5.1 concerns a transition from a state that is specified by the value of the vector of previous potential outcomes, $\boldsymbol{Y}_-(\boldsymbol{d})$. When $Y_T(\boldsymbol{d})$ is binary, $E[Y_T(\boldsymbol{d})|\boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-, \boldsymbol{X} = \boldsymbol{x}]$ can be viewed as a generalization of the transition probability. As a simple example, with $L = T-1$, one may be interested in a transition to one state when all previous potential outcomes have stayed in the other state until $T-1$. When $L = 1$ with $\boldsymbol{Y}_-(\boldsymbol{d}) = Y_{T-1}(\boldsymbol{d})$, the transition-specific ATE becomes $\Pr[Y_T(\boldsymbol{d}) = 1|Y_{T-1}(\boldsymbol{d}) = 0] - \Pr[Y_T(\tilde{\boldsymbol{d}}) = 1|Y_{T-1}(\tilde{\boldsymbol{d}}) = 0]$ introduced in Section 3. This is a particular example of the treatment effect on the transition probability. The treatment effects on transitions have been studied by, e.g., Abbring and Van den Berg (2003), Heckman and Navarro (2007), Fredriksson and Johansson (2008) and Vikström et al. (2018).[15] Let $Y_t(d_t) \equiv \mu_t(Y_{t-1}, d_t, X_t, U_t(d_t))$ be the *period-specific potential outcome* at time t. Since $Y_{t-1} = Y_{t-1}(\boldsymbol{D}^{t-1})$, the period-specific potential outcome can be expressed as $Y_t(d_t) = Y_t(\boldsymbol{D}^{t-1}, d_t)$ using the usual potential outcome. As a corollary of the result above, we also identify a related parameter that specifies the previous state by the observed outcome: $E[Y_T(1) - Y_T(0)|Y_{T-1} = y_{T-1}]$.

Corollary 5.1. *Under Assumptions C, SX, RS and SP, for each y_{T-1}, $E[Y_T(1)|y_{T-1}, \boldsymbol{x}] - E[Y_T(0)|y_{T-1}, \boldsymbol{x}]$ is identified for $\boldsymbol{x} \in \mathcal{X}(\boldsymbol{d}) \cap \mathcal{X}(\tilde{\boldsymbol{d}})$.*

The corollary is derived by observing that $Y_T(d_T) = Y_T(\boldsymbol{D}^{T-1}, d_T)$, and thus

$$\begin{aligned}&E[Y_T(d_T)|y_{T-1}, \boldsymbol{x}]\\ &= \sum_{\boldsymbol{d}^{T-1} \in \mathcal{D}^{T-1}} \Pr[\boldsymbol{D}^{T-1} = \boldsymbol{d}^{T-1}|\boldsymbol{x}] E[Y_T(\boldsymbol{d}^{T-1}, d_T)|Y_{T-1}(\boldsymbol{d}^{T-1}) = y_{T-1}, \boldsymbol{D}^{T-1} = \boldsymbol{d}^{T-1}, \boldsymbol{x}],\end{aligned}$$

where each $E[Y_T(\boldsymbol{d}^{T-1}, d_T)|Y_{T-1}(\boldsymbol{d}^{T-1}) = y_{T-1}, \boldsymbol{d}^{T-1}, \boldsymbol{x}]$ is identified from the iteration at $t = T-1$ in the proof of Theorem 5.1 by taking $Y_-(\boldsymbol{d}) = Y_{T-1}(\boldsymbol{d}^{T-1})$.

6 Partial Identification

Suppose Assumption SP does not hold in that X_t does not exhibit sufficient rectangular variation, or that there is no X_t that is excluded from the selection equation at time t. In this case, we partially identify the ARSF's, ATE's and $\boldsymbol{d}^*(\boldsymbol{x})$ (or $\boldsymbol{d}^\dagger(\boldsymbol{x})$).

We briefly illustrate the calculation of the bounds on the ARSF $E[Y_T(\boldsymbol{d})|\boldsymbol{x}]$ when the sufficient rectangular variation is not guaranteed; the case where X_t does not exist at all can be dealt in a similar manner, and so is omitted. For each $E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{y}^{t-1}, \boldsymbol{d}^{t-1}, d_t']$ in the proof of Theorem 4.1, we can calculate its upper and lower bounds depending on the sign of $\mu_t(y_{t-1}, 1, x_t) - \mu_t(y_{t-1}, 0, \tilde{x}_t)$, which is identified in Lemma 4.1. Note that, in the context of this section, $\tilde{x}_t$ does *not* necessarily differ from x_t. For example, for the lower bound on

[14] As before, the parameters in Theorem 5.1 and Corollary 5.1 below can be defined for any given period instead of the terminal period T. The identification analysis of such parameters is essentially the same, and thus omitted.

[15] The definition of the treatment effect on the transition probability in this paper differs from those defined in the literature on duration models, e.g., that in Vikström et al. (2018). Since Vikström et al. (2018)'s main focus is on Y_t that is irreversible, they define a different treatment parameter that yields a specific interpretation under dynamic selection; see their paper for details. In addition, they assume sequential randomization and that treatments are assigned earlier than the transition of interest.

$E[Y_T(\boldsymbol{d})|\boldsymbol{x}] = E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}]$, suppose $\mu_t(y_{t-1}, d_t, x_t) - \mu_t(y_{t-1}, d'_t, x'_t) \geq 0$ for given y_{t-1}, where x'_t is allowed to equal x_t. Then, by the definition of the $\boldsymbol{U}$-set and under Assumption RS, it satisfies that $\mathcal{U}^T(\boldsymbol{d}, y_T; \boldsymbol{x}) \supseteq \mathcal{U}^T(d'_t, \boldsymbol{d}_{-t}, y_T; x'_t, \boldsymbol{x}_{-t})$, conditional on $Y_{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{x}^{t-1}) = y_{t-1}$. Therefore, we have a lower bound on as $E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{y}^{t-1}, \boldsymbol{d}^{t-1}, d'_t]$ as

$$\begin{aligned}
& E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{y}^{t-1}, \boldsymbol{d}^{t-1}, d'_t] \\
= & \Pr[\boldsymbol{U}(\boldsymbol{d}) \in \mathcal{U}^T(\boldsymbol{d}, 1; \boldsymbol{x}) | \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1}) \in \mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{y}^{t-1}), \boldsymbol{V}^t \in \mathcal{V}^t(\boldsymbol{d}^{t-1}, d'_t, \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1}))] \\
\geq & \Pr\left[\boldsymbol{U}(d'_t, \boldsymbol{d}_{-t}) \in \mathcal{U}^T(d'_t, \boldsymbol{d}_{-t}, 1; x'_t, \boldsymbol{x}_{-t}) \;\middle|\; \begin{array}{c} \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1}) \in \mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{y}^{t-1}), \\ \boldsymbol{V}^t \in \mathcal{V}^t(\boldsymbol{d}^{t-1}, d'_t, \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1})) \end{array} \right] \\
= & E[Y_T(d'_t, \boldsymbol{d}_{-t})|x'_t, \boldsymbol{x}_{-t}, \boldsymbol{z}, \boldsymbol{y}^{t-1}, \boldsymbol{d}^{t-1}, d'_t]. \qquad (6.1)
\end{aligned}$$

Then, it is possible to calculate the lower bounds on $E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}]$ using the iterative scheme introduced in the proof of Theorem 4.1. That is, at each iteration, we take the previous iteration's lower bound as given, expand each main term in (A.3) as before, and apply (6.1) for necessary terms.

Lastly, depending on the signs of the ATE's, we can construct bounds on $\boldsymbol{d}^*(\boldsymbol{x})$ (or $\boldsymbol{d}^\dagger(\boldsymbol{x})$), which will be expressed as strict subsets of $\mathcal{D}$. The partial identification of the optimal regimes may not yield sufficiently narrow bounds unless there are a sufficient number of ATE's whose bounds are informative about their signs. In general, however, the informativeness of bounds truly depends on the policy questions. Note that $\mathcal{D}$ is a discrete set. Even though the bounds may not be informative about the optimal regime, they may still be useful from the planner's perspective if they can help her exclude a few suboptimal regimes, i.e., $\boldsymbol{d}^\circ$ such that $E[Y_T(\boldsymbol{d})|\boldsymbol{x}] \geq E[Y_T(\boldsymbol{d}^\circ)|\boldsymbol{x}]$ for some $\boldsymbol{d}$.

7 Subsequences of Treatments

An important extension of the model introduced in this paper is to the case where treatments do not appear in every period, while the outcomes are constantly observed. For example, institutionally, there may only be a one-shot treatment at the beginning of time or a few treatments earlier in the horizon, or there may be evenly spaced treatment decisions with a lower frequency than outcomes. A potential outcome that corresponds to this situation can be defined as a function of a certain subsequence $\boldsymbol{d}_-$ of $\boldsymbol{d}$. Let $\boldsymbol{d}_- \equiv (d_{t_1}, ..., d_{t_K}) \in \mathcal{D}_- \subseteq \{0,1\}^K$ be a $1 \times K$ subvector of $\boldsymbol{d}$, where $t_1 < t_2 < \cdots < t_K \leq T$ and $K < T$. Then, the potential outcomes $Y_t(\boldsymbol{d}_-)$ and the associated structural functions are defined as follows: Let $\boldsymbol{d}_-^{t_k} \equiv (d_{t_1}, ..., d_{t_k})$. A potential outcome in the period when a treatment exists is expressed using a switching regression model as

$$Y_{t_k}(\boldsymbol{d}_-) = Y_{t_k}(\boldsymbol{d}_-^{t_k}) = \mu_{t_k}(Y_{t_k-1}(\boldsymbol{d}_-^{t_{(k-1)}}), d_{t_k}, X_{t_k}, U_{t_k}(d_{t_k}))$$

for $k \geq 1$ with $Y_{t_1-1}(\boldsymbol{d}_-^{t_0}) = Y_{t_1-1}$, and a potential outcome when there is no treatment is expressed as

$$Y_t(\boldsymbol{d}_-) = Y_t(\boldsymbol{d}_-^{t_k}) = \mu_t(Y_{t-1}(\boldsymbol{d}_-^{t_k}), U_t)$$

for t such that $t_k < t < t_{(k+1)}$ $(1 \leq k \leq K-1)$. Lastly, $Y_t(\boldsymbol{d}_-) = Y_t = \mu_t(Y_{t-1}, U_t)$ for $t < t_1$ and $Y_t(\boldsymbol{d}_-) = Y_t(\boldsymbol{d}_-^{t_K}) = \mu_t(Y_{t-1}(\boldsymbol{d}_-^{t_K}), U_t)$ for $t > t_K$. Each structural model at the time of no treatment is a plain dynamic model with a lagged dependent variable. Let $T = 4$ and $\boldsymbol{d}_- = (d_1, d_3)$ for illustration. Then the sequence of potential outcomes can be expressed as

$$\begin{aligned} Y_4(\boldsymbol{d}_-) &= Y_4(\boldsymbol{d}^3) = \mu_4(Y_3(\boldsymbol{d}^3), U_4), \\ Y_3(\boldsymbol{d}_-) &= Y_3(\boldsymbol{d}^3) = \mu_3(Y_2(d_1), d_3, X_3, U_3(d_3)), \\ Y_2(\boldsymbol{d}_-) &= Y_2(d_1) = \mu_2(Y_1(d_1), U_2), \\ Y_1(\boldsymbol{d}_-) &= Y_1(d_1) = \mu_1(Y_0, d_1, X_1, U_1(d_1)). \end{aligned}$$

The selection equations are of the following form: For $k \geq 1$,

$$D_{t_k} = 1\{\pi_{t_k}(Y_{t_k-1}, D_{t_{(k-1)}}, Z_{t_k}) \geq V_{t_k}\},$$

where the lagged outcome and the *latest* treatment enter each equation. The observable variables are $(\boldsymbol{Y}, \boldsymbol{D}_-, \boldsymbol{X}_-, \boldsymbol{Z}_-)$.[16]

Now all the parameters introduced in Section 3 can be readily modified by replacing $\boldsymbol{d}$ with $\boldsymbol{d}_-$ for some $\boldsymbol{d}_-$; we omit the definitions for the sake of brevity. Moreover, the identification analysis of Section 4 can be easily modified in accordance with the extended setting. Let $\boldsymbol{U}_-(\boldsymbol{d}_-) \equiv (U_{t_1}(d_{t_1}), ..., U_{t_K}(d_{t_K}))$ and let $\boldsymbol{U}(\boldsymbol{d}_-)$ be the vector of all the outcome unobservables that consists of $\boldsymbol{U}_-(\boldsymbol{d}_-)$ and $\{U_t\}_{t \in \{1,...,T\} \setminus \{t_1,...,t_K\}}$.

Assumption C′. *The distribution of* $(\boldsymbol{U}_-(\boldsymbol{d}_-), \boldsymbol{V}_-)$ *has strictly positive density with respect to Lebesgue measure on* $\mathbb{R}^{2K}$.

Assumption SX′. $(\boldsymbol{Z}_-, \boldsymbol{X}_-)$ *and* $(\boldsymbol{U}(\boldsymbol{d}_-), \boldsymbol{V}_-)$ *are independent.*

Let $\boldsymbol{d}_{-,-t_k}$ be $\boldsymbol{d}_-$ without the t_k-th element.

Assumption RS′. *For each* t_k *and* $\boldsymbol{d}_{-,-t_k}$, $\{\boldsymbol{U}_-(d_{t_k}, \boldsymbol{d}_{-,-t_k})\}_{d_{t_k}}$ *are identically distributed conditional on* $(\boldsymbol{U}^{t_k-1}(\boldsymbol{d}_-^{t_{(k-1)}}), \boldsymbol{V}_-^{t_k})$.

Under these modified assumptions, Lemma 4.1 is now only relevant for $t = t_k$. Restrict the definitions of $\mathcal{X}_t(d_t; \boldsymbol{x}_{-t}, \boldsymbol{z}_{-t})$ in (4.5) and $\mathcal{X}_t(d_t; \boldsymbol{x}_{-t})$ to hold only for $t = t_k$.

Assumption SP′. *For each* t_k *and* d_{t_k}, $\Pr[X_{t_k} \in \mathcal{X}_{t_k}(d_{t_k}; \boldsymbol{x}_{-,-t_k}, \boldsymbol{z}_{-,-t_k}) | \boldsymbol{x}_{-,-t_k}, \boldsymbol{z}_{-,-t_k}] > 0$ *almost everywhere.*

Let $\mathcal{X}_-(\boldsymbol{d}_-) \equiv \{\boldsymbol{x}_- : x_{t_k} \in \mathcal{X}_{t_k}(d_{t_k}; \boldsymbol{x}_{-,-t_k}) \text{ for some } (x_{t_{(k+1)}}, ..., x_{t_K}), \text{ for } k \geq 1\}$.

Theorem 7.1. *Under Assumptions C′, SX′, RS′ and SP′,* $E[Y_T(\boldsymbol{d}_-)|\boldsymbol{x}_-]$ *is identified for* $\boldsymbol{d}_- \in \mathcal{D}_-$, $\boldsymbol{x}_- \in \mathcal{X}_-(\boldsymbol{d}_-)$.

Corollary 7.1. *Under Assumptions C′, SX′, RS′ and SP′,* $E[Y_T(\boldsymbol{d}_-) - Y_T(\tilde{\boldsymbol{d}}_-)|\boldsymbol{x}_-]$ *is identified for* $\boldsymbol{d}_-, \tilde{\boldsymbol{d}}_- \in \mathcal{D}_-$ *and* $\boldsymbol{x}_- \in \mathcal{X}_-(\boldsymbol{d}_-) \cap \mathcal{X}_-(\tilde{\boldsymbol{d}}_-)$, *and* $\boldsymbol{d}_-^*(\boldsymbol{x}_-)$ *and* $\boldsymbol{d}_-^\dagger(\boldsymbol{x}_-)$ *are identified for* $\boldsymbol{x}_- \in \bigcap_{\boldsymbol{d}_- \in \mathcal{D}_-} \mathcal{X}_-(\boldsymbol{d}_-)$.

[16] It may be the case that X_t is observed whenever Y_t is observed, and thus is included in the Y_t-equations for $t \neq t_k$ as well. We ignore that case here.

8 Conclusions

In this paper, we consider identification in a nonparametric model for dynamic treatments and outcomes. We introduce a sequence of selection models, replacing the assumption of sequential randomization, which may be hard to justify under partial compliance or in observational settings. We consider treatment and outcome processes of general forms, and avoid making strong assumptions on distribution and functional forms, nor assumptions on rationality. We show that the treatment parameters and optimal treatment regimes are point identified under the two-way exclusion restriction and sequential rank similarity. We argue that the reverse exclusion restriction is a useful alternative tool for empirical researchers who seek identification in this type of nonseparable models with endogeneity. This source of variation may especially be easy to find and justify in a dynamic setting as in this paper. When the reverse exclusion restriction is violated, we show how to characterize bounds on these parameters.

The identification analysis is constructive and naturally suggests an estimation procedure by the sample analog principle. Standard approaches in the nonparametric estimation literature can be use to estimate quantities as $E[Y_T|\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{d}]$ and $P[\boldsymbol{D} = \boldsymbol{d}|\boldsymbol{x}, \boldsymbol{z}]$. A few remarks are worth making. First, a dimensionality problem needs to be addressed in estimation, since we consider multiple (although short) periods and multiple covariates. Recent techniques for dimension reduction, such as the LASSO, can be employed to address this problem. In our specific context, however, we may want to use methods such as the group LASSO (Yuan and Lin (2006)) that can obey certain grouped structure in the set of conditioning variables. Consider $\boldsymbol{X} = (X_1, ..., X_T)$ where X_t may contain multiple covariates. Then, for instance, if sparsity is more likely to hold across covariates but not across time, we may want to select a covariate across all periods if we are to select that covariate at all. Second, the estimation procedure is in principle two-step as we need to estimate h in the first step. Therefore, in conducting inference, the sampling error from the first-stage estimation of h should be reflected, as in other nonparametric two-step estimations.

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A Proofs

A.1 High-Level Conditions for Assumption SX

As discussed in Remark 4.3, Assumption SX is a sufficient condition for high-level conditions for the proofs of Lemma 4.1 and Theorem 4.1.

Lemma A.1. *Assumption SX implies the following: (i)* $(Z_t, X_t) \perp (U_t(d_t), V_t)|\boldsymbol{Z}^{t-1}, \boldsymbol{X}^{t-1}$*; (ii)* $Z_t \perp (\boldsymbol{U}(\boldsymbol{d}), \boldsymbol{V}^t)|\boldsymbol{Z}^{t-1}, \boldsymbol{X}_{-t}$ *and* $X_t \perp (\boldsymbol{U}(\boldsymbol{d}), \boldsymbol{V}^t)|\boldsymbol{Z}^{t-1}, \boldsymbol{X}_{-t}$.

In the proofs below, we use these high-level conditions. Therefore, some of the intermediate results we obtain in the proofs are slightly different from the ones described in the main text for which Assumption SX is directly applied.

A.2 Proof of Lemma 4.1

We first define the $\boldsymbol{U}$-set and $\boldsymbol{V}$-set. The $\boldsymbol{U}$-set is defined in the main text. Realizing the dependence of $Y_{s-1}(\boldsymbol{d}^{s-1}, \boldsymbol{x}^{s-1})$ on $(\boldsymbol{U}^{s-1}(\boldsymbol{d}^{s-1}), \boldsymbol{x}^{s-1}, \boldsymbol{d}^{s-1})$, let

$$\pi_s^*(\boldsymbol{U}^{s-1}(\boldsymbol{d}^{s-1}), \boldsymbol{x}^{s-1}, \boldsymbol{d}^{s-1}, z_s) \equiv \pi_s(Y_{s-1}(\boldsymbol{d}^{s-1}, \boldsymbol{x}^{s-1}), d_{s-1}, z_s),$$

and define the set of $\boldsymbol{V}^t$ as

$$\mathcal{V}^t(\boldsymbol{d}^t, \boldsymbol{u}^{t-1}) \equiv \mathcal{V}^t(\boldsymbol{d}^t, \boldsymbol{u}^{t-1}; \boldsymbol{z}^t, \boldsymbol{x}^{t-1}) \equiv \{\boldsymbol{V}^t : d_s = 1\{V_s \leq \pi_s^*(\boldsymbol{u}^{s-1}, \boldsymbol{x}^{s-1}, \boldsymbol{d}^{s-1}, z_s)\} \text{ for all } s \leq t\}$$

for $t \geq 2$. Fix $t \geq 3$. Given (4.1), consider the case $\Pr[D_t = 1|\boldsymbol{z}^t, \boldsymbol{x}^{t-1}, \boldsymbol{d}^{t-1}, y_{t-1}] > \Pr[D_t = 1|\tilde{z}_t, \boldsymbol{z}^{t-1}, \boldsymbol{x}^{t-1}, \boldsymbol{d}^{t-1}, y_{t-1}]$; the opposite case is symmetric. Using the definitions of the sets above, we have

$$\begin{aligned} &\Pr[D_t = 1|\boldsymbol{z}^t, \boldsymbol{x}^{t-1}, \boldsymbol{d}^{t-1}, y_{t-1}] \\ =&\Pr[V_t \leq \pi_t(y_{t-1}, d_{t-1}, z_t)|\boldsymbol{z}^t, \boldsymbol{x}^{t-1}, \mathcal{V}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{U}^{t-2}(\boldsymbol{d}^{t-2})), \mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, y_{t-1})] \\ =&\Pr[V_t \leq \pi_t(y_{t-1}, d_{t-1}, z_t)|\boldsymbol{z}^{t-1}, \boldsymbol{x}^{t-1}, \mathcal{V}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{U}^{t-2}(\boldsymbol{d}^{t-2})), \mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, y_{t-1})], \end{aligned}$$

where the last equality is given by Assumption SX and Lemma A.1(i). Note that the sets $\mathcal{V}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{U}^{t-2}(\boldsymbol{d}^{t-2}))$ and $\mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, y_{t-1})$ do not change with the change in z_t. Therefore, a parallel expression can be derived for $\Pr[D_t = 1|\tilde{z}_t, \boldsymbol{z}^{t-1}, \boldsymbol{x}^{t-1}, \boldsymbol{d}^{t-1}, y_{t-1}]$. Let $\pi_t \equiv (y_{t-1}, d_{t-1}, z_t)$ and $\tilde{\pi}_t \equiv (y_{t-1}, d_{t-1}, \tilde{z}_t)$ for abbreviation. Then, under Assumption C,

$$\begin{aligned} 0 <& \Pr[D_t = 1|\boldsymbol{z}^t, \boldsymbol{x}^{t-1}, \boldsymbol{d}^{t-1}, y_{t-1}] - \Pr[D_t = 1|\tilde{z}_t, \boldsymbol{z}^{t-1}, \boldsymbol{x}^{t-1}, \boldsymbol{d}^{t-1}, y_{t-1}] \\ =& \Pr[V_t \leq \pi_t|\boldsymbol{z}^{t-1}, \boldsymbol{x}^{t-1}, \mathcal{V}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{U}^{t-2}(\boldsymbol{d}^{t-2})), \mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, y_{t-1})] \\ &- \Pr[V_t \leq \tilde{\pi}_t|\boldsymbol{z}^{t-1}, \boldsymbol{x}^{t-1}, \mathcal{V}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{U}^{t-2}(\boldsymbol{d}^{t-2})), \mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, y_{t-1})], \end{aligned}$$

which implies $\pi_t > \tilde{\pi}_t$. Next, we have

$$\begin{aligned} &\Pr[Y_t = 1, D_t = 1|\boldsymbol{z}^t, \boldsymbol{x}^t, \boldsymbol{d}^{t-1}, y_{t-1}] \\ =&\Pr[U_t(1) \leq \mu_t(y_{t-1}, 1, x_t), V_t \leq \pi_t|\boldsymbol{z}^{t-1}, \boldsymbol{x}^{t-1}, \mathcal{V}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{U}^{t-2}(\boldsymbol{d}^{t-2})), \mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, y_{t-1})] \end{aligned}$$

by Assumption SX and Lemma A.1(i). Again, note that $\mathcal{V}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{U}^{t-2}(\boldsymbol{d}^{t-2}))$ and $\mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, y_{t-1})$ do not change with the change in (z_t, x_t), which is key. Therefore, similar expressions can be derived for the other terms involved in h_t, and we have

$$
\begin{aligned}
&h_t(z_t, \tilde{z}_t, x_t, \tilde{x}_t; \boldsymbol{z}^{t-1}, \boldsymbol{x}^{t-1}, \boldsymbol{d}^{t-1}, y_{t-1}) \\
=&\Pr[U_t(1) \leq \mu_t(y_{t-1}, 1, x_t), \tilde{\pi}_t \leq V_t \leq \pi_t | \boldsymbol{z}^{t-1}, \boldsymbol{x}^{t-1}, \mathcal{V}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{U}^{t-2}(\boldsymbol{d}^{t-2})), \mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, y_{t-1})] \\
&- \Pr[U_t(0) \leq \mu_t(y_{t-1}, 0, \tilde{x}_t), \tilde{\pi}_t \leq V_t \leq \pi_t | \boldsymbol{z}^{t-1}, \boldsymbol{x}^{t-1}, \mathcal{V}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{U}^{t-2}(\boldsymbol{d}^{t-2})), \mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, y_{t-1})],
\end{aligned}
$$

the sign of which identifies the sign of $\mu_t(y_{t-1}, 1, x_t) - \mu_t(y_{t-1}, 0, \tilde{x}_t)$ by Assumption RS. The case $t \leq 2$ can be shown analogously with $\mathcal{V}^1(d_1) \equiv \mathcal{V}^1(d_1; z_1) \equiv \{V_1 : d_1 = 1\{V_1 \leq \pi_1(0, 0, z_1)\}\}$. □

A.3 Proof of Theorem 4.1

As the first step of identifying $E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}]$ for given $\boldsymbol{d} = (d_1, ..., d_T)$, $\boldsymbol{x} = (x_1, ..., x_T)$ and $\boldsymbol{z} = (z_1, ..., z_T)$, we apply the result of Lemma 4.1. Fix $t \geq 2$ and $y_{t-1} \in \{0, 1\}$. Suppose x_t' is such that $\mu_t(y_{t-1}, d_t, x_t) = \mu_t(y_{t-1}, d_t', x_t')$ with $d_t' \neq d_t$ by applying Lemma 4.1. The existence of x_t' is guaranteed by Assumption SP, as $x_t \in \mathcal{X}_t(d_t, y_{t-1}; \boldsymbol{x}_{-t}, \boldsymbol{z}_{-t}) \subset \mathcal{X}_t(d_t; \boldsymbol{x}_{-t}, \boldsymbol{z}_{-t})$. The implication of $\mu_t(y_{t-1}, d_t, x_t) = \mu_t(y_{t-1}, d_t', x_t')$ for relevant $\boldsymbol{U}$-sets is as follows: By the definition of the $\boldsymbol{U}$-set, $\boldsymbol{U} \in \mathcal{U}(\boldsymbol{d}, y_T; \boldsymbol{x})$ is equivalent to $\boldsymbol{U} \in \mathcal{U}^T(d_t', \boldsymbol{d}_{-t}, y_T; x_t', \boldsymbol{x}_{-t})$ *conditional on* $Y_{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{x}^{t-1}) = y_{t-1}$ for all $\boldsymbol{x}_{-t}$ and $\boldsymbol{d}_{-t}$. Analogous to the $\boldsymbol{U}$-set defined earlier, define

$$
\mathcal{U}^t(\boldsymbol{d}^t, \boldsymbol{y}^t) \equiv \mathcal{U}^t(\boldsymbol{d}^t, \boldsymbol{y}^t; \boldsymbol{x}^t) \equiv \{\boldsymbol{U}^t(\boldsymbol{d}^t) : y_s = Y_s(\boldsymbol{d}^s, \boldsymbol{x}^s) \text{ for all } s \leq t\}.
$$

Then, for $t \geq 2$,

$$
\begin{aligned}
&E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}^{t-1}, \boldsymbol{y}^{t-1}, \boldsymbol{d}^{t-1}, d_t'] \\
=&\Pr\left[\boldsymbol{U}(\boldsymbol{d}) \in \mathcal{U}^T(\boldsymbol{d}, 1; \boldsymbol{x}) \;\middle|\; \begin{array}{c} \boldsymbol{x}, \boldsymbol{z}^{t-1}, \\ \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1}) \in \mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{y}^{t-1}), \\ \boldsymbol{V}^t \in \mathcal{V}^t(\boldsymbol{d}^{t-1}, d_t', \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1})) \end{array} \right] \\
=&\Pr\left[\boldsymbol{U}(\boldsymbol{d}) \in \mathcal{U}^T(\boldsymbol{d}, 1; \boldsymbol{x}) \;\middle|\; \begin{array}{c} \boldsymbol{x}_{-t}, \boldsymbol{z}^{t}, \\ \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1}) \in \mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{y}^{t-1}), \\ \boldsymbol{V}^t \in \mathcal{V}^t(\boldsymbol{d}^{t-1}, d_t', \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1})) \end{array} \right],
\end{aligned}
\tag{A.1}
$$

where the last equality follows from Assumption SX and Lemma A.1(ii). Then, by Assumption RS and the discussion above, (A.1) is equal to

$$
\begin{aligned}
&\Pr\left[\boldsymbol{U}(d_t', \boldsymbol{d}_{-t}) \in \mathcal{U}^T(d_t', \boldsymbol{d}_{-t}, 1; x_t', \boldsymbol{x}_{-t}) \;\middle|\; \begin{array}{c} \boldsymbol{x}_{-t}, \boldsymbol{z}^{t}, \\ \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1}) \in \mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{y}^{t-1}), \\ \boldsymbol{V}^t \in \mathcal{V}^t(\boldsymbol{d}^{t-1}, d_t', \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1})) \end{array} \right] \\
=&\Pr\left[\boldsymbol{U}(d_t', \boldsymbol{d}_{-t}) \in \mathcal{U}^T(d_t', \boldsymbol{d}_{-t}, 1; x_t', \boldsymbol{x}_{-t}) \;\middle|\; \begin{array}{c} x_t', \boldsymbol{x}_{-t}, \boldsymbol{z}^{t}, \\ \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1}) \in \mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{y}^{t-1}), \\ \boldsymbol{V}^t \in \mathcal{V}^t(\boldsymbol{d}^{t-1}, d_t', \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1})) \end{array} \right] \\
=&E[Y_T(d_t', \boldsymbol{d}_{-t})|x_t', \boldsymbol{x}_{-t}, \boldsymbol{z}^t, \boldsymbol{y}^{t-1}, \boldsymbol{d}^{t-1}, d_t'],
\end{aligned}
\tag{A.2}
$$

where the first equality is by Assumption SX and Lemma A.1. We use the result (A.2) in the next step.

First, note that $E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}^T, \boldsymbol{y}^{T-1}, \boldsymbol{d}^T] = E[Y_T|\boldsymbol{x}, \boldsymbol{z}^T, \boldsymbol{y}^{T-1}, \boldsymbol{d}^T]$ is trivially identified for any generic values $(\boldsymbol{d}, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{y}^{T-1})$. We prove by means of mathematical induction. For given $2 \leq t \leq T-1$, suppose $E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}^t, \boldsymbol{y}^{t-1}, \boldsymbol{d}^t]$ is identified for any generic values $(\boldsymbol{d}, \boldsymbol{x}, \boldsymbol{z}^t, \boldsymbol{y}^{t-1})$, and consider the identification of

$$\begin{aligned} E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}^{t-1}, \boldsymbol{y}^{t-2}, \boldsymbol{d}^{t-1}] = & \Pr[D_t = d_t|\boldsymbol{x}, \boldsymbol{z}^{t-1}, \boldsymbol{y}^{t-2}, \boldsymbol{d}^{t-1}]E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}^{t-1}, \boldsymbol{y}^{t-2}, \boldsymbol{d}^{t-1}, d_t] \\ & + \Pr[D_t = d_t'|\boldsymbol{x}, \boldsymbol{z}^{t-1}, \boldsymbol{y}^{t-2}, \boldsymbol{d}^{t-1}]E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}^{t-1}, \boldsymbol{y}^{t-2}, \boldsymbol{d}^{t-1}, d_t']. \end{aligned} \tag{A.3}$$

The first main term $E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}^{t-1}, \boldsymbol{y}^{t-2}, \boldsymbol{d}^{t-1}, d_t]$ in (A.3) is identified, by integrating over $y_{t-1} \in \{0, 1\}$ the quantity $E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}^{t-1}, \boldsymbol{y}^{t-1}, \boldsymbol{d}^t]$, which is assumed to be identified in the previous iteration since it is equal to $E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}^t, \boldsymbol{y}^{t-1}, \boldsymbol{d}^t]$ by Assumption SX and Lemma A.1(ii). The remaining unknown term in (A.3) satisfies

$$\begin{aligned} & E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}^{t-1}, \boldsymbol{y}^{t-2}, \boldsymbol{d}^{t-1}, d_t'] \\ = & \Pr[Y_{t-1} = 1|\boldsymbol{x}, \boldsymbol{z}^{t-1}, \boldsymbol{y}^{t-2}, \boldsymbol{d}^{t-1}, d_t']E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}^{t-1}, (\boldsymbol{y}^{t-2}, 1), \boldsymbol{d}^{t-1}, d_t'] \\ & + \Pr[Y_{t-1} = 0|\boldsymbol{x}, \boldsymbol{z}^{t-1}, \boldsymbol{y}^{t-2}, \boldsymbol{d}^{t-1}, d_t']E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}^{t-1}, (\boldsymbol{y}^{t-2}, 0), \boldsymbol{d}^{t-1}, d_t']. \end{aligned}$$

By applying (A.2) to the unknown terms in this expression, we have

$$E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}^{t-1}, \tilde{\boldsymbol{y}}^{t-1}, \boldsymbol{d}^{t-1}, d_t'] = E[Y_T(d_t', \boldsymbol{d}_{-t})|x_t', \boldsymbol{x}_{-t}, \boldsymbol{z}^t, \tilde{\boldsymbol{y}}^{t-1}, \boldsymbol{d}^{t-1}, d_t'] \tag{A.4}$$

for each $\tilde{\boldsymbol{y}}^{t-1}$, which is identified from the previous iteration. Therefore, $E[Y_T(\boldsymbol{d})|\boldsymbol{x}, \boldsymbol{z}^{t-1}, \boldsymbol{y}^{t-2}, \boldsymbol{d}^{t-1}]$ is identified. Note that when $t = 2$, $\boldsymbol{Y}^0$ is understood to mean there is no conditioning. Lastly, when $t = 1$,

$$E[Y_T(\boldsymbol{d})|\boldsymbol{x}] = \Pr[D_1 = d_1|\boldsymbol{x}]E[Y_T(\boldsymbol{d})|\boldsymbol{x}, d_1] + \Pr[D_1 = d_1'|\boldsymbol{x}]E[Y_T(\boldsymbol{d})|\boldsymbol{x}, d_1'].$$

Noting that $Y_0 = 0$, suppose x_1' is such that $\mu_1(0, d_1, x_1) = \mu_1(0, d_1', x_1')$ with $d_1' \neq d_1$ by applying Lemma 4.1. Then,

$$\begin{aligned} E[Y_T(\boldsymbol{d})|\boldsymbol{x}, d_1'] &= \Pr[\boldsymbol{U}(\boldsymbol{d}) \in \mathcal{U}^T(\boldsymbol{d}, 1; \boldsymbol{x})|\boldsymbol{x}_{-1}, z_1, V_1 \in \mathcal{V}^1(d_1')] \\ &= \Pr[\boldsymbol{U}(d_1', \boldsymbol{d}_{-1}) \in \mathcal{U}^T(d_1', \boldsymbol{d}_{-1}, 1; x_1', \boldsymbol{x}_{-1})|\boldsymbol{x}_{-1}, z_1, V_1 \in \mathcal{V}^1(d_1')] \\ &= E[Y_T(d_1', \boldsymbol{d}_{-1})|x_1', \boldsymbol{x}_{-1}, z_1, d_1'], \end{aligned}$$

by Assumption SX and Lemma A.1(ii), which is identified from the previous iteration for $t = 2$. Therefore, $E[Y_T(\boldsymbol{d})|\boldsymbol{x}]$ is identified. □

A.4 Proof of Theorem 5.1

We analyze the identification of $E[Y_T(\boldsymbol{d})|\boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-, \boldsymbol{x}, \boldsymbol{z}]$. Since

$$E[Y_T(\boldsymbol{d})|\boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-, \boldsymbol{x}, \boldsymbol{z}] = \Pr[Y_T(\boldsymbol{d}) = 1, \boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-|\boldsymbol{x}, \boldsymbol{z}]/\Pr[\boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-|\boldsymbol{x}, \boldsymbol{z}],$$

we identify each term in the fraction. For each term, the proof is parallel to that of Theorem 4.1. Let $\tilde{\boldsymbol{y}}_- \equiv (y_1, ..., y_{t_{\tilde{L}}})$ be a subvector (not necessarily strict) of $\boldsymbol{y}$, where $t_1 < t_2 < \cdots < t_{\tilde{L}} \leq T$ and $\tilde{L} \leq T$; e.g., when $\tilde{L} = T$, $\tilde{\boldsymbol{y}}_- = \boldsymbol{y}$. Generalizing the $\boldsymbol{U}$-sets introduced in Section 4, define

$$\mathcal{U}^{t_{\tilde{L}}}(\boldsymbol{d}^{t_{\tilde{L}}}, \tilde{\boldsymbol{y}}_-) \equiv \mathcal{U}^{t_{\tilde{L}}}(\boldsymbol{d}^{t_{\tilde{L}}}, \tilde{\boldsymbol{y}}_-; \boldsymbol{x}^{t_{\tilde{L}}}) \equiv \{\boldsymbol{U}^{t_{\tilde{L}}}(\boldsymbol{d}^{t_{\tilde{L}}}) : y_s = Y_s(\boldsymbol{d}^s, \boldsymbol{x}^s) \text{ for all } s \in \{t_1, ..., t_{\tilde{L}}\}\}.$$

In the first part of the proof, we identify $\Pr[Y_T(\boldsymbol{d}) = 1, \boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_- | \boldsymbol{x}, \boldsymbol{z}]$. Take $\tilde{\boldsymbol{Y}}_-(\boldsymbol{d}) = (\boldsymbol{Y}_-(\boldsymbol{d}), Y_T(\boldsymbol{d}))$ with realization $\tilde{\boldsymbol{y}}_- = (\boldsymbol{y}_-, 1)$. For simplicity, we directly use Assumption SX without invoking Lemma A.1(ii). Then, for $2 \leq t \leq T-1$, we have

$$\begin{aligned}
&\Pr[Y_T(\boldsymbol{d}) = 1, \boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-, \boldsymbol{Y}^{t-1} = \boldsymbol{y}^{t-1}, \boldsymbol{D}^t = (\boldsymbol{d}^{t-1}, d_t') | \boldsymbol{x}, \boldsymbol{z}] \\
&= \Pr[\boldsymbol{U}(\boldsymbol{d}) \in \mathcal{U}^T(\boldsymbol{d}, \boldsymbol{y}_-, 1; \boldsymbol{x}), \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1}) \in \mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{y}^{t-1}), \boldsymbol{V}^t \in \mathcal{V}^t(\boldsymbol{d}^{t-1}, d_t', \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1}))] \\
&= \Pr\left[\begin{array}{c} \boldsymbol{U}(d_t', \boldsymbol{d}_{-t}) \in \mathcal{U}^T(d_t', \boldsymbol{d}_{-t}, \boldsymbol{y}_-, 1; x_t', \boldsymbol{x}_{-t}), \\ \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1}) \in \mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{y}^{t-1}), \\ \boldsymbol{V}^t \in \mathcal{V}^t(\boldsymbol{d}^{t-1}, d_t', \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1})) \end{array}\right] \\
&= \Pr[Y_T(d_t', \boldsymbol{d}_{-t}) = 1, \boldsymbol{Y}_-(d_t', \boldsymbol{d}_{-t}) = \boldsymbol{y}_-, \boldsymbol{Y}^{t-1} = \boldsymbol{y}^{t-1}, \boldsymbol{D}^t = (\boldsymbol{d}^{t-1}, d_t') | x_t', \boldsymbol{x}_{-t}, \boldsymbol{z}],
\end{aligned} \tag{A.5}$$

where the second equality uses x_t' such that $\mu_t(y_{t-1}, d_t, x_t) = \mu_t(y_{t-1}, d_t', x_t')$ by applying Lemma 4.1. First, $\Pr[Y_T(\boldsymbol{d}) = 1, \boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-, \boldsymbol{Y}^{T-1} = \boldsymbol{y}^{T-1}, \boldsymbol{D}^T = \boldsymbol{d}^T | \boldsymbol{x}, \boldsymbol{z}] = \Pr[Y_T = 1, \boldsymbol{Y}_- = \boldsymbol{y}_-, \boldsymbol{Y}^{T-1} = \boldsymbol{y}^{T-1}, \boldsymbol{D}^T = \boldsymbol{d}^T | \boldsymbol{x}, \boldsymbol{z}]$ is trivially identified for any generic values $(\boldsymbol{d}, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{y}^{T-1})$. For given $2 \leq t \leq T-1$, suppose $\Pr[Y_T(\boldsymbol{d}) = 1, \boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-, \boldsymbol{Y}^{t-1} = \boldsymbol{y}^{t-1}, \boldsymbol{D}^t = \boldsymbol{d}^t | \boldsymbol{x}, \boldsymbol{z}]$ is identified for any generic values $(\boldsymbol{d}, \boldsymbol{x}, \boldsymbol{z}, \boldsymbol{y}^{t-1})$, and consider identification of

$$\begin{aligned}
&\Pr[Y_T(\boldsymbol{d}) = 1, \boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-, \boldsymbol{Y}^{t-2} = \boldsymbol{y}^{t-2}, \boldsymbol{D}^{t-1} = \boldsymbol{d}^{t-1} | \boldsymbol{x}, \boldsymbol{z}] \\
&= \Pr[Y_T(\boldsymbol{d}) = 1, \boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-, \boldsymbol{Y}^{t-2} = \boldsymbol{y}^{t-2}, \boldsymbol{D}^t = (\boldsymbol{d}^{t-1}, d_t) | \boldsymbol{x}, \boldsymbol{z}] \\
&\quad + \Pr[Y_T(\boldsymbol{d}) = 1, \boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-, \boldsymbol{Y}^{t-2} = \boldsymbol{y}^{t-2}, \boldsymbol{D}^t = (\boldsymbol{d}^{t-1}, d_t') | \boldsymbol{x}, \boldsymbol{z}].
\end{aligned} \tag{A.6}$$

The first term in the expression is identified, by summing over y_{t-1} the quantity $\Pr[Y_T(\boldsymbol{d}) = 1, \boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-, \boldsymbol{Y}^{t-1} = \boldsymbol{y}^{t-1}, \boldsymbol{D}^t = \boldsymbol{d}^t | \boldsymbol{x}, \boldsymbol{z}]$, which is identified from the previous iteration. The second unknown term in (A.6) satisfies

$$\begin{aligned}
&\Pr[Y_T(\boldsymbol{d}) = 1, \boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-, \boldsymbol{Y}^{t-2} = \boldsymbol{y}^{t-2}, \boldsymbol{D}^t = (\boldsymbol{d}^{t-1}, d_t') | \boldsymbol{x}, \boldsymbol{z}] \\
&= \Pr[Y_T(\boldsymbol{d}) = 1, \boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-, \boldsymbol{Y}^{t-1} = (\boldsymbol{y}^{t-2}, 1), \boldsymbol{D}^t = (\boldsymbol{d}^{t-1}, d_t') | \boldsymbol{x}, \boldsymbol{z}] \\
&\quad + \Pr[Y_T(\boldsymbol{d}) = 1, \boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-, \boldsymbol{Y}^{t-1} = (\boldsymbol{y}^{t-2}, 0), \boldsymbol{D}^t = (\boldsymbol{d}^{t-1}, d_t') | \boldsymbol{x}, \boldsymbol{z}].
\end{aligned} \tag{A.7}$$

But note that, by (A.5), each term in (A.7) satisfies

$$\begin{aligned}
&\Pr[Y_T(\boldsymbol{d}) = 1, \boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-, \boldsymbol{Y}^{t-1} = \tilde{\boldsymbol{y}}^{t-1}, \boldsymbol{D}^t = (\boldsymbol{d}^{t-1}, d_t') | \boldsymbol{x}, \boldsymbol{z}] \\
&= \Pr[Y_T(d_t', \boldsymbol{d}_{-t}) = 1, \boldsymbol{Y}_-(d_t', \boldsymbol{d}_{-t}) = \boldsymbol{y}_-, \boldsymbol{Y}^{t-1} = \tilde{\boldsymbol{y}}^{t-1}, \boldsymbol{D}^t = (\boldsymbol{d}^{t-1}, d_t') | x_t', \boldsymbol{x}_{-t}, \boldsymbol{z}]
\end{aligned} \tag{A.8}$$

for each $\tilde{\boldsymbol{y}}^{t-1}$, which is identified from the previous iteration. Therefore, $\Pr[Y_T(\boldsymbol{d}) = 1, \boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-, \boldsymbol{Y}^{t-2} = \boldsymbol{y}^{t-2}, \boldsymbol{D}^{t-1} = \boldsymbol{d}^{t-1}|\boldsymbol{x}, \boldsymbol{z}]$ is identified. Lastly, when $t = 1$,

$$\begin{aligned}\Pr[Y_T(\boldsymbol{d}) = 1, \boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-|\boldsymbol{x}, \boldsymbol{z}] =& \Pr[Y_T(\boldsymbol{d}) = 1, \boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-, D_1 = d_1|\boldsymbol{x}, \boldsymbol{z}] \\ &+ \Pr[Y_T(\boldsymbol{d}) = 1, \boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-, D_1 = d_1'|\boldsymbol{x}, \boldsymbol{z}].\end{aligned}$$

The first term is identified from the iteration for $t = 2$. Noting that $Y_0 = 0$, suppose x_1' is such that $\mu_1(0, d_1, x_1) = \mu_1(0, d_1', x_1')$ with $d_1' \neq d_1$ by Lemma 4.1. Then, similarly to (A.5),

$$\begin{aligned}&\Pr[Y_T(\boldsymbol{d}) = 1, \boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-, D_1 = d_1'|\boldsymbol{x}, \boldsymbol{z}] \\ =& \Pr[Y_T(d_1', \boldsymbol{d}_{-1}) = 1, \boldsymbol{Y}_-(d_1', \boldsymbol{d}_{-1}) = \boldsymbol{y}_-, D_1 = d_1'|x_1', \boldsymbol{x}_{-1}, \boldsymbol{z}],\end{aligned}$$

which is also identified from the previous iteration for $t = 2$. Therefore $\Pr[Y_T(\boldsymbol{d}) = 1, \boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-|\boldsymbol{x}, \boldsymbol{z}]$ is identified.

In the second part of the proof, we identify $\Pr[\boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-|\boldsymbol{x}, \boldsymbol{z}]$. Take $\tilde{\boldsymbol{Y}}_-(\boldsymbol{d}) = \boldsymbol{Y}_-(\boldsymbol{d}) \equiv (Y_{t_1}(\boldsymbol{d}), ..., Y_{t_L}(\boldsymbol{d}))$ with realization $\tilde{\boldsymbol{y}}_- = \boldsymbol{y}_-$. Then, for $2 \leq t \leq t_L - 1$, we can show the following equivalence, analogous to (A.5):

$$\begin{aligned}&\Pr[\boldsymbol{Y}_-(\boldsymbol{d}) = \boldsymbol{y}_-, \boldsymbol{Y}^{t-1} = \boldsymbol{y}^{t-1}, \boldsymbol{D}^t = (\boldsymbol{d}^{t-1}, d_t')|\boldsymbol{x}, \boldsymbol{z}] \\ =& \Pr[\boldsymbol{U}^{t_L}(\boldsymbol{d}^{t_L}) \in \mathcal{U}^{t_L}(\boldsymbol{d}^{t_L}, \boldsymbol{y}_-; \boldsymbol{x}^{t_L}), \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1}) \in \mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{y}^{t-1}), \boldsymbol{V}^t \in \mathcal{V}^t(\boldsymbol{d}^{t-1}, d_t', \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1}))] \\ =& \Pr\left[\begin{array}{c}\boldsymbol{U}^{t_L}(d_t', \boldsymbol{d}_{-t}^{t_L}) \in \mathcal{U}^{t_L}(d_t', \boldsymbol{d}_{-t}^{t_L}, \boldsymbol{y}_-; x_t', \boldsymbol{x}_{-t}^{t_L}), \\ \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1}) \in \mathcal{U}^{t-1}(\boldsymbol{d}^{t-1}, \boldsymbol{y}^{t-1}), \\ \boldsymbol{V}^t \in \mathcal{V}^t(\boldsymbol{d}^{t-1}, d_t', \boldsymbol{U}^{t-1}(\boldsymbol{d}^{t-1}))\end{array}\right] \\ =& \Pr[\boldsymbol{Y}_-(d_t', \boldsymbol{d}_{-t}) = \boldsymbol{y}_-, \boldsymbol{Y}^{t-1} = \boldsymbol{y}^{t-1}, \boldsymbol{D}^t = (\boldsymbol{d}^{t-1}, d_t')|x_t', \boldsymbol{x}_{-t}, \boldsymbol{z}].\end{aligned}$$

The rest of the proof is an immediate modification of the iterative argument in the first part, and hence is omitted. □