Aggregative Efficiency of Bayesian Learning in Networks*

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Abstract

We consider a sequential social-learning environment with rational agents and Gaussian private signals. Agents observe some subset of their predecessors, and we focus on the efficiency of private-signal aggregation on different observation networks. In equilibrium, actions are a log-linear function of observations and admit a signal-counting interpretation. The fraction of available signals incorporated into the group consensus ("*aggregative efficiency*") and hence the speed of social learning depend on the extent of *informational confounding* in the network. Agents who do not observe all predecessors optimally discount neighbors' behavior to avoid over-counting early movers' confounding actions. We show how to compute every agent's accuracy on any network. When agents move in generations and observe some members of the previous generation in a symmetric manner, we derive an exact expression for aggregative efficiency as a function of the network parameters. Each generation aggregates fewer than two extra signals in the long run, even when generations are arbitrarily large. When agents observe all predecessors from the previous generation, no more than three signals are aggregated per generation starting from the third generation, and larger generations lead to a slower learning rate.

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1 Introduction

In many economic environments, information about an unknown state of the world is dispersed among a society of agents. As people take actions based on their private signals and their observations of social neighbors, the process of social learning gradually aggregates decentralized information into a group consensus.

How does the underlying social network influence the efficiency of this information aggregation? Even if two networks both lead to the correct group consensus asymptotically, agents might aggregate almost all previous signals in one network but make use of only a small fraction of these signals in the other. Although people learn completely in the long run in both networks, they learn at different rates and can experience different levels of welfare.

The economic theory literature contains a large body of work on Bayesian models of sequential social learning, where privately informed individuals move in turn and draw rational inferences from their observations. These papers have largely focused on long-run learning outcomes, and less is known about how the social network affects the rate of learning. As Golub and Sadler (2016)'s recent survey points out:

> "A significant gap in our knowledge concerns short-run dynamics and rates of learning in these models. [...] The complexity of Bayesian updating in a network makes this difficult, but even limited results would offer a valuable contribution to the literature."

The present paper investigates the impact of the social network on the efficiency of private-signal aggregation, and hence on the rate of rational sequential learning. We work with the canonical sequential social-learning model, but make two richness assumptions to address some of the "complexity" that Golub and Sadler (2016) mention. We assume the state is binary and agents have Gaussian private signals about the state. We also suppose that agents have rich actions, so players exactly infer their neighbors' beliefs through their behavior. This rich-signals, rich-actions world strips away some other obstructions to efficient learning (considered by Harel, Mossel, Strack, and Tamuz (2020); Molavi, Tahbaz-Salehi, and Jadbabaie (2018); Rosenberg and Vieille (2019) and others) and isolates the role of the social network.

In general, the observation network creates informational confounds for social learning even with rich action spaces. Suppose an agent only observes the actions of a pair of neighbors who have both seen the action of an even earlier mover. From the agent's perspective, this unobserved early action confounds the informational content of her two neighbors' behavior, as the observation network makes it impossible to fully incorporate the neighbors' private information without over-weighting the early mover's private information. Rational agents solve a signal-extraction problem to decide how to optimally combine their observations and signals. Networks differ in the severity of such informational confounds, and thus Bayesian social learning can aggregate information more or less efficiently.

We find that the unique equilibrium of the social-learning game has a log-linear form. We characterize the equilibrium strategy profile that solves agents' signal-extraction problems and give a procedure to compute the finite-agent accuracy of social learning in any network. The equilibrium action of each agent is distributed as if she sees some (possibly non-integer) number of independent private signals. This signal-equivalence property characterizes action distributions up to a single parameter measuring the accuracy of beliefs. It also lets us define learning efficiency in any network in terms of the fraction of private signals that are consolidated in equilibrium, a quantity we call "*aggregative efficiency.*" Networks that lead to faster rates of learning are precisely those that impound larger fractions of available signals into the social consensus.

We show that aggregative efficiency allows non-trivial welfare comparisons across network structures. Networks with higher aggregative efficiency reach any utility threshold earlier when private signals are not too precise, and they are ranked strictly higher by sufficiently patient social-welfare functions. By contrast, all network structures satisfying a mild condition are equally optimal under the "infinitely patient" social-welfare function that evaluates networks on their long-run learning outcome alone.

As the main application, we quantify the information loss due to confounding in a class of *generations networks*. Agents are arranged into generations of size K and each agent in generation t observes some subset of her generation $t-1$ predecessors. This network structure could correspond to actual generations in families, or successive cohorts in settings like firms or universities. How well can agents learn when they only observe the actions from the recent past, but not the choices from long ago or from their contemporaries? A broad insight is that these networks cannot sustain much learning: even if generation sizes are large, additional generations after the first contribute very little extra information.

We first study the speed of learning in "maximal generations networks" where each agent in generation t observes the actions of all predecessors in generation $t-1$. Society learns completely in the long run for every K, but aggregative efficiency is worse with larger K. We also show that no matter the size of the generations, social learning accumulates no more than three signals per generation starting with the third generation, and no more than two signals per generation asymptotically.

More generally, we consider any *symmetric* inter-generational observation structure — all agents observe the same number of neighbors and all pairs of distinct agents in the same generation share the same number of common neighbors. We derive a simple formula for the aggregative efficiency as a function of the network parameters. This expression quantifies how adding more links to the network trades off the increased number of social observations against the lower informational content of each observation, due to the extra confounding. The result also implies that the same long-run bound of two signals aggregated per generations holds for all networks in this class and for all generation sizes. An arbitrarily small fraction of available signals is included in the social consensus and agents learn arbitrarily slowly relative to the efficient rate, even though there is a feasible (but non-equilibrium) strategy profile that is eventually more accurate than aggregating K_0 signals per generation for every $K_0 < K$. We discuss an economic application of our results to the value of mentorship in speeding up learning within organizations.

In the appendix, we study the effect of adding links on learning dynamics in arbitrary networks (which need not have the generations structure). We show that society learns completely in the long run if and only if late enough agents have arbitrarily long *observational paths*. As a result, adding links to an observation network can only (weakly) improve its long-run learning outcome. But, the same is not true for the rate of learning. In a special class of networks without confounding, adding links speeds up learning and improves every agent's accuracy. In general, however, agents can become less accurate in networks with additional links, even when those new links do not introduce new *intransitivities* into the network. Extra observations can harm agents, even without creating any additional confounds.

1.1 Related Literature

We study rational learning in a sequential model (as first introduced by Banerjee (1992) and Bikhchandani, Hirshleifer, and Welch (1992)) with network observations. Acemoglu, Dahleh, Lobel, and Ozdaglar (2011) and Lobel and Sadler (2015) show that in sequential-learning environments similar to our model, rational agents learn the true state asymptotically under mild conditions on the network. We instead focus on finite-time learning accuracy and the speed of learning in different networks.

Harel, Mossel, Strack, and Tamuz (2020) study a setting where a fixed group of agents repeatedly receive signals and choose actions each period, learning from each others' past actions. Like in our generations network, they find that the rate of learning can be equivalent to perfectly observing an arbitrarily small fraction of private signals. The mechanism behind their result, "rational groupthink," relies on coarse communication — agents have a finite action space and may get trapped in a wrong consensus for an extended period of time, because small changes in individual beliefs that do not lead to taking a different action are unobservable to other group members. In fact, social learning would proceed at the efficient rate if actions were rich. We highlight a different mechanism for inefficient aggregation of decentralized information: an observation network that generates informational confounds can also lead to rates of learning far below the optimum even in a setting with rich actions.

The coarseness of the action space serves as the primary obstruction to the efficient rate of social learning in several other papers. Rosenberg and Vieille (2019) consider rational sequential learning with binary actions and relate properties of the private signal distribution to whether the speed of learning achieves a particular benchmark. Hann-Caruthers, Martynov, and Tamuz (2018) compare the rates of learning from past binary actions versus past signals. By contrast, we study network-based obstructions to achieving the efficient rate of learning and characterize this rate asymptotically in some examples, by making stronger assumptions on the informational environment.

Another group of papers point out that sequential social learning can be slow when information about the state derives from myopic agents' information-acquisition choices. In settings where agents pay for experiments and observe the actions but not the signal realizations of their predecessors, Burguet and Vives (2000) show that costly information acquisition slows down learning relative to exogenous signals, while Mueller-Frank and Pai (2016) and Lomys (2019) show that equilibrium learning is slower than the social planner's solution. Liang and Mu (2020) prove that slow learning obtains even in a setting where myopic agents see predecessors' signal realizations. We abstract away from this source of slow learning by letting agents have exogenous signals, following most of the literature on sequential social learning. This allows us to derive more substantial results and comparative statics about how different networks influence the rate of learning.

To the best of our knowledge, Lobel, Acemoglu, Dahleh, and Ozdaglar (2009) is the only other paper that considers how the rate of rational sequential learning varies with the observation network. In a binary-actions model, they compare two specific network structures where each agent has one neighbor: either their immediate predecessor, or a random past agent drawn uniformly. We give an expression for the equilibrium accuracy of every agent on arbitrary fixed networks — in particular we allow for general neighborhood sizes. Informational confounds among social observations, the key obstacle to fast learning that we identify, only appear in networks where agents observe two or more neighbors.

Several papers calculate speed of learning under naive updating heuristics instead of rational learning, e.g., Ellison and Fudenberg (1993) and Molavi, Tahbaz-Salehi, and Jadbabaie (2018). In the DeGroot updating model, Golub and Jackson (2012) show that speed of learning is determined by a simple network statistic that also measures the amount of homophily in the network.

In a different class of social-learning models where a finite set of agents repeatedly observe their neighbors on a fixed network and simultaneously choose actions every period, Gale and Kariv (2003) and Goyal (2012) have compared learning dynamics on specific networks to highlight a possible trade-off between the accuracy of the long-run group consensus and the speed of convergence to said consensus. In the sequential social-learning model we study, this trade-off does not appear since rational agents learn correctly in the long run on all networks satisfying mild conditions (Proposition 3).

Board and Meyer-ter-Vehn (2020) also study the role of the social network in a continuous-time product adoption model featuring random entry times and perfectly informative private signals. They show that starting from a network where none of i's direct neighbors share common indirect neighbors, adding links among i's neighbors always leads to slower adoption for i. These additional links would not affect i's learning in a sequential social-learning model, since they do not generate what we call informational confound — that is, multiple neighbors of i learning from a common source that i does not observe.

2 Model

There are two equally likely states of the world, $\omega \in \{0, 1\}$. An infinite sequence of agents indexed by $i \in \mathbb{N}_+$ move in order, each acting once. On her turn, agent i observes a *private signal* $s_i \in \mathbb{R}$ and the actions of her *neighbors*, $N(i) \subseteq \{1, ...i-1\}$. Agent i then chooses an *action* $a_i \in [0, 1]$ to maximize the expectation of

$$u_i(a_i, \omega) \coloneqq -(a_i - \omega)^2$$

given her belief about ω. So, she will choose the action equal to the probability she assigns to the event $\{\omega = 1\}$.

We consider a Gaussian information structure where private signals (s_i) are conditionally i.i.d. given the state. We have $s_i \sim \mathcal{N}(1, \sigma^2)$ when $\omega = 1$ and $s_i \sim \mathcal{N}(-1, \sigma^2)$ when $\omega = 0$, where $\mathcal{N}(a, b^2)$ is the normal distribution with mean a and variance b^2, and $0 < 1/\sigma^2 < \infty$ is the private signal precision.

Agents' neighbors are defined by a deterministic network with adjacency matrix M. We put $M_{i,j} = 1$ if $j \in N(i)$ and $M_{i,j} = 0$ otherwise. The network M is common knowledge.

With the network M fixed, let $d_i := |N(i)|$ denote the number of i's neighbors. A *strategy* for agent i is a function $A_i : [0, 1]^{d_i} \times \mathbb{R} \to [0, 1]$, where $A_i(a_{j(1)}, ..., a_{j(d_i)}, s_i)$ specifies i's play after observing actions $a_{j(1)}, ..., a_{j(d_i)}$ from neighbors $N(i) = \{j(1), ..., j(d_i)\}$ and when own private signal is s_i.[1] Given a profile of strategies $(A_i)_{i \in \mathbb{N}_+}$, observation $(a_{j(1)}, ..., a_{j(d_i)}, s_i)$ is *on-path* if it has positive density under the profile. A perfect Bayesian equilibrium (*equilibrium* for short) is a strategy profile $(A_i^*)_{i \in \mathbb{N}_+}$ so that for all i and for all on-path observations of i, A_i^* maximizes the Bayesian expected utility given the (well-defined) posterior belief about ω. We will see that in any equilibrium, $s_i \mapsto A_i^*(a_{j(1)}, ..., a_{j(d_i)}, s_i)$ is a surjective function onto $(0,1)$ for all i and $a_{j(1)}, ..., a_{j(d_i)}$. So an observation is on-path in equilibrium if and only if all observed actions are interior.

The sequential nature of the social-learning game implies there is a unique equilibrium. Agent 1 who has no social observations must use the same strategy $A_1^*(s_1)$ in all equilibria. So agent 2 also only has one equilibrium strategy A_2^*, as the behavior of agent 1 is unique across all equilibria. Proceeding inductively, there is a unique equilibrium profile $(A_i^*)_{i \in \mathbb{N}_+}$.

3 Equilibrium

3.1 Linearity of Equilibrium

We will find it convenient to work with the following log-transformations of variables: $\tilde{s}_i := \ln\left(\frac{\mathbb{P}[\omega=1|s_i]}{\mathbb{P}[\omega=0|s_i]}\right)$, $\tilde{a}_i := \ln\left(\frac{a_i}{1-a_i}\right)$. We call $\tilde{s}_i$ the *log-signal* of i and $\tilde{a}_i$ the *log-action* of i. These changes are bijective, so it is without loss to use the log versions. Write $\tilde{A}_i^*(\tilde{a}_{j(1)}, ..., \tilde{a}_{j(d_i)}, \tilde{s}_i)$ for i's equilibrium log-strategy: the (unique) equilibrium map between the log-actions of i's neighbors and i's own log-signal to i's log-action.

In this section, we show that every $\tilde{A}_i^*$ is a linear function of its arguments, with coefficients that only depend on the network M and not on the precision of private signals. We also show that there exist constants $(r_i)_{i \in \mathbb{N}_+}$ with $r_i \leq i$ so that in equilibrium, (a_i, ω) is jointly distributed as-if i chooses a_i solely based on r_i independent private signals.[2] The constants r_i depend on the network and may be interpreted as the number of signals that social learning on M aggregates by agent i. This gives a sufficient statistic to compare society's short-run accuracy on different networks.

In general, the behavior of i's neighbors are correlated even after conditioning on the state. Intuitively, i would like to put enough weight on the actions of her neighbors to incorporate their private signals, but doing so would also over-count the signals of the earlier agents observed by several members of $N(i)$ but not by i. The social network M thus creates an informational confound that generally prevents i from fully extracting the signals of $N(i)$.

[1] It is without loss for equilibrium analysis to focus on pure strategies, since agents are never indifferent between two actions in equilibrium.

[2] The constants r_i need not be integers, and we will formalize the meaning this claim for non-integer r_i in Definition 1.

The equilibrium strategy of i represents the optimal aggregation of her neighbors' actions. The next result shows the optimal aggregation is linear and gives an explicit expression for the coefficients. All proofs are in the Appendix.

Proposition 1. *For each agent i with $N(i) = \{j(1), ..., j(d_i)\}$, there exist constants $(\beta_{i,j(k)})_{k=1}^{d_i}$ so that i's equilibrium log-strategy is given by*

$$\tilde{A}_i^*(\tilde{a}_{j(1)}, ..., \tilde{a}_{j(d_i)}, \tilde{s}_i) = \tilde{s}_i + \sum_{k=1}^{d_i} \beta_{i,j(k)} \tilde{a}_{j(k)}.$$

The vector of coefficients $\vec{\beta}_i$ is given by

$$\vec{\beta}_i = 2 \left(\mathbb{E}[(\tilde{a}_{j(1)}, ..., \tilde{a}_{j(d_i)}) \mid \omega = 1] \times \text{Cov}[\tilde{a}_{j(1)}, ..., \tilde{a}_{j(d_i)} \mid \omega = 1]^{-1} \right).$$

These coefficients do not depend on the conditional variance of the private signals $1/\sigma^2$.

The interpretation of the inverse covariance matrix in $\vec{\beta}_i$ is that i rationally discounts the actions of two neighbors $j(1)$ and $j(2)$ if their actions are correlated in equilibrium.

For general private signal distributions, models of Bayesian updating in networks have tractability issues, as Golub and Sadler (2016) point out. The key lemma to proving Proposition 1 is the following property of the Gaussian information structure in our model, which ensures that i's observations have a jointly Gaussian distribution conditional on ω. This permits us to study optimal inference in closed form.

Lemma 1. *For each i, the log-signal $\tilde{s}_i$ has a Gaussian distribution conditional on ω, with $\mathbb{E}[\tilde{s}_i \mid \omega = 0] = -2/\sigma^2$, $\mathbb{E}[\tilde{s}_i \mid \omega = 1] = 2/\sigma^2$, and $\text{Var}[\tilde{s}_i \mid \omega = 0] = \text{Var}[\tilde{s}_i \mid \omega = 1] = 4/\sigma^2$.*

Proposition 1 implies that we may find weights $(w_{i,j})_{i \geq j}$ so that the realizations of equilibrium log-actions are related to the realizations of log-signals by $\tilde{a}_i = \sum_{j=1}^{i} w_{i,j} \tilde{s}_j$. Let W be the matrix containing all such weights. Since none of the $\vec{\beta}_i$ vectors depends on σ^2, neither does W.

Proposition 1 leads to an inductive procedure to compute the coefficients in the unique equilibrium profile and the matrix W. We start with the first row of W, $W_1 = (1, 0, 0, ...)$. Proceeding iteratively, once the first $i - 1$ rows of W have been constructed, we know the weights that each of i's neighbor's log-actions $\tilde{a}_{j(k)}$ puts on different log-signals, hence we can compute $\mathbb{E}[(\tilde{a}_{j(1)}, ..., \tilde{a}_{j(d_i)}) \mid \omega = 1]$ and $\text{Cov}[\tilde{a}_{j(1)}, ..., \tilde{a}_{j(d_i)} \mid \omega = 1]$. We can find $\vec{\beta}_i$ using Proposition 1, and hence construct the i-th row of W.

3.2 Measure of Accuracy

We would like to evaluate networks in terms of their short-run social-learning accuracy, so as to compare the rates of Bayesian learning on different networks. Towards a measure of accuracy, imagine that agent i's only information about ω consists of $n \in \mathbb{N}_+$ independent private signals. Then, the Bayesian i would play the log-action equal to the sum of the n log-signals, so by Lemma 1 her behavior would follow the conditional distributions $\tilde{a}_i \sim \mathcal{N}\left(\pm n \cdot \frac{2}{\sigma^2}, n \cdot \frac{4}{\sigma^2}\right)$, with the positive and negative means in states $\omega = 1$ and $\omega = 0$ respectively. We quantify learning accuracy using distributions of this form that allow for non-integer n.

Definition 1. Social learning *aggregates* $r \in \mathbb{R}_+$ *signals by agent* i if the equilibrium log-action $\tilde{a}_i$ has the conditional distributions $\mathcal{N}\left(\pm r \cdot \frac{2}{\sigma^2}, r \cdot \frac{4}{\sigma^2}\right)$ in the two states. If this holds for some $r \in \mathbb{R}_+$, then we say i's behavior has a *signal-counting interpretation*.

When agents use a non-equilibrium strategy profile, in general the conditional distributions of $\tilde{a}_i$ need not equal $\mathcal{N}\left(\pm r \cdot \frac{2}{\sigma^2}, r \cdot \frac{4}{\sigma^2}\right)$ for any r, even when the profile is log-linear. Indeed, if this profile results in i putting weights $(w_{i,j})_{j \leq i}$ on log-signals $(\tilde{s}_j)_{j \leq i}$, then $\tilde{a}_i$ has a signal-counting interpretation if and only if $\sum_{j=1}^{i} w_{i,j} = \sum_{j=1}^{i} w_{i,j}^2$.

But as the next result shows, the equilibrium log-actions always admit a signal-counting interpretation on any network.

Proposition 2. *There exist* $(r_i)_{i \geq 1}$ *so that social learning aggregates* r_i *signals by agent* i*. These* $(r_i)_{i \geq 1}$ *depend on the network* M*, but not on private signal precision.*

We can use $(r_i)_{i \geq 1}$ as a measure of how the network M affects the speed of rational information aggregation in our social-learning setting. An alternative interpretation is that $r_i/i \in [0, 1]$ measures the fraction of all available signals that get incorporated into the social consensus by agent i, with some signals lost during social learning due to informational confounding.

Definition 2. If $\lim_{i \to \infty}(r_i/i)$ exists, it is called the *aggregative efficiency* of the network.

The aggregative efficiency measures the fraction of signals in the entire society that individuals manage to aggregate under social learning. Networks that induce faster social learning in the long run are equivalently those with higher levels of aggregative efficiency.

The signal-counting interpretation of behavior is closely identified with the rational learning rule. Even if all of i's predecessors are rational, one can show that i's log-action does not admit a signal-counting interpretation under "generic" log-linear strategies. Conversely, a rational agent's behavior always admits a signal-counting interpretation even when her predecessors use arbitrary non-rational log-linear strategies.

Corollary 1. *Fix arbitrary log-linear strategies for agents $i < I$, that is i's log-action is $\beta_{i,0}\tilde{s}_i + \sum_{k=1}^{d_i} \beta_{i,j(k)}\tilde{a}_{j(k)}$ for any constants $(\beta_{i,j(k)})_{k=0}^{d_i}$ where $N(i) = \{j(1), ..., j(d_i)\}$. If agent I plays the best response against the strategies of $i < I$, then I's behavior has a signal-counting interpretation.*

This result provides one way to extend the definitions of r_i and aggregative efficiency to analyze the rate of social learning under any log-linear heuristic. For a given heuristic, consider a rational outside observer who has no private signal and who only sees the i-th heuristic learner's action. It follows from Corollary 1 that this observer's log-action has the conditional distributions $\mathcal{N}\left(\pm r_i \cdot \frac{2}{\sigma^2}, r_i \cdot \frac{4}{\sigma^2}\right)$ for some r_i. Here r_i measures the informativeness of the heuristic learner i's behavior in the units of private signals and leads to an upper-bound on i's utility.

3.3 Long-Run Learning

Before turning to results about finite-time accuracy, we develop two equivalent necessary and sufficient conditions for long-run learning in our setting. We say society *learns completely in the long run* if (a_i) converges to ω in probability. For a given network M, write $\overline{PL}(i) \in \mathbb{N}$ to refer to the length of the longest path in M originating from i (this length is 0 if $N(i) = \varnothing$).

Proposition 3. *The following are equivalent: (1) $\lim_{i\to\infty} \overline{PL}(i) = \infty$; (2) $\lim_{i\to\infty} \left[\max_{j\in N(i)} j\right] = \infty$; (3) society learns completely in the long run.*

Condition (1) of Proposition 3 says society learns completely in the long run if and only late enough agents have arbitrarily long observational paths. In fact, the proof of the result shows $r_i \geq \overline{PL}(i) + 1$ in all networks. Condition (2) is the analog of Acemoglu, Dahleh, Lobel, and Ozdaglar (2011)'s *expanding observations* property for a deterministic network. It says if we consider the most recent neighbor observed by each agent, then this sequence of most recent neighbors tends to infinity. Acemoglu, Dahleh, Lobel, and Ozdaglar (2011) show that expanding observations is necessary and sufficient for long-run learning in a random-networks model with rich signals and binary actions. With continuous actions, the same result is a consequence of Proposition 2.

Proposition 3 tells us that whether society learns in the long run is not a useful criterion for comparing different networks in this setting, as the conditions that guarantee long-run learning are very mild. We will instead focus on comparing $(r_i)_{i\geq 1}$ and aggregative efficiency across different networks. Section 5 shows that aggregative efficiency comparisons translate into two kinds of welfare comparisons.

4 Rate of Learning in Generations Networks

As an application of Section 3's characterization results, we study the speed of rational learning in *generations networks*. Agents are sequentially arranged into generations of size K, with agents within each generation placed into *positions* 1 through K. Agents in the first generation (i.e., $i = 1, ..., K$) have no neighbors. A collection of *observation sets*, $\Psi_k \subseteq \{1, ..., K\}$ for $k = 1, ..., K$ define the network M for agents in later generations. The agent in position k in generation $t \geq 2$ observes agents in positions Ψ_k from generation $t-1$ (and no agents from any other generation).That is, for $i = (t-1)K + k$ where $t \geq 2$ and $1 \leq k \leq K$, network M has $N(i) = \{(t-2)K + \psi : \psi \in \Psi_k\}$. [3] Figure 1 shows an example with $K = 3$.

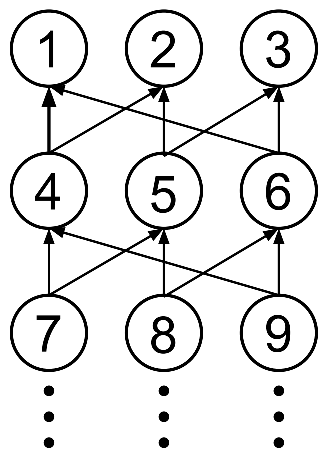

Figure 1: A generations network with $K = 3$ agents per generation and the observation sets $\Psi_1 = \{1, 2\}$, $\Psi_2 = \{2, 3\}$, and $\Psi_3 = \{1, 3\}$.

4.1 Full Observations and the Role of Generation Size

We first focus on the *maximal generations network* where $\Psi_k = \{1, ..., K\}$ for all k, so agents in generation t for $t \geq 2$ have all agents in generation $t-1$ as their neighbors.[4] The next result relates the generation size K to the speed of signal aggregation.

[3] Stolarczyk, Bhardwaj, Bassler, Ma, and Josić (2017) study a related model where only the first generation observes private signals. Their main results characterize when no information gets lost between generations, i.e., social learning is completely efficient.

[4] This network is similar to the "multi-file" treatment in the laboratory experiment of Eyster, Rabin, and Weizsacker (2018), except agents only observe the actions of the immediate past generation, not those of all previous generations. In the multi-file treatment, unlike in the maximal generations network, Bayesian agents can perfectly infer the private signals of all previous movers in equilibrium.

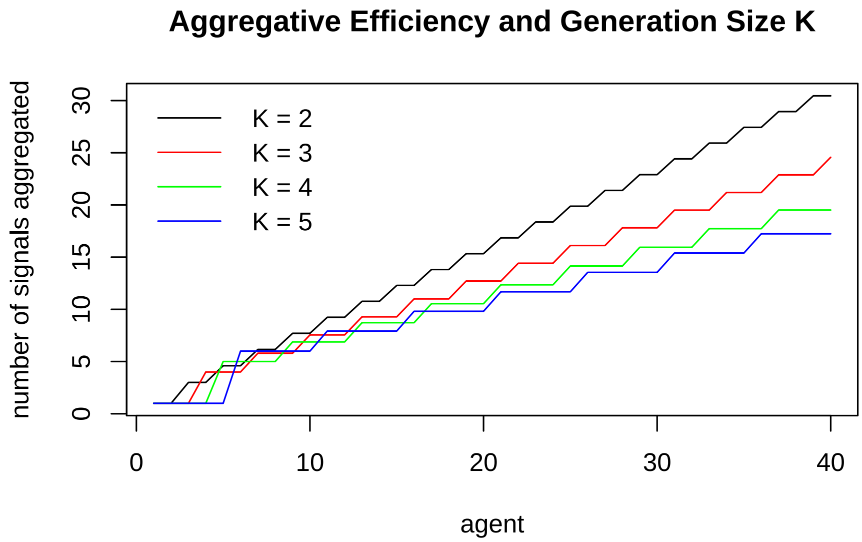

Figure 2: Number of signals aggregated by social learning in maximal generations networks with different numbers of agents per generation, $K \in \{2, 3, 4, 5\}$.

Proposition 4. *In the maximal generations network with any $K \geq 1$, society learns completely in the long run. We have $\lim_{i\to\infty}(r_i/i) = \frac{(2K-1)}{K^2}$, so aggregative efficiency is lower with larger K, and social learning aggregates no more than two signals per generation asymptotically for any K. For any K and any agents i, i' in generation t and $t-1$ with $t \geq 3$, $r_i - r_{i'} \leq 3$.*

Proposition 4 contains two parts. First, it shows that even though society learns completely with any K, the aggregative efficiency is lower with higher K. Indeed, if $K = 1$, then every agent perfectly incorporates all past private signals and the speed of social learning is the highest possible. Not only does this result about the aggregative efficiency imply an asymptotic ranking on the speed of learning, but the same comparative statics about speed also hold numerically for all agents $i \geq 16$ when comparing among $K \in \{2, 3, 4, 5\}$, as shown in Figure 2.

Second, Proposition 4 bounds the number of signals that social learning aggregates per generation in the maximal generations network. The proof of Proposition 3 shows $r_i \geq \overline{PL}(i) + 1$ in all networks and thus provides a lower bound of 1: each agent i in generation t has $\overline{PL}(i) = t - 1$. Proposition 4 shows this lower bound is not too far from the actual learning rate. No matter how large K is, social learning aggregates fewer than two signals per generation asymptotically. There is also a short-run version of this result: starting with generation 3, fewer than three signals are aggregated per generation for any K. For K large, these bounds of two or three signals per generation constitute an arbitrarily small fraction of available signals.

4.2 Partial Observations and Aggregative Efficiency

Proposition 4 shows that social learning aggregates fewer than two signals per generation asymptotically on maximal generations networks with any K. We now provide an exact expression for the aggregative efficiency in a broad class of generations networks with more general observation sets. In particular, this result will imply the same two signals per generation bound holds for all networks in this larger class.

We only impose one regularity assumption on the observation sets $(\Psi_k)_k$: symmetry.

Definition 3. The observation sets are *symmetric* if all agents observe $d \geq 1$ neighbors and all pairs of agents in the same generation share c common neighbors, i.e. $|N(i)| = d$ for all $i > K$ and $|N(i_1) \cap N(i_2)| = c$ whenever $i_1 = (t-1)K + k_1$ and $i_2 = (t-1)K + k_2$ for some $t \geq 2$ and $1 \leq k_1 < k_2 \leq K$ distinct.

To give a class of examples symmetric networks, fix any non-empty subset $E \subseteq \{1, ..., K\}$, and let $(\Psi_k)_k$ be such that for all $1 \leq k \leq K$, $\Psi_k = E$. To interpret, E represents the prominent positions in the society, and agents only observe predecessors in these prominent positions from the past generation. The maximal generations network represents the special case of $E = \{1, ..., K\}$. For another example, suppose $K \geq 2$ and each agent observes a different subset of $K - 1$ predecessors from the previous generation. Specifically, $\Psi_k = \{1, ..., K\} \backslash \{k-1\}$ for $2 \leq k \leq K$, and $\Psi_1 = \{1, ..., K-1\}$. This network is symmetric with $d = K - 1$ and $c = K - 2$. (The network in Figure 1 has this structure, with $d = 2$ and $c = 1$.) More generally, for every $c \geq 1$ and $d = mc + 1$ where m is a positive integer, there exists a symmetric network with parameters d, c and $K = (d^2 - 1)/c$ (Jørgensen, 2001).

Theorem 1. *Suppose the observation sets $(\Psi_k)_k$ are symmetric, with every agent observing d neighbors and every pair of agents in the same generation sharing c common neighbors. Then*[5]

$$\lim_{i \to \infty} (r_i / i) = \left(1 + \frac{d^2 - d}{d^2 - d + c}\right) \frac{1}{K}.$$

Theorem 1 gives the exact aggregative efficiency for a broader class of generations networks and quantifies the information loss due to confounding. Provided $c \geq 1$, the number of signals aggregated per generation is strictly increasing in d and strictly decreasing in c, with the interpretation that more observations speed up the rate of learning per generation

[5] With the convention $0/0 = 0$.

but more confounding slows it down, all else equal. Theorem 1 specializes to the expression for aggregative efficiency in Proposition 4 by letting $d = c = K$.

In fact, the maximal generations network leads to the slowest per-generation rate of learning among all symmetric $(\Psi_k)_k$ where each agent observes K neighbors. Theorem 1 also implies that the aggregative efficiency of the network where agents observe all predecessors from the past generation is identical to that of the network where each agent observes all members of the previous generation except the agent in the position immediately before theirs (i.e., society learns as quickly in the maximal generations network with $K = 3$ as in the network from Figure 1). The extra social observations in the first network exactly cancel out the reduced informational content of each observation, due to the more severe informational confounds in equilibrium.

Even though the maximal generations network leads to the worst aggregative efficiency conditional on the number of observations, Theorem 1 nevertheless provides a uniform learning-rate bound of two signals per generation across all symmetric generations networks, as $\frac{d^2-d}{d^2-d+c} \leq 1$. To provide some intuition for this bound, imagine that instead of observing their predecessors, all agents in generation t observe a common set of n independent signals, in addition to their own private signal. We can show an agent in generation $t+1$ who observes d of these generation t predecessors puts a weight of $\frac{n+1}{dn+1}$ on each of their log-actions, and aggregates $\frac{n(2d-1)+1}{nd+1}$ more signals than they do. As $n \to \infty$, the number of extra signals aggregated approaches $\frac{2d-1}{d} \leq 2$. In any generations network for late enough t, each generation t agent's social observation constitutes a highly informative signal of the state (i.e., $n \to \infty$), but different agents can have different observations. This somewhat alleviates the informational confounding for generation $t+1$, but is limited by the fact that all agents' actions are approaching perfect correlation when $t \to \infty$. Even agents with very different neighborhoods end up observing highly correlated information in the long run, so no network in the class we consider aggregates more than two signals per generation asymptotically.

The uniformly slow speed of signal aggregation is an inefficiency generated by decentralized social learning, not an inherent limitation of the generations structure. To illustrate this point, we show there exist feasible (but non-equilibrium) log-linear strategies so that agents are asymptotically more accurate than aggregating K_0 signals per generation for every $K_0 < K$.

We consider a slightly more restricted class of networks.

Definition 4. The observation sets $(\Psi_k)_k$ are *strongly connected* if for every $1 \leq k_1 \leq k_2 \leq K$, there exist t_1, t_2 so that $t_1 K + k_1$ is connected to $t_2 K + k_2$ in M.

This rules out the cases such as when the second agent in every generation is always excluded from the indirect neighborhood of the first agent of every future generation, which would mean agents in the first position cannot aggregate more than $K-1$ signals per generation.

We introduce a new measure of accuracy. Agent n's action is *more accurate than r signals* if $\mathbb{P}[a_n > 0.5 \mid \omega = 1] > \mathbb{P}[A_r > 0.5 \mid \omega = 1]$ and $\mathbb{P}[a_n < 0.5 \mid \omega = 0] > \mathbb{P}[A_r < 0.5 \mid \omega = 0]$, where the log transform of A_r has conditional distributions $\tilde{A}_r \sim \mathcal{N}(\pm r \cdot \frac{2}{\sigma^2}, r \cdot \frac{4}{\sigma^2})$ in the two states. That is, n's action is more likely to lean towards the correct state than the action of someone who observes r independent signals. While this definition applies even for non-equilibrium strategies that do not lead to $\tilde{a}_n$ having the conditional distributions $\mathcal{N}(\pm r_n \cdot \frac{2}{\sigma^2}, r_n \cdot \frac{4}{\sigma^2})$, if some such r_n existed then the definition would be equivalent to $r_n > r$.

Proposition 5. *Suppose the observation sets $(\Psi_k)_k$ are strongly connected and symmetric with $c \geq 1$. There is a log-linear strategy profile such that, for every positive real number $K_0 < K$, there exists a corresponding T so that for all $t \geq T$ and $1 \leq k \leq K$, the action of agent $(t-1)K + k$ is more accurate than $(t-1)K_0$ signals.*

As K grows large, Theorem 1 and Proposition 5 combine to say that in strongly connected and symmetric generations networks with $c \geq 1$, individuals only manage to aggregate an arbitrarily small fraction of the private signals that can be feasibly aggregated by a social planner using a log-linear strategy. The idea behind the construction is that the social planner can counteract the muddling of private signals when a group of individuals share common social observations by asking each agent to put extra weight on her own private signal in choosing her action.[6]

4.3 Application: Value of Mentorship

We provide an economic application of our results in terms of the value of mentors who share their private signals with mentees in the next generation.

Many organizations with cohort structures, such as universities and firms, have mentorship programs that pair newcomers with members of a previous cohort. Our results suggest that one benefit of such programs is that mentors provide information that helps newcomers interpret others' actions, thus increasing the speed of learning within the organization.

Formally, we model a mentor as sharing her private signal with a mentee in the subsequent generation. Equivalently, the mentor could share a sufficient statistic describing her best estimate of the state based on her social observations. If we begin with the maximal generations network and add mentorship relationships in this way, learning is nearly efficient.

[6] If non-log-linear strategies are allowed, then the social planner can achieve close to perfect information aggregation in every generation using exotic strategies that encode individuals' signals far into the decimal expansions of their actions, for example.

Corollary 2. *Suppose each agent observes the actions of all members of the previous generation and the private signal of any member of the previous generation. Then $r_i > i - K$ for all i, and therefore the aggregative efficiency is 1.*

If an agent observes the actions of the previous generation along with one of their private signals, she can calculate the common confounding information and fully compensate for this confound. In networks with large K, showing each agent just one extra signal (of someone from the previous generation) increases aggregative efficiency from nearly 0 to 1.

In the context of the application, incumbents in the organization act based on private information and shared organizational knowledge. A newcomer ignorant of the organization knowledge cannot fully separate these two forces that shape others' behavior. But by describing her perspective, a mentor can help a newcomer interpret everyone else's behavior, removing the informational confound and extracting the private information that underlie these predecessors' actions. A related force is described in management literature:

> "Mentors can be powerful socializing agents as an individual adjusts to a new job or organization. As protégés learn about their roles within the organization, mentors can help them correctly interpret their experiences within the organization's expectations and culture." – Chao (2007)

Our result formalizes this intuition in a social-learning environment. Our stylized model of mentorship abstracts away from many of its other benefits (e.g., the expertise of the mentor in terms of being able to generate more precise signals than the mentee), and shows how the "interpretive" value of mentorship improves learning within the organization.

If each mentor instead generates a new, independent private signal for their mentee, rather than sharing the realization of their own private signal from the past, then social learning does not speed up very much. Compared to a world without mentoring, this intervention would at most double the number of signals aggregated by each agent. In organizations with large cohorts, mentors who share their personal experience increase the rate of social learning much more than mentors who generate new signals. This shows that Corollary 2 relies critically on the "interpretative" channel of mentoring: the key is not so much that the mentor provides an extra signal about the state of the world, but that this signal clarifies other people's behavior and allows the mentee to extract more information from said behavior.

5 Aggregative Efficiency and Welfare Comparisons Across Networks

Let $v_i := \mathbb{E}[u_i(a_i^*, \omega)]$ denote the expected equilibrium welfare of agent i, and recall that $-0.25 < v_i < 0$ for every i on any network and with any private signal precision $0 < 1/\sigma^2 < \infty$. If society learns completely on a network, then $\lim_{i\to\infty} v_i = 0$. Given a threshold level $\underline{v} \in (-0.25, 0)$ of utility, we might ask when does social learning first attain $v_i \geq \underline{v}$. We say social learning *strongly attains* $\underline{v}$ by agent I if I is the smallest integer such that $v_i \geq \underline{v}$ for all $i \geq I$. We say social learning *weakly attains* $\underline{v}$ by agent i if i is the earliest agent with $v_i \geq \underline{v}$ (but the expected utilities of some later agents may fall below $\underline{v}$).

The next result shows that when signals are not too precise, a network with a higher aggregative efficiency strongly attains any such utility threshold strictly earlier than a network with lower aggregative efficiency weakly attains the same threshold.

Proposition 6. *Suppose in networks M and M', social learning aggregates $(r_i)_{i\geq 1}$ and $(r_i')_{i\geq 1}$ signals by agent i, respectively, with $\lim_{i\to\infty}(r_i/i) > \lim_{i\to\infty}(r_i'/i) > 0$. For every utility threshold $\underline{v} \in (-0.25, 0)$, there exists a bound $\tau > 0$ on private signal precision so that whenever $0 < 1/\sigma^2 \leq \tau$, social learning strongly attains $\underline{v}$ by agent I in M and weakly attains $\underline{v}$ by agent i' in M', with $I < i'$.*

Now fix the signal precision and consider the expected welfare profiles $(v_i)_{i\geq 1}$ and $(v_i')_{i\geq 1}$ in two networks M and M' that both lead to complete social learning. A planner could compare these two profiles through a social welfare function Λ with $\Lambda(v) = \sum_{i=1}^{\infty} \lambda_i v_i + \lambda_\infty(\lim_{i\to\infty} v_i)$, where $\lambda_1, \lambda_2, ..., \lambda_\infty \geq 0$ is a summable sequence of welfare weights that combine utilities across agents. Here λ_∞ is the welfare weight on "the end of time," and comparing two networks based on whether they lead to complete social learning corresponds to an "infinitely patient" Λ_∞ with the weights $\lambda_i = 0$ for all $i \in \mathbb{N}_+$ and $\lambda_\infty = 1$. A social welfare function Λ_T is called *T-patient* if $\lambda_i = 0$ for all $i < T$ and $\lambda_i > 0$ for all finite $i \geq T$. That is, the planner is blind to the welfare of the first $T - 1$ agents, but strictly cares about the welfare of all later agents. One example is $\lambda_i = \delta^{i-T}$ for $i \geq T$ where the welfare of agents later than T are discounted at rate $\delta \in (0, 1)$. For large T, we can interpret a T-patient social welfare function as corresponding to a "very patient" but not "infinitely patient" planner. The next result implies that all very patient planners will rank M and M' based on their aggregative efficiency, even though the degenerate limiting case of the infinitely patient planner is indifferent between them.

Proposition 7. *Suppose in networks M and M', social learning aggregates $(r_i)_{i\geq 1}$ and $(r_i')_{i\geq 1}$ signals by agent i, respectively, with $\lim_{i\to\infty}(r_i/i) > \lim_{i\to\infty}(r_i'/i)$. There exists a $\underline{T} \in \mathbb{N}_+$*

such that for all $T \geq \underline{T}$, any T-patient social welfare function Λ_T is strictly higher on M than on M'.

6 Conclusion

This paper presents a tractable model of sequential social learning that lets us study how quickly rational agents learn on different observation networks. Generally, observation networks with intransitivities confound the informational content of neighbors' behavior and slow down learning. Rational agents face an optimal signal-extraction problem, whose solution takes a log-linear form in our environment. For a class of symmetric networks where agents move in generations, additional observations speed up learning but extra confounding slows it down. Confounding severely limits the rate of signal aggregation — on any network in this class, social learning aggregates no more than two signals per generation in the long run, even for arbitrarily large generations.

We derive an analytic expression of the aggregative efficiency in all such networks and quantify the information loss due to confounding. This allows us to make precise comparisons about the rate of learning and welfare across different networks, where additional links may trade off extra observations against the reduced informational content of each observation.

We have focused on how the network structure affects the speed of social learning and abstracted away from many other sources of learning-rate inefficiency. These other sources may realistically co-exist with the informational-confounding issues discussed here and complicate the analysis. For instance, even though the complete network allows agents to exactly infer every predecessor's private signal, it could lead to worse informational free-riding incentives in settings where agents must pay for the precision of their private signals (e.g., Ali (2018)), compared to networks where agents have fewer observations. Studying the trade-offs and/or interactions between network-based information confounds and other obstructions to fast learning could lead to fruitful future work.

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Appendix

A Proofs

A.1 Proof of Lemma 1

Proof. We show that $\tilde{s}_i = \frac{2}{\sigma^2} s_i$. This is because

$$\tilde{s}_i = \ln\left(\frac{\mathbb{P}[\omega = 1|s_i]}{\mathbb{P}[\omega = 0|s_i]}\right) = \ln\left(\frac{\mathbb{P}[s_i|\omega = 1]}{\mathbb{P}[s_i|\omega = 0]}\right) = \ln\left(\frac{\exp\left(\frac{-(s_i-1)^2}{2\sigma^2}\right)}{\exp\left(\frac{-(s_i+1)^2}{2\sigma^2}\right)}\right)$$
$$= \frac{-(s_i^2 - 2s_i + 1) + (s_i^2 + 2s_i + 1)}{2\sigma^2} = \frac{2}{\sigma^2} s_i.$$

The result then follows from scaling the conditional distributions of s_i, $(s_i \mid \omega = 1) \sim \mathcal{N}(1, \sigma^2)$ and $(s_i \mid \omega = 0) \sim \mathcal{N}(-1, \sigma^2)$. □

A.2 Proof of Proposition 1

Proof. Agent 1 does not observe any predecessors, so clearly $\tilde{A}_1^*(\tilde{s}_1) = \tilde{s}_1$. Suppose by way of induction that the equilibrium strategies of all agents $j \leq I - 1$ are linear. Then each $\tilde{a}_j$ for $j \leq I - 1$ is a linear combination of $(\tilde{s}_\ell)_{\ell=1}^I$, which by Lemma 1 are conditionally Gaussian with conditional means $\pm 2/\sigma^2$ in states $\omega = 1$ and $\omega = 0$ and conditional variance $4/\sigma^2$ in each state. This implies $(\tilde{a}_{j(1)}, ..., \tilde{a}_{j(n_I)})$ have a conditional joint Gaussian distribution with $(\tilde{a}_{j(1)}, ..., \tilde{a}_{j(n_I)}) \sim \mathcal{N}(\vec{\mu}, \Sigma)$ conditional on $\omega = 1$, and t $(\tilde{a}_{j(1)}, ..., \tilde{a}_{j(n_I)}) \sim \mathcal{N}(-\vec{\mu}, \Sigma)$ conditional on $\omega = 0$, where $\vec{\mu} = \mathbb{E}[(\tilde{a}_{j(1)}, ..., \tilde{a}_{j(d_i)})' \mid \omega = 1]$ and $\Sigma = \mathrm{Cov}[\tilde{a}_{j(1)}, ..., \tilde{a}_{j(d_i)} \mid \omega = 1]$.

From the the multivariate Gaussian density, (writing $(\tilde{a}_{j(1)}, ..., \tilde{a}_{j(n_I)})' = \vec{a}$),

$$\ln\left(\frac{\mathbb{P}[\tilde{a}_{j(1)}, ..., \tilde{a}_{j(n_I)} \mid \omega = 1]}{\mathbb{P}[\tilde{a}_{j(1)}, ..., \tilde{a}_{j(n_I)} \mid \omega = 0]}\right) = \ln\left(\frac{\exp(-\frac{1}{2}(\vec{a} - \vec{\mu})'\Sigma^{-1}(\vec{a} - \vec{\mu}))}{\exp(-\frac{1}{2}(\vec{a} + \vec{\mu})'\Sigma^{-1}(\vec{a} + \vec{\mu}))}\right)$$
$$= \vec{a}'\Sigma^{-1}\vec{\mu} + \vec{\mu}'\Sigma^{-1}\vec{a}$$

which is $2(\vec{\mu}'\Sigma^{-1}) \cdot (\tilde{a}_{j(1)}, ..., \tilde{a}_{j(n_I)})'$ because Σ is symmetric. This then shows agent I's equilibrium strategy must also be linear, completing the inductive step. This argument also gives the explicit form of $\vec{\beta}_{I,\cdot}$.

For the final statement, we first prove a lemma.

Lemma A.1. *Let $\hat{W}$ be the submatrix of W with rows $N(i)$ and columns $\{1, ..., i-1\}$. Then $\vec{\beta}_i = \vec{\mathbf{1}}'_{(i-1)} \times \hat{W}'(\hat{W}\hat{W}')^{-1}$ and the i-th row of W is $W_i = \left((\vec{\beta}'_{i,\cdot} \times \hat{W}), 1, 0, 0, ...\right)$.*

Proof. Suppose $N(i) = \{j(1), ..., j(d_i)\}$ with $j(1) < ... < j(d_i)$. By Lemma 1 and construction of $\hat{W}$, we have $\mathbb{E}[\tilde{a}_{j(k)} \mid \omega = 1] = \frac{2}{\sigma^2}\sum_{\ell=1}^{i-1}\hat{W}_{k,\ell}$. So, $\mathbb{E}[(\tilde{a}_{j(1)}, ..., \tilde{a}_{j(d_i)}) \mid \omega = 1] = \frac{2}{\sigma^2}(\hat{W} \cdot \mathbf{1}_{(i-1)})' = \frac{2}{\sigma^2}\mathbf{1}'_{(i-1)}\hat{W}'$. Also, again by Lemma 1 and construction of $\hat{W}$, we can calculate that for $1 \leq k_1 \leq k_2 \leq d_i$, $\text{Cov}[\tilde{a}_{j(k_1)}, \tilde{a}_{j(k_2)} \mid \omega = 1] = \frac{4}{\sigma^2}\sum_{\ell=1}^{i-1}(\hat{W}_{k_1,\ell}\hat{W}_{k_2,\ell})$, meaning $\text{Cov}[\tilde{a}_{j(1)}, ..., \tilde{a}_{j(d_i)} \mid \omega = 1] = \frac{4}{\sigma^2}\hat{W}\hat{W}'$. It then follows from what we have shown above that $\vec{\beta}_{i,\cdot} = 2 \cdot \frac{2}{\sigma^2}\mathbf{1}'_{(i-1)}\hat{W}' \times \left[\frac{4}{\sigma^2}\hat{W}\hat{W}'\right]^{-1} = \vec{\mathbf{1}}'_{(i-1)} \times \hat{W}'(\hat{W}\hat{W}')^{-1}$.

Since i puts weight 1 on $\tilde{s}_i$ and weights $\vec{\beta}_{i,\cdot}$ on $(\tilde{a}_{j(1)}, ..., \tilde{a}_{j(d_i)})' = \hat{W} \times (\tilde{s}_1, ..., \tilde{s}_{i-1})'$, this shows the first $i-1$ elements in the row W_i must be $\vec{\beta}'_{i,\cdot} \cdot \hat{W}$ while the i-th element is 1. □

To prove the final statement of Proposition 1, $W_1 = (1, 0, 0, ...)$ does not depend on σ^2. The same applies to $\vec{\beta}_{1,\cdot}$. By way of induction, suppose rows W_i and vectors $\vec{\beta}_{i,\cdot}$ do not depend on σ^2 for any $i \leq I$. If $\hat{W}$ is the submatrix of W with rows $N(I+1)$, then since $N(I+1) \subseteq \{1, ..., I\}$, by the inductive hypothesis $\hat{W}$ must be independent of σ^2. Thus the same independence also applies to $\vec{\beta}_{I+1,\cdot}$ since this vector only depends on $\hat{W}$ by the result just derived. In turn, since W_{I+1} is only a function of $\vec{\beta}'_{I+1,\cdot}$ and $\hat{W}$, and these terms are independent of σ^2 as argued before, same goes for W_{I+1}, completing the inductive step. □

A.3 Proof of Proposition 2

Proof. It suffices to show that $\mathbb{E}[\tilde{a}_i \mid \omega = 1] = \frac{1}{2}\text{Var}\left[\tilde{a}_i \mid \omega = 1\right]$. By Proposition 1, $\tilde{a}_i = \tilde{s}_i + \sum_{k=1}^{d_i}\beta_{i,j(k)}\tilde{a}_{j(k)}$. From Lemma 1, we have $\mathbb{E}[\tilde{s}_i \mid \omega = 1] = \frac{1}{2}\text{Var}\left[\tilde{s}_i \mid \omega = 1\right]$. Furthermore, $\tilde{s}_i$ is independent from $\sum_{k=1}^{d_i}\beta_{i,j(k)}\tilde{a}_{j(k)}$, as the latter term only depends on $\tilde{s}_1, ..., \tilde{s}_{i-1}$. So we need only show $\mathbb{E}[\sum_{k=1}^{d_i}\beta_{i,j(k)}\tilde{a}_{j(k)} \mid \omega = 1] = \frac{1}{2}\text{Var}\left[\sum_{k=1}^{d_i}\beta_{i,j(k)}\tilde{a}_{j(k)} \mid \omega = 1\right]$

Let $\vec{\mu} = \mathbb{E}[(\tilde{a}_{j(1)}, ..., \tilde{a}_{j(d_i)})' \mid \omega = 1]$ and $\Sigma = \text{Cov}[\tilde{a}_{j(1)}, ..., \tilde{a}_{j(d_i)} \mid \omega = 1]$. Using the expression for $\vec{\beta}_{i,\cdot}$ from Proposition 1, $\mathbb{E}\left[\sum_{k=1}^{d_i}\beta_{i,j(k)}\tilde{a}_{j(k)} \mid \omega = 1\right] = 2\left(\vec{\mu}'\Sigma^{-1}\right) \cdot \vec{\mu}$. Also,

$$\begin{aligned}\text{Var}\left[\sum_{k=1}^{d_i}\beta_{i,j(k)}\tilde{a}_{j(k)} \mid \omega = 1\right] &= \left(2\vec{\mu}'\Sigma^{-1}\right)\Sigma\left(2\vec{\mu}'\Sigma^{-1}\right)' \\ &= 4\vec{\mu}'\Sigma^{-1}\vec{\mu}\end{aligned}$$

using the fact that Σ is a symmetric matrix. This is twice $\mathbb{E}\left[\sum_{k=1}^{d_i}\beta_{i,j(k)}\tilde{a}_{j(k)} \mid \omega = 1\right]$ as desired. □

A.4 Proof of Corollary 1

Proof. When $i < I$ use log-linear strategies, each $\tilde{a}_i$ is some linear combination of $(\tilde{s}_\ell)_{\ell \le I-1}$. Thus, $(\tilde{a}_j)_{j\in N(I)}$ are conditionally jointly Gaussian, $(\tilde{a}_j)_{j\in N(I)} \mid \omega \sim \mathcal{N}(\pm\vec{\mu}, \Sigma)$. This is sufficient for the the proofs of Propositions 1 and 2 to go through, implying that the $\tilde{a}_I$ maximizing I's expected utility using the information in $(\tilde{a}_j)_{j\in N(I)}$ is a log-linear strategy and has a signal-counting interpretation. □

A.5 Proof of Proposition 3

We first state and prove an auxiliary lemma.

Lemma A.2. *For any* $0 < \epsilon < 0.5$,

$$\mathbb{P}[a_i > 1-\epsilon \mid \omega = 1] = 1 - \Phi\left(\frac{\ln\left(\frac{1-\epsilon}{\epsilon}\right) - r_i\frac{2}{\sigma^2}}{\sqrt{r_i\frac{2}{\sigma}}}\right),$$

where Φ *is the standard Gaussian distribution function. This expression is increasing in* r_i *and approaches 1. Also,*

$$\mathbb{P}[a_i < \epsilon \mid \omega = 0] = \Phi\left(\frac{\ln\left(\frac{1-\epsilon}{\epsilon}\right) + r_i\frac{2}{\sigma^2}}{\sqrt{r_i\frac{2}{\sigma}}}\right).$$

This expression is increasing in r_i *and approaches 1.*

Proof. Note that $a_i > 1-\epsilon$ if and only if $\tilde{a}_i > \ln\left(\frac{1-\epsilon}{\epsilon}\right) > 0$. Given that $(\tilde{a}_i \mid \omega = 1) \sim \mathcal{N}\left(r_i \cdot \frac{2}{\sigma^2}, r_i \cdot \frac{4}{\sigma^2}\right)$ by Proposition 2, the expression for $\mathbb{P}[a_i > 1-\epsilon \mid \omega = 1]$ follows. To see that it is increasing in r_i, observe that $\frac{d}{dr_i}\frac{\ln\left(\frac{1-\epsilon}{\epsilon}\right) - r_i\frac{2}{\sigma^2}}{\sqrt{r_i\frac{2}{\sigma}}}$ has the same sign as

$$\frac{-2}{\sigma^2}(\sqrt{r_i}\frac{2}{\sigma^2}) - (\ln\left(\frac{1-\epsilon}{\epsilon}\right) - r_i\frac{2}{\sigma^2})(\frac{1}{2}r_i^{-0.5}\frac{2}{\sigma}) = -\frac{2}{\sigma^3}\sqrt{r_i} - \ln\left(\frac{1-\epsilon}{\epsilon}\right)r_i^{-0.5}\frac{1}{\sigma} < 0.$$

Also, it is clear that $\lim_{r_i\to\infty}\frac{\ln\left(\frac{1-\epsilon}{\epsilon}\right) - r_i\frac{2}{\sigma^2}}{\sqrt{r_i\frac{2}{\sigma}}} = -\infty$, hence $\lim_{r_i\to\infty}\mathbb{P}[a_i > 1-\epsilon \mid \omega = 1] = 1$. The results for $\mathbb{P}[a_i < \epsilon \mid \omega = 0]$ follow from analogous arguments. □

We now turn to the proof of Proposition 3.

Proof. By Proposition 2, there exist $(r_i)_{i\ge 1}$ so that social learning aggregates r_i signals by agent i. We first show that society learns completely in the long run if and only if $\lim_{i\to\infty} r_i = \infty$. Let $\epsilon' > 0$ be given and suppose $\lim_{i\to\infty} r_i = \infty$. Putting $\epsilon = \min(\epsilon', 0.4)$, we get that $\mathbb{P}[|a_i - \omega| < \epsilon \mid \omega = 1] \to 1$ and $\mathbb{P}[|a_i - \omega| < \epsilon \mid \omega = 0] \to 1$ since the two expressions in Lemma A.2 increase in r_i and approach 1, hence also $\mathbb{P}[|a_i - \omega| < \epsilon^{'}] \to 1$. So society learns completely in the long run. Conversely, if $r_i < K < \infty$ for infinitely many i, then by Lemma A.2 we will get that $\mathbb{P}[|a_i - \omega| < 0.1 \mid \omega = 1]$ are bounded by $1 - \Phi\left(\frac{\ln(9) - K\frac{2}{\sigma^2}}{\sqrt{K}\frac{2}{\sigma}}\right)$ for these i, hence society does not learn completely in the long run.

Next, we show that Conditions (1) and (2) in the proposition are both necessary and sufficient conditions for $\lim_{i\to\infty} r_i = \infty$.

Condition (1): $\lim_{i\to\infty} \overline{PL}(i) = \infty$.

Necessity: Suppose $\lim_{i\to\infty} r_i = \infty$. For $\ell \in \mathbb{N}$, let $I(\ell) := \{i : \overline{PL}(i) = \ell\}$. We show by induction that $I(\ell)$ is finite for all $\ell \in \mathbb{N}$. For every $i \in I(0)$, $r_i = 1$, so $\lim_{i\to\infty} r_i = \infty$ implies $|I(0)| < \infty$. Now suppose $|I(\ell)| < \infty$ for all $\ell \leq L$. If $i \in I(L+1)$, then every j that can be reached along M from i must belong to $I(\ell)$ for some $\ell \leq L$. The subnetwork containing i is therefore a subset of $\cup_{\ell=0}^{L} I(\ell)$, a finite set by the inductive hypothesis. Thus $r_i \leq 1 + \sum_{\ell=0}^{L} |I(\ell)|$ for all $i \in I(L+1)$. So $\lim_{i\to\infty} r_i = \infty$ implies $I(L+1)$ is finite, completing the inductive step and proving $I(\ell)$ is finite for all ℓ. Hence $\lim_{i\to\infty} \overline{PL}(i) = \infty$.

Sufficiency: First note if $j \in N(i)$, then $r_i \geq r_j + 1$. This is because in equilibrium, $\tilde{a}_j \sim \mathcal{N}\left(\pm r_j \cdot \frac{2}{\sigma^2}, r_j \cdot \frac{4}{\sigma^2}\right)$ conditional on the two states, and furthermore $\tilde{a}_j$ is conditionally independent of s_i. So, $\tilde{a}_j + \tilde{s}_i$ is a possibly play for i, which would have the conditional distributions $\mathcal{N}\left(\pm(r_j + 1) \cdot \frac{2}{\sigma^2}, (r_j + 1) \cdot \frac{4}{\sigma^2}\right)$ in the two states. If $r_i < r_j + 1$, then i would have a profitable deviation by choosing $\tilde{a}_i = \tilde{a}_j + \tilde{s}_i$ instead, since it follows from Lemma A.2 that a log-action that aggregates more signals leads to higher expected payoffs.

Condition (2): $\lim_{i\to\infty} \left[\max_{j\in N(i)} j\right] = \infty$.

Necessity: If Condition (2) is violated, there exists some $\bar{j} < \infty$ so that there exist infinitely many i's with $N(i) \subseteq \{1, ..., \bar{j}\}$. The subnetwork containing any such i is a subset of $\{1, ..., \bar{j}\}$, so $r_i \leq \bar{j} + 1$. We cannot have $\lim_{i\to\infty} r_i = \infty$.

Sufficiency: Construct an increasing sequence $C_1 \leq C_2 \leq ...$ as follows. Condition (2) implies there exists C_1 so that $\max_{j\in N(i)} j \geq 1$ for all $i \geq C_1$. So, $\overline{PL}(i) \geq 1$ for all $i \geq C_1$. Suppose $C_1 \leq ... \leq C_n$ are constructed with the property that $\overline{PL}(i) \geq k$ for all $i \geq C_k$, $k = 1, ..., n$. Condition (2) implies there exists C_{n+1} so that $\max_{j\in N(i)} j \geq C_n$ for all $i \geq C_{n+1}$. But since all $j \geq C_n$ have $\overline{PL}(j) \geq n$ by the inductive hypothesis, all $i \geq C_{n+1}$ must have $\overline{PL}(i) \geq n + 1$, completing the inductive step. This shows $\lim_{i\to\infty} \overline{PL}(i) = \infty$. By the sufficiency of Condition (1) for $\lim_{i\to\infty} r_i = \infty$, we see that Condition (2) implies the same. □

A.6 Proof of Theorem 1

Proof. If $d = 1$, then exactly one signal is aggregated per generation so $r_i/K \to 1$ as required. Also, if $c = 0$, then we must have $d = 1$. From now on we assume $d \geq 2$ and $c \geq 1$.

Lemma A.3. *For $d \geq 2$, each generation t and each $i \neq i'$ in generation t, $\text{Var}\,[\tilde{a}_i \mid \omega = 1]$ and $\text{Cov}\,[\tilde{a}_i, \tilde{a}_{i'} \mid \omega = 1]$ depend only on t and not on the identities of i or i', which we call Var_t and Cov_t, respectively. Similarly, for i in generation t and each $j \in N(i)$, the weight $\beta_{i,j}$ depends only on t, which we call β_t.*

Proof. The results hold by inductively applying the symmetry condition. Clearly they are true for $t = 2$. Suppose they are true for all $t \leq T$. For an agent i in generation $t = T + 1$, the inductive hypothesis implies $\text{Var}[\tilde{a}_j \mid \omega = 1]$ is the same for all $j \in N(i)$,$\mathbb{E}[\tilde{a}_j \mid \omega = 1]$ is the same for all $j \in N(i)$ (by using Proposition 2, and all pairs $j, j' \in N(i)$ with $j \neq j'$ have the same conditional covariance. Thus by Proposition 1, i places the same weight, say β_t, on all neighbors. □

So we have

$$\text{Var}[\tilde{a}_i \mid \omega = 1] = \frac{4}{\sigma^2} + \beta_t^2(d\text{Var}_{t-1} + (d^2 - d)\text{Cov}_{t-1})$$

for all i in generation t, and

$$\text{Cov}[\tilde{a}_i, \tilde{a}_{i'} \mid \omega = 1] = \beta_t^2(c\text{Var}_{t-1} + (d^2 - c)\text{Cov}_{t-1})$$

for all agents $i \neq i'$ in generation t. This shows the claims for $t = T + 1$.

Taking the difference of the two expressions for Var_t and Cov_t gives:

$$\text{Var}_t - \text{Cov}_t = \frac{4}{\sigma^2} + \beta_t^2(d - c)(\text{Var}_{t-1} - \text{Cov}_{t-1}). \tag{1}$$

We now require two auxiliary lemmas.

Lemma A.4. *Consider the Markov chain on $\{1, ..., K\}$ with state transition matrix p, with $p_{i,j} = \mathbb{P}[i \to j] = 1/d$ if $j \in \Psi_i$, 0 otherwise. Suppose $(\Psi_k)_k$ is symmetric with $c \geq 1$. Then $p_i^\infty := \lim_{t\to\infty}(p^t)_i \in [0, 1]^K$ exists, and it is the same for all $1 \leq i \leq K$.*

Proof. For existence of p_i^∞, consider the decomposition of the Markov chain into its communication classes, $C_1, ..., C_L \subseteq \{1, ..., K\}$. Without loss suppose the first L' communication classes are closed and the rest are not.

We show that each closed communication class is aperiodic when $(\Psi_k)_k$ is symmetric and $c, d \geq 1$. Let $i \in C_\ell$ for $1 \leq \ell \leq L'$. Let $\Psi_i = \{j_1, ..., j_d\}$. If $i \in \Psi_i$, then i's periodicity is 1. Otherwise, $\Psi_i \subseteq C_\ell$ since C_ℓ is closed, so for every $1 \leq h \leq d$ there exists a cycle of some length Q_h starting at i, where the h-th such cycle is $i \to j_h \to ... \to i$. Since $c \geq 1$, i and j_1 share a common neighbor, which must be j_{h^*} for some $1 \leq h^* \leq d$. We can therefore construct a cycle of length $Q_{h^*}+1$ starting at i, $i \to j_1 \to j_{h^*} \to ... \to i$. Since cycle lengths Q_{h^*} and $Q_{h^*}+1$ are coprime, i's periodicity is 1.

By standard results (see e.g., Billingsley (2013)) there exist $\nu_\ell^*, 1 \leq \ell \leq L^{'}$, so that $\lim_{t\to\infty}(p^t)_i = \nu_\ell^*$ whenever $i \in C_\ell$. If $i \notin \cup_{1\leq\ell\leq L^{'}} C_\ell$, then starting the process at i, almost surely the process enters one of the closed communication classes eventually. This shows $\lim_{t\to\infty}(p^t)_i$ exists and is equal to $\sum_{\ell=1}^{L^{'}} q_\ell \nu_\ell^*$, where q_ℓ is the probability that the process started at i enters C_ℓ before any other closed communication class.

To prove that p_i^∞ is the same for all i, we inductively show that for all $i \neq j$, $\| p_i^\infty - p_j^\infty \|_{\max} \leq \left(\frac{d-c}{d}\right)^t$ for all $t \geq 1$. Since $c \geq 1$, this would show that in fact $p_i^\infty = p_j^\infty$ for all i, j.

For the base case of $t = 1$, enumerate $\Psi_i = \{n_1, ..., n_c, n_{c+1}, ..., n_d\}$, $\Psi_j = \{n_1, ..., n_c, n^{'}_{c+1}, ..., n^{'}_d\}$ where all $n_1, ..., n_d, n^{'}_{c+1}, ..., n^{'}_d \in \{1, ..., K\}$ are distinct. Then

$$p_i^\infty = \frac{1}{d}\left(\sum_{k=1}^{c} p_{n_k}^\infty\right) + \frac{1}{d}\left(\sum_{k=c+1}^{d} p_{n_k}^\infty\right),$$

$$p_j^\infty = \frac{1}{d}\left(\sum_{k=1}^{c} p_{n_k}^\infty\right) + \frac{1}{d}\left(\sum_{k=c+1}^{d} p_{n_k^{'}}^\infty\right),$$

so

$$\begin{aligned}\| p_i^\infty - p_j^\infty \|_{\max} &\leq \frac{1}{d}\sum_{k=c+1}^{d} \| p_{n_k}^\infty - p_{n_k^{'}}^\infty \|_{\max} \\ &\leq \frac{d-c}{d} \cdot 1\end{aligned}$$

where the 1 comes from $\| x - y \|_{\max} \leq 1$ for any two distributions x, y.

The inductive step just replaces the bound $\| x - y \|_{\max} \leq 1$ with $\| p_{n_k}^\infty - p_{n_k^{'}}^\infty \|_{\max} \leq \left(\frac{d-c}{d}\right)^{t-1}$ from the inductive hypothesis. □

Lemma A.5. $\beta_t \to 1/d$.

Proof. By Proposition 2, we can compute that:

$$\beta_{t+1} = \frac{\text{VAR}_t}{\text{VAR}_t + (d-1)\text{COV}_t} \geq \frac{1}{d}.$$

It is therefore sufficient to show that $\text{VAR}_t/\text{COV}_t \to 1$. The weight $w_{i,i'}$ that an agent i in generation t places on the private signal of an agent $i^{'}$ in generation $t - \tau$ is equal to the product of $\prod_{j=1}^{\tau} \beta_{t+1-j}$ and the number of paths from i to i' in the network M.

We can compute the number of paths as follows. Consider a Markov chain with states $\{1, \ldots, K\}$ and state transition probabilities $\mathbb{P}[k_1 \to k_2] = 1/d$ if $k_2 \in \Psi_{k_1}$, $\mathbb{P}[k_1 \to k_2] = 0$. The number of paths from i in generation t to j in generation $t - \tau$ is equal to d^τ times the probability that the state is j after τ periods.

By Lemma A.4, there exists a stationary distribution $\pi^* \in \mathbb{R}_+^K$ with $\sum_{k=1}^K \pi_k^* = 1$ of the Markov chain. Given $\epsilon > 0$, we can choose τ_0 such that the number of paths from i in generation t to $j = (\tau - 1)K + k$ in generation τ is in $[d^\tau(\pi_k^* - \epsilon), d^\tau(\pi_k^* + \epsilon)]$ for all t and all $\tau \geq \tau_0$.

Fixing distinct agents i and i' in generation t:

$$\text{Var}_t = \frac{4}{\sigma^2} + \frac{4}{\sigma^2} \sum_{\tau=1}^{t-1} \sum_{k=1}^{K} w_{i,(t-\tau)K+k}^2 \text{ and } \text{Cov}_t = \frac{4}{\sigma^2} \sum_{\tau=1}^{t-1} \sum_{k=1}^{K} w_{i,(t-\tau)K+k} w_{i',(t-\tau)K+k}.$$

We want to show that

$$\text{Var}_t / \text{Cov}_t = \frac{1 + \sum_{\tau=1}^{t-1} \sum_{k=1}^K w_{i,(t-\tau)K+k}^2}{\sum_{\tau=1}^{t-1} \sum_{k=1}^K w_{i,(t-\tau)K+k} w_{i',(t-\tau)K+k}} \to 1.$$

Take $\epsilon > 0$ smaller than π_k^* for all k. For $\tau \geq \tau_0$, we have

$$w_{i,(t-\tau)K+k} w_{i',(t-\tau)K+k} \geq (d^\tau \prod_{j=1}^{\tau} \beta_{t+1-j})^2 (\pi_k^* - \epsilon)^2 \text{ and } w_{i,(t-\tau)K+k}^2 \leq (d^\tau \prod_{j=1}^{\tau} \beta_{t+1-j})^2 (\pi_k^* + \epsilon)^2$$

The covariance grows at least linearly in t since each $\beta \geq 1/d$, while the contribution from periods $t - \tau_0 + 1, \ldots, t$ is bounded and therefore lower order. Thus,

$$\limsup_{t\to\infty} \text{Var}_t / \text{Cov}_t \leq \limsup_{t\to\infty} \frac{\sum_{k=1}^K \sum_{\tau=\tau_0}^{t-1} (d^\tau \prod_{j=1}^{\tau} \beta_{t+1-j})^2 (\pi_k^* + \epsilon)^2}{\sum_{k=1}^K \sum_{\tau=\tau_0}^{t-1} (d^\tau \prod_{j=1}^{\tau} \beta_{t+1-j})^2 (\pi_k^* - \epsilon)^2} \leq \max_{1 \leq k \leq K} \frac{(\pi_k^* + \epsilon)^2}{(\pi_k^* - \epsilon)^2}.$$

Since ϵ is arbitrary, this completes the proof of the lemma. □

We return to the proof of Theorem 1. Fix small $\epsilon > 0$. By Lemma A.5, we can choose T such that $\beta_t \leq \frac{1+\epsilon}{d}$ for all $t \geq T$. Therefore, $\beta_t^2(d - c) \leq \frac{(1+\epsilon)^2}{d^2}(d - c)$ for $t \geq T$. Consider the contraction map $\varphi(x) = \frac{4}{\sigma^2} + \frac{(1+\epsilon)^2}{d^2}(d - c)x$. Iterating Equation (1) starting with $t = T$, we find that $\text{Var}_t - \text{Cov}_t \leq \varphi^{(t-T)}(\text{Var}_T - \text{Cov}_T)$, so this shows

$$\limsup_{t\to\infty} (\text{Var}_t - \text{Cov}_t) \leq \frac{4}{\sigma^2} \cdot \frac{d^2}{d^2 - (1+\epsilon)^2 d + (1+\epsilon)^2 c}$$

where the RHS is the fixed point of φ. Since this holds for all small $\epsilon > 0$,we get $\limsup_{t\to\infty}(\text{Var}_t -$

$\text{Cov}_t) \leq \frac{4}{\sigma^2}\frac{d^2}{d^2-d+c}$.

At the same time, $\beta_t \geq \frac{1}{d}$ for all t. Consider the contraction map $\varphi(x) = \frac{4}{\sigma^2} + \frac{1}{d^2}(d-c)x$. Iterating Equation (1) starting with $t = 1$, we find that $\text{Var}_t - \text{Cov}_t \geq \varphi^{(t-1)}(\text{Var}_1 - \text{Cov}_1)$, so this shows

$$\liminf_{t\to\infty}(\text{Var}_t - \text{Cov}_t) \geq \frac{4}{\sigma^2}\cdot\frac{d^2}{d^2-d+c}$$

where the RHS is the fixed point of φ. Combining with the result before, we get $\lim_{t\to\infty}(\text{Var}_t - \text{Cov}_t) = \frac{4}{\sigma^2}\cdot\frac{d^2}{d^2-d+c}$.

Using Proposition 2, we have $\text{Var}_{t+1} = 2(\beta_{t+1}d(\text{Var}_t/2) + 2/\sigma^2)$, so

$$\begin{aligned}
\text{Var}_{t+1} - \text{Var}_t &= (\beta_{t+1}d - 1)\text{Var}_t + \frac{4}{\sigma^2} \\
&= (\frac{d\text{Var}_t}{\text{Var}_t + (d-1)\text{Cov}_t} - 1)\text{Var}_t + \frac{4}{\sigma^2} \\
&= (\frac{d\text{Var}_t}{d\text{Var}_t - (d-1)(\text{Var}_t - \text{Cov}_t)} - 1)\text{Var}_t + \frac{4}{\sigma^2}
\end{aligned}$$

Using $\lim_{t\to\infty}(\text{Var}_t - \text{Cov}_t) = \frac{4}{\sigma^2}\cdot\frac{d^2}{d^2-d+c}$, we conclude

$$\lim_{t\to\infty}(\text{Var}_{t+1} - \text{Var}_t) = \lim_{t\to\infty}\left(\frac{\text{Var}_t}{\text{Var}_t - \frac{4}{\sigma^2}\frac{d^2-d}{d^2-d+c}} - 1\right)\text{Var}_t + \frac{4}{\sigma^2}.$$

Since $\text{Var}_t \to \infty$, we get $\lim_{t\to\infty}\left(\frac{\text{Var}_t}{\text{Var}_t - \frac{4}{\sigma^2}\frac{d^2-d}{d^2-d+c}} - 1\right)\text{Var}_t = \frac{4}{\sigma^2}\frac{d^2-d}{d^2-d+c}$ using Taylor expansion. So $\lim_{t\to\infty}(\text{Var}_{t+1} - \text{Var}_t) = \frac{4}{\sigma^2}\left(\frac{d^2-d}{d^2-d+c} + 1\right)$, implying $r_i = \left(1 + \frac{d^2-d}{d^2-d+c}\right)\frac{i}{K} + o(i)$. So $\lim_{i\to\infty}(r_i/i) = \left(1 + \frac{d^2-d}{d^2-d+c}\right)\frac{1}{K}$. □

A.7 Proof of Proposition 4

Proof. Regardless of K, for each agent i in generation t, $\overline{PL}(i) = t-1$, so $\lim_{i\to\infty}\overline{PL}(i) = \infty$. By Proposition 3, society learns completely in the long run. The expression for r_i comes from specializing Theorem 1 (whose proof does not depend on Proposition 4) to the case of $d = c = K$. Observe $\frac{(2K-1)}{K^2}\cdot K = (2K-1)/K < 2$ for any $K \geq 1$.

To bound r_i starting with the 3rd generation, we first establish a lemma that expresses $\vec{\beta}_{i,\cdot}$ in closed-form for an agent i in generation $t+1$. Let $\tilde{a}_{\text{sum}}$ be the sum of the log-actions played in generation $t-1$ in equilibrium. By the linearity of equilibrium (Proposition 1), there must exist some $\mu_{\text{sum}}, \sigma^2_{\text{sum}} > 0$ so that the conditional distributions of $\tilde{a}_{\text{sum}}$ in the two states are $\mathcal{N}(\pm\mu_{\text{sum}}, \sigma^2_{\text{sum}})$.

Lemma A.6. *Each element in $\vec{\beta}_{i,\cdot}$ is* $\left(\frac{\mu_{sum}^2}{\sigma_{sum}^2}+\frac{1}{\sigma^2}\right)/\left(K\frac{\mu_{sum}^2}{\sigma_{sum}^2}+\frac{1}{\sigma^2}\right)$.

Proof. An application of Proposition 1 shows each agent j in generation t aggregates $\tilde{a}_{\text{sum}}$ and own private signal $\tilde{s}_j$ according to $\tilde{a}_j = 2\cdot\frac{\mu_{\text{sum}}}{\sigma_{\text{sum}}^2}\tilde{a}_{\text{sum}}+\tilde{s}_j$.

Next, consider the problem of someone in generation $t+1$ who observes the log-actions $\tilde{a}_j$ of the K agents $j=(t-1)K+k$ for $1\le k\le K$ from generation t. By symmetry, i places the same weight on these K log-actions in equilibrium. To find this weight, we calculate

$$\mathbb{E}\left[\sum_{k=1}^{K}\tilde{a}_{(t-1)K+k}\mid\omega=1\right]=2K\frac{\mu_{\text{sum}}^2}{\sigma_{\text{sum}}^2}+2K\frac{1}{\sigma^2}$$

$$\text{VAR}\left[\sum_{k=1}^{K}\tilde{a}_{(t-1)K+k}\mid\omega=1\right]=K\cdot\left(4\cdot\frac{\mu_{\text{sum}}^2}{\sigma_{\text{sum}}^2}+4\cdot\frac{1}{\sigma^2}\right)+K\cdot(K-1)\cdot4\cdot\frac{\mu_{\text{sum}}^2}{\sigma_{\text{sum}}^2}$$

So by Proposition 1,

$$\beta_{i,j}=\frac{2\cdot\left(2K\frac{\mu_{\text{sum}}^2}{\sigma_{\text{sum}}^2}+2K\frac{1}{\sigma^2}\right)}{K\cdot\left(4\cdot\frac{\mu_{\text{sum}}^2}{\sigma_{\text{sum}}^2}+4\cdot\frac{1}{\sigma^2}\right)+K\cdot(K-1)\cdot4\cdot\frac{\mu_{\text{sum}}^2}{\sigma_{\text{sum}}^2}}=\frac{\frac{\mu_{\text{sum}}^2}{\sigma_{\text{sum}}^2}+\frac{1}{\sigma^2}}{K\frac{\mu_{\text{sum}}^2}{\sigma_{\text{sum}}^2}+\frac{1}{\sigma^2}}$$

for every $j=(t-1)K+k$ for $1\le k\le K$, as desired. □

Consider an agent i in generation t. From Proposition 2, there is some $x_{old}>0$ so that $\tilde{a}_i\sim\mathcal{N}(\pm x_{old},2x_{old})$ conditional on the two states. In fact, from Proposition 1, $x_{old}=2\cdot\frac{\mu_{\text{sum}}^2}{\sigma_{\text{sum}}^2}+\frac{2}{\sigma^2}$. For an agent in generation $t+1$, using the same argument and applying the formula for $\vec{\beta}_{i,\cdot}$ from Lemma A.6, we have

$$x_{new}=\frac{2K(\frac{\mu_{\text{sum}}^2}{\sigma_{\text{sum}}^2}+\frac{1}{\sigma^2})^2}{K\frac{\mu_{\text{sum}}^2}{\sigma_{\text{sum}}^2}+\frac{1}{\sigma^2}}+\frac{2}{\sigma^2}.$$

A hypothetical agent who observes $\tilde{a}_{\text{sum}}$ (the sum of log-actions in generation $t-1$) with conditional distributions $\mathcal{N}(\pm\mu_{\text{sum}},\sigma_{\text{sum}}^2)$ and three independent private signals would play a log-action with conditional distributions $\mathcal{N}(\pm y,2y)$ where

$$y=\left[2\frac{\mu_{\text{sum}}^2}{\sigma_{\text{sum}}^2}+\frac{6}{\sigma^2}\right]+\frac{2}{\sigma^2}.$$

We have

$$(y - x_{new}) \cdot (K\frac{\mu_{\text{sum}}^2}{\sigma_{\text{sum}}^2} + \frac{1}{\sigma^2}) = \left[2\frac{\mu_{\text{sum}}^2}{\sigma_{\text{sum}}^2} + \frac{6}{\sigma^2}\right] \cdot \left[K\frac{\mu_{\text{sum}}^2}{\sigma_{\text{sum}}^2} + \frac{1}{\sigma^2}\right] - 2K(\frac{\mu_{\text{sum}}^2}{\sigma_{\text{sum}}^2} + \frac{1}{\sigma^2})^2$$
$$=(2+6K) \cdot \frac{\mu_{\text{sum}}^2}{\sigma_{\text{sum}}^2} \cdot \frac{1}{\sigma^2} + \frac{6}{\sigma^4} - 4K \cdot \frac{\mu_{\text{sum}}^2}{\sigma_{\text{sum}}^2} \cdot \frac{1}{\sigma^2} - 2K\frac{1}{\sigma^4}$$
$$\geq 2K\frac{1}{\sigma^2}\left(\frac{\mu_{\text{sum}}^2}{\sigma_{\text{sum}}^2} - \frac{1}{\sigma^2}\right).$$

We must have $\mathbb{P}[\tilde{a}_{\text{sum}} > 0 \mid \omega = 1] \geq \mathbb{P}[\tilde{s}_1 > 0 \mid \omega = 1]$, a probability that just depends on the ratio of the mean and standard deviation. So $\frac{\mu_{\text{sum}}}{\sigma_{\text{sum}}} \geq \frac{1}{\sigma}$, i.e. $\frac{\mu_{\text{sum}}^2}{\sigma_{\text{sum}}^2} \geq \frac{1}{\sigma^2}$. Hence the difference above is positive. This shows $x_{new} - x_{old} \leq 3 \cdot \frac{2}{\sigma^2}$. □

A.8 Proof of Proposition 5

Proof. Consider a Markov process with states $\{1, ..., K\}$ and state transition probabilities $\mathbb{P}[k_1 \to k_2] = 1/|\Psi_{k_1}|$ if $k_2 \in \Psi_{k_1}$, $\mathbb{P}[k_1 \to k_2] = 0$ otherwise. (Each Ψ_k is non-empty, since the observation sets are strongly connected.) This process is irreducible by strong connectivity. Also, since the observation sets are symmetric with $c \geq 1$, the proof of Lemma A.4 implies the process is aperiodic. By standard results (see e.g., Billingsley (2013)), there exists a stationary distribution $\pi^* \in \mathbb{R}_{++}^K$ with $\sum_{k=1}^K \pi_k^* = 1$, such that $\lim_{\tau\to\infty}(M_\Psi)^\tau \vec{e}_k = \pi^*$ for every $1 \leq k \leq K$, where $\vec{e}_k \in \mathbb{R}^K$ is a vector with 1 in position k and 0 in other positions, and M_Ψ is the stochastic matrix for the Markov process.

For $t \geq 1$, $1 \leq k \leq K$, abbreviate agent $i = (t-1)K + k$ as $[t, k]$. Consider the strategy profile where agent $[1, k]$ puts weight $1/\pi_k^*$ on her log-signal, while agent $[t, k]$ for $t \geq 2$ puts weight $1/|\Psi_k|$ on each observed log-action and weight $1/\pi_k^*$ on her log-signal. The weight that $[t, k]$ puts on the log-signal of $[t', k']$ for $t' < t$ is $(1/\pi_{k'}^*) \cdot ((M_\Psi)^{t-t'}\vec{e}_k)_{k'}$. Noting this quantity only depends on the difference $t - t'$ and on k, k', we abbreviate it as $c_{t-t',k,k'}$ and observe that $\max_{k,k'} |c_{\tau,k,k'} - 1| \to 0$ as $\tau \to \infty$, since $\lim_{\tau\to\infty}(M_\Psi)^\tau \vec{e}_k = \pi^*$ for every k.

We show that under this strategy profile, $\tilde{a}_i$ with $i = [t, k]$ has the conditional distributions $\mathcal{N}(\pm((t-1)K + o(i))\frac{2}{\sigma^2}, ((t-1)K + o(i))\frac{4}{\sigma^2})$. Let $\epsilon > 0$ be given and we show for all large enough $i = [t, k]$, $|\mathbb{E}[\tilde{a}_i \mid \omega = 1]/(2/\sigma^2) - ((t-1)K)| < \epsilon i$. This is because there is T so that $\max_{k,k'} |c_{\tau,k,k'} - 1| < \epsilon/4$ for all $\tau \geq T$, which shows

$$|\mathbb{E}[\tilde{a}_i \mid \omega = 1]/(2/\sigma^2) - ((t-1)K)| \leq (\epsilon/4)(t-1-T)K + \max_{k,k',\tau<T} |c_{\tau,k,k'} - 1| \cdot (TK) + 1/\pi_k^*.$$

Because there are finitely many values of $c_{\tau,k,k'}$ with $\tau < T$, the maximum $\max_{k,k',\tau<T} |c_{\tau,k,k'} - 1|$ is constant in i. Thus the bound is a constant term in i plus a term no larger than $(\epsilon/4) \cdot i$.

By similar reasoning,

$$|\mathrm{Var}[\tilde{a}_i \mid \omega = 1]/(4/\sigma^2) - ((t-1)K)| \leq (\epsilon/2 + \epsilon^2/16)(t-1-T)K + \max_{k,k',\tau<T} |c^2_{\tau,k,k'} - 1| \cdot (TK) + (1/\pi^*_k)^2.$$

The bound is a constant term in i plus a term no larger than $(2\epsilon/3) \cdot i$ for ϵ near 0.

Let $K_0 < K$ be given. If $\tilde{A}$ has the conditional distributions $\mathcal{N}(\pm(t-1)K_0 \cdot \frac{2}{\sigma^2}, (t-1)K_0 \cdot \frac{4}{\sigma^2})$ in the two states, then $\mathbb{P}[A > \frac{1}{2} \mid \omega = 1] = \Phi(\sqrt{(t-1)K_0}/\sigma)$. Pick some $\epsilon > 0$ so that $\frac{K-\epsilon}{\sqrt{K+\epsilon}} > \sqrt{K_0}$. There corresponds T so that for for $i = [t,k]$ with $t \geq T$ and $1 \leq k \leq K$, $\mathbb{E}[\tilde{a}_i \mid \omega = 1] \geq (t-1)(K-\epsilon)\frac{2}{\sigma^2}$ and $\mathrm{Var}[\tilde{a}_i \mid \omega = 1] \leq (t-1)(K+\epsilon)\frac{4}{\sigma^2}$, so $\mathbb{P}[a_i > \frac{1}{2} \mid \omega = 1] \geq \Phi(\sqrt{(t-1)} \cdot (K-\epsilon)/(\sqrt{K+\epsilon}\sigma))$, so i is more accurate than $(t-1)K_0$ signals. □

A.9 Proof of Corollary 2

Proof. We claim that for any agent i in generation t, the action $\tilde{a}_i$ is equal to the sum of $\tilde{s}_i$ and $\tilde{s}_j$ for all agents j in generations $1, \ldots, t-1$. The proof is by induction on t. The claim holds for the first generation because all agents in the first generation choose $\tilde{a}_i = \tilde{s}_i$.

Consider an agent in generation t. By the inductive hypothesis, she observes neighbors' actions $\tilde{a}_j = \tilde{s}_j + \sum_{j' \leq (t-2)K} \tilde{s}_{j'}$ for all j in generation $t-1$ and observes s_j for one such j. Therefore, she can compute $\sum_{j' \leq (t-2)K} \tilde{s}_{j'}$ and $\tilde{s}_j$ for all j in generation $t-1$. Since these signals are independent and she has access to no information about other signals from her generation, she chooses

$$\tilde{a}_i = \tilde{s}_i + \sum_{j \leq (t-1)K} \tilde{s}_j.$$

By induction, we have $r_i = K(t-1) + 1 > i - K$ for all agents in generation t. □

A.10 Proof of Proposition 6

We first show that expected utility is increasing in r_i.

Lemma A.7. *Agent i's expected utility is a strictly increasing function of r_i.*

Proof. Let $r_i > r_i' \geq 1$. Consider an agent j who observes two conditionally independent Gaussian signals of the state, s_A and s_B. When $\omega = 1$, $s_A \sim \mathcal{N}(1, \sigma^2/r_i')$ and $s_B \sim \mathcal{N}(1, \sigma^2/(r_i - r_i'))$. When $\omega = 0$, $s_A \sim \mathcal{N}(-1, \sigma^2/r_i')$ and $s_B \sim \mathcal{N}(-1, \sigma^2/(r_i - r_i'))$. If this agent chooses an action a_j using only s_A, then the conditional distributions of the log-action are $\tilde{a}_j \sim \mathcal{N}(\pm r_i' \cdot \frac{2}{\sigma^2}, r_i' \cdot \frac{4}{\sigma^2})$. If the agent instead chooses an action a_j^* using both s_A and s_B, then the conditional distributions of the log-action are $\tilde{a}_j^* \sim \mathcal{N}(\pm r_i \cdot \frac{2}{\sigma^2}, r_i \cdot \frac{4}{\sigma^2})$, by the conditional independence of s_A and s_B. Action a_j^* gives strictly higher expected utility to j than action a_j since it is based on an extra informative signal, and this implies i has strictly higher expected utility when social learning aggregates r_i instead of $r_i^{'}$ signals. □

We now prove Proposition 6.

Proof. From the hypotheses, there exist $0 < \rho_L < \rho_H$ and a finite $\underline{I}$ so that $r_i \geq \rho_H i$ and $r_i^{'} \leq \rho_L i$ for all $i \geq \underline{I}$. Without loss we can choose $\underline{I} > \frac{\rho_H}{\rho_H - \rho_L}$. Let $R := \max_{i \leq \underline{I}} r_i^{'} < \infty$.

We choose τ so that for any $0 < 1/\sigma^2 \leq \tau$, an agent who aggregates R signals has expected utility strictly lower than $\underline{v}$. To see this is possible, note that we can choose $\epsilon > 0$ small enough so that $-(1-\epsilon)(0.5-\epsilon)^2 < \underline{v}$. Find $\zeta > 0$ so that if $y \leq \zeta$, then $\frac{\exp(y)}{1+\exp(y)} \leq 0.5 + \epsilon$. Suppose some agent j's log-action has the conditional distributions $\tilde{a}_j \sim \mathcal{N}(\pm R \cdot \frac{2}{\sigma^2}, R \cdot \frac{4}{\sigma^2})$. Then $\mathbb{P}[\tilde{a}_j > \zeta \mid \omega = 1] \to 0$ as $1/\sigma^2 \to 0$, since ζ is $\frac{\sigma^2\delta - 2R}{2R\sigma}$ standard deviations above the mean when $\omega = 1$, a quantity that tends to infinity as $\sigma \to \infty$. But whenever $\mathbb{P}[\tilde{a}_j > \zeta \mid \omega = 1] \leq \epsilon$, j's conditional expected payoff when $\omega = 1$ is bounded above by $\mathbb{P}[a_j \leq 0.5 + \epsilon \mid \omega = 1] \cdot (-(0.5-\epsilon)^2) \leq -(1-\epsilon)(0.5-\epsilon)^2$, and symmetrically the same goes for j's conditional expected payoff when $\omega = 0$.

For a given $1/\sigma^2 \leq \tau$, let $i^{''}$ be the least integer in the set $\{\underline{I}+1, \underline{I}+2, ...\}$ such that $\rho_L i^{''}$ signals lead to an expected utility of at least $\underline{v}$. This $i^{''}$ exists since $\rho_L > 0$. Utility $\underline{v}$ is weakly attained by no earlier than $i^{''}$ in network $M^{'}$. This is because $M^{'}$ cannot weakly attain $\underline{v}$ before agent $\underline{I}+1$ by construction of τ, while agents $i^{'} \geq \underline{I}+1$ and later aggregate no more than $\rho_L i^{'}$ signals on network $M^{'}$ and their utilities are strictly increasing in the number of signals aggregated by Lemma A.7. On the other hand, M strongly attains $\underline{v}$ by no later than $I = i^{''} - 1$. This is because $\rho_H(i^{''}-1) - \rho_L i^{''} = (\rho_H - \rho_L)i^{''} - \rho_H \geq (\rho_H - \rho_L)\underline{I} - \rho_H > 0$ by choice of $\underline{I}$, so $r_i \geq \rho_L i^{''}$ for all $i \geq i^{''} - 1$. We again appeal to Lemma A.7 to deduce all agents $i^{''} - 1$ and later in M have expected utilities at least $\underline{v}$. □

A.11 Proof of Proposition 7

Proof. As in the proof of Proposition 6, there exists some I so that $r_i > r_i^{'}$ for all $i \geq I$. Now let $T = I$. Since welfare is a strictly increasing function in r by Lemma A.7, network M leads to strictly higher welfare than $M^{'}$ for all agents $i \geq I$. □

B When Does Adding Links Improve Accuracy?

For two observation networks M and $M^\bullet$, write $M^\bullet \geq M$ if $M^\bullet$ can be generated from M by adding links, that is $M^\bullet_{j,k} \geq M_{j,k}$ for all j, k. By Proposition 3, adding links leads to weakly better *asymptotic* learning outcomes — if the conditions for complete long-run learning are satisfied for M, then the same holds for $M^\bullet$. But, when does adding links improve *finite-time* accuracy?

We show that agent i is more accurate in network $M^\bullet$ than in network M if both networks are *transitive at* i: that is, whenever $j \in N(i)$ and $k \in N(j)$, we have $k \in N(i)$. We also highlight that *intransitivities* — that is, sequences of links $i_n \to i_{n-1}, i_{n-1} \to i_{n-2}, ..., i_2 \to i_1$ such that $i_n \not\to i_1$ — form a key obstacle to obtaining higher accuracy on denser networks. Accuracy may decrease for some agents if the new links create new intransitivities. Further, accuracy may decrease even in the absence of new intransitivities, if the baseline network M already contains intransitivities.

Proposition B.1. *Suppose $M^\bullet \geq M$ and both networks are transitive at i. Then r_i is weakly higher on $M^\bullet$ than on M.*

The proof of Proposition B.1 shows that for any network that is transitive at i, $W_{i,j} = 1$ for $j \in \bar{N}(i) \cup \{i\}$ and $W_{i,j} = 0$ otherwise — that is, i perfectly incorporates the private signals of all agents she indirectly observes. That is, r_i is equal to the number of agents indirectly observed by i. The denser network $M^\bullet$ improves i's accuracy because it expands i's indirect neighborhood.

We show by example that the same conclusion does not hold if $M^\bullet$ generates new intransitivities relative to M.

Example B.1. Consider the networks $M, M_2^\bullet, ..., M_{n-1}^\bullet$ in Figure B.1.

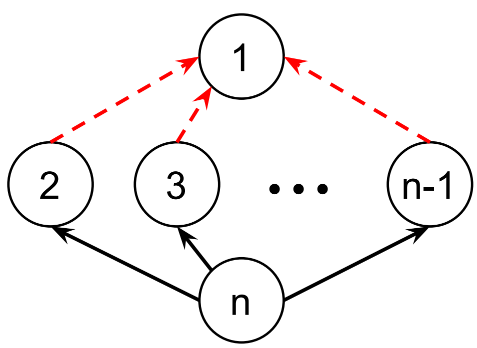


Figure B.1: The black links define a transitive network M with n agents. For $k \in \{2, ..., n-1\}$, adding the $k-1$ red links from agents $2, ..., k$ to agent 1 creates a new network $M_k^\bullet$ that is no longer transitive for agent n. For $k \geq 5$, agent n has strictly lower accuracy on $M^\bullet$ than M.

In network M, agent n perfectly incorporates the private signals of neighbors $2, ..., n-1$ and social learning aggregates $r_n = n-1$ signals. In network $M_k^\bullet$ for $2 \leq k \leq n-1$, the additional links expand n's indirect neighborhood relative to M, but also create informational confounds through intransitivities. In the new network, n cannot disentangle the private signals of her neighbors from the unobserved signal s_1 that serves as a common influence for her neighbors' behavior.

In the equilibrium on network $M_k^\bullet$, n puts weight $\frac{2}{k}$ on the log-actions of $2, ..., k$, thus social learning aggregates $r_n^{(k)} = 4 \cdot \frac{k-1}{k} + n - k$ signals by agent n. We have that $r_n^{(2)} > r_n$, so the first red link helps and allows n to incorporate all private signals. But $r_n^{(k)}$ is strictly decreasing in k. In particular, $r_n^{(k)}$ is strictly smaller than r_n whenever $k \geq 5$. Adding four or more red links to the original network M strictly harms n's welfare.

Suppose the baseline network M already contains some intransitivities. The next examples shows that adding links may decrease some agent's accuracy even if the new links do not create new intransitivities. In particular, links can harm agents without creating any new confounds simply by changing the weights on existing confounds.

Example B.2. Consider the networks M and $M^\bullet$ in Figure B.2.

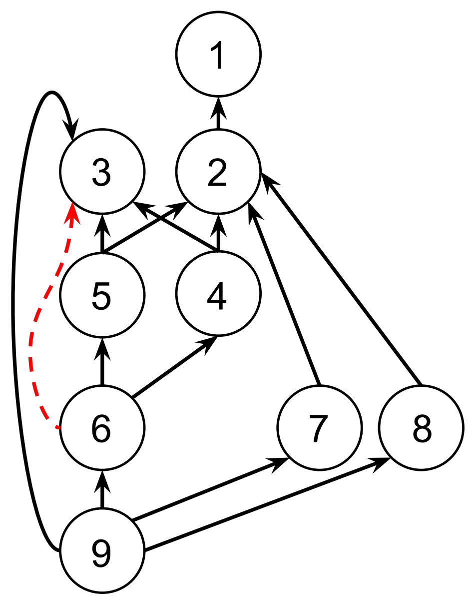


Figure B.2: Adding the new link in red does not create new intransitivities, but nevertheless decreases agent 9's accuracy.

Intransitivities exist both in the old network M defined by the black links, and in the new network $M^\bullet$ that adds the one red link. Even though the newly added link does not generate any additional intransitivities, we have $r_9 = \frac{3681}{533} \approx 6.91$ in the old network and $r_9 = \frac{1977}{287} \approx 6.89$ in the new network, so socially learning aggregates fewer signals by agent 9 in $M^\bullet$.

Agent 9 becomes less accurate in $M^\bullet$ because agent 6's new link causes her to change her equilibrium play in a way that generates negative informational externality for agent 9. In both M and $M^\bullet$, agent 6 cannot fully incorporate the private signals of agents 4 and 5 without over-counting the private signals of agents 1 and 2. In network M, agent 6 puts weight $\frac{4}{7}$ on the log-actions of agents 4 and 5, thus weight $\frac{8}{7}$ on $\tilde{s}_1$ and $\tilde{s}_2$. In network $M^\bullet$, agent 6 puts a higher weight $\frac{3}{5}$ on the log-actions of agents 4 and 5, because she can now subtract off part of the informational confound using her observation of agent 3. This change in her equilibrium strategy means her over-weighting of $\tilde{s}_1$ and $\tilde{s}_2$ is exacerbated, with these log-signals each receiving weight $\frac{6}{5}$. At the same time, $\tilde{s}_1$ and $\tilde{s}_2$ also confound agent 9's inference about the private signals of agents 7 and 8. Agent 9 finds it harder to incorporate agent 6's private signal in $M^\bullet$, because $\tilde{a}_6$ now contains a more severe over-counting of $\tilde{s}_1$ and $\tilde{s}_2$. The change in agent 6's play on $M^\bullet$ does not taken into account the welfare of agent 9, who has a different signal-extraction problem that involves worse confounding by $\tilde{s}_1$ and $\tilde{s}_2$.

B.1 Proofs for Appendix B

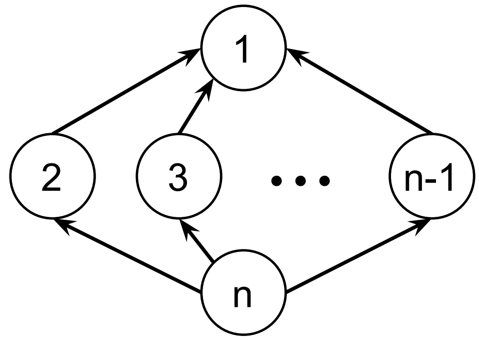
B.1.1 Proof of Proposition B.1

Proof. We first show that on any network transitive at i, the equilibrium strategy of i is such that $\tilde{a}_i = \sum_{j\in\bar{N}(i)\cup\{i\}} \tilde{s}_j$. Clearly, $\tilde{a}_i$ cannot put any weight on the log-signals of agents not in $\bar{N}(i)\cup\{i\}$, for information outside of the sub-network containing i cannot reach i. Also, if feasible, $\sum_{j\in\bar{N}(i)\cup\{i\}} \tilde{s}_j$ is the optimal signal aggregation for i. For every $j \in \bar{N}(i)$, we have $N(j) \subseteq N(i)$. Since i knows j's linear equilibrium strategy $\tilde{A}^*_j((\tilde{a}_k)_{k\in N(j)}, \tilde{s}_j)$, i can identify $\tilde{s}_j$ by calculating $\tilde{a}_j - \sum_{k\in N(j)} \beta_{j,k}\tilde{a}_k$. Therefore i can identify the sum $\sum_{j\in\bar{N}(i)} \tilde{s}_j$ using her neighbors' actions.

Combined with Proposition 2, this shows r_i on any network transitive at i is equal to the cardinality of $\bar{N}(i)$ plus one. Agent i must have a larger indirect neighborhood on $M^\bullet$ if $M^\bullet \geq M$. □

B.1.2 Proof of Example B.1

Proof. First, we show $\vec{\beta}_{n,j} = \frac{2}{n-1}$ for $j = 2, ..., n-1$ in the following network.



We apply Lemma A.1 to calculate $\vec{\beta}_{n,\cdot}$. The submatrix $\hat{W}$ of W with rows $\{2, ..., n-1\}$ and columns $\{1, ..., n-1\}$ is

$$\hat{W} = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 1 & 0 & 1 & 0 & \dots \\ 1 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

So we get

$$(\hat{W}\hat{W}')^{-1} = \begin{pmatrix} 2 & 1 & 1 & \dots \\ 1 & 2 & 1 & \dots \\ 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} = I_{n-2} - \frac{1}{n-1}\text{Ones}_{n-2}$$

where I_{n-2} is the $(n-2) \times (n-2)$ identity matrix and Ones_{n-2} is the $(n-2) \times (n-2)$ matrix of all 1's. So,

$$\begin{aligned} \hat{W}'(\hat{W}\hat{W}')^{-1} &= \hat{W}' - \frac{1}{n-1}\hat{W}'\text{Ones}_{n-2} \\ &= \begin{pmatrix} 1 & 1 & 1 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} - \frac{1}{n-1}\begin{pmatrix} n-2 & n-2 & n-2 & \dots \\ 1 & 1 & 1 & \dots \\ 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{aligned}$$

where the dimension of each matrix is $(n-1) \times (n-2)$. The sum of each column is $2 - \frac{2n-4}{n-1} = \frac{2}{n-1}$, which is $\vec{\beta}_{n,j}$ for $j = 2, ..., n-1$.

In the equilibrium on network $M_k^{\bullet}$, agent n puts weight 1 on each of $\tilde{a}_{k+1}, ..., \tilde{a}_{n-1}$ and weight $2/k$ on $\tilde{a}_2, ..., \tilde{a}_k$ by comparison to the network above.

We have $r_n^{(2)} = n > r_n$, while $\frac{d}{dk}(4\frac{k-1}{k} + n - k) = \frac{4}{k^2} - 1 < 0$ for $k > 2$. This shows $r_n^{(k)}$

is strictly decreasing in k for $k \geq 2$. Finally, $r_n - r_n^{(k)} = k^2 - 5k + 4$ is a convex quadratic function with zeroes at 1 and 4. So $r_n = r_n^{(4)}$ while $r_n > r_n^{(k)}$ for all $k \geq 5$. □

B.1.3 Proof of Example B.2

Proof. For the network where agent 6 does not observe agent 3, by Lemma A.1 we find

$$\vec{\beta}_{6,\cdot} = \vec{\mathbf{1}}'_{(5)} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \times \left(\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \right)^{-1} = \begin{pmatrix} \frac{4}{7} & \frac{4}{7} \end{pmatrix}.$$

Then, letting

$$\hat{W} := \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{8}{7} & \frac{8}{7} & \frac{8}{7} & \frac{4}{7} & \frac{4}{7} & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

we get

$$\vec{\beta}_{9,\cdot} = \vec{\mathbf{1}}'_{(8)} \hat{W}'(\hat{W}\hat{W}')^{-1} = \vec{\mathbf{1}}'_{(8)} \hat{W}' \frac{1}{533} \begin{pmatrix} 853 & -280 & 128 & 128 \\ -280 & 245 & -112 & -112 \\ 128 & -112 & 371 & -162 \\ 128 & -112 & -162 & 371 \end{pmatrix} = \frac{1}{533} \begin{pmatrix} 61 \\ 413 \\ 131 \\ 131 \end{pmatrix},$$

and hence we can calculate the 9th row of W and r_9.

For the network where agent 6 observes agent 3, note that agent 6 can recover $\tilde{s}_1 + \tilde{s}_2 + \tilde{s}_4$ and $\tilde{s}_1 + \tilde{s}_2 + \tilde{s}_5$ using $\tilde{a}_4 - \tilde{a}_3$ and $\tilde{a}_5 - \tilde{a}_3$. Thus, the weights that agent 6 puts on $\tilde{a}_4$ and $\tilde{a}_5$ are the same as in a network where agents 4 and 5 only observe agent 2. This can be computed by Lemma A.1:

$$\vec{\mathbf{1}}'_{(5)} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \times \left(\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \right)^{-1} = \begin{pmatrix} \frac{3}{5} & \frac{3}{5} \end{pmatrix},$$

which shows $\vec{\beta}_{6,\cdot} = (-\frac{1}{5}, \frac{3}{5}, \frac{3}{5})'$. Then, letting

$$\hat{W} := \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{6}{5} & \frac{6}{5} & 1 & \frac{3}{5} & \frac{3}{5} & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

we get

$$\vec{\beta}_{9,\cdot} = \vec{\mathbf{1}}'_{(8)}\hat{W}'(\hat{W}\hat{W}')^{-1} = \vec{\mathbf{1}}'_{(8)}\hat{W}'\frac{1}{287}\begin{pmatrix} 412 & -125 & 60 & 60 \\ -125 & 125 & -60 & -60 \\ 60 & -60 & 201 & -86 \\ 60 & -60 & -86 & 201 \end{pmatrix} = \frac{1}{287}\begin{pmatrix} 72 \\ 215 \\ 69 \\ 69 \end{pmatrix},$$

and hence we can calculate the 9th row of W and r_9. □