Unobserved Mechanism Design: Equal Priority Auctions

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Intro

- tl;dr.
- Eye tracking - people don't read webpages.
- User agreements.
- Marketing research (Dickson and Sawyer, 1990) - 50% didn't know price, 25% didn't realize it was on special.

eBay

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[5]: ax = sns.distplot(normalized_bid)
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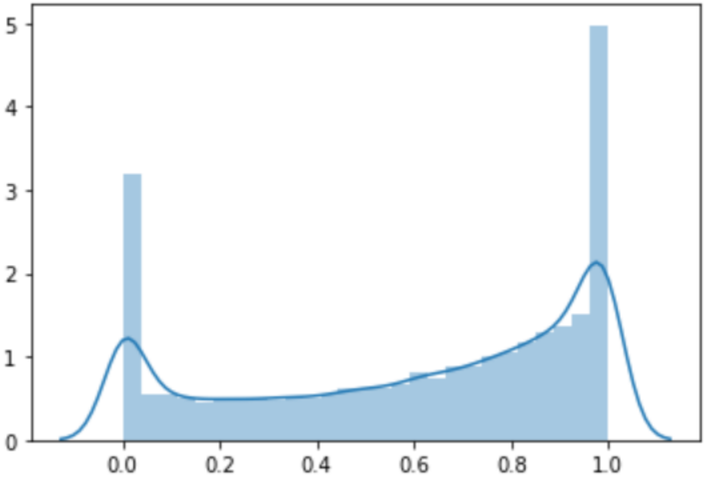


Setup

- Conventional single unit iid valuations environment.
- With some probability a buyer is uninformed of seller's mechanism and information type is private - multi-dimensional type space.
- Incentives regarding information type are in one direction only - informed buyers can pretend to be uninformed, but uninformed buyers can't act like informed.

Result

- The equilibrium in which the seller's expected revenue is the highest is an 'equal priority auction': informed buyers with middle valuations have the same allocation priority as uninformed buyers.
- So long as the probability with which each bidder is uninformed is strictly positive, there is a strictly positive probability that informed buyers will trade at a price that doesn't depend on any of the informed buyers' valuations.

Model

- 1 seller with 0 reservation value, n buyers with iid valuations drawn from F on $[0,1]$, with strictly positive density f.
- If buyer with valuation v accepts offer p, payoffs are $v-p$ for buyer and p for seller.
- Profit function $\pi(w) = w(1-F(w))$ is strictly concave, with a unique maximizer r^*.
- Concavity of π implies the virtual valuation function

$$\phi(v) = v - (1-F(v))/f(v)$$

is increasing above r^*, with $\phi(r^*) = 0$.

Mechanism

- Message space is $\mathcal{M}$ for all bidders - embeds $[0,1] \times [0,1]$, for example browsing history.
- $\mathcal{M}$ is common knowledge.
- A mechanism is a triple $\{\mathcal{M}, p, q\}$ where p and q are vector valued functions mapping elements of $\mathcal{M}^n$ to a price offer and probability for each bidder.
- Mechanism generates offers instead of transfers - buyer receiving an offer may not accept it.

Game

- The seller commits to a mechanism (for example a computer program) and publishes it.
- Each buyer independently observes the mechanism with probability $1 - \alpha$ then sends a message in $\mathcal{M}$ to the seller.
- The seller runs his or her program which sends an offer to one of the buyers.
- If the offer is rejected there is no trade.
- If the offer is accepted by the buyer who receives it, a trade occurs at that price.

Heuristic

- We model unobserved mechanism design as a game of imperfect information.
- The solution concept is perfect Bayesian equilibrium.
- There are no pure strategy equilibria.
- Informed - a revelation principle leads to direct mechanisms.
- Uninformed - restrict to babbling by uninformed (uncommunicative equilibrium) and amend direct mechanisms.

Permutation

- We will restrict to symmetric equilibria.
- Denote number of uninformed as m.
- Reorder n buyers such that the first $n-m$ of them are informed.
- For each $v = (v_1, \ldots, v_n) \in [0,1]^n$, and for each $i = 1, \ldots, n-m$, let

$$\rho_m^i(v) = (v_i, v_2, \ldots, v_{i-1}, v_1, v_{i+1}, \ldots, v_{n-m}, v_{n-m+1}, \ldots, v_n\}.$$

Direct mechanisms

A direct mechanism $\delta = \left\{ \left(q_m^\epsilon, p_m^\epsilon\right)_{m=0}^{n-1}, \left(q_m^\mu, p_m^\mu\right)_{m=1}^{n} \right\}$, where $q_m^\tau, p_m^\tau : [0,1]^n \to [0,1]$, $\tau = \epsilon, \mu$, satisfy

- $(q_m^\tau(v), p_m^\tau(v))$, $\tau = \epsilon, \mu$, are invariant to $(v_{n-m+1}, \dots, v_n)$;
- $(q_m^\epsilon, p_m^\epsilon)$ are invariant to permutations of $(v_2, \dots, v_{n-m})$, and (q_m^μ, p_m^μ) are invariant to permutations of $(v_1, \dots, v_{n-m})$;
- for all v and for all m,

$$\sum_{i=1}^{n-m} q_m^\epsilon\left(\rho_m^i(v)\right) + m q_m^\mu(v) \leq 1.$$

Informed

- Expected trading probability and price for an informed with valuation w, first conditional on m, and then expectations over m:

$$Q^{\epsilon}(w) = \sum_{m=0}^{n-1} B(m; n-1, \alpha)\mathbb{E}_v \left\{ q_m^{\epsilon}(v) | v_1 = w \right\},$$
$$P^{\epsilon}(w) = \sum_{m=0}^{n-1} B(m; n-1, \alpha)\mathbb{E}_v \left\{ q_m^{\epsilon}(v) p_m^{\epsilon}(v) | v_1 = w \right\},$$

where $B(m; n-1, \alpha) = \binom{n-1}{m} (1-\alpha)^{n-1-m}\alpha^m$.

- Payoff is

$$U^{\epsilon}(w) = wQ^{\epsilon}(w) - P^{\epsilon}(w).$$

Uninformed

- Payoff to an uninformed with valuation w is

$$U^{\mu}(w) = \sum_{m=0}^{n-1} B(m; n-1, \alpha)\mathbb{E}_v \left\{ q^{\mu}_{m+1}(v) \max \left[w - p^{\mu}_{m+1}(v), 0 \right] \right\}.$$

- We keep the max operator because uninformed may not accept an offer.

Incentive compatible mechanism

- δ is incentive compatible for informed with respect to valuations, if
$$U^{\epsilon}(w) = \int_0^w Q^{\epsilon}(x)\, dx,$$
and $Q^{\epsilon}(\cdot)$ is non-decreasing.
- δ is incentive compatible if it is incentive compatible for informed with respect to valuations, and if
$$U^{\epsilon}(w) \geq U^{\mu}(w)$$
for every w.

Seller's revenue

- Assume δ is incentive compatible.
- Seller's revenue from informed is

$$n(1-\alpha)\int_0^1 Q^{\epsilon}(w)\,\phi(w)f(w)dw.$$

- Seller's revenue from uninformed is given by

$$\sum_{m=1}^{n} B(m;n,\alpha)\mathbb{E}_v\left\{mq_m^{\mu}(v)\pi\left(p_m^{\mu}(v)\right)\right\}.$$

- δ is optimal if it maximizes sum of revenues from informed and uninformed.

Revelation principle

Theorem

(i) For any symmetric uncommunicative equilibrium, there exists an incentive compatible direct mechanism δ^* that achieves the equilibrium expected revenue and is optimal. (ii) Any optimal direct mechanism δ^* can be used to construct an equilibrium.

- Existence in (i) follows from standard revelation principle with respect to informed, and optimality follows from seller's option to deviate without being detected by uninformed.
- Construction in (ii) uses 'password' mechanism.

Equal Priority Auction

Direct mechanisms that can be implemented as second-price auctions with pooling.

- Offer to uninformed is independent of valuations of informed.
- For allocations among informed, there is a pooling interval of valuations, with strict separation above and below the interval.
- An informed bidder has the same allocation priority as uninformed if his valuation is in the pooling interval, higher priority if his valuation is above, and lower priority if his valuation is below.

Parameterization

An equal priority auction consists of:

- r - 'reserve price'.
- t - price offer t to uninformed.
- v_- and v_+ - upper and lower bound of an interval of valuations.
- Assume $r \leq t \leq v_- \leq v_+$.

Allocation and offer

Denote number of uninformed as m, and number of bids on $[v_-, v_+]$ as k.

- If $m \geq 1$ and the highest bid is no greater than v_+, then the seller makes an offer t to each uninformed buyer and an offer v_- to each informed bidder who bid in the interval $[v_-, v_+]$ with probability $1/(m+k)$.
- Otherwise, the seller makes an offer to the highest bidder, given by (b is the second highest bid)

$$\begin{cases} b & b > v_+ \\ r & m = 0; b < r \\ b & m = 0; b \in (r, v_-) \\ \frac{v_- + (m+k)v_+}{m+k+1} & \text{otherwise.} \end{cases}$$

EPA as a direct mechanism

- Probability $Q^{\epsilon}(w)$ with which an informed bidder with valuation w receives an offer is

$$\begin{cases} 0 & \text{if} \quad w < r \\ (1-\alpha)^{n-1}F^{n-1}(w) & \text{if} \quad w \in [r, v_-) \\ \chi(v_-, v_+) & \text{if} \quad w \in [v_-, v_+] \\ \sum_{m=0}^{n-1} B(m; n-1, \alpha) F^{n-1-m}(w) & \text{if} \quad w > v_+, \end{cases}$$

where $\chi(v_-, v_+)$ is given by

$$\sum_{m=0}^{n-1} B(m; n-1, \alpha) \sum_{k=0}^{n-1-m} B_k^{n-1-m}(v_-, v_+)/(m+k+1),$$

with

$$B_k^{n-1-m}(v_-, v_+) = \binom{n-1-m}{k} (F(v_+) - F(v_-))^k F^{n-1-m-k}(v_-).$$

Incentive compatibility

- Payoff to an informed buyer with valuation w is

$$U^{\epsilon}(w) = \int_0^w Q^{\epsilon}(x)dx.$$

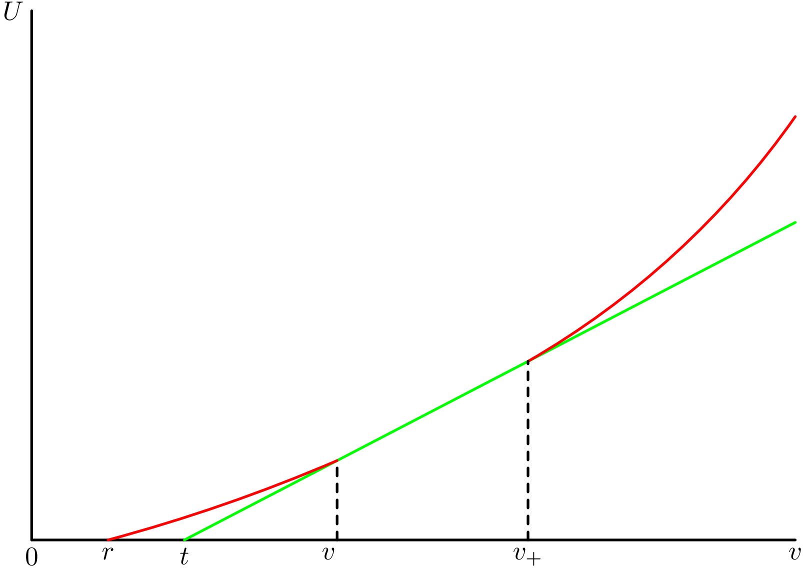
- Payoff to an uninformed buyers with valuation w is

$$U^{\mu}(w) = \chi(v_-, v_+)\max[w - t, 0].$$

- Incentive compatible if

$$U^{\epsilon}(v_-) \geq U^{\mu}(v_-).$$

Payoffs

- 

Optimal EPA problem

- Choose $r \leq t \leq v_- \leq v_+$ to maximize

$$n(1-\alpha)\int_0^1 Q^\epsilon(x)\,\phi(x)f(x)dx + n\alpha\chi(v_-,v_+)\pi(t)$$

subject to

$$\int_0^{v_-} Q^\epsilon(x)\,dx \geq \chi(v_-,v_+)\,(v_- - t)\,.$$

First order conditions

Necessary conditions for an equal priority auction $\{r, t, v_-, v_+\}$ to be revenue maximizing in the class of equal priority auctions are

$$\int_0^{v_-} Q^\epsilon(x)\,dx = \chi(v_-, v_+)\,(v_- - t)\,;$$

$$(1-\alpha)(v_- - t)(\phi(v_+) - \phi(v_-))f(v_-) - \alpha(\pi(t) - \phi(v_-))$$
$$= (1-\alpha)((\pi(v_-) - \pi(v_+)) - (F(v_+) - F(v_-))\phi(v_+));$$

$$\phi(r)f(r) + (\phi(v_+) - \phi(v_-))f(v_-) = 0;$$

$$\alpha\pi'(t) + (1-\alpha)(\phi(v_+) - \phi(v_-))f(v_-) = 0.$$

Characteristics

- $v_- < v_+$: pooling happens so long as $\alpha > 0$.
- $r < r^* < t$: reserve price for $m = 0$ is lowered and offer to uninformed raised.
- $\phi(v_+) < \pi(t)$: informed just above v_+ has higher priority than uninformed even though virtual value is less than 'outside option' $\pi(t)$.
- Informed with low valuations benefit from presence of uninformed, and uninformed with high valuations are hurt by presence of informed.

Main Theorem

Theorem

Suppose $\pi(\cdot)$ *is strictly concave. An optimal equal priority auction is an optimal direct mechanism.*

- By our revelation principle we can construct an equilibrium for the unobserved mechanism game that has the same payoffs.
- The proof uses Lagrangian relaxation.
- Recall that a direct mechanism δ consists of a series of functions $(q_m^{\epsilon}, p_m^{\epsilon})_{m=0}^{n-1}$ and $(q_m^{\mu}, p_m^{\mu})_{m=1}^{n}$.
- By strict concavity of $\pi(\cdot)$, in any optimal direct mechanism $p_m^{\mu}(v)$ is constant - denote it as p^{μ}.

Optimal direct mechanism problem

- Define

$$Q^{\mu} = \sum_{m=0}^{n-1} B(m; n-1, \alpha) \mathbb{E}_v \left\{ q^{\mu}_{m+1}(v) \right\}$$

- Optimal direct mechanism problem: choose $(q^{\epsilon}_m, p^{\epsilon}_m)^{n-1}_{m=0}$, $(q^{\mu}_m)^{n}_{m=1}$, and p^{μ}, to maximize

$$n(1-\alpha) \int_0^1 \{ Q^{\epsilon}(w) \phi(w) f(w) dw \} + n\alpha Q^{\mu} \pi(p^{\mu})$$

subject to feasibility, $Q^{\epsilon}(\cdot)$ non-decreasing, and for all w,

$$\int_0^w Q^{\epsilon}(x)\, dx \geq Q^{\mu} \max[w - p^{\mu}, 0] .$$

Relaxed Lagrangian

- Let $\lambda(\cdot)$ be a non-negative function on $[0,1]$.
- The relaxed problem is to maximize

$$n(1-\alpha)\int_0^1 \left\{Q^{\epsilon}(w)\,\phi(w)f(w)dw\right\} + n\alpha Q^{\mu}\pi\left(p^{\mu}\right)$$
$$+\int_0^1 \lambda(w)\left\{\int_0^w Q^{\epsilon}(x)\,dx - Q^{\mu}\max\left[w-p^{\mu},0\right]\right\}dw.$$

Revenue versus incentive

- Disaggregating by m, define

$$Q^{\epsilon}_m(w) = \mathbb{E}_v \left\{ q^{\epsilon}_m(v) | v_1 = w \right\}, \ \ Q^{\mu}_m = \mathbb{E}_v \left\{ q^{\mu}_{m+1}(v) \right\}.$$

- For each $w \in [0, 1]$, define

$$K^{\epsilon}(w) = n(1 - \alpha)\phi(w) + \int_w^1 \lambda(x) dx / f(w);$$

$$K^{\mu} = n\alpha\pi(p^{\mu}) - \int_0^1 \lambda(x) \max[x - p^{\mu}, 0] dx.$$

- Integrating by parts, rewrite Lagrangian as

$$\sum_{m=0}^{n-1} B(m; n-1, \alpha) \int_0^1 K^{\epsilon}(w) f(w) Q^{\epsilon}_m(w) dw$$

$$+ \sum_{m=0}^{n-1} B(m; n-1, \alpha) K^{\mu} Q^{\mu}_{m+1}.$$

Construction of multiplier function

- Fix an optimal equal priority auction $\{r, t, v_-, v_+\}$.
- Let $\lambda(\cdot)$ be such that $\lambda(w) = 0$ for all $w \notin [v_-, v_+]$, and $K^\epsilon(w) = K^\epsilon$ for all $w \in [v_-, v_+]$.
- Use first order conditions for $\{r, t, v_-, v_+\}$ to show the direct mechanism given by $\{r, t, v_-, v_+\}$ maximizes Lagrangian subject to feasibility and non-decreasing $Q^\epsilon(\cdot)$.
- $\{r, t, v_-, v_+\}$ then solves original problem.

Point-wise maximization

- Concavity of $\pi(\cdot)$ and first order condition imply Lagrangian is maximized by $p^\mu = t$.
- First order condition with respect to r implies Lagrangian is maximized when $m = 0$.
- First order conditions with respect to v_- and v_+ imply

$$\frac{B(m; n-1, \alpha)}{n-m} K^\epsilon = \frac{B(m-1; n-1, \alpha)}{m} K^\mu.$$

- Concavity of $\pi(\cdot)$ implies $K^\epsilon(w) > K^\epsilon$ for $w > v_+$, and $K^\epsilon(w) < K^\epsilon$ for $w < v_-$.