Iterative Weak Dominance and Interval-Dominance Supermodular Games*

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Abstract

This paper extends Milgrom and Robert's treatment of supermodular games in two ways. It points out that their main characterization result holds under a weaker assumption. It refines the arguments to provide bounds on the set of strategies that survive iterative deletion of weakly dominated strategies. I derive the bounds by iterating the best-response correspondence. I give conditions under which they are independent of the order of deletion of dominated strategies. The results have implications for equilibrium selection and dynamic stability in games. *Journal of Economic Literature* Classification Numbers: C72, D81.

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1 Introduction

Milgrom and Roberts [11] and Vives [20] provide useful analyses of the class of supermodular games introduced by Topkis [18]. In a supermodular game, each player's strategy set is partially ordered and there are strategic complementarities that cause a player's best response to be increasing in opponents' strategies. Milgrom and Roberts and Vives describe many applications of the games in the class.

Milgrom and Roberts [11] and Vives [20] demonstrate that supermodular games have a largest and smallest equilibrium. Milgrom and Roberts demonstrate that these extreme equilibria can be obtained by iterating the best-response correspondence and characterize the set of strategies that survive iterative deletion of strictly dominated strategies. This paper extends these insights in two ways. First, it broadens the class of games for which the basic conclusions hold. Second, it provides parallel results for strategies that survive iterative deletion of weakly dominated strategies. The new result forms the basis for equilibrium selection arguments.

I enlarge the class of supermodular games by replacing an increasing-difference condition used by Milgrom and Roberts and Vives with a weaker condition, interval dominance, introduced by Quah and Strulovici [16].

Section 3 points out a small generalization of the basic result of Milgrom and Roberts characterizing the set of strategies that survive iterative deletion of strictly dominated strategies. Section 4 extends the results to weak dominance. The analysis trades off using a more restrictive solution concept (deleting weakly dominated strategies instead of strongly dominated strategies) with analyzing a broader class of games. Section 5 discusses the implications of the characterization result for comparative statics and dynamics. Section 6 discusses applications. This section describes two classes of games that fail to be supermodular and have large sets of equilibria that survive iterative deletion of strongly dominated strategies. I demonstrate that analogs of the methods introduced to study supermodular games partially extend to these games. Appendix B contains proofs omitted from the main text.

2 Preliminaries

There is a finite set of players. I denotes the player set. Each player has a strategy set X_i with typical element x_i. $X = \Pi_{i \in I} X_i$ is the set of strategy profiles. I denote by x_{-i} the strategies of Player i's opponents. Each strategy set is partially ordered by $\geq_i$; $\geq$ denotes the product order derived from the $\geq_i$ (so that $x \geq x'$ if and only if $x_i \geq_i x_i$ for all i). Denote Player i's utility function by $u_i(x_i, x_{-i})$. Denote by $u = (u_i)_{i \in I}$ the set of utility functions. A game in ordered normal form is $\Gamma = (I, X, u, \geq)$.

Consider a set X with a partial order $\geq$ that is transitive, reflexive, and antisymmetric. To make the paper self contained, I place standard definitions (lattice, chain, order continuity, supermodularity, strong set order) in Appendix A.

The paper uses weaker versions of basic single-crossing properties. I review the basic ideas and then discuss them in a strategic setting.

Definition 1. *Given two lattices X and Y, a function $f : X \times Y \to \mathbb{R}$ has increasing differences in its two arguments x and y if for all $x'' \geq x'$, the difference $f(x'', y) - f(x', y)$ is nondecreasing in y.*

This paper replaces increasing differences with weaker assumptions. There are several ways to weaken the increasing-differences property. The next definition is standard.

Definition 2. *Given two lattices X and Y, a function $f : X \times Y \to \mathbb{R}$ satisfies the single-crossing property in its two arguments x and y if for all $y'' > y'$, $x'' > x'$,*

$$f(x'', y') \geq (>) f(x', y') \implies f(x'', y'') \geq (>) f(x', y''). \tag{1}$$

Single crossing is also more restrictive than necessary.

Definition 3. *Given two lattices X and Y. A function $f : X \times Y \to \mathbb{R}$ satisfies the interval-dominance (ID) property in its two arguments x and y if for all $y'' > y'$, $x'' > x'$, (1) holds whenever $f(x'', y') \geq f(x, y')$ for all $x \in [x', x'']$.*

Quah and Strulovici [16] introduce Condition (ID) and derives basic properties. Quah and Strulovici [15] contains additional results, included detailed discussion of the implications of (ID) when X is multidimensional. It is apparent that increasing differences implies single crossing which in turn implies interval dominance. It is straightforward to confirm that the converse implication does not hold.

The paper introduces and uses variations on Condition (ID) to study an application. I defer these discussions to when they are needed in Section 6.

Definition 4. *The game $\Gamma = (I, X, u, \geq)$ is an interval-dominance supermodular (ID-supermodular) game if, for each $i \in I$:*

(A1) X is a complete lattice;

(A2) $u_i : X \to \mathbb{R}$ is order upper semi-continuous in x_i for fixed x_{-i}; u_i order upper continuous in x_{-i} for fixed x_i; and u_i is bounded above;

(A3) u_i is supermodular in x_i for fixed x_{-i};

(A4) u_i satisfies the interval-dominance property in x_i and x_{-i} on all interval sublattices of X.

The distinction between supermodular and ID-supermodular games is that (A4) replaces the condition that u_i has increasing differences.

A useful preliminary observation is Topkis's Monotonicity Theorem.

Fact 1. *Let X be a lattice and X a partially ordered set. Let $f(x, y) : X \times Y \to \mathbb{R}$. Suppose that $f(\cdot)$ is supermodular in x for given y. For any sublattice $X' \subset X$, let $M(A_1) \equiv \arg\max_{z \in X'} f(z, y)$. $M(X')$ is a sublattice of X.*

In the context of games, the first part of the theorem states that the set of best replies form a sublattice when the payoff function is supermodular in a player's strategy. This result is part of the Topkis Monotonicity Theorem as stated in Milgrom and Roberts [11].

An important property of supermodular games is monotonicity of the best-reply correspondence.

Fact 2. *Let Γ be an ID-supermodular game. Let $J = J_1 \times \cdots \times J_I$ be an interval sublattice of X. If $x''_{-i} \geq x'_{-i}$, then*

$$\arg\max_{x_i \in J_i} u_i(x_i, x''_{-i}) \geq \arg\max_{x_i \in J_i} u_i(x_i, x'_{-i}).$$

Fact 2 generalizes a result of Milgrom and Shannon [12, Theorem 4] that assumes the single-crossing crossing property rather than (ID) and a result of Quah and Strulovici [16, Theorem 1] that assumes that X is a subset of $\mathbb{R}$. Quah and Strulovici [15, Theorem 1] proves Fact 2. As Milgrom and Shannon note, the lemma holds if one replaces the assumption of super modularity with the weaker assumption of quasi supermodularity.[1]

3 Iterative Deletion of Strictly Dominated Strategies

This section presents a small generalization of Milgrom and Roberts [11, Theorem 5].

In order to formulate the result, let $\hat{X} \subset X$. Define a mapping Z from subsets of X to subsets of X by:

$$Z_i(\hat{X}) = \{x_i \in X_i : \text{ for all } x'_i \in X_i \text{ there exists } \hat{x} \in \hat{X} \text{ such that } u_i(x_i, \hat{x}_{-i}) \geq u_i(x'_i, \hat{x}_{-i})\},$$

and $Z(\hat{X}) = \{(z_1, \ldots, z_I) : z_i \in Z_i(\hat{X})\}$. Strategies in $Z_i(\hat{X})$ are best replies to strategies in $\hat{X}_{-i}$. Let $\overline{Z}(\hat{X})$ denote the interval $[\inf(Z(\hat{X})), \sup(Z(\hat{X}))]$. The process of iteratively deleting strictly dominated strategies starts with $X^0 = X$ and lets $X^t = Z(X^{t-1})$. A strategy $x_i \in X_i$ is **serially undominated** if $x_i \in Z_i(X^t)$ for all t.

Theorem 1. *Let Γ be an ID-supermodular game. For each player i, there exist largest and smallest serially undominated strategies, $\underline{x}_i$ and $\overline{x}_i$. Moreover, the strategy profiles $\{\underline{x}_i : i \in I\}$ and $\{\overline{x}_i : i \in I\}$ are pure Nash equilibrium profiles.*

Theorem 1 is Milgrom and Roberts's Theorem 5 from under the assumption of interval dominance rather than increasing differences. The theorem follows from the next lemma. I include a proof of the lemma to identify precisely where I relax Milgrom and Roberts's condition.

Lemma 1. *Let $\underline{z}, \overline{z} \in X$ be profiles such that $\underline{z} \leq \overline{z}$, let $\underline{B}_i(x)$ and $\overline{B}_i(x)$ denote the smallest and largest best responses for i to $x \in X$, and let $\underline{B}(x)$ and $\overline{B}(x)$ denote the collections $\underline{B}_i(x)$ and $\overline{B}_i(x)$, $i \in I$. Then $\sup Z([\underline{z}, \overline{z}]) = \overline{B}(\overline{z})$ and $\inf Z([\underline{z}, \overline{z}]) = \underline{B}(\underline{z})$, and $Z([\underline{z}, \overline{z}]) = [\underline{B}(\underline{z}), \overline{B}(\overline{z})]$.*

Proof of Lemma 1. The largest and smallest best responses are well defined by Fact 1. By definition, $\underline{B}(\underline{z})$ and $\overline{B}(\overline{z})$ are in $Z([\underline{z}, \overline{z}])$, and thus $[\underline{B}(\underline{z}), \overline{B}(\overline{z})] \subset Z([\underline{z}, \overline{z}])$. Suppose $z \notin [\underline{B}(\underline{z}), \overline{B}(\overline{z})]$ and, in particular, suppose $z_i \ngeq z_i^* \equiv \underline{B}_i(\underline{z})$. I claim that $z_i \notin Z([\underline{z}, \overline{z}])$ because z_i is strongly dominated by $z_i \vee z_i^*$. For any $x_i \in [z_i, z_i^* \vee z_i)$,

$$u_i(x_i \vee z_i^*, \underline{z}_{-i}) - u_i(x_i, \underline{z}_{-i}) \geq u_i(z_i^*, \underline{z}_{-i}) - u_i(x_i \wedge z_i^*, \underline{z}_{-i}) > 0, \tag{2}$$

[1] A function is quasi-supermodular if $f(x) \geq f(x \wedge y)$ implies $f(x \vee y) \geq f(y)$ and $f(x) > f(x \wedge y)$ implies $f(x \vee y) > f(y)$.

where the first inequality follows from supermodularity and the second from the definition of z_i^*.

It follows from (2) that for any $x_i \in [\underline{z}_i, \overline{z}_i]$,

$$u_i(x_i \vee z_i^*, \underline{z}_{-i}) > u_i(x_i, \underline{z}_{-i}). \quad (3)$$

Furthermore, if $x_i \in [z_i, z_i \vee z_i^*)$, then $x_i \vee z_i^* = z_i \vee z_i^*$ and Inequality (3) implies that for $x_i \in [z_i, z_i \vee z_i^*)$,

$$u_i(z_i \vee z_i^*, \underline{z}_{-i}) > u_i(x_i, \underline{z}_{-i}). \quad (4)$$

It follows from (ID), (3), and (4) that if $z_i \not\geq z_i^*$ then

$$u_i(z_i \vee z_i^*, z_{-i}) > u_i(z_i, z_{-i}) \text{ for all } z_{-i} \in [\underline{z}_{-i}, \overline{z}_{-i}]. \quad (5)$$

An analogous argument applies to show that if $z_i \not\leq \overline{B}_i(\overline{z})$, then z_i is strictly dominated. ■

It is straightforward to show that (3) follows from quasi-supermodularity when $x_i \not\geq z_i^*$, so the lemma holds if the weaker assumption of quasisupermodularity replaces (A3) in the definition of (ID)-supermodular games.

Milgrom and Roberts [11, Theorem 5] state and prove this result for supermodular games. The proof above follows their proof. They derive Inequality (2) and then complete the proof by pointing out that increasing differences implies

$$u_i(z_i \vee \hat{z}_i, z_{-i}) - u_i(z_i, z_{-i}) \geq u_i(z_i \vee \hat{z}_i, \underline{z}_{-i}) - u_i(z_i, \underline{z}_{-i}) \quad (6)$$

provided that $z_{-i} \geq \underline{z}_{-i}$. The lemma follows from (2) and (6). I simply point out that the (ID) condition is sufficient for the result.

Milgrom and Roberts use the lemma to prove the theorem. Their proof goes through without modification.

4 Iterative Deletion of Weakly Dominated Strategies

Modifications of the proofs of Lemma 1 and Theorem 1 allow us to establish descriptions of the set of strategies that survive iterative deletion of weakly dominated strategies.

Definition 5. *Given a game $\Gamma = (I, X, u, \geq)$ and subsets $X_i' \subset X_i$, with $X' = \Pi_{i \in I} X_i'$, player i's strategy $x_i \in X_i'$ is weakly dominated relative to X_i' if there exists $z_i \in X_i'$ such that $u_i(x_i, x_{-i}) \leq u_i(z_i, x_{-i})$ for all $x_{-i} \in X_{-i}'$, with strict inequality for at least one $x_{-i} \in X_{-i}'$.*

Weak dominance will typically delete more strategies than strong dominance. Hence it has the potential to provide more restrictive predictions. I analyze the implications of applying iterative deletion of weakly dominated strategies instead of iterated deletion of strongly dominated strategies. This section studies **iterative interval deletion of weakly dominated strategies**. The procedure iteratively removes weakly dominated strategies beginning with a game $\Gamma^0 = (I, X^0, u, \geq)$ in which $X^0 = [\underline{x}^0, \overline{x}^0]$ and constructs games $\Gamma^k = (I, X^k, u, \geq)$ where $X^k = [\underline{x}^k, \overline{x}^k]$ is the smallest set such that all strategies in $X^{k-1} \setminus X^k$ are weakly dominated with respect to X^{k-1}. I will describe the set of strategies that survive this process, that is, the set of strategies that are in X^k for all k. It is possible that different ways of deleting weakly dominated strategies will lead to different limit sets. I reference results that identify games in which the order of deletion is essentially unimportant.

The procedure that iteratively deletes dominated strategies works by assuming that existing strategies are in an interval and then finding a (potentially smaller) interval of strategies that are undominated. It is possible that some strategies are weakly dominated but not strictly dominated. If this happens, then the process of iterative deletion of weakly dominated strategies will lead to a small set of surviving strategies. In this section, I point out how to modify Milgrom and Robert's arguments to apply to weak dominance. In Section 6, I discuss examples in which weak dominance in fact, is more selective than strong dominance and in which it is possible to use the arguments of supermodular games to characterize a refined set of equilibria.

Theorem 2. *Let Γ be a finite ID-supermodular game. For each Player i, there exist largest and smallest strategies that survive iterative interval deletion of weakly dominated strategies, $\underline{x}_i$ and $\overline{x}_i$. Moreover, the strategy profiles $\{\underline{x}_i : i \in I\}$ and $\{\overline{x}_i : i \in I\}$ are pure Nash equilibrium profiles.*

Theorem 2 extends Theorem 1 to weak dominance. I have added the assumption that Γ is finite. I explain the importance of this assumption after the proof.

The theorem requires two preliminary results.

Let $\underline{W}_i(x)$ denote the smallest best response to x and let $\overline{W}_i(x)$ denote the largest best response to x and let $\underline{W}(x)$ and $\overline{W}(x)$ denote the collections $\underline{W}_i(x)$ and $\overline{W}_i(x)$, $i \in I$. Let $E_i(x_i) = \{z_i \in X_i : u_i(x_i, z_{-i}) = u_i(z_i, z_{-i})$ for all $z_{-i} \in X_{-i}\}$.

Lemma 2. *Let Γ be an ID-supermodular game. Let $\underline{z}, \overline{z} \in X$ be profiles such that $\underline{z} \leq \overline{z}$. There exist largest and smallest strategies that are not weakly dominated. These strategies are, respectively, the largest element in $E_i(\underline{W}_i(\overline{z}))$ and the smallest element in $E_i(\overline{W}_i(\underline{z}))$.*

The way to construct the smallest strategy that is not weakly dominated for Player i is to consider the set of strategies that are best responses to the lowest strategy in $[\underline{z}, \overline{z}]$. If there are multiple best responses, the interval-dominance property suggests that the largest of the best responses performs at least as well than other best responses against higher strategies. This observation makes the largest best response to the smallest strategy a candidate for smallest strategy that is not weakly dominated. In fact, there may be other, smaller, strategies that are equivalent to the largest best response to $\underline{z}_{-i}$ in the sense that these strategies yield identical payoffs against all strategies in $[\underline{z}_{-i}, \overline{z}_{-i}]$. The proof of Lemma 2 shows that there exists a smallest strategy that is equivalent to the largest best response to $\underline{z}_{-i}$ and that this strategy is the smallest strategy that is not weakly dominated. The details are in Appendix B.

Let $\underline{z}, \overline{z} \in X$ be profiles such that $\underline{z} \leq \overline{z}$. Let $\overline{E}_i(x)$ denote the largest element of $E_i(x)$ and $\underline{E}_i(x)$ denote the smallest element of $E_i(x)$. Let $\underline{E}(x)$ and $\overline{E}(x)$ denote the collections $\underline{E}_i(x)$ and $\underline{E}_i(x)$. So, for example,

$$\underline{E}(x) = \{\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_I) : \tilde{x}_i \in \underline{E}_i(x) \text{ for all } i = 1, \ldots, I\}.$$

Define

$$\underline{s}_i = \inf\{x_i \in [\underline{z}_i, \overline{z}_i] : x_i \text{ is not weakly dominated in } [\underline{z}, \overline{z}]\}$$

and

$$\overline{s}_i = \sup\{x_i \in [\underline{z}_i, \overline{z}_i] : x_i \text{ is not weakly dominated in } [\underline{z}, \overline{z}]\}.$$

Now let $Z_i^w([\underline{z}, \overline{z}]) = [\underline{s}_i, \overline{s}_i]$ and

$$Z^w([\underline{z}, \overline{z}]) = \{x = (x_1, \ldots, x_I) : x_i \in Z_i^w(\ [\underline{z}, \overline{z}]) \ \text{ for all } i = 1, \ldots I\}.$$

Finally let $\overline{Z}^w([\underline{z}, \overline{z}])$ denote the interval $[\inf(Z^w([\underline{z}, \overline{z}]), \sup(Z^w([\underline{z}, \overline{z}]))]$.

Lemma 2 implies the following result.

Lemma 3. *Let Γ be an ID-supermodular game. Let $\underline{z}, \overline{z} \in X$ be profiles such that $\underline{z} \leq \overline{z}$. Then* $\sup Z^w([\underline{z}, \overline{z}]) = \overline{E}(\underline{W}(\overline{z}))$ *and* $\inf Z^w([\underline{z}, \overline{z}]) = \underline{E}(\overline{W}(\underline{z}))$*, and* $\overline{Z}([\underline{z}, \overline{z}]) = [\underline{E}(\overline{W}(\underline{z})), \overline{E}(\underline{W}(\overline{z}))]$.

Lemma 3 parallels Lemma 1. The first difference is that if $z_i \not\geq z_i^* \equiv \overline{E}(\underline{W}(\overline{z}))$, there is no guarantee that $z_i \vee z_i^*$ strictly dominates z_i. It is possible that $z_i \wedge z_i^*$ is a best response to $\underline{z}_{-i}$. Hence the second inequality in (2) could be weak. The second difference is that one can use weak dominance rather than strict dominance to delete a strategy. So one need only establish that $u_i(z_i \vee z_i^*, z{-}i) > u_i(z_i, z_{-i})$ for some $z_{-i} \in [\underline{z}, \overline{z}]$. This follows from the definition of z_i^*.

Proof of Theorem 2. The proof of the theorem follows the proof of Theorem 1. One applies Lemma 3 to obtain a decreasing sequence of intervals $[\underline{y}^k, \overline{y}^k]$ such that strategies outside of these intervals are weakly dominated. By monotonicity, $\lim_{k\to\infty} \underline{y}^k$ and $\lim_{k\to\infty} \overline{y}^k$ exist. Denote the limits by $\underline{y}$ and $\overline{y}$ respectively. It is straightforward to show that these limits are Nash Equilibrium profiles. In finite games (where the process of deleting strategies terminates after a finite number of iterations), it follows by construction that $\underline{y}$ and $\overline{y}$ are not weakly dominated by any strategy in $[\underline{y}, \overline{y}]$. From Lemma 3, it follows that anything that survives iterated deletion of weakly dominated strategies must be inside the interval.

The process described only removes strategies outside of the interval $[\underline{y}^k, \overline{y}^k]$. Consequently, it is possible that there are strategies in the interval $[\underline{y}, \overline{y}]$ that are weakly dominated. When the strategy set is finite, it must be the case that $\underline{y}_i$ and $\overline{y}_i$ remain undominated even if additional strategies are deleted. To see this notice that, by construction $\underline{y}_i$ is a best response to $\underline{y}_{-i}$ and the only other best responses to $\underline{y}_{-i}$ in $[\underline{y}_i, \overline{y}_i]$ are equivalent to $\underline{y}_i$. Consequently, $\underline{y}_i$ can only be weakly dominated if $\underline{y}_j$ is deleted for $j \neq i$. Hence no procedure can delete $\underline{y}_i$. Similarly, $\overline{y}_i$ cannot be deleted. This completes the proof of Theorem 2. ■

Theorem 2 uses the assumption that strategy sets are finite. This assumption guarantees that the iterative deletion process terminates in a finite number of steps and, consequently, that $\underline{y}$ and $\overline{y}$ are not weakly dominated. The next example demonstrates that the bounds obtained through the iterative process may be weakly dominated in games in which X_i are infinite.

Example 1. *Consider a three player game in which* $X_1 = [0,1]$ *and* $X_i = [0,2]$ *for* $i = 2,3$*;* $u_1(x) = x_1(x_2 - 1)$*,* $u_i(x) = x_1x_2x_3 - x_i^3/3$ *for* $i = 2,3$*. In this case* $\overline{y}^k = (1, 2^{2^{-k}}, 2^{2^{-k}})$ *and* $\underline{y}^k = (0,0,0)$*. It follows that* $\overline{y} = (1,1,1)$ *and* $\underline{y} = (0,0,0)$*. Both* $\overline{y}$ *and* $\overline{y}$ *are Nash equilibria, but* $\overline{y}$ *is weakly dominated with respect to strategies in* $[\underline{y}, \overline{y}]$*.*

Theorem 2 applies to a particular procedure for removal of weakly dominated strategies. Unlike iterative deletion of strictly dominates strategies, the outcome of iterative deletion of weakly dominated strategies may depend on the procedure. On the other hand, for some interesting classes of games, deletion of weakly dominated strategies is essentially independent of the procedure.

Marx and Swinkels [10] show that if a game satisfies the transfer of decision maker indifference (TDI) property, then two "full"[2] procedures for deleting weakly dominated strategies are the same up to the additional or removal of redundant strategies and a renaming of strategies. The TDI property states that if (given the behavior of the other players) Player i is indifferent between two strategies given the behavior of opponents, then all other players are also indifferent between the Player i's choice of strategies. TDI is restrictive, but can be shown to hold in interesting applications including (generically) the examples described in the Section 6.

5 Additional Properties

5.1 Dynamics

Milgrom and Roberts [11] show that there is relationship between adaptive dynamics and supermodular games. To do this, they consider a time-dependent strategy profile $x(t)$. They let $P(T,t)$ denote the strategies played between times T and t: $P(T,t) = \{x(s) : s \in [T,t)\}$ and say that $\{x(t)\}$ is a process **consistent with adaptive dynamics** if for all T there exists T' such that for all $t > T'$, $x(t) \in \overline{Z}([\inf P(T,t), \sup P(T,t)])$. They define $\underline{x} = \inf S$, $\overline{x} = \sup S$, $\underline{B}_k(x) = \underline{B}(\underline{B}^{k-1}(x))$, and $\overline{B}_k(x) = \overline{B}(\overline{B}^{k-1}(x))$ and show (in Theorem 8) that whenever $\{x(t)\}$ is a process consistent with adaptive dynamics in a supermodular game, for all k there exists T_k such that for all $t > T_k$, $x(t) \in [\underline{B}_k(\underline{x}), \overline{B}_k(\overline{x})]$.

The condition that a process is consistent with adaptive dynamics guarantees that strategies played at time t are best replies to strategies played in the not-too distant past. The conclusion of the theorem is that any process consistent with adaptive dynamics must eventually stop playing strictly dominated strategies and therefore converge to the interval of strategies with lower bound equal to the smallest Nash equilibrium and upper bound equal to the largest Nash equilibrium. This result is a direct consequence of Lemma 1 and holds true for ID-supermodular games. It is straightforward to modify the result to conclude that a more restrictive class of adaptive dynamics converges to the smaller set of strategies identified in Theorem 2.

The process $\{x(t)\}$ is **consistent with cautious adaptive dynamics** if for all T there exists T' such that for all $t > T'$, $x(t) \in \overline{Z}^w([\inf P(T,t), \sup P(T,t)])$.[3] Let $\underline{H}_k(x) = \underline{E}(\overline{W}(\underline{H}^{k-1}(x)))$, and $\overline{H}_k(x) = \overline{H}(\underline{W}(\overline{H}^{k-1}(x)))$.

[2]A "full" procedure stops only if it reaches a stage where there are no weakly dominated strategies.

[3]I use "cautious" in the sense of cautious rationalizability of Pearce [14]. The notion is that the

Theorem 3. *If $\{x(t)\}$ is a process consistent with cautious adaptive dynamics in a supermodular game, then for all k there exists T_k such that for all $t > T_k$, $x(t) \in [\underline{B}_k(\underline{x}), \overline{B}_k(\overline{x})]$.*

Theorem 3 is a direct consequence of Lemma 3.

Echenique [4] presents a modification of the procedure used to find upper and lower bounds in the proofs of Theorems 1 and 2 to provide an algorithm that finds all pure-strategy Nash equilibria in games in supermodular games. One can interpret the algorithm as a dynamic process. Consequently, there exist adaptive processes that reach Nash equilibria that do not survive iterative deletion of weakly dominated strategies. This result does not contradict Theorem 3. Instead it indicates that procedures that reach Nash equilibria that do not survive iterative deletion of weakly dominated strategies are not cautious. A critical issue is whether it is plausible to restrict attention to cautious dynamics. I believe that the correct answer is "it depends." On one hand, Cabrales and Ponti [2] and Gale, Binmore, and Samuelson [5] present examples of plausible evolutionary dynamics that converge to outcomes that that use weakly dominated strategies. On the other hand, Dubey, Haimanko, and Zapechelnyuk [3] introduce **pseudo-potential games**. A pseudo-potential game is a game for which there exists function $\phi : X \to \mathbb{R}$ such that $\arg\max_{x_i \in X_i} \phi(x_i, x_{-i}) \subset \arg\max_{x_i \in X_i} u_i(x_i, x_{-i})$. Dubey, Haimanko, and Zapechelnyuk [3] give conditions under which games with complementarities are pseudo-potential games. Their results imply that finite, two-player (ID) supermodular games are pseudo-potential games. Dubey, Haimanko, and Zapechelnyuk identify several properties of pseudo-potential games, including the property that there are no best-response cycles in generic, finite pseudo-potential games. This property guarantees convergence of best reply dynamics. Weak dominance has interesting implications only for games with non-generic payoffs.[4] Cautiously adaptive dynamics provide a way to extend these results to non-generic games.

5.2 Comparative Statics

In order to ask comparative statics questions, assume that there is a partially ordered set of parameters P and there is a parameterized family of games $\{\Gamma(p)\}_{p \in P}$ where $\Gamma(p) = \{I, X, u(\cdot; p), \geq\}$ where $u : X \times P \to \mathbb{R}^I$.

Theorem 4. *If $\{\Gamma(p)\}_{p \in P}$ is a family of ID-supermodular games and u_i has increasing differences in x_i and p for fixed x_{-i} then the largest and smallest strategies that survive iterative interval deletion of weakly dominated strategies, $\underline{x}_i(p)$ and $\overline{x}_i(p)$ are nondecreasing functions of p.*

The proof of this result is a straightforward modification Theorem 6 in Milgrom and Roberts [11]. The proof, which is in Appendix B, requires verification that $\underline{H}$ are $\overline{H}$ are monotonic.

Milgrom and Roberts [11, Theorem 7] given conditions under which it is possible to compare payoffs of different equilibria.

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adaptive process is a best response to beliefs that place positive probability on all "recently" used strategies.

[4]The applications I study have non-generic normal-form payoffs because they are derived from games with a fixed dynamic structure.

Theorem 5. *Let $\Gamma = (I, X, u, \geq)$ be an ID-supermodular game. Let $\underline{x}_i$ and $\overline{x}_i$ denote the smallest and largest elements of X_i, and suppose y and z are two equilibria with $y \geq z$. (1) If $u_i(\underline{x}_i, x_{-i})$ is increasing in x_{-i}, then $u_i(y) \geq u_i(z)$. (2) If $u_i(\overline{x}_i, x_{-i})$ is decreasing in x_{-i}, then $u_i(y) \leq u_i(z)$. If the condition in (1) holds for some subset of players I_1 and the condition in (2) holds for the remainder $I \setminus I_1$, then the largest equilibrium is the most preferred equilibrium for the players in I_1, and the least preferred for the remaining players.*

This result holds in my setting, but one variation is worth noting. If Condition 1 in the theorem holds, then the largest Nash Equilibrium is Pareto dominant (in the set of Nash Equilibria). It is possible that strategies used in this equilibrium do not survive iterative deletion of weakly dominated strategies. The upper bound in Theorem 2 may therefore not be the Pareto dominant Nash Equilibrium. Instead it will be (in finite games), the Pareto-dominant Nash Equilibrium in strategies that survive iterative deletion of weakly dominated strategies. Milgrom and Roberts discuss an interesting classes of games (games with positive spillovers) in which equilibria are Pareto ranked. The literature treats the largest Nash Equilibrium as salient in these games. For typical specifications of these games, the largest equilibrium is also an equilibrium that survives iterative deletion of weakly dominated strategies. Theorem 5 suggests that in more general settings it may be the that Pareto-efficient Nash equilibrium fails to survive iterative deletion of weakly dominated strategies.

5.3 Quasisupermodularity

This paper concentrates on weakening the monotonicity condition (increasing differences) used by Milgrom and Roberts. Theorem 1 merely replaces increasing differences with interval dominance. Theorem 2 extends the result – again with the weaker condition – to iterative weak dominance. In the same way, one can replace the supermodularity assumption with quasi-supermodularity. They compare values of two quantities, which are in turn the difference between between a function evaluated at a higher and a lower point. Supermodularity and increasing differences require that the first quantity is greater than the second. Quasi-supermodularity and single crossing (interval dominance) require the weaker condition that the first quantity is non-negative (positive) whenever the first one is non-negative (positive). It is the second implication that is needed for the main results. That is, Theorems 1 and 2 hold if payoff functions are quasi-supermodular. I chose not to state the more general results because I know of no application in which payoffs are quasi-supermodular but not supermodular.

5.4 Identification

There is a literature that estimates supermodular games. For example, Uetake and Watanabe [19] use the bounds constructed in Milgrom and Roberts [Theorem 5][11] to generate moment inequalities. I believe that the same techniques would apply to estimate strategies that satisfy the refinement (surviving iterated deletion of weakly dominates strategies). The bounds constructed in Theorem 2 would replace those in Theorem 1.[5] This kind of study would be consistent with research by Aradillas-Lopez and Tamer [1], which compares the identification power of rationalizability to Nash Equilibria and Molinari and Rosen [13] who estimate level-k rationality in a supermodular game.

There is an econometric literature that tries to identify and test monotone comparative statics in supermodular games. There are two basic approaches. The first approach (for example,Lazzati [7], and Uetake and Watantabe [19]) is to impose monotonicity and study the restrictions imposed by a solution concept (Nash equilibrium or rationalizability) on data. One could ask this question instead requiring the solution only use strategies that survive iterative deletion of weakly dominated strategies. Theorem 2 suggests new bounds on strategies that would replace the restrictions the literature has provided for rationalizability.

Another approach imposes no a priori restrictions and asks when a data set is consistent with equilibrium behavior in a supermodular game. Lazzati [7], Lazzati, Quah, and Shirai [8] provide a necessary and sufficient condition for a data set to be consistent with Nash equilibrium behavior in a supermodular game with a one-dimensional strategy space. A natural modification of the question is to ask whether the data set is consistent with equilibrium behavior in weakly undominated strategies in a ID-supermodular game.

6 Applications

Extending the results about supermodular games from strong to weak dominance is more than a curiosity only if there exist interesting games under which the assumptions of the previous section hold and the arguments reduce the set of predictions. An ideal application would be a ID-supermodular game that is not supermodular, in which weak dominance arguments have more power to refine the set of equilibria than strong dominance arguments, and in which insights about the structure of equilibria available from the results in this paper have substantive interest.

This section describes two applications. One characteristic that the examples share is a sequential structure. They add a round of strategic behavior to an underlying game. This kind of game is a natural place to expect weak dominance to play a role as weak dominance can place restrictions on off-the-path behavior.

The examples demonstrate that there are games that are not supermodular, but have some of the structure of supermodular games. It will be clear that strong dominance arguments do not restrict the predictions. Detailed analysis of the implications of weak dominance require specialized arguments and appear in separate papers.

Each of the applications is imperfect because the games are not ID-supermodular. In each case, I must extent the theory somewhat so that it applies.

[5] One limitation of the approach is that Uetake and Watanabe focus on one-dimensional strategies spaces. The (ID)-supermodular games that I identify in Section 6 in which weak dominance has selection power involve multidimensional strategy spaces.

6.1 Cheap Talk

Kartik and Sobel [6] apply techniques like those in this paper to provide a selection argument in simple cheap-talk games. In the cheap-talk game, nature selects $t \in T$; one player, the Sender (S), learns t and sends a message $m \in M$; the other player, the Receiver (R), takes an action $a \in A$ in response to m. A strategy for S is a mapping $\sigma : T \to M$. A strategy for R is a mapping $\alpha : M \to A$. Assume that M is a finite, ordered, set and that A and T are equal to the unit interval. Assume that there is a prior distribution on types; for convenience assume that the prior is finitely supported and $p(t)$ is the probability that the type is t. Payoffs depend only on a and t. The payoff to Player i when t is the Sender's type and a is the action of the Receiver is $U^i(a,t)$. Assume that $U^i(\cdot)$ is twice continuously differentiable, with negative second derivative with respect to a and positive cross partial. With this structure, order R strategies in the natural way: $\alpha'' \succ \alpha'$ if $\alpha''(m) \geq \alpha'(m)$ for all m. Order S strategies "backwards" so that $\sigma'' \succ \sigma'$ if and only if $\sigma''(t) \leq \sigma'(t)$ for all t.[6] The payoff functions for the cheap-talk game are $u_S(\sigma,\alpha) = EU^S(\alpha(\sigma(t)),t)$ and $u_R(\alpha,\sigma) = EU^R(\alpha(\sigma(t),t))$, where the expectation is taken using the prior on types. It is straightforward to check that this game satisfies the TDI condition of Marx and Swinkels [10].

In this subsection, I describe several properties of this class of games and show how the general results provide some insight into the structure of their equilibria.

Lemma 4. *For $i = 1, 2$, $u_i(\cdot)$ is supermodular in x_i for fixed x_{-i} in cheap-talk games.*

Lemma 4 follows from a straightforward argument, which appears in Appendix B.

Without further assumptions best responses will not have any monotonicity properties in the basic cheap-talk game. For example, suppose that $U^R(a,t) = -(a-t)^2$, and the prior is uniform on $\{0, 1/N, \ldots, k/N, \ldots, 1\}$ for some even number N. Assume that M contains messages m_0 and m_1 with $m_0 < m_1$. If the Sender always sends m_0, then it is a best response for the Receiver to respond to m_0 with .5 and all other messages with 0. Denote this strategy by α^{**}. Let

$$\sigma(t) = \begin{cases} 1 & \text{if } t \in [0, .5] \\ 0 & \text{if } t \in (.5, 1] \end{cases} \text{ and } \alpha(m) = \begin{cases} 1 & \text{if } m_0 \\ 0 & \text{otherwise} \end{cases}.$$

The Receiver prefers $\alpha \wedge \alpha^{**}$ to α when S always sends m_0, but R's preferences reverse when S plays σ. Consequently interval dominance does not hold. One can confirm that S's preferences violate interval dominance and that the violations do not depend on the choice of order over S's strategies.

Best response correspondences do have some monotonicity properties for a restricted version of the cheap-talk game. Henceforth consider a **monotonic restriction** of the cheap-talk game. In the monotonic restriction, the Sender and Receiver are restricted to monotonic strategies (σ is monotonic if $t'' > t'$ implies $\sigma(t'') \geq \sigma(t')$; α is monotonic if $m'' > m'$ implies that $\alpha(m'') \geq \alpha(m')$). See Kartik and Sobel [6] for a justification of the monotonic restriction. I call the monotonic restriction of a cheap-talk game a **monotone cheap-talk game**.

[6] This ordering guarantees that R's best response increases when S's strategy increases.

Even with the restriction to monotonic strategies, the cheap-talk game does not satisfy increasing differences.

To see that the Sender's payoff does not satisfy increasing differences, let $\sigma(t) \equiv 0$ and $\sigma'(t) \equiv 1$ so that $u_S(\sigma, \alpha) - u_S(\sigma', \alpha) = E[U^S(\alpha(0), t) - U^S(\alpha(1), t)]$. The right-hand side is not monotonic in $\alpha(0)$ (or in $\alpha(1)$), so the increasing difference condition does not hold.

To see that the Receiver's payoff does not satisfy increasing differences, let $\alpha'(t) \equiv 1$. Hence $u_R(\alpha, \sigma)$ does not depend on σ. Fix a message $\tilde{m}$ and let

$$\alpha(m) = \begin{cases} 0 & \text{if } m \leq \tilde{m} \\ 1 & \text{if } m > \tilde{m} \end{cases}$$

so that $u_R(\alpha, \sigma) - u_R(\alpha', \sigma)$. Increasing σ will change $u_R(\alpha', \sigma)$ but these changes need not be monotonic.

In general, the Receiver's preferences do not satisfy (ID). To see this, let m_0 denote the lowest message and suppose that $\sigma'(t) > m_0$ for all t, while $\sigma''(t) \equiv m_0$. It follows that $\sigma'' \succ \sigma'$. It is straightforward to construct σ', α and α^* such that $u^R(\alpha \vee \alpha^*, \sigma') > u^R(\alpha, \sigma')$ but $u^R(\alpha \vee \alpha^*, \sigma'') < u^R(\alpha, \sigma'')$. For example, let σ' be a separating strategy; let α^* be a best response to σ'; and let $\alpha(m) = \arg\max \sum_t U_R(a,t)p(t)$ for all m.

Consequently, the general results about ID-supermodular games do not apply to this example. In order to use the characterization results, I must weaken the (ID) property.

Definition 6. *Let X and Y be lattices. Assume $x^* \in \arg\max_{x \in X} f(x, y')$, and $x^{**} \in \arg\max_{x \in X} f(x, y'')$. A function $f : X \times Y \to \mathbb{R}$ satisfies the weak generalized interval-dominance property (WID) in its two arguments on the set $X \times Y$ if for all $y'' > y'$,*

$$f(x' \vee t, y') \geq f(x', y') \implies$$
$$\exists \tilde{t} \leq t, f(\tilde{t}, y') \geq f(x', y'), \quad \text{such that } f(x' \vee \tilde{t}, y'') \geq f(x', y'') \tag{7}$$

and

$$f(x' \wedge t, y'') \geq f(x', y'') \implies$$
$$\exists \tilde{t} \geq t, f(\tilde{t}, y'') \leq f(x', y''), \quad \text{such that } f(x' \wedge \tilde{t}, y') \geq f(x', y'). \tag{8}$$

The (WID) condition is weaker than (ID). Appendix C proves this result and introduces related concepts. One way to get an intuition for (WID) is to compare it to single crossing, which requires Condition (7) and (8) to hold when $\tilde{t} = x'$.[7]

(ID) and (WID) are both conditions that relate to how solutions to $\max_x u_i(x, y)$ change with the parameter y. Fact 2 states that in an ID-supermodular game, Player i's set of best responses are increases in x_{-i}, where "increasing" is interpreted in the sense of the strong set order. If $u_i(\cdot)$ satisfies (WID), then best responses are increasing in a weaker sense.

The next result describes a property of (WID). The proposition uses the following notation: $x^{**} \in \arg\max f(x, y'')$, $x^* \in \arg\max f(x, y')$, $\overline{x}^{**} = \max \arg\max f(x, y'')$, $\underline{x}^* = \min \arg\max f(x, y')$.

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[7]The analog to (8) in the definition of single crossing is implied by Condition 1.

Proposition 1. *Let X and Y be lattices. If the function $f : X \times Y \to \mathbb{R}$ satisfies (WID), then*

$$x^{**} \vee \underline{x}^{*} \in \arg\max f(x, y'') \text{ and } x^{*} \wedge \overline{x}^{**} \in \arg\max f(x, y'). \tag{9}$$

Appendix C contains a proof of Proposition 1. The conclusion of Proposition 1 certainly holds when $\arg\max u_i(\cdot, x_{-i})$ is increasing in the strong set order (provided that there exist solutions to the maximization problems). It is straightforward to confirm that monotonicity property in the proposition is actually weaker.

I say that a game $\Gamma = (I, X, u, \geq)$ is WID-supermodular if it satisfies Conditions (A1)-(A3) in Definition 4 and Condition (A4) is replaced by the requirement that u_i satisfies (WID) in x_i and x_{-i} on all interval sublattices of X.

The class of WID-supermodular games is interesting because monotone cheap-talk games are WID-supermodular and because the equilibria of these games have some of the important properties of ID-supermodular games and because . The remainder of this subsection reports results that confirm these claims. I show first that monotone cheap-talk games and WID-supermodular. I conclude the subsection (Theorem 6) with the observation that equilibria that survive iterated deletion of weakly dominated strategies have nice bounds in WID-supermodular games.

Lemma 5. *Receiver's preferences in a monotone cheap-talk game satisfy (WID).*

Similarly, the Sender's preferences also satisfy (WID) but not (ID).

Lemma 6. *Sender's preferences in a monotone cheap-talk game satisfy (WID).*

The three preliminary results of this subsection combine to establish the following proposition.

Proposition 2. *Monotone cheap-talk games are (WID)-supermodular.*

Proposition 2 is useful because it is possible to generalize Theorem 2. Although I am unable to prove an analog to Lemma 2 for WID-supermodular games, the following result holds for WID-supermodular games.

Theorem 6. *Let Γ be a WID-supermodular game with strategy space $[\underline{z}, \overline{z}]$. For each player i, there exist pure Nash equilibrium profiles $\underline{x}_i$ and $\overline{x}_i$ such that all strategies that survive iterative interval deletion of weakly dominated strategies are contained in $[\underline{x}_i, \overline{x}_i]$. Moreover, there exist an increasing sequences $\{\underline{y}^n\}_{n=1}^{\infty}$ and a decreasing sequence $\{\overline{y}^n\}_{n=1}^{\infty}$ where $\underline{y}^1 = \underline{z}$ and $\overline{y}^1 = \overline{z}$; for $n > 1$, $\underline{y}_i^n = \underline{W}_i(\underline{y}^{n-1})$ and $\overline{y}_i^n = \overline{W}_i(\overline{y}^{n-1})$; and $\underline{x} = \lim_{n\to\infty} \underline{y}^n$, and $\overline{x} = \lim_{n\to\infty} \overline{y}^n$.*

I omit the proof for Theorem 6, which parallels the proof of Theorem 2. The result states that one can bound the set of strategies that survive iterative interval deletion of weakly dominated strategies by taking best responses to the largest and smallest strategies surviving. Proposition 2 demonstrates that Theorem 6 applies to monotone cheap talk games. These games typically have multiple Nash equilibria. Kartik and Sobel [6] establish that in a widely studied class of monotone cheap-talk games there is a unique equilibrium outcome that survives iterative deletion of weakly dominates strategies (in any order). Consequently the bounds in Theorem 6 are (strictly) more restrictive than the bounds in Theorem 2.

6.2 Investment Games

This subsection presents an example in which the (WID) fails, but the logic of the weak dominance argument still applies. To accommodate this example, I extend the concept of an ID-supermodular game.

Definition 7. *Let $\Gamma = (I, X, u, \geq)$ be a game and suppose that there exists a j such that $X_j = X'_j \times X''_j$, where X'_j and X''_j are complete lattices. The game $\Gamma = (I, X, u, \geq)$ is a interval-dominance supermodular (ID-supermodular) game conditional on X'_j if, for each $i \neq j \in I$:*

(A1) X is a complete lattice;

(A2) for each $i \in I$, $u_i : X \to \mathbb{R}$ is order upper-semicontinuous in x_i for fixed x_{-i}; u_i is order upper continuous in x_{-i} for fixed x_i; and u_i is bounded above;

(A3) for each $i \in I$, u_i is supermodular in x_i for fixed x_{-i};

(A4′) for each $i \neq j \in I$ u_i satisfies the interval-dominance property in x_i and x_{-i} on all interval sublattices of X. and

(A5) u_j satisfies the interval-dominance property in x''_j and (x_{-j}, x'_j) on all interval sublattices of X.

I refer to (A5) as conditional interval dominance of u_j (given x'_j).

If Γ is an ID-supermodular game conditional on X'_j, then one can apply dominance arguments component by component. That is, one can fix $x'_j \in X'_j$ and construct a decreasing sequence $[\underline{y}^k, \overline{y}^k]$ such that the X'_j component of $\underline{y}^k$ and $\overline{y}^k$ is equal to x'_j for all k such that strategies outside of these intervals (X'_j fixed) are weakly dominated. Even when a game fails to be ID-supermodular, the dominance arguments apply in part. In this subsection, I identify a class of games that are ID-supermodular conditional on a set, but not ID-supermodular. These arguments are the basis of a uniqueness result on games with pre-play communication (Sobel [17]).

Consider a two-player game in which Player 1 first makes an observable investment ($k \in K$) and then both players simultaneously make decisions ($(x_1, x_2) \in \tilde{X}_1 \times X_2$). I assume that $K, \tilde{X}_1, X_2 \subset [0, 1]$ and that $X_1 = K \times \tilde{X}_1$. The strategy set for Player 1 consists of pairs $(k, x_1) \in X_1$. Strategies for Player 2 are mappings from K into X_2. As in the previous subsection, limit attention to monotonic strategies for Player 2 (if $k'' > k'$, then $x_2(k'') \geq x_2(k')$). Preferences in the game are derived from utility functions for the subgame in which players simultaneously make decisions: $U_i : \tilde{X}_1 \times X_2 \to \mathbb{R}$. Payoff functions for the extended game take the form $u_i((k, x_1), x_2) = U_i(x_1, x_2(k)) + \lambda_i k$. Order Player 1's strategies by the order on X_1 induced by the standard order on $[0, 1] \times [0, 1]$. Order Player 2's strategies $x''_2 \geq x'_2$ if and only if $x''_2(k) \geq x'_2(k)$ for all K. Provided that the underlying game has generic payoffs (for example, $U_i(x) = U_i(x')$ implies $x = x'$), then the investment games satisfies the TDI condition of Marx and Swinkels [10].

It is possible to interpret this setting as a model of communication about intentions. Under this interpretation, Player 1 first sends a signal to Player 2 and then they play a two-player simultaneous move game. An alternative interpretation is that k is an investment that Player 1 makes prior to the players participating in a two-player game. For the first interpretation, $\lambda_i = 0$ for both i. For the second interpretation, $\lambda_1 < 0 = \lambda_2$.

Definition 8. *An investment game is a* **regular investment game** *if the underlying game determined by* (U_1, U_2) *is a supermodular game (so that* u_i *satisfies increasing differences and is supermodular for* $i = 1, 2$*);* $x_2(\cdot)$ *is monotonic; and* $U_2(x_1, \cdot)$ *is single-peaked in its second argument for all* x_1*.*[8]

Appendix B contains routine verifications of the following three properties.

Lemma 7. *For* $i = 1, 2$*,* $u_i(\cdot)$ *is supermodular in* x_i *for fixed* x_{-i} *in regular investment games.*

Lemma 8. *In a regular investment game, preferences for Player 2 satisfy interval dominance.*

Lemma 9. *In a regular investment game, Player 1's preferences satisfy conditional ID* given* k *for all* k*.*

The lemmas of this subsection combine to establish the following proposition.

Proposition 3. *Regular investment games games are (ID*)-supermodular games conditional on* K*.*

Proposition 3 does not guarantee that one can use weak dominance arguments to refine predictions in monotone cheap-talk games. Sobel [17] establishes that a unique equilibrium outcome survives iterative deletion of weakly dominates strategies (in any order) in a class of games with pre-play communication.

A regular investment game need not satisfy (WID*) for Player one. In order to satisfy (WID*) it must be the case that there exists $(\tilde{k}^*, \tilde{x}_1^*) \leq (k^*, x_1^*)$ such that $u_1((\tilde{k}^*, \tilde{x}_1^*), x_2') \geq u_1((k, x_1), x_2')$ for all (k, x_1) and

$$u_1((k, x_1) \vee (\tilde{k}^*, \tilde{x}_1^*), x_2') \geq u_1((k, x_1), x_2') \implies u_1((k, x_1) \vee (\tilde{k}^*, \tilde{x}_1^*), x_2'') \geq u_1((k, x_1), x_2''). \tag{10}$$

Suppose that $U_1(x_1^*, x_2'(k^*)) > U_1(x_1, x_2)$ unless $(x_1, x_2) = (x_1^*, x_2'(k^*))$. When $x_1 = x_1^*$, $k \geq k^*$, $k < k^*$, and $x_2''(k^*) > x_2''(k) = x_2'(k^*) > x_2'(k)$. Under these assumptions (10) reduces to

$$U_1(x_1^*, x_2'(k^*)) \geq U_1(x_1^*, x_2'(k)) \implies U_1(x_1^*, x_2''(k^*)) \geq U_1(x_1^*, x_2'(k^*)). \tag{11}$$

The implication in (11) fails when $x_2''(k^*) > x_2'(k^*)$.

[8]The function $U_2(x_1, \cdot)$ is single peaked in its first argument if there exists $x_2^*(x_1)$ such that $U_2(x_1, x_2)$ is increasing in x_2 for $x_2 < x_2^*(x_1)$ and decreasing in x_2 for $x_2 > x_2^*(x_1)$.

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Appendix A: Definitions

Following Milgrom and Roberts, I define several basic concepts.

Definition 9. *Given $T \subset X$, $\bar{b} \in X$ is called an upper bound for T if $\bar{b} \geq x$ for all $x \in T$; it is the supremum of T (denoted $\sup(T)$) if it is an upper bound and for all upper bounds b of T, $b \geq \bar{b}$. Lower bounds and infimums are defined analogously. A point x is a maximal element of X if there is no $y \in X$ such that $y > x$ (that is, no y such that $y \geq x$ but not $x \geq y$); it is the largest element of X if $x \geq y$ for all $y \in X$. Minimal and smallest elements are defined similarly.*

Definition 10. *The set X is a lattice if for each two point set $\{x, y\} \subset X$, there is a supremum for $\{x, y\}$ (denoted $x \vee y$ and called the join of x and y) and an infimum (denoted $x \wedge y$ and called the meet of x and y) in X. The lattice is complete if for all nonempty subsets $T \subset X$, $\inf(T) \in X$ and $\sup(T) \in X$. An interval is a set of the form $[x, y] \equiv \{z : y \geq z \geq x\}$.*

Definition 11. *A sublattice T of a lattice X is a subset of X that is closed under $\wedge$ and $\vee$. An interval sublattice T of a lattice X is a sublattice of X of the form $[\underline{x}, \overline{x}]$ for some $\underline{x}, \overline{x} \in X$, $\underline{x} \leq \overline{x}$. A complete sublattice T is a sublattice such that the infimum and supremum of every subset of T is in T.*

Definition 12. *A chain $C \subset X$ is a totally ordered subset of X, that is, for any $x \in C$ and $y \in C$, $x \geq y$ or $y \geq x$.*

Definition 13. *Given a complete lattice X, a function $f : X \to \mathbb{R}$ is order continuous if it converges along every chain C (in both the increasing and decreasing directions), that is, if $\lim_{x \in C, x \downarrow \inf C} f(x) = f(\inf(C))$ and $\lim_{x \in C, x \uparrow \sup C} f(x) = f(\sup(C))$. It is order upper-semicontinuous if $\limsup_{x \in C, x \downarrow \inf C} f(x) \leq f(\inf(C))$ and $\liminf_{x \in C, x \uparrow \sup C} f(x) \leq f(\sup(C))$.*

Definition 14. *A function $f : X \to \mathbb{R}$ is supermodular if for all $x, y \in X$,*

$$f(x) + f(y) \leq f(x \wedge y) + f(x \vee y). \tag{12}$$

Definition 15. *The set S'' dominates S' in the strong set order (written $S'' \geq S'$) if $x^* \in S'$ and $x^{**} \in S''$ imply that $x^* \wedge x^{**} \in S'$ and $x^* \vee x^{**} \in S''$.*

Appendix B: Proofs

The Appendix contains proofs that did not appear in the main text.

Proof of Lemma 2. Let $\underline{w}_i = \overline{W}_i(\underline{z})$ be the largest best response to the smallest strategy profile for Player i. It follows from (ID) that any $x_i \leq \underline{w}_i$ is either weakly dominated by $\underline{w}_i$ or equivalent to $\underline{w}_i$ in the sense that $u_i(x_i, x_{-i}) = u_i(\underline{w}_i, x_{-i})$ for all $x_{-i} \in [\underline{z}_{-i}, \overline{z}_{-i}]$.

I claim that $E_i(\underline{w}_i)$ is a lattice. If $x, x' \in E_i(\underline{w}_i)$, then $x_i \vee x_i'$ and $x_i \wedge x_i'$ are best responses to $\underline{z}_{-i}$. Hence $x_i \vee x_i' \leq \underline{w}_i$ by the definition of $\underline{w}_i$. Consequently, by (ID), $x_i \vee x_i' \in E_i(\underline{w}_i)$. Furthermore, $u_i(x_i \wedge x_i', z_{-i}) \leq u_i(x_i \vee x_i', z_{-i})$ for all $z_{-i} \in [\underline{z}_{-i}, \overline{z}_{-i}]$ by (ID). It follows that

$$2u_i(x_i \vee x_i', z_{-i}) \geq u_i(x_i \vee x_i', z_{-i}) + u_i(x_i \wedge x_i', z_{-i}) \geq u_i(x_i, z_{-i}) + u_i(x_i', z_{-i}) = 2u_i(x_i \vee x_i', z_{-i}), \tag{13}$$

where the second inequality follows from supermodularity and the equation follows because $x_i, x_i', x_i \vee x_i' \in E_i(\underline{w}_i)$. Consequently the first inequality in (13) must be an equation and $x_i \wedge x_i' \in E_i(\underline{w}_i)$ by supermodularity.

Since $E_i(\underline{w}_i)$ is a lattice, it has a smallest element, w_i^*. I claim that w_i^* is the smallest strategy that is not weakly dominated. First observe that if $x_i < w_i^*$, then x_i is weakly dominated. This claim follows because, by definition, x_i is less than $\underline{w}_i$ but not equivalent to $\underline{w}_i$. So it must be weakly dominated by $\underline{w}_i$. Second take any $z_i \not\geq w_i^*$. I claim that $z_i \vee w_i^*$ weakly dominates z_i. The claim follows because

$$u_i(z_i \vee w_i^*, z_{-i}) - u_i(z_i, z_{-i}) \geq u_i(w_i^*, z_{-i}) - u_i(z_i \wedge w_i^*, z_{-i})$$

by supermodularity and w_i^* weakly dominates $z_i \vee w_i^*$ by the first observation (since $w_i^* > z_i \wedge w_i^*$). Finally, to show that w_i^* is not weakly dominated, assume that w_i^* is weakly dominated by some w_i and argue to a contraction. If w_i weakly dominates w_i^*, then it must also weakly dominate $\underline{w}_i$, since w_i^* and $\underline{w}_i$ are equivalent. But since $\underline{w}_i$ is the largest best response to $\underline{z}$, $w_i \leq \underline{w}_i$. Consequently w_i is equivalent to $\underline{w}_i$ or weakly dominated by $\underline{w}_i$. If w_i is equivalent to $\underline{w}_i$ it cannot weakly dominates w_i^*. If w_i is weakly dominated by $\underline{w}_i$, then it must also be weakly dominated by w_i^*, so it cannot weakly dominate w_i^*. It follows that w_i^* is the smallest of $[\underline{z}_i, \overline{z}_i]$ that is not weakly dominated. A similar argument demonstrates that there is a largest element of $[\underline{z}_i, \overline{z}_i]$ that is not weakly dominated.

Proof of Theorem 4. Let $\overline{H}(x,p)$ be the largest strategy that is equivalent to the smallest best response to x. I claim that $\overline{H}(x,p)$ is nondecreasing in p. Suppose that $p' > p$. Let $\hat{z}_i' = \overline{H}_i(x,p')$ and $\hat{z}_i = \overline{H}_i(x,p)$. Since the set of best responses is a lattice and the best response set is monotonic by Fact 2, $\hat{z}_i \wedge \hat{z}_i'$ is a best response to x given p. Since $\hat{z}_i \wedge \hat{z}_i' \leq \hat{z}_i$, it must be equal to $\hat{z}_i$. It follows that

$$\begin{aligned} u_i(\hat{z}_i{}' \vee \hat{z}_i, z_{-i}, p') - u_i(\hat{z}_i{}', z_{-i}, p') &\geq \\ u_i(\hat{z}_i, z_{-i}, p') - u_i(\hat{z}_i{}' \wedge \hat{z}_i, z_{-i}, p') &\geq \\ u_i(\hat{z}_i, z_{-i}, p) - u_i(\hat{z}_i{}' \wedge \hat{z}_i, z_{-i}, p) &= 0, \end{aligned} \tag{14}$$

where the first inequality follows from supermodularity, the second inequality follows from the interval dominance property (in p). It follows from (14) that $\hat{z}_i{}' \vee \hat{z}_i = \overline{H}(x,p')$. It must be that $\hat{z}_i{}' \geq \hat{z}_i$. This establishes that $\overline{H}_i(x,p)$ (and hence $\overline{H}(x,p)$) is nondecreasing. Every Nash equilibrium satisfies $\overline{H}(x,\tau) \geq x$. By Tarksi's Fixed Point Theorem, $\overline{x}(p) = \sup\{x : \overline{H}(x,p) \geq x\}$ is a fixed point of $\overline{H}(\cdot,p)$, so it is the largest Nash equilibrium. Since $\overline{H}(x,\cdot)$ is nondecreasing, $\overline{H}(x,\cdot)$ is nondecreasing. A similar argument applies to the smallest equilibrium. ■

Proof of Lemma 4.

$$\begin{aligned}
u_S(\sigma \vee \sigma', \alpha) + u_S(\sigma \wedge \sigma', \alpha) &= E[U^S(\alpha(\min\{\sigma(t), \sigma'(t)\}), t) + U^S(\alpha(\max\{\sigma(t), \sigma'(t)\}), t)] \\
&= E[U^S(\alpha(s(t)), t) + U^S(\alpha(\sigma'(t)), t)] \\
&= u_S(\sigma, \alpha) + u_S(\sigma', \alpha)
\end{aligned}$$

$$\begin{aligned}
u_R(\alpha \vee \alpha', \sigma) + u_R(\alpha \wedge \alpha', \sigma) &= E[U^R(\max\{\alpha(\sigma(t)), \alpha'(\sigma(t))\}, t) + U^R(\min\{\alpha(\sigma(t)), \alpha'(\sigma(t))\}, t)] \\
&= E[U^R(\alpha(\sigma(t)), t) + U^R(\alpha'(\sigma(t)), t)] \\
&= u_R(\alpha, \sigma) + u_R(\alpha', \sigma)
\end{aligned}$$

■

Proof of Lemma 5. Assume $u_R(\alpha \vee \alpha^*, \sigma') \geq u_R(a, s')$. Let $\tilde{\alpha}^* = \min \arg\max_\alpha u_R(\alpha, \sigma')$ be the smallest best response to σ'. From Proposition 5 (Appendix C), it sufficient to show that if $\sigma'' \succ \sigma'$, then $u_R(\alpha \vee \tilde{\alpha}^*, \sigma'') \geq u_R(\tilde{\alpha}, \sigma'')$. Let $\mu_\sigma(\cdot \mid m)$ be the posterior distribution over t given $\sigma(t) = m$. The posterior is well defined if there exists t such that $\sigma(t) = m$. It suffices to prove that, for all m in the image of $\sigma''(\cdot)$,

$$\sum_t U^R(\alpha \vee \tilde{\alpha}(m), t)\mu_{\sigma''}(t \mid m) \geq \sum_t U^R(\alpha(m), t)\mu_{\sigma''}(t \mid m). \tag{15}$$

If $\sigma'(t) < m$ for all t, then $\sigma''(t) < m$ for all t (recall that $\sigma'' \succ \sigma'$ implies $\sigma''(t) \leq \sigma'(t)$ for all t). If there exists t such that $\sigma'(t) = m$, then $\sigma'' \succ \sigma'$ implies that $\mu_{\sigma''}$ stochastically dominates $\mu_{\sigma'}$. Since $\tilde{\alpha}^*(m)$ solves $\max \sum_t U^R(a, t)\mu_{\sigma'}(t \mid m)$, it follows from the supermodularity of u_R that the solution to $\max \sum_t U^R(\tilde{\alpha}(m), t)\mu_{\sigma''}(t \mid m)$ is greater than $\tilde{\alpha}^*(m)$ and by concavity of $U_R(\cdot, t)$ that $\alpha(m) \leq \tilde{\alpha}(m)$ implies that inequality (15) holds. If $\sigma'(t) > m$ for all t, then $\tilde{\alpha}(m) = 0$ by definition and Inequality (15) holds.

It remains to consider the case in which there does not exist t such that $\sigma'(t) = m$, but $\sigma'(t) < m$ for some t. In this case, define $\underline{m}$ to be

$$\max\{m' < m : \text{ there exists } t \text{ such that } \sigma'(t) = m'\}.$$

It follows that $\tilde{\alpha}^*(m)$ solves $\max \sum_t U^R(a, t)\mu_{\sigma'}(t \mid \underline{m})$. Let $\bar{t} = \max\{t : \sigma'(t) \leq \underline{m}\}$. Since $\sigma'(t) = \underline{m}$ for some t, $\bar{t}$ is well defined. Furthermore, $\tilde{\alpha}(m) \leq \arg\max U^R(a, \bar{t})$. Since $\sigma'' \succ \sigma'$, $\mu_{\sigma''}(t \mid m) = 0$ if $t < \bar{t}$. Hence $\tilde{\alpha}^*(m) \leq \arg\max \sum_t u_R(\alpha(m), t)\mu_{\sigma''}(t \mid m)$ and so (15) holds.

A symmetric argument establishes that if $\sigma'' \succ \sigma'$, $u_R(\alpha \wedge \alpha^{**}, \sigma'') \geq u_R(\alpha, \sigma'')$, then $u_R(\alpha \wedge \tilde{\alpha}^{**}, \sigma') \geq u_R(\alpha, \sigma')$ (when $\tilde{\alpha}^{**}$ is the largest best response to σ''). ■

Proof of Lemma 6. Assume that $u_S(\alpha', \sigma^* \vee \sigma) \geq u_S(\alpha', \sigma)$. Let $\tilde{\sigma}^* = \min \arg\max U^S(\alpha', \sigma)$ be the smallest best response to α'. From Proposition 5, it suffices to show that if $\alpha'' \succ \alpha'$, then $u_S(\alpha'', \sigma \vee \tilde{\sigma}) \geq u_S(\alpha'', \sigma)$. It suffices to show that, for all t, $\sigma(t) > \tilde{\sigma}^*(t)$

implies that $U^S(\alpha''(\tilde{\sigma}^*(t)), t) \geq U^S(\alpha''(\sigma(t)), t))$. If $\sigma(t) > \tilde{\sigma}^*(t)$, then by definition of $\tilde{\sigma}^*$, $U^S(\alpha'(\tilde{\sigma}^*(t), t) > U^S(\alpha'(\sigma(t)), t)$. The inequality must be strict because $\tilde{\sigma}$ is the smallest best response (so type t sends the highest message that leads to the maximum available payoff) and $\tilde{\sigma}^*(t) < \sigma(t)$. It follows from concavity of $U^S(\cdot, t)$ that $U^S(\alpha''(\tilde{\sigma}^*(t)), t) \geq U^S(\alpha''(\sigma(t)), t))$. Notice that this inequality may be weak (if $\alpha''(\tilde{\sigma}^*(t)) = \alpha''(\sigma(t)))$ so that (WID) does not hold.

A symmetric argument establishes that if $\alpha'' \succ \alpha'$, $u_S(\alpha'', \sigma^{**} \wedge \sigma) \geq u_S(\alpha'', \sigma)$ implies that $u_S(\alpha'', \tilde{\sigma}^{**} \wedge \sigma) \geq u_S(\alpha'', \sigma)$, where $\tilde{\sigma}^{**} = \max \arg\max u_S(\alpha'', \sigma)$. ■

Proof of Lemma 7. Assume without loss of generality that $k \geq k'$,

$$\begin{aligned} u_1((k, \tilde{x}_1) \vee (k', \tilde{x}_1''), x_2) + u_1((k, \tilde{x}_1) \wedge (k', \tilde{x}_1'), x_2) &= \\ U_1(\tilde{x}_1 \vee \tilde{x}_1', x_2(k)) + U_1(\tilde{x}_1 \wedge \tilde{x}_1', x_2(k')) + \lambda_1(k + k'). \end{aligned}$$

Note that

$$\begin{aligned} & U_1(\tilde{x}_1 \vee \tilde{x}_1', x_2(k)) + U_1(\tilde{x}_1 \wedge \tilde{x}_1', x_2(k')) + \lambda_1(k + k') = \\ & U_1(\tilde{x}_1 \vee \tilde{x}_1', x_2(k)) + U_1(\tilde{x}_1 \wedge \tilde{x}_1', x_2(k)) + (U_1(\tilde{x}_1 \wedge \tilde{x}_1', x_2(k')) - U_1(\tilde{x}_1 \wedge \tilde{x}_1', x_2(k))) + \lambda_1(k + k') \geq \\ & U_1(\tilde{x}_1, x_2(k)) + U_1(\tilde{x}_1', x_2(k)) + U_1(\tilde{x}_1 \wedge \tilde{x}_1', x_2(k')) - U_1(\tilde{x}_1 \wedge \tilde{x}_1', x_2(k)) + \lambda_1(k + k') = \\ & u_1((k, \tilde{x}_1), x_2) + u_1((k', \tilde{x}_1'), x_2) + U_1(\tilde{x}_1', x_2(k)) - U_1(\tilde{x}_1', x_2(k')) + U_1(\tilde{x}_1 \wedge \tilde{x}_1', x_2(k')) - U_1(\tilde{x}_1 \wedge \tilde{x}_1', x_2(k)) \geq \\ & u_1((k, \tilde{x}_1), x_2) + u_1((k', \tilde{x}_1'), x_2) \end{aligned}$$

where the first inequality follows from the supermodularity of $U_1(\cdot, y)$ and the second inequality follows from

$$U_1(\tilde{x}_1', x_2(k)) - U_1(\tilde{x}_1', x_2(k')) + U_1(\tilde{x}_1 \wedge \tilde{x}_1', x_2(k')) - U_1(\tilde{x}_1 \wedge \tilde{x}_1', x_2(k)) \geq 0$$

(because $U_1(\cdot)$ satisfies increasing differences, $x_2(\cdot)$ is monotonic, and $k \geq k'$). This establishes supermodularity of $u_1(\cdot)$.

Supermodularity of u_2 is straightforward:

$$\begin{aligned} u_2((k, \tilde{x}_1), x_2 \vee x_2') + u_2((k, \tilde{x}_1), x_2 \wedge x_2') &= U_2(\tilde{x}_1, \max\{x_2(k), x_2'(k)\}) + U_2(\tilde{x}_1, \min\{x_2(k), x_2'(k)\}) \\ &= u_2((k, \tilde{x}_1), x_2) + u_2((k, \tilde{x}_1), x_2'). \end{aligned}$$

■

Proof of Lemma 8. Fix a strategy $(k', \tilde{x}_1')$ for Player 1. Let x_2^* be a best response to this strategy. I will show that if $u_2((k', \tilde{x}_1'), x_2 \vee x_2^*) \geq u_2((k', \tilde{x}_1'), x_2)$ and $(k'', \tilde{x}_1'') \geq (k', \tilde{x}_1')$, then there exists $\tilde{x}_2^*$ such that $\tilde{x}_2^* \leq x_2^*$ and $\tilde{x}_2^*$ is a best response to $(k', \tilde{x}_1')$ such that $u_2((k'', \tilde{x}_1''), x_2 \vee \tilde{x}_2^*) \geq u_2((k'', \tilde{x}_1''), x_2)$. The result then follows from Proposition 4. To do this, it suffices to show that

$$U_2(x_1', x_2 \vee \tilde{x}_2^*(k')) \geq U_2(x_1', x_2(k')) \implies U_2(x_1'', x_2 \vee \tilde{x}_2^*(k'')) \geq U_2(x_1'', x_2(k'')). \quad (16)$$

If $x_2(k') \geq x_2^*(k')$, (16) clearly holds (with $\tilde{x}_2^* = x_2^*$). If $x_2(k') < x_2^*(k')$, then set

$$\tilde{x}_2^*(k) = \begin{cases} \min\{x_2^*(k), \max\{x_2(k), x_2^*(k')\}\} & \text{if } k \geq k' \\ x_2^*(k) & \text{if } k < k' \end{cases}.$$

To establish (16) it suffices to show

$$U_2(\tilde{x}_1'', \tilde{x}_2^*(k'')) \geq U_2(\tilde{x}_1'', x_2(k'')) \tag{17}$$

whenever $\tilde{x}_2^*(k'') > x_2(k'')$. It follows from the definition of $\tilde{x}_2(\cdot)$ that if $\tilde{x}_2^*(k'') > x_2(k'')$, then $\tilde{x}_2^*(k'') = x_2^*(k')$. By definition, $x_2^*(k')$ is a best response to $\tilde{x}_1'$. Increasing differences (of $U_2(\cdot)$), implies that the best response(s) to x_1'' is greater than $x_2^*(k')$. Consequently, Inequality (17) follows because $U_2(\tilde{x}_1', \cdot)$ is single peaked and $x_2^*(k') = \tilde{x}_2^*(k'') > x_2(k'')$.

A symmetric argument establishes the other part of the definition of (ID). ■

Proof of Lemma 9. To show that the game satisfies conditional ID for Player 1, fix a strategy x_2' for Player 2 and let $x_2'' > x_2'$. Let $(k^*, \tilde{x}_1^*)$ satisfy $u_1((k^*, \tilde{x}_1^*), x_2') \geq u_1((k^*, \tilde{x}_1), x_2')$ for all $\tilde{x}_1$. I want to show that

$$u_1((\tilde{k}^*, \tilde{x}_1) \vee (\tilde{k}^*, \tilde{x}_1^*), x_2') \geq u_1((k^*, \tilde{x}_1), x_2') \implies u_1((k^*, \tilde{x}_1) \vee (\tilde{k}^*, \tilde{x}_1^*), x_2'') \geq u_1((k^*, \tilde{x}_1), x_2''). \tag{18}$$

The implication follows directly from the interval dominance property of $U_1(\cdot)$ since in that case

$$u_1((k^*, \tilde{x}_1) \vee (k^*, \tilde{x}_1^*), x_2) = U_1(\tilde{x}_1 \vee \tilde{x}_1^*, x_2(k^*)) \text{ and } u_1((k^*, \tilde{x}_1), x_2) = U_1(\tilde{x}_1, x_2(k^*))$$

for all x_2. Consequently (18) is equivalent to

$$U_1(\tilde{x}_1 \vee \tilde{x}_1^*, x_2'(k^*)) \geq U_1(\tilde{x}_1, x_2'(k^*)) \implies U_1(\tilde{x}_1 \vee \tilde{x}_1^*, x_2''(k^*)) \geq U_1(\tilde{x}_1, x_2''(k^*)).$$

■

Appendix C

This appendix clarifies the connection between the WID and ID conditions. I begin by introducing a new concept and then I show its relationship to (ID). I then introduce another concept and show that it is equivalent to (WID). The new definitions are transparently nested, making it clear that (ID) implies (WID). Finally, I prove that (WID) implies that best responses are monotonic in a way that is implied by (ID). Throughout I will assume that X and Y are lattices, $f(\cdot)$ is a function $f : X \times Y \to \mathbb{R}$, and $\arg\max_{x \in J} f(x, y)$ is nonempty for all intervals $J \subset X$ and $y \in Y$.

Definition 16. *Assume $x^* \in \arg\max_{x \in X} f(x, y')$, and $x^{**} \in \arg\max_{x \in X} f(x, y'')$. A function $f : X \times Y \to \mathbb{R}$ satisfies the revised interval-dominance property (RID) in its two arguments on the set $X \times Y$ if for all $y'' \geq y'$,*

$$f(x' \vee x^*, y') \geq f(x', y') \implies f(x' \vee x^*, y'') \geq f(x', y'')$$

and

$$f(x' \wedge x^{**}, y'') \geq f(x', y'') \implies f(x' \wedge x^{**}, y') \geq f(x', y').$$

(RID) is an awkward condition because it relies on conditions defined in terms of x^*. It is a useful formulation for some of the arguments in Appendix B. Letting $x'' = x' \vee x^*$, $x'' \geq x'$ and the conditions in Definition 16 are implied by single crossing. Definition 16 imposes the condition less often than single crossing. The next result demonstrates that (RID) is a reformulation of (ID).[9]

Proposition 4. *Let X and Y be lattices. A function $f : X \times Y \to \mathbb{R}$ satisfies (ID) if and only if it satisfies (RID) on all intervals $[x', x''] \subset X$.*

Proof of Proposition 4. First I show that (RID) implies (ID). If $f(x'', y') \geq f(x, y')$ for all $x \in [x', x'']$, then $x'' \in \arg\max_{x \in [x', x'']} f(x, y')$. It follows that $f(x'' \vee x, y') \geq f(x, y')$ and so (RID) implies that $f(x'', y'') \geq f(x, y'')$ for $x \in [x', x'']$. If, furthermore, $f(x', y'') \geq f(x'', y'')$, then $x' \in \arg\max_{x \in [x', x'']} f(x, y'')$ so (RID) implies that $f(x', y') \geq f(x'', y')$. Consequently, if $f(x'', y') > f(x', y')$ then $f(x'', y'') > f(x', y'')$. It follows that if (RID) holds on all intervals, then (ID) holds.

Next I show that (ID) implies (RID). Fix an interval $[x', x''] \subset X$. Let $x^* \in \arg\max_{x \in [x', x'']} f(x, y')$ and $x^{**} \in \arg\max_{x \in [x', x'']} f(x, y'')$.

Let $f(\hat{x} \vee x^*, y') \geq f(\hat{x}, y')$ for some $\hat{x} \in [x', x'']$.

It follows from supermodularity of $f(\cdot)$ that for any $x \in X$,

$$f(x \vee x^*, y') + f(x \wedge x^*, y') \geq f(x, y') + f(x^*, y') \tag{19}$$

for all $x \in X$. Since $x, x^* \in [x', x'']$ implies that $x \wedge x^* \in [x', x'']$, it follows from the definition of x^* that $f(x^*, y') \geq f(x \wedge x^*, y')$ for all $x \in [x', x'']$. Inequality (19) implies that

$$f(x \vee x^*, y') \geq f(x, y') \tag{20}$$

for all $x \in [x', x'']$. Since $\hat{x} \in [x', x'']$, (20) implies

$$f(x \vee x^*, y') \geq f(x, y') \tag{21}$$

for all $x \in [\hat{x}, \hat{x} \vee x^*]$. Since $x \in [\hat{x}, \hat{x} \vee x^*]$ implies that $x \vee x^* = \hat{x} \vee x^*$, it follows that $f(x \vee x^*, y') = f(\hat{x} \vee x^*, y')$. Consequently (21) implies that $f(\hat{x} \vee x^*, y') \geq f(\hat{x}, y')$ for all $x \in [\hat{x}, \hat{x} \vee x^*]$ and therefore, by (ID), $f(\hat{x} \vee x^*, y'') \geq f(x, y'')$.

A similar argument establishes the symmetric implication. ■

The next definition parallels (RID).

Definition 17. *Assume $y'' > y'$, $x^* \in \arg\max_{x \in X} f(x, y')$, and $x^{**} \in \arg\max_{x \in X} f(x, y'')$. A function $f : X \times Y \to \mathbb{R}$ satisfies the revised weak interval-dominance property (RWID) in its two arguments on the set $X \times Y$ if*

$$f(x' \vee x^*, y') \geq f(x', y') \implies$$

$$\exists \tilde{x}^* \in \arg\max_{x \in X} f(x, y'),\ \tilde{x}^* \leq x^*,\ \textit{such that}\ f(x' \vee \tilde{x}^*, y'') \geq f(x', y'') \tag{22}$$

[9] I owe this argument to an anonymous referee.

and

$$f(x' \wedge x^{**}, y'') \geq f(x', y'') \implies$$
$$\exists \tilde{x}^{**} \in \arg\max_{x \in X} f(x, y''),\ \tilde{x}^{**} \geq x^{**},\ \textit{such that } f(x' \wedge \tilde{x}^{**}, y') \geq f(x', y'). \qquad (23)$$

It is clear that (RID) implies (RWID). Condition (WID) is plainly stronger than Condition (RWID). The next result shows that they are equivalent. Propositions 4 and 5 implies that (ID) implies (WID).

Proposition 5. *Let X and Y be lattices. Assume $y'' > y'$, $x^* \in \arg\max_{x \in X} f(x, y')$, and $x^{**} \in \arg\max_{x \in X} f(x, y'')$. A function $f : X \times Y \to \mathbb{R}$ satisfies the revised weak interval-dominance property (WRID) in its two arguments on the set $X \times Y$ if and only if it satisfies (WID) in its two arguments on the set $X \times Y$.*

Proof of Proposition 5. If (WID) holds, then (RWID) clearly holds. I want to show that if $f(x \vee z, y') \geq f(x, y')$, then $f(x \vee \tilde{z}, y'') \geq f(x, y'')$ for $\tilde{z} \leq z$. Let $z^* = \min \arg\max_{w \in [x \wedge z, z]} f(w, y')$. It follows that $x \wedge z^* \in [x \wedge z, z]$ so $f(z^*, y') \geq f(x \wedge z^*, y')$. It follows from supermodularity that $f(x \vee z^*, y') \geq f(x, y')$. Hence (RWID) implies that $f(x \vee z^*, y'') \geq f(x, y'')$. Since $z^* \leq z$ and $f(z^*, y') \geq f(z, y')$, it follows that (WID) holds. ■

Proposition 1 (stated in the text) shows that (WID) implies that solutions to parameterized optimizations are increasing in a sense that is weaker than the strong set order. The proposition uses the following notation: $x^{**} \in \arg\max f(x, y'')$, $x^* \in \arg\max f(x, y')$, $\overline{x}^{**} = \max \arg\max f(x, y'')$, $\underline{x}^* = \min \arg\max f(x, y')$.

Proof of Proposition 1.
By definition, $f(\underline{x}^*, y') \geq f(\underline{x}^* \wedge x^{**}, y')$ and hence, by supermodularity, $f(\underline{x}^* \vee x^{**}, y') \geq f(x^{**}, y')$. It follows from (WRID) that $f(\underline{x}^* \vee x^{**}, y'') \geq f(x^{**}, y'')$. A similar argument shows that when (WRID) holds, $\overline{x}^{**} \wedge x^* \in \arg\max f(x, y')$. ■

Proposition 1 is a variation on Fact 2. Both results demonstrate how assumptions of $f(\cdot)$ make it possible to evaluate how the set of solutions to the parameterized optimization problem $\max_{x \in J} f(x, y)$ change with the parameter y. Fact 2 demonstrates that supermodularity and (ID) combine to guarantee that maximizers are increasing with respect to the strong set order. Proposition 1 demonstrates that supermodularity and (WID) combine to guarantee that maximizers are increasing in the weaker sense captured by (9).[10]

LiCalzi and Veinott [9] present several variations on single-crossing conditions. Corollary 11 contains results that demonstrate different ways in which these conditions can

[10] The relationship induced by the conditions in (9) need not be transitive. That is, it is possible for $\arg\max f(x, y_1)$ to be distinct from $\arg\max f(x, y_2)$ and for (9) to hold both when $(y_1, y_2) = (y', y'')$ and when $(y_1, y_2) = (y'', y')$.

lead to monotone comparative statics with respect to different ways to order sets. These results are in the spirit of Proposition 1 but are distinct.