

# Skewed Noise\*

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March 18, 2015

We study the attitude of decision makers to skewed noise. For a binary lottery that yields the better outcome with probability  $p$ , we identify noise around  $p$  with a compound lottery that induces a distribution over the exact value of the probability and has an average value  $p$ . We propose and characterize a new notion of skewed distributions, and use a recursive non-expected utility model to provide conditions under which rejection of symmetric noise implies rejection of skewed to the left noise as well. We demonstrate that rejection of these types of noises does not preclude acceptance of some skewed to the right noise, in agreement with recent experimental evidence. We apply the model to study random allocation problems (one-sided matching) and show that it can predict systematic preference for one allocation mechanism over the other, even if the two agree on the overall probability distribution over assignments. The model can also be used to address the phenomenon of ambiguity seeking in the context of decision making under uncertainty.

JEL Classification number: D81, C78

Keywords: Skewed distributions, recursive non-expected utility, ambiguity seeking, one-sided matching.

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\*We thank Faruk Gul, Yoram Halevy, Efe Ok, and Rakesh Vohra, as well as the co-editor and four anonymous referees, for their comments and help. We thank Francesca Toscano and Zhu Zhu of Boston College for helpful research assistance.

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# 1 Introduction

Standard models of decision making under risk assume that individuals obey the reduction of compound lotteries axiom, according to which a decision maker is indifferent between any multi-stage lottery and the simple lottery that induces the same probability distribution over final outcomes. Experimental and empirical evidence suggest, however, that this axiom is often violated (see, among others, Kahneman and Tversky [25], Bernasconi and Loomes [5], Conlisk [13], Harrison, Martinez-Correa, and Swarthout [22], and Abdellaoui, Klibanoff, and Placido [1]). Individuals may have preferences over gambles with identical probability distributions over final outcomes if they differ in the form of the timing of resolution of uncertainty. Also, individuals may view ambiguity as lotteries over the true values of the probabilities and may simply like or dislike not knowing their exact values. For example, subjects are typically not indifferent between betting on a known probability  $p$  and betting on a known distribution over the value of that probability even when the mean probability of the distribution is  $p$ .

Halevy [21] and recently Miao and Zhong [32], for example, consider preferences over two-stage lotteries and demonstrate that individuals are averse to the introduction of symmetric noise, that is, symmetric mean-preserving spread into the first-stage lottery. One rationale for this kind of behavior is that the realizations in a symmetric noise cancel out each other and simply create an undesired confusion in evaluation. On the other hand, asymmetric noises, and in particular positively skewed ones, may be desirable. Boney [6] conducted an experiment in which decision makers had to choose one of three investment plans. In all three prospects, the overall probability of success (which results in a prize  $\bar{x} = \$200$ ) is  $p = 0.2$ , and with the remaining probability the investment fails and the decision maker receives  $\underline{x} = \$0$ . Option  $A$  represents an investment plan in which the decision maker is confident about the probability of success. In  $B$  and  $C$ , on the other hand, the probability of success is uncertain. Prospect  $B$  (resp.,  $C$ ) represents a negatively (positively) skewed distribution around  $p$  in which it is very likely that the true probability slightly exceeds (falls below)  $p$  but it is also possible, albeit unlikely, that the true probability is much lower (higher). Boney's main finding is that decision makers are not indifferent between the three prospects and that most prefer  $C$  to  $A$  and  $A$  to  $B$ . Moreover, these preferences are robust to different values of  $\bar{x}$ ,  $\underline{x}$ , and  $p$ .

In Boney's experiment, the underlying probability of success  $p$  was the

same in all three options. In a recent experiment, which we discuss in more detail in Section 3, Abdellaoui, Klibanoff, and Placido [1] found strong evidence that aversion to compound risk (i.e., noise) is an increasing function of  $p$ . In particular, their results are consistent with a greater aversion to negatively skewed noise around high probabilities than to positively skewed noise around small probabilities.

In this paper we propose a model that can accommodate the behavioral patterns discussed above. For a binary lottery that yields the better outcome with probability  $p$ , we identify noise around  $p$  with a two-stage lottery that induces a distribution over the exact value of the probability and has an average value  $p$ . We introduce and characterize a new notion of skewness, and use a version of Segal's [38] recursive non-expected utility model to outline conditions under which a decision maker who always rejects symmetric noise will also reject any negatively skewed noise (for instance, will prefer option  $A$  to  $B$  in the example above) but may seek some positively skewed noise.

We suggest two applications. First, we apply our model to study a simple variant of the house allocation problem (or one-sided matching), where the goal is to look for a systematic way of assigning a set of indivisible objects to a group of individuals having preferences over these objects. We demonstrate that different mechanisms, which agree on the overall probability distribution over assignments and hence are being treated equivalently in the standard model, induce different compound lotteries. Comparing two familiar mechanisms, versions of the random serial dictator and of the random top cycle, our model predicts systematic preference for the latter over the former for a large set of parameters, while permitting the opposite preferences when one type of the goods is scarce, but almost everyone prefers it over the alternative type (see Section 4 for details).

Our second application shows that our model can be used to address the recently documented phenomenon of ambiguity seeking in the context of decision making under uncertainty. The recursive model we study here was first suggested by Segal [37] as a formal way to analyze attitudes towards ambiguity. Under this interpretation, ambiguity is identified as a two-stage lottery, where the first stage captures the decision makers subjective uncertainty about the true probability distribution over the states of the world, and the second stage determines the probability of each outcome, conditional on the probability distribution that has been realized. Our model permits the co-existence of aversion to symmetric ambiguity (as in Ellsberg's famous paradox) and ambiguity seeking, especially in situations where the decision

maker anticipates a bad outcome, yet believes that there is a small chance that things are not as bad as they seem. In this case, he might not want to know the exact values of the probabilities.

The fact that the recursive evaluation of two-stage lotteries in Segal's model is done using non-expected utility functionals is key to our analysis. It is easy to see that if the decision maker uses the same expected utility functional in each stage he will be indifferent to noise. In Section 6 we further show that a version of the recursive model in which the two stages are evaluated using different expected utility functionals (Kreps and Porteus [26], Klibanoff, Marinacci, and Mukerji [25]) cannot accommodate the co-existence of rejecting all symmetric noise while still accepting some positively skewed noise.

In this paper, we confine our attention to the analysis of attitudes to noise related to the probability  $p$  in the binary prospect which pays  $\bar{x}$  with probability  $p$  and  $\underline{x}$  otherwise, where  $\bar{x} > \underline{x}$ . In reality the decision maker may face lotteries with many outcomes and the probabilities of receiving each of them may be uncertain. We prefer to deal only with binary lotteries since when there are many outcomes their probabilities depend on each other and therefore skewed noise over the probability of one event may affect noises over other probabilities in too many ways. This complication is avoided when there are only two outcomes — whatever the decision maker believes about the probability of receiving  $\bar{x}$  completely determines his beliefs regarding the probability of receiving  $\underline{x}$ . Note that while the underlying lottery is binary, the noise itself (that is, the distribution over the value of  $p$ ) may have many possible values or may even be continuous.

The rest of the paper is organized as follows: Section 2 describes the analytical framework and introduces notations and definitions that will be used in our main analysis. Section 3 studies attitudes towards asymmetric noises and states our main behavioral result. Section 4 and Section 5 are devoted to applications. Section 6 comments on the relationship of our paper to other models. All proofs are relegated to the appendix.

## 2 The model

### 2.1 Preferences

Fix two monetary outcomes  $\bar{x} > \underline{x}$ . The underlying lottery we consider is the binary prospect  $(\bar{x}, p; \underline{x}, 1 - p)$ , which pays  $\bar{x}$  with probability  $p$  and  $\underline{x}$  otherwise. We identify this lottery with the number  $p \in [0, 1]$  and analyze noise around  $p$  as a two-stage lottery, denoted by  $\langle p_1, q_1; \dots; p_n, q_n \rangle$ , that yields with probability  $q_i$  the lottery  $(\bar{x}, p_i; \underline{x}, 1 - p_i)$ ,  $i = 1, 2, \dots, n$ , and satisfies  $\sum_i p_i q_i = p$ . Let

$$\mathcal{L}_2 = \{ \langle p_1, q_1; \dots; p_n, q_n \rangle : p_i, q_i \in [0, 1], i = 1, 2, \dots, n, \text{ and } \sum_i q_i = 1 \}.$$

Let  $\succeq$  be a complete and transitive preference relation over  $\mathcal{L}_2$ , which is represented by  $U : \mathcal{L}_2 \rightarrow \mathfrak{R}$ . Throughout the paper we confine our attention to preferences that admit the following representation:

$$U(\langle p_1, q_1; \dots; p_n, q_n \rangle) = V(c_{p_1}, q_1; \dots; c_{p_n}, q_n) \quad (1)$$

where  $V$  is a functional over simple (finite support) one-stage lotteries over the interval  $[\underline{x}, \bar{x}]$  and  $c$  is a certainty equivalent function (not necessarily the one obtained from  $V$ ).<sup>1</sup> According to this model, the decision maker evaluates a two-stage lottery  $\langle p_1, q_1; \dots; p_n, q_n \rangle$  recursively. He first replaces each of the second-stage lotteries with its certainty equivalent,  $c_{p_i}$ . This results in a simple, one-stage lottery over the certainty equivalents,  $(c_{p_1}, q_1; \dots; c_{p_n}, q_n)$ , which he then evaluates using the functional  $V$ .<sup>2</sup> We assume throughout that  $V$  is monotonic with respect to first-order stochastic dominance and continuous with respect to the weak topology.

There are several reasons that lead us to study this special case of  $U$ . First, it explicitly captures the sequentiality aspect of two-stage lotteries, by distinguishing between the evaluations made in each stage ( $V$  and  $c$  in the first and second stage, respectively). Second, it allows us to state our results using familiar and easy to interpret conditions that are imposed on

<sup>1</sup>The function  $c : [0, 1] \rightarrow \mathfrak{R}$  is a certainty equivalent function if for some  $W$  over one-stage lotteries,  $W(c_p, 1) = W(\bar{x}, p; \underline{x}, 1 - p)$ .

<sup>2</sup>The functional  $V$  thus represents some underlying complete and transitive binary relation over simple lotteries, which is used in the first stage to evaluate lotteries over the certainty equivalents of the second stage. To avoid confusion with the main preferences over  $\mathcal{L}_2$ , we will impose all the assumptions in the text directly on  $V$ .

the functional  $V$ , which do not necessarily carry over to a general  $U$ . Finally, the model is a special case of the recursive non-expected utility model of Segal [38]. This will facilitate the comparison of our results with other models (see, for example, Section 6).

We identify simple lotteries with their cumulative distribution functions, denoted by capital letters ( $F, G$ , and  $H$ ). Denote by  $\mathcal{F}$  the set of all cumulative distribution functions of simple lotteries over  $[\underline{x}, \bar{x}]$ . We assume that  $V$  satisfies the assumptions below (specific conditions on  $c$  will be discussed only in the relevant section). These assumptions are common in the literature on decision making under risk.

**Definition 1** *The function  $V$  is quasi concave if for any  $F, G \in \mathcal{F}$  and  $\lambda \in [0, 1]$ ,*

$$V(F) \geq V(G) \implies V(\lambda F + (1 - \lambda)G) \geq V(G).$$

Quasi concavity implies preference for randomization among equally valued prospects. Together with risk aversion ( $V(F) \geq V(G)$  whenever  $G$  is a mean preserving spread of  $F$ ), quasi concavity implies preference for portfolio diversification (Dekel [15]), which is an important feature when modeling markets of risky assets.<sup>3</sup>

Following Machina [28], we assume that  $V$  is smooth, in the sense that it is Fréchet differentiable, defined as follows.

**Definition 2** *The function  $V : \mathcal{F} \rightarrow \mathfrak{R}$  is Fréchet differentiable if for every  $F \in \mathcal{F}$  there exists a local utility function  $u_F : [\underline{x}, \bar{x}] \rightarrow \mathfrak{R}$ , such that for every  $G \in \mathcal{F}$ ,*

$$V(G) - V(F) = \int u_F(x) d[G(x) - F(x)] + o(\|G - F\|)$$

where  $\|\cdot\|$  is the  $L_1$ -norm.

For fixed  $x > y > z$ , let  $(p, q)$  represent the distribution of the lottery  $(z, p; y, 1 - p - q; x, q)$ , where  $(p, q) \in \mathfrak{R}_+^2$  and  $p + q \leq 1$ . Such lotteries are

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<sup>3</sup>The evidence regarding the validity of quasi concavity is supportive yet inconclusive: while the experimental literature that documents violations of linear indifference curves (see, for example, Coombs and Huang [14]) found deviations in both directions, that is, either preference for or aversion to randomization, both Sopher and Narramore [40] and Dwenger, Kubler, and Weizsacker [17] found explicit evidence in support of quasi concavity.

represented in a Marschak-Machina triangle (see panel (i) of Fig. 1). The bold curves are the indifference curves of  $V$ . Quasi concavity implies that these curves are convex. The dotted parallels to the tangent line at the distribution  $F$  to the indifference curve through this point are indifference curves of  $u_F(\cdot)$ .

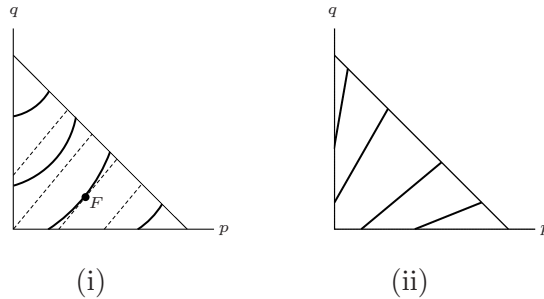


Figure 1: Indifference curves

The following assumption is taken from Machina [28]:

**Definition 3** *The Fréchet differentiable functional  $V$  satisfies Hypothesis II if for every  $x \in [\underline{x}, \bar{x}]$ ,*

$$-\frac{u_G''(x)}{u_G'(x)} \geq -\frac{u_F''(x)}{u_F'(x)}$$

*whenever  $G$  first-order stochastically dominates  $F$ .*

In the Marschak-Machina triangle, Hypothesis II means that indifference curves are “fanning out” (see panel (ii) of Fig. 1). The term  $-\frac{u_F''(\cdot)}{u_F'(\cdot)}$  is analogous to the Arrow-Pratt measure of risk aversion and has the same interpretation as in expected utility theory. Hypothesis II was suggested by Machina as a behavioral regularity that can address known violations of expected utility, such as the Allais paradox.<sup>4</sup>

<sup>4</sup>While fanning out over some range is necessary to address Allais paradox, the evidence on global fanning out is mixed. Although fanning out in the lower region of the triangle is frequently observed, such pattern is inconsistent with many studies that document “fanning in” indifference curves in the other region. See Camerer [8] for a comprehensive survey of these findings.

For the purpose of our analysis, we only need a weaker notion of Hypothesis II, which requires that property to hold just for degenerate lotteries (i.e., Dirac measures), denoted by  $\delta_y$ . Formally,

**Definition 4** *The Fréchet differentiable functional  $V$  satisfies Weak Hypothesis II if for every  $x$  and for every  $y > z$ ,*

$$-\frac{u''_{\delta_y}(x)}{u'_{\delta_y}(x)} \geq -\frac{u''_{\delta_z}(x)}{u'_{\delta_z}(x)}.$$

## 2.2 Skewed distributions

Our aim in this paper is to analyze attitude to skewed noise, that is, to noise that is not symmetric around its mean. For that we need first to formally define the notion of a skewed distribution. To simplify notation and terminology we only analyze skewness to the left, but all definitions and results can be made with skewness to the right.

For a distribution  $F$  on  $[\underline{x}, \bar{x}]$  with expected value  $\mu$  and for  $\delta > 0$ , let

- $\eta_1(F, \delta) = \int_{\underline{x}}^{\mu-\delta} F(x)dx.$
- $\eta_2(F, \delta) = \int_{\mu+\delta}^{\bar{x}} [1 - F(x)]dx.$

**Definition 5** *The lottery  $X$  with the distribution  $F$  on  $[\underline{x}, \bar{x}]$  and expected value  $\mu$  is skewed to the left (or negatively skewed) if for every  $\delta > 0$ ,  $\eta_1(F, \delta) \geq \eta_2(F, \delta)$ , that is, if the area below  $F$  between  $\underline{x}$  and  $\mu - \delta$  is larger than the area above  $F$  between  $\mu + \delta$  and  $\bar{x}$  (see Fig. 2).*

The usefulness of this new notion of skewness will become clear in Section 3.1, where we discuss the proof of our main behavioral result. For now we only demonstrate that it is stronger than a possible alternative definition, according to which the lottery  $X$  with the distribution  $F$  and expected value  $\mu$  is skewed to the left if  $\int_{\underline{x}}^{\bar{x}} (y - \mu)^3 dF(y) \leq 0$ .

**Proposition 1** *If  $X$  with distribution  $F$  and expected value  $\mu$  is skewed to the left as in Definition 5, then for all odd  $n$ ,  $\int_{\underline{x}}^{\bar{x}} (y - \mu)^n dF(y) \leq 0$ .<sup>5</sup>*

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<sup>5</sup>The converse of Proposition 1 is false. For example, let  $F$  be the distribution of the lottery  $(-10, \frac{1}{10}; -2, \frac{1}{2}; 0, \frac{4}{35}; 7, \frac{2}{7})$ . Note that its expected value  $\mu$  is zero. Moreover,  $E[(X - \mu)^3] = -6 < 0$  and  $E[(X - \mu)^{2n+1}]$  is decreasing with  $n$ , which means that all odd moments of  $F$  are negative. Nevertheless, the area below the distribution from  $-10$  to  $-5$  is  $\frac{1}{2}$ , but the area above the distribution from  $5$  to  $10$  is  $\frac{4}{7} > \frac{1}{2}$ , which means that  $F$  is not skewed to the left according to Definition 5.



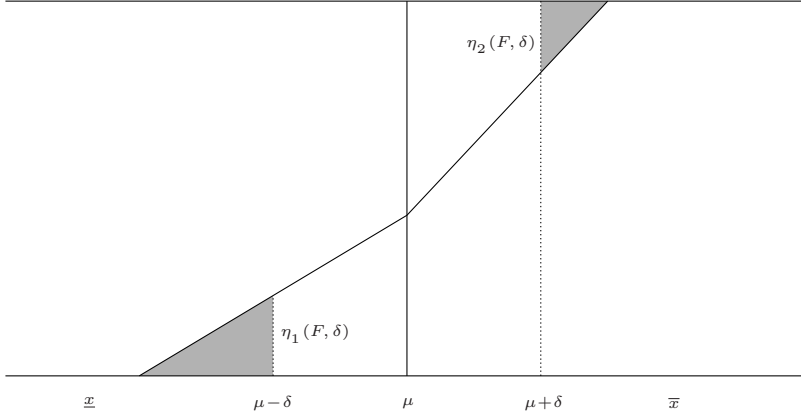


Figure 2: Definition 5,  $\eta_1(F, \delta) \geq \eta_2(F, \delta)$

### 3 Asymmetric noise

We now discuss the main topic of the paper, namely the attitude of decision makers to skewed noise. Recall our notation for two-stage lotteries of the form  $\langle p_1, q_1; \dots; p_m, q_m \rangle$ , where  $p_i = (\bar{x}, p_i; \underline{x}, 1 - p_i)$  and  $\bar{x} > \underline{x}$ .

**Definition 6** *We say that the decision maker rejects symmetric noise if for all  $p \in (0, 1)$ , for all  $\alpha \leq \min\{p, 1 - p\}$ , and for all  $\varepsilon \leq \frac{1}{2}$ ,*

$$\langle p, 1 \rangle \succeq \langle p - \alpha, \varepsilon; p, 1 - 2\varepsilon; p + \alpha, \varepsilon \rangle.$$

As before, we assume that the preference relation  $\succeq$  over  $\mathcal{L}_2$  can be represented as in eq. (1) by  $U(\langle p_1, q_1; \dots; p_n, q_n \rangle) = V((c_{p_1}, q_1; \dots; c_{p_n}, q_n))$ , where  $V$  is a functional over simple lotteries and  $c$  is a certainty equivalent function.

**Theorem 1** *Suppose that  $V$  is quasi concave, Fréchet differentiable, and satisfies Weak Hypothesis II. If the decision maker always rejects symmetric noise, then  $\langle p, 1 \rangle \succeq \langle p_1, q_1; \dots; p_n, q_n \rangle$  whenever  $\sum_i p_i q_i = p$  and the distribution of  $(p_1, q_1; \dots; p_n, q_n)$  is skewed to the left.*

*Furthermore, there are  $V$  and  $c$  such that the decision maker rejects all symmetric and negatively skewed noise, yet accepts some positively skewed noise (i.e., prefers it to  $\langle p, 1 \rangle$ ).*

The first part of Theorem 1 provides conditions under which if the decision maker always rejects symmetric noise, then he will also reject negatively-skewed noise. The conditions on  $V$  are familiar in the literature and, as we have pointed out in the introduction and will further discuss in Section 5, rejection of symmetric noise is empirically supported. The theoretical link between attitudes toward symmetric and negatively skewed noise will be useful in the applications we consider in subsequent sections. The second part suggests that such behavior is consistent with preference for some positively-skewed noise. The distinction between positive and negative skewness is the basis for our analysis, and as we argue, is also supported by empirical evidence. It is this part of the theorem that distinguishes our model from other known preferences over compound lotteries that cannot accommodate rejections of all symmetric noise with acceptance of some positively-skewed noise (Section 6).

In Example 1 below we go further and introduce a family of functionals for which we can provide sufficient conditions for acceptance of some positively skewed noise. In particular, for every  $p > 0$ , if the probability  $q$  of receiving  $(\bar{x}, p; \underline{x}, 1 - p)$  is sufficiently small, then the decision maker will prefer the noise  $\langle p, q; 0, 1 - q \rangle$  over receiving the lottery  $(\bar{x}, pq; \underline{x}, 1 - pq)$  for sure. To guarantee this property, we show that for the functional form of this example, the first non-zero derivative of  $V(c_{pq}, 1) - V(c_p, q; 0, 1 - q)$  with respect to  $q$  at  $q = 0$  is negative (see Appendix A). Note that while Theorem 1 is independent of the function  $c$ , the specification of  $c$  is crucial for this result.

**Example 1** *Let  $V(c_{p_1}, q_1; \dots; c_{p_n}, q_n) = E[w(c_p)] \times E[c_p]$ , where  $w(x) = \frac{\zeta x - x^\zeta}{\zeta - 1}$  and  $c_p = \beta p + (1 - \beta)p^\kappa$ .<sup>6</sup> These functions satisfy all the assumptions of Theorem 1, and there is an open neighborhood of  $(\beta, \zeta, \kappa) \in \mathfrak{R}^3$  for which for every  $p > 0$  there exists a sufficiently small  $q > 0$  such that  $\langle p, q; 0, 1 - q \rangle \succeq \langle pq, 1 \rangle$ .*

Theorem 1 does not restrict the location of the skewed distribution, but it is reasonable to find skewed to the left distributions over the value of the probability  $p$  when  $p$  is high, and skewed to the right distributions when  $p$  is low. The theorem is thus consistent with the empirical observation that decision makers reject skewed to the left distributions concerning high probability of a good event, but seek such distributions when the probability of the good event is low.

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<sup>6</sup>The function  $V$  belongs to the quadratic utility model of Chew, Epstein, and Segal [10].

Our results can explain some of the findings in a recent paper by Abdellaoui, Klibanoff, and Placido [1]. For three different compound lotteries, subjects were asked for their compound lottery premium (as in Dillenberger [16]), that is, the maximal amount they are willing to pay to replace a compound lottery with its binary, single-stage counterpart. The underlying binary lottery yields €50 with probability  $p$  and 0 otherwise. The three two-stage lotteries were  $\langle 0.5, \frac{1}{6}; 0, \frac{5}{6} \rangle$ ,  $\langle 1, \frac{5}{22}; 0.5, \frac{12}{22}; 0, \frac{5}{22} \rangle$ , and  $\langle 1, \frac{5}{6}; 0.5, \frac{1}{6} \rangle$ , with base probabilities of winning  $p = \frac{1}{12}$ ,  $p = \frac{1}{2}$ , and  $p = \frac{11}{12}$ , respectively. They found that the compound lottery premium is an increasing function of  $p$ . Other studies too provide evidence for the pattern of more compound risk aversion for high probabilities than for low probabilities, and even for compound risk seeking for low probabilities (see, for example, Kahn and Sarin [24] and Viscusi and Chesson [42]). Following Theorem 1, we argue that it is not only the magnitude of the probabilities that drive their results, but the fact that in the three lotteries above, noise is positively skewed, symmetric, and negatively skewed, respectively. Indeed, in a recent paper, Masatlioglu, Orhun, and Raymond [30] found that individuals exhibit a strong preference for positively skewed noise over negatively skewed ones.

### 3.1 Outline of the proof of Theorem 1

We discuss only the first part of the theorem, according to which rejection of symmetric noise implies rejection of negatively skewed noise (the second part is proved by showing that the functional form in Example 1 satisfies all the required properties). In the recursive model, rejection of symmetric noise implies that for any  $p$ , the local utility of  $V$  at  $\delta_{c_p}$  prefers  $\langle p, 1 \rangle$  to  $\langle p - a, \frac{1}{2}; p + a, \frac{1}{2} \rangle$ . By Weak Hypothesis II, this ranking prevails also when evaluated using the local utility at  $\delta_{c_{p^*}}$ , for  $p^* > p$ . Pick a lottery  $\langle p, 1 \rangle$ . By Theorem 2 below, any skewed to the left noise  $Q$  around  $p$  can be obtained as the limit of *left* symmetric splits (Definition 7). By Weak Hypothesis II, each such split will be rejected when evaluated using the local utility at  $\delta_{c_p}$  and, by Fréchet differentiability,  $Q$  itself will also be rejected. Quasi concavity then implies that it will be rejected globally, that is,  $\langle p, 1 \rangle \succeq Q$ .

We conclude by stating Theorem 2, which is a mathematical result of an independent interest.

**Definition 7** *Let  $\mu$  be the expected value of a lottery  $X$ . Lottery  $Y$  is obtained from  $X$  by a left symmetric split if  $Y$  is the same as  $X$ , except for*

that one of the outcomes  $x$  of  $X$  such that  $x \leq \mu$  was split into  $x + \alpha$  and  $x - \alpha$ , each with half of the probability of  $x$ .

**Theorem 2**

1. If the lottery  $Y = (y_1, p_1; \dots; y_n, p_n)$  with expected value  $\mu$  is skewed to the left, then there is a sequence of lotteries  $X_i$ , each with expected value  $\mu$ , such that  $X_1 = (\mu, 1)$ ,  $X_i \rightarrow Y$ , and  $X_{i+1}$  is obtained from  $X_i$  by a left symmetric split. Moreover, it can be done such that in each step the size of the spread is bounded by  $\max_i y_i - \mu$ .
2. Consider the sequence  $\{X_i\}$  of lotteries where  $X_1 = (\mu, 1)$  and  $X_{i+1}$  is obtained from  $X_i$  by a left symmetric split. Then the distributions  $F_i$  of  $X_i$  converge and the limit distribution  $F$  is skewed to the left.

To illustrate the theorem, consider two examples, one where the procedure terminates in a finite number of steps and one where it does not. In both cases we move from a degenerate lottery  $X$  to a skewed to the left binary lottery  $Y$  with the same expected value as  $X$ . For the first example, let  $X = (3, 1)$  and  $Y = (0, \frac{1}{4}; 4, \frac{3}{4})$  and obtain

$$X = (3, 1) \rightarrow (2, \frac{1}{2}; 4, \frac{1}{2}) \rightarrow (0, \frac{1}{4}; 4, \frac{1}{4} + \frac{1}{2}) = Y.$$

For a sequence that does not terminate, let  $X = (5, 1)$  and  $Y = (0, \frac{1}{6}; 6, \frac{5}{6})$ . Here we obtain

$$\begin{aligned} X &= (5, 1) \rightarrow (4, \frac{1}{2}; 6, \frac{1}{2}) \rightarrow (2, \frac{1}{4}; 6, \frac{3}{4}) \rightarrow (0, \frac{1}{8}; 4, \frac{1}{8}; 6, \frac{3}{4}) \rightarrow \dots \\ &\quad (0, \frac{1}{2} \sum_1^n \frac{1}{4^i}; 4, \frac{1}{2 \cdot 4^n}; 6, \frac{1}{2} + \sum_1^n \frac{1}{4^i}) \rightarrow \dots (0, \frac{1}{6}; 6, \frac{5}{6}) = Y. \end{aligned}$$

The main difficulty in proving part 1 of Theorem 2 is the fact that whereas outcomes to the left of  $\mu$  can be manipulated, any split of probabilities that lands some probability to the right of  $\mu$  must hit its exact place according to  $Y$ , as we will not be able to touch it later again.

**Remark 1** The two parts of Theorem 2 do not create a simple if and only if statement, because the support of the limit distribution  $F$  in part 2 need not be finite. On the other hand, part 1 of the theorem does not hold for continuous distributions. By the definition of left symmetric splits, if the probability of  $x > \mu$  in  $X_i$  is  $p$ , then for all  $j > i$ , the probability of  $x$  in  $X_j$  must be at least  $p$ . It thus follows that the distribution  $F$  cannot

be continuous above  $\mu$ . However, it can be shown that if  $F$  with expected value  $\mu$  is skewed to the left, then there is a sequence of finite skewed to the left distributions  $F_n$ , each with expected value  $\mu$ , such that  $F_n \rightarrow F$ . This enables us to use Theorem 2 even for continuous distributions.

**Remark 2** Menezes, Geiss, and Tressler [31] characterize a notion of increasing downside risk by combining a mean-preserving spread of an outcome below the mean followed by a mean-preserving contraction of an outcome above the mean, in a way that the overall result is a transfer of risk from the right to the left of a distribution, keeping the variance intact. Distribution  $F$  has more downside risk than distribution  $G$  if one can move from  $G$  to  $F$  in a sequence of such mean-variance-preserving transformations. Menezes et al. [31] do not provide a definition (and a characterization as in our Theorem 2) of a skewed to the left distribution. Observe that our characterization involves a sequence of only symmetric left splits, starting in the degenerate lottery that puts all the mass on the mean. In particular, our splits are not mean-variance-preserving and occur only in one side of the mean.

## 4 Allocation Mechanisms

In this section we apply our results to the comparison of two known allocation mechanisms of indivisible goods. We demonstrate that agents with preferences as studied in this paper may systematically prefer one mechanism to the other, even though both mechanisms are considered to be the same in standard models, in the sense that they induce the same probability distribution over successful matchings.

Consider the following variant of the house allocation problem (Hylland and Zeckhauser [23]). Let  $N = \{1, 2, \dots, n\}$  be a group of individuals and assume that there are  $n$  goods to be allocated among them. The goods are of two types,  $g_1$  and  $g_2$ , and we denote by  $t_1$  and  $t_2 = n - t_1$  the number of units of each type. Each of the  $n$  individuals has the same stochastic preferences, where with probability  $q$  he prefers  $g_1$  to  $g_2$  (independently of the preferences of other group members). We normalize payoffs so that the utility from the desired outcome is 1 and the utility from the other outcome is zero.

Many important goods are allocated using randomizing devices. These include, among others, the allocation of public schools, course schedule, or

dormitory rooms to students, and shifts, offices, or tasks to workers. We consider two familiar mechanisms, each consists of two stages.

- Random Top Cycle (TC): In the first stage, the allocation of the goods among the agents is randomly determined, so that for  $j = 1, 2$ , the probability of person  $i$  to hold good of type  $g_j$  is  $\frac{t_j}{n}$ . In the second stage, the entire profile of preferences is revealed. Those who like their holding will keep it. The rest will trade according to the following schedule: If  $k$  people holding one type of good and  $\ell \leq k$  people holding the other type are unhappy with their holdings, then the latter  $\ell$  will trade and get their desired outcome, while  $\ell$  out of the former  $k$  will be selected at random and get their preferred option. The other  $k - \ell$  will keep their undesired outcome.<sup>7</sup>
- Random Serial Dictatorship (SD): In the first stage the order of the agents is randomly determined, so that the probability of person  $i$  to be in place  $j = 1, \dots, n$  is  $\frac{1}{n}$ . In the second stage, the entire profile of preferences is revealed. The agents then choose the goods according to the order determined in the first stage. A person will get his desired outcome if when his turn arrives such a unit is still available.

We adopt an ex ante perspective. In the TC procedure, person  $i$  is facing the following two stage lottery. With probability  $\frac{t_j}{n}$  he will receive good  $j$ . Once he knows his holding, he will be able to compute the probability that he'll eventually obtain his desired outcome, denoted  $r_j$ . Using our notation, he'll face the lottery over probabilities  $\langle r_1, \frac{t_1}{n}; r_2, \frac{t_2}{n} \rangle$ . Similarly, once a person knows his position  $j$  in the queue in the SD procedure, he'll be able to calculate the probability he'll win his desired outcome  $s_j$ . This procedure thus translates into the lottery  $\langle s_1, \frac{1}{n}; \dots; s_n, \frac{1}{n} \rangle$ .

The literature on one-sided and two-sided matching (for a recent survey, see Abdulkadiroğlu and Sönmez [3]) typically maintains the assumption that agents are only interested in the overall probability they'll receive their desired outcome. This leads to some results, showing the equivalence of different randomized mechanisms (Abdulkadiroğlu and Sönmez [2]; see also Pathak and Sethuraman [33]). In particular, Abdulkadiroğlu and Sönmez's [2] results

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<sup>7</sup>This is a variant of the classic top cycle mechanism. It can, equivalently, be formulated more closely to the familiar top cycle, as a problem of matching with indifferences and using a specific tie-breaking rule. Since the environment we consider is simple, we maintain our formulation and slightly abuse the title "cycle."

imply that both TC and SD lead to the same overall probability of success, that is,  $r_1 t_1 + r_2 t_2 = \sum_j s_j$ .

Recently, Budish et al. [7] pointed out that even if two lotteries are ex-ante equivalent, one may still prefer one to the other because of the nature of the probabilities they assign to different ex-post outcomes; for example, one may wish to control for envy ex-post. We show here that our analysis of preferences for lotteries over probabilities imply some clear cut results regarding preferences over the aforementioned mechanisms.

Suppose for example that  $q = \frac{1}{2}$  and  $g_1 = g_2$ . This of course does not mean that everyone will be satisfied, which will be the case only if exactly half of the population prefer each option. Under TC, receiving  $g_1$  or  $g_2$  yields exactly the same lottery and hence the same probability of getting the desired outcome. TC therefore leads to the degenerate lottery over probabilities  $\langle p, 1 \rangle$ , for some  $p \in (0, 1)$ . Under RD, the first half of the group will be able to choose their desired outcomes regardless of their preferences. The next person cannot be certain that his desires will be satisfied, as it may happen that he and everyone above him will have the same preferences. Denote the obtained distribution over probabilities by  $F$ . As before (see [2]), its expected value is  $p$ . Since for every  $\delta$ ,  $1 - F(p + \delta) \geq \frac{1}{2} \geq F(p - \delta)$ , it follows that  $F$  is skewed to the left. By Theorem 1, TC is preferred to RD. Observe that for this result we didn't need to compute the exact lotteries induced by TC and SD.

In the formal analysis to follow, we confine attention to the case of large (continuum) economies. Let  $p$  be the proportion of the  $g_1$  units and assume, without loss of generality, that  $p \geq \frac{1}{2}$ . As before, let  $q$  be the probability that an individual prefers  $g_1$  to  $g_2$ .

Consider first the case  $p < q$ . In the TC mechanism, each individual receives  $g_1$  with probability  $p$ , hence  $qp$  individuals will receive  $g_1$  and will be satisfied with it. The other  $(1 - q)p$  who received  $g_1$  will try to replace it with  $g_2$ . Of the  $1 - p$  who received  $g_2$ ,  $q(1 - p)$  will try to replace it with  $g_1$ . Since  $q > p$ ,  $(1 - q)p < q(1 - p)$  and therefore the  $(1 - q)p$  individuals who want to replace  $g_1$  with  $g_2$  will be able to do so. In other words, all those who received  $g_1$  are satisfied. Of the  $1 - p$  who received  $g_2$ ,  $(1 - q)(1 - p)$  will keep it, and out of the other  $q(1 - p)$ ,  $\frac{(1 - q)p}{q(1 - p)}$  will be able to replace  $g_2$  with  $g_1$ . Their probability of success is therefore

$$(1 - q) + q \frac{(1 - q)p}{q(1 - p)} = \frac{(1 - q)}{(1 - p)} < 1.$$

The TC mechanism thus yields the lottery over probabilities of receiving the desired outcome given by  $X_1 = \langle 1, p; \frac{(1-q)}{(1-p)}, 1-p \rangle$ .

Consider now the SD mechanism (still assuming  $p < q$ ). Out of the first  $\frac{p}{q}$ ,  $q \times \frac{p}{q} = p$  will choose  $g_1$  and  $(1-q) \times \frac{p}{q} < 1-p$  will choose  $g_2$ . As  $g_1$  is exhausted by the first  $\frac{p}{q}$ , the other  $\frac{q-p}{q}$  will be able to satisfy their desires only if they prefer  $g_2$  to  $g_1$ . The probability of this event is  $1-q$ . SD thus leads to the lottery over probabilities given by  $Y_1 = \langle 1, \frac{p}{q}; 1-q, \frac{q-p}{q} \rangle$ .

Similarly, if  $p > q$  then TC leads to the lottery  $X_2 = \langle 1, 1-p; \frac{q}{p}, p \rangle$  and SD leads to  $Y_2 = \langle 1, \frac{1-p}{1-q}; q, \frac{p-q}{1-q} \rangle$ . Observe that independently of the relative magnitude of  $p$  and  $q$ , the expected probability of receiving the preferred good is the same under both mechanisms. Also observe that if  $p = q$ , then in a large economy both TC and SD will yield (almost) everyone his desired outcome for sure.

For the main result in this section, we look at preferences that satisfy the assumptions of Theorem 1 and in addition satisfy the following property:

**Definition 8** *The relation  $\succeq$  displays aversion to larger left-symmetric noise if for every  $p_i$  below the average probability  $\bar{p}$ , which is obtained with probability  $q_i$ , and for every  $\varepsilon' > \varepsilon \geq 0$ ,*

$$\langle \dots; p_i - \varepsilon, \frac{q_i}{2}; p_i + \varepsilon, \frac{q_i}{2}; \dots \rangle \succeq \langle \dots; p_i - \varepsilon', \frac{q_i}{2}; p_i + \varepsilon', \frac{q_i}{2}; \dots \rangle.$$

We assumed that  $p \geq \frac{1}{2}$ . It is easy to verify that for all  $q \geq p$ ,  $Y_1$  is a mean preserving spread of  $X_1$ , and for all  $p \geq q$ ,  $Y_2$  is a mean preserving spread of  $X_2$ . As argued before, this may not be enough to guarantee preference for the  $X$  lotteries over the  $Y$  lotteries. The next theorem analyzes these preferences.

**Theorem 3** *Suppose that  $\succeq$  satisfies the assumptions of Theorem 1 and displays aversion to larger left-symmetric noise (Definition 8). If  $q \geq 2p - 1$ , then in large economies TC is preferred to SD.*

Since  $p \geq \frac{1}{2}$ ,  $X_1 = \langle 1, p; \frac{(1-q)}{(1-p)}, 1-p \rangle$  is skewed to the left and  $X_2 = \langle 1, 1-p; \frac{q}{p}, p \rangle$  is skewed to the right. For  $p < q < 1$ ,  $\frac{p}{q} > \frac{1}{2}$  and therefore  $Y_1 = \langle 1, \frac{p}{q}; 1-q, \frac{q-p}{q} \rangle$  is skewed to the left. As long as  $\frac{1-p}{1-q} \geq \frac{1}{2}$ ,  $Y_2 = \langle 1, \frac{1-p}{1-q}; q, \frac{p-q}{1-q} \rangle$  is skewed to the left. However, when  $p > 2p - 1 > q$ , we obtain that  $\frac{1-p}{1-q} < \frac{1}{2}$  and  $Y_2$  too is skewed to the right. Consider now the



case where  $p$  is large and  $q$  is small.<sup>8</sup> In this case most of the outcomes are of one type, but individuals are not likely to like it. Both  $X_2$  and  $Y_2$  are then lotteries that yield with high probability a bad outcome and a good outcome with a small probability. Following Theorem 1, we know that there are functionals that satisfy our assumptions but that for sufficiently small  $q$  and sufficiently large  $p$ , both mechanisms will be considered better than a (hypothetical) mechanism that will offer the desired outcome with the (same) expected probability  $1 - p + q$ .

The lottery  $Y_2$  is a mean preserving spread of  $X_2$ . It yields the good outcome (probability 1) with a higher probability than  $X_2$ , but its bad outcome is worse than the bad outcome under  $X_2$ . Moving from  $X_2$  to  $Y_2$  thus increases the small probability of the good probability (1) at the cost of (slightly) reducing the desirability of the bad probability. This has the flavor of our arguments for preference for a lottery that yields with a small probability a good chance of getting a good outcome over a simple lottery that yields the good with the expected probability. The following proposition establishes this connection.

**Proposition 2** *The following two statements are equivalent:*

1. *For a sufficiently small  $q$ ,  $\langle 1, q; 0, 1 - q \rangle \succeq \langle q, 1 \rangle$ .*
2. *For a sufficiently small  $q$  and sufficiently large  $p$ ,  $Y_2 \succeq X_2$  (that is, SD is preferred to TC).*

Once again, the preferences of Example 1 are consistent with our analysis (see Appendix A). These preferences satisfy all the assumptions of Theorem 3, except for the regions  $2p - 1 < q < p^2$  and  $\frac{2p}{1+p} < q < 2p - p^2$ . Nevertheless, we show by means of numerical analysis that on these regions too TC is preferred to SD. Following Proposition 2, for a sufficiently high  $p$  and sufficiently small  $q$ , the preferences of Example 1 rank SD above TC.

## 5 Ambiguity aversion and seeking

Ambiguity aversion is one of the most investigated phenomena in decision theory. Consider the classic Ellsberg [18] thought experiment: subjects are

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<sup>8</sup>This case is not covered by Theorem 3, where Case 2.3 in its proof strongly depends on the assumption that  $q \geq 2p - 1$ .

presented with two urns. Urn 1 contains 100 red and black balls, but the exact color composition is unknown. Urn 2 has exactly 50 red and 50 black balls in it. Subjects are asked to choose an urn from which a ball will be drawn, and to bet on the color of this ball. If a bet on a specific urn is correct the subject wins \$100, zero otherwise. Let  $C_i$  be the bet on a color (Red or Black) draw from Urn  $i$ . Ellsberg predicted that most subjects will be indifferent between  $R_1$  and  $B_1$  as well as between  $R_2$  and  $B_2$ , but will strictly prefer  $R_2$  to  $R_1$  and  $B_2$  to  $B_1$ . While, based on symmetry arguments, it seems plausible that the number of red balls in urn 1 equals the number of black balls, Urn 1 is ambiguous in the sense that the exact distribution is unknown whereas urn 2 is risky, as the probabilities are known. An ambiguity averse decision maker will prefer to bet on the risky urn to bet on the ambiguous one. Ellsberg's predictions were confirmed in many experiments.<sup>9</sup>

The recursive model we study here was first suggested by Segal [37] as a formal way to capture ambiguity aversion.<sup>10</sup> Under this interpretation, ambiguity is identified as two-stage lotteries. The first stage captures the decision maker's uncertainty about the true probability distribution over the states of the world (the true composition of the urn in Ellsberg's example), and the second stage determines the probability of each outcome, conditional on the probability distribution that has been realized. Holding the prior probability distribution over states fixed, an ambiguity averse decision maker prefers the objective (unambiguous) simple lottery to any (ambiguous) compound one. Note that according to Segal's model, preferences over ambiguous prospects are induced from preferences over the compound lotteries that reflect the decision maker's beliefs. That is, the first stage is imaginary and corresponds to the decision maker's subjective beliefs over the values of the true probabilities.

While Ellsberg-type behavior seems intuitive and is widely documented, there are situations where decision makers actually prefer not to know the probabilities with much preciseness. Becker and Bronwson [4, fnt. 4] describe

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<sup>9</sup>A comprehensive reference to the evidence on Ellsberg-type behavior, and on attitude towards ambiguity in general, can be found in Peter Wakker's annotated bibliography, posted at his website under <http://people.few.eur.nl/wakker/refs/webrfrncs.doc>.

<sup>10</sup>There are many other ways to model ambiguity aversion. Prominent examples include Choquet expected utility (Schmeidler [36]), maximin expected utility (Gilboa and Schmeidler [20]), variational preferences (Maccheroni, Marinacci, and Rustichini [29]),  $\alpha$ -maxmin (Ghirardato, Maccheroni, and Marinacci [19]), and the smooth model of ambiguity aversion (Klibanoff, Marinacci, and Mukerji [27]).

a conversation where Ellsberg himself suggested that people may prefer ambiguity with respect to a low-probability good event. There is indeed a growing experimental literature that challenges the assumption that people are globally ambiguity averse. See, among others, Chew, Miao, and Zhong [11], and van de Kuilen and Wakker [41]. Consider the following situation. A person suspects that there is a high probability that he will face a bad outcome (severe loss of money, serious illness, criminal conviction, etc.). Yet he believes that there is some (small) chance things are not as bad as they seem (Federal regulations will prevent the bank from taking possession of his home, it is really nothing, they won't be able to prove it). These beliefs might emerge, for example, from consulting with a number of experts (such as accountants, doctors, lawyers) who disagree in their opinions; the vast majority of which are negative but some believe the risk is much less likely. Does the decision maker really want to know the exact probabilities of these events? The main distinction between the sort of ambiguity in Ellsberg's experiment and the ambiguity in the last examples is that the latter is asymmetric and, in particular, positively skewed. On the other hand, if the decision maker expects a good outcome with high probability, he would probably prefer to know this probability for sure, rather than knowing that there is actually a small chance that things are not that good. In other words, asymmetric but negatively skewed ambiguity may well be undesired.

To illustrate attitudes towards skewed ambiguity, reconsider Boiney's [6] experiment given in the introduction.<sup>11</sup> A decision maker can choose one of three investment plans, for each of which he believes that the overall probability that the investment be successful is the same,  $p$ . The decision maker prefers an investment plan  $A$ , in which he is confident about its probability of success over plan  $B$  that describes a negatively skewed distribution around  $p$ , where it is very likely that the true probability slightly exceeds  $p$  but it is also possible (albeit unlikely) that the true probability is very low. Yet, he prefers Option  $C$ , which describes a positively skewed distribution around  $p$ , where it is very likely that the true probability falls slightly below  $p$  but there is a small chance it is actually a decent probability, over both  $A$  and  $B$ .

Our model is consistent with this type of behavior. Let  $(\bar{x}, p; \underline{x}, 1 - p)$

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<sup>11</sup>This example is slightly different from the intuition given in the previous paragraph; it demonstrates different ambiguity attitude towards different patterns of noise around a given average value, rather than towards skewed to the right (left) noise around low (high) average probability.

denote a conceivable probability distribution over the two possible states (success, prize  $\bar{x}$ , and failure, prize  $\underline{x} < \bar{x}$ ). For some  $q_1, q_2 \geq \frac{1}{2}$  and  $p_1 < p_2 < p < p_3 < p_4$  such that  $q_1 p_3 + (1 - q_1) p_1 = q_2 p_2 + (1 - q_2) p_4 = p$ , the three investment options  $A$ ,  $B$ , and  $C$  can be described as the two-stage lotteries  $\langle p, 1 \rangle$ ,  $\langle p_3, q_1; p_1; 1 - q_1 \rangle$ , and  $\langle p_2, q_2; p_4, 1 - q_2 \rangle$ , respectively. Since  $B$  is skewed to the left, the Hypothesis of Theorem 1 implies that  $A$  is preferred to  $B$ . And since  $C$  is skewed to the right, we have no implication for the ranking of  $C$  and  $A$ . In particular, the model permits having strict preference for  $C$  (for instance, using the functional form of Example 1), in line with our intuition and with the observed data.<sup>12</sup>

## 6 Relations to the literature

The proofs of the theorems presented in this paper are quite complicated, and it is therefore natural to ask whether simpler models can deliver the same results.

Consider first a recursive model in which the decision maker is an expected utility maximizer in each of the two stages. Let  $u$  and  $v$  be the vNM utility functions over outcomes in the first and in the second stages. The decision maker evaluates a two-stage lottery  $\langle p_1, q_1; \dots; p_n, q_n \rangle$  recursively by

$$U(\langle p_1, q_1; \dots; p_n, q_n \rangle) = \sum_i q_i u(v^{-1}(E_v[p_i])) \quad (2)$$

where  $v^{-1}(E_v[p])$  is the certainty equivalent of lottery  $p$  calculated using the function  $v$ . In the context of temporal lotteries, this model is a special case of the one studied by Kreps and Porteus [26]. In the context of ambiguity, this is the model of Klibanoff, Marinacci, Mukerji [27]. We now argue that while this model is consistent with the rejection of all symmetric noise, it cannot accommodate preferences that reject all symmetric noise while still accepting some positively skewed noise. To see this, note that by eq. (2), for any  $p$ , the value of the noise  $\langle p + \alpha, \frac{1}{2}; p - \alpha, \frac{1}{2} \rangle$  is

$$\frac{1}{2} u(v^{-1}[(p + \alpha)v(x) + (1 - p - \alpha)v(y)]) +$$

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<sup>12</sup>Boiney [6] used a two-stage lottery as an operational definition of ambiguity. His experiment involves prospects like the ones describe in the text, with the specification  $\bar{x} = \$200$ ,  $\underline{x} = \$0$ ,  $p = 0.2$ ,  $q_1 = q_2 = 0.9$ ,  $p_1 = 0.02$ ,  $p_2 = 0.18$ ,  $p_3 = 0.22$ , and  $p_4 = 0.38$ . His data strongly supports the ranking  $C \succ A \succ B$ . He further showed that this ranking is robust for a wide range of parameters.

$$\frac{1}{2}u\left(v^{-1}[(p-\alpha)v(x)+(1-p+\alpha)v(y)]\right).$$

The value of the simple lottery  $(\bar{x}, p; \underline{x}, 1-p)$  is<sup>13</sup>

$$u\left(v^{-1}[pv(\bar{x})+(1-p)v(\underline{x})]\right).$$

Rejection of symmetric noise implies that for any  $p$  and  $\alpha$  in the relevant range,

$$\begin{aligned} u\left(v^{-1}[pv(\bar{x})+(1-p)v(\underline{x})]\right) &\geq & (3) \\ &\frac{1}{2}u\left(v^{-1}[(p+\alpha)v(\bar{x})+(1-p-\alpha)v(\underline{x})]\right) + \\ &\frac{1}{2}u\left(v^{-1}[(p-\alpha)v(\bar{x})+(1-p+\alpha)v(\underline{x})]\right). \end{aligned}$$

Pick any two numbers  $a > b$  in  $[0, 1]$  and note that by setting  $p = \frac{1}{2}(a+b)$  and  $\alpha = \frac{1}{2}(a-b)$ , inequality (3) is equivalent to the requirement that

$$\begin{aligned} u\left(v^{-1}\left[\frac{1}{2}(a+b)v(\bar{x})+(1-\frac{1}{2}(a+b))v(\underline{x})\right]\right) &\geq \\ &\frac{1}{2}u\left(v^{-1}[av(\bar{x})+(1-a)v(\underline{x})]\right) + \frac{1}{2}u\left(v^{-1}[bv(\bar{x})+(1-b)v(\underline{x})]\right). \end{aligned}$$

Since  $a$  and  $b$  are arbitrary, this inequality should hold for all such pairs. This is the case if and only if the function  $u \circ v^{-1}$  is mid-point concave, which by continuity implies that  $u \circ v^{-1}$  is concave. But then the decision maker would reject any noise.

This feature of the recursive expected utility model implies that it cannot accommodate the pattern of preferences over allocation mechanisms of Section 4. In particular, since the lottery over probabilities induced by SD is a mean preserving spread of the one induced by TC, the recursive expected utility model predicts that independently of the values of  $p$  and  $q$ , a decision maker who rejects symmetric noise will always prefer TC to SD. In contrast, as we establish in Theorem 3 and Proposition 2, while rejection of symmetric noise in our model implies preference for TC over SD for many possible

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<sup>13</sup>Alternatively, the simple lottery can be translated into the two-stage lottery that pays  $\delta_{\bar{x}}$  with probability  $p$  and  $\delta_{\underline{x}}$  with probability  $1-p$ , where  $\delta_x$  is the distribution of the lottery that yields  $x \in [\underline{x}, \bar{x}]$  with probability one. Its recursive value is

$$pu\left(v^{-1}[v(\bar{x})]\right) + (1-p)u\left(v^{-1}[v(\underline{x})]\right) = pu(\bar{x}) + (1-p)u(\underline{x}).$$

The result in this section is independent of this specification; in the analysis below, rejecting symmetric noises would imply that  $u \circ v^{-1}$  is convex, which again implies rejection of any noise.

combinations of  $p$  and  $q$ , it does not preclude preferring SD to TC, especially when the induced lotteries over probabilities are skewed to the right.

Dillenberger [16] studied a property called preferences for one-shot resolution of uncertainty, which, in the language of this paper, means that the decision maker rejects *all* noise. Dillenberger confined his attention to recursive preferences over two-stage lotteries in which the certainty equivalent functions in the second stage are calculated using the same  $V$  that applied in the first stage.<sup>14</sup> He showed an equivalence between preferences for one-shot resolution of uncertainty and a static property called negative certainty independence (NCI). Recently, Cerreia Vioglio, Dillenberger, and Ortoleva [9] derived a utility representation — termed cautious expected utility — of all preferences that satisfy NCI together with basic rationality postulates.

Our analysis in this paper is based on functionals  $V$  that are Fréchet differentiable. Segal and Spivak [39] defined a preference relation as exhibiting second-order (resp., first-order) risk aversion if the derivative of the implied risk premium on a small, actuarially fair gamble vanishes (resp., does not vanish) as the size of the gamble converges to zero.<sup>15</sup> If  $V$  is Fréchet differentiable, then it satisfies second-order risk aversion. There are interesting preference relations that are not Fréchet differentiable (and that do not satisfy second-order risk aversion), e.g., the rank-dependent utility model of Quiggin [34]. The analysis of attitudes towards skewed noise in the recursive model for general first-order risk aversion preferences is not vacuous, but is significantly different than the one presented in this paper and will be developed in future work.

## Appendix: Proofs

**Proof of Proposition 1:** Let the lottery  $Y$  be obtained from the lottery  $Z$  by a left symmetric split and denote by  $x$  their common mean. For example, the outcome  $z_i \leq x$  with probability  $p_i$  of  $Z$  is split into  $z_i - \alpha$  and  $z_i + \alpha$ , each with probability  $\frac{p_i}{2}$ . Denote the distributions of  $Y$  and  $Z$  by  $F$  and  $G$ . Since for  $t < 0$  and odd  $n$ ,  $t^n$  is a concave function, it follows that if  $z_i + \alpha \leq x$ ,

<sup>14</sup>This is known as the time neutrality axiom (Segal [38]).

<sup>15</sup>Formally, if  $\pi(t)$  is the amount of money that an agent would pay to avoid the non-degenerate gamble  $x + t\tilde{\varepsilon}$ , where  $E(\tilde{\varepsilon}) = 0$ , then  $\pi(t)$  is  $O(t)$  and  $o(t)$  for first and second-order risk averse preferences, respectively.

then

$$\begin{aligned} & \int_{\underline{x}}^{\bar{x}} (t-x)^n dF(t) - \int_{\underline{x}}^{\bar{x}} (t-x)^n dG(t) = \\ & \frac{p_i}{2} [(z_i - \alpha - x)^n + (z_i + \alpha - x)^n] - p_i (z_i - x)^n \leq 0. \end{aligned} \quad (4)$$

If  $z_i + \alpha > x$  we need to manipulate eq. (4) a little further. Let  $\xi = z_i - x$  and obtain

$$\begin{aligned} & \frac{p_i}{2} [(z_i - \alpha - x)^n + (z_i + \alpha - x)^n] - p_i (z_i - x)^n = \\ & \frac{p_i}{2} [(\xi - \alpha)^n + (\xi + \alpha)^n] - p_i \xi^n = \\ & \frac{p_i}{2} \xi^n + \frac{p_i}{2} \sum_{j=1}^{\frac{n-1}{2}} \binom{n}{2j-1} \xi^{2j-1} \alpha^{n-2j+1} - \frac{p_i}{2} \sum_{j=0}^{\frac{n-1}{2}} \binom{n}{2j} \xi^{2j} \alpha^{n-2j} + \\ & \frac{p_i}{2} \xi^n + \frac{p_i}{2} \sum_{j=1}^{\frac{n-1}{2}} \binom{n}{2j-1} \xi^{2j-1} \alpha^{n-2j+1} + \frac{p_i}{2} \sum_{j=0}^{\frac{n-1}{2}} \binom{n}{2j} \xi^{2j} \alpha^{n-2j} - p_i \xi^n = \\ & p_i \sum_{j=1}^{\frac{n-1}{2}} \binom{n}{2j-1} \xi^{2j-1} \alpha^{n-2j+1} \leq 0 \end{aligned}$$

where the last inequality follows by the fact that  $\xi \leq 0$ . Since  $X$  with expected value  $\mu$  is skewed to the left it follows by Theorem 2 that it can be obtained as the limit of a sequence of left symmetric splits. At  $\delta_\mu$  (the distribution of  $(\mu, 1)$ ),  $\int_{\underline{x}}^{\bar{x}} (y - \mu)^n d\delta_\mu = 0$ . The claim follows by the fact that each left symmetric split reduces the value of the integral.  $\blacksquare$

**Proof of Theorem 1:** Let  $c_p$  be the certainty equivalent of the lottery  $(\bar{x}, p; \underline{x}, 1 - p)$ . The two-stage lottery  $\langle p - \alpha, \varepsilon; p, 1 - 2\varepsilon; p + \alpha, \varepsilon \rangle$  translates in the recursive model into the lottery  $(c_{p-\alpha}, \varepsilon; c_p, 1 - 2\varepsilon; c_{p+\alpha}, \varepsilon)$ . Since the decision maker always rejects symmetric noise, it follows that the local utility  $u_{\delta_{c_p}}$  satisfies

$$u_{\delta_{c_p}}(c_p) \geq \frac{1}{2} u_{\delta_{c_p}}(c_{p-\alpha}) + \frac{1}{2} u_{\delta_{c_p}}(c_{p+\alpha}).$$

By Weak Hypothesis II, for every  $r \geq p$ ,

$$u_{\delta_{c_r}}(c_p) \geq \frac{1}{2} u_{\delta_{c_r}}(c_{p-\alpha}) + \frac{1}{2} u_{\delta_{c_r}}(c_{p+\alpha}). \quad (5)$$

Consider first the lottery over the probabilities given by  $Q = \langle p_1, q_1; \dots; p_m, q_m \rangle$  where  $\sum q_i p_i = p$  (we deal with the distributions with non-finite support at the end of the proof). If  $Q$  is skewed to the left, then it follows by Theorem 2 that there is a sequence of lotteries  $Q_i = \langle p_{i,1}, q_{i,1}, \dots, p_{i,n_i}, q_{i,n_i} \rangle \rightarrow Q$  such that  $Q_1 = \langle p, 1 \rangle$  and for all  $i$ ,  $Q_{i+1}$  is obtained from  $Q_i$  by a left symmetric split. For each  $i$ , let  $\tilde{Q}_i = (c_{p_{i,1}}, q_{i,1}; \dots; c_{p_{i,n_i}}, q_{i,n_i})$ . Suppose  $p_{i,j}$  is split into  $p_{i,j} - \alpha$  and  $p_{i,j} + \alpha$ . By eq. (5), as  $p > p_{i,j}$ ,

$$\begin{aligned} E[u_{\delta_p}(\tilde{Q}_i)] &= \\ & q_{i,j} u_{\delta_{c_p}}(c_{p_{i,j}}) + \sum_{m \neq j} q_{i,m} u_{\delta_{c_p}}(c_{p_{i,m}}) \geq \\ & \frac{1}{2} q_{i,j} u_{\delta_{c_p}}(c_{p_{i,j}-\alpha}) + \frac{1}{2} q_{i,j} u_{\delta_{c_p}}(c_{p_{i,j}+\alpha}) + \sum_{m \neq j} q_{i,m} u_{\delta_{c_p}}(c_{p_{i,m}}) = \\ & E[u_{\delta_{c_p}}(\tilde{Q}_{i+1})]. \end{aligned}$$

As  $Q_i \rightarrow Q$ , and as for all  $i$ ,  $u_{\delta_{c_p}}(c_p) \geq E[u_{\delta_p}(\tilde{Q}_i)]$ , it follows by continuity that  $u_{\delta_{c_p}}(c_p) \geq E[u_{\delta_{c_p}}(\tilde{Q})]$ . By Fréchet Differentiability

$$\left. \frac{\partial}{\partial \varepsilon} V\left(\varepsilon \tilde{Q} + (1 - \varepsilon) \delta_{c_p}\right) \right|_{\varepsilon=0} \leq 0.$$

Quasi-concavity now implies that  $V(\delta_{c_p}) \geq V(\tilde{Q})$ , or  $\langle p, 1 \rangle \succeq Q$ . Finally, as preferences are continuous, it follows by Remark 1 that the theorem holds for all  $Q$ , even if its support is not finite.

For the second part of the theorem, we show in Appendix A that the functional form in Example 1 satisfies all the assumptions of the theorem and always accepts some positively skewed noise.  $\blacksquare$

**Proof of Theorem 2:** Lemma 1 proves part 1 of the theorem for binary lotteries  $Y$ . After a preparatory claim (Lemma 2), the general case of this part is proved in Lemma 3 for lotteries  $Y$  with  $F_Y(\mu) \geq \frac{1}{2}$ , and for all lotteries in Lemma 4. That this can be done with bounded shifts is proved in Lemma 5. Part 2 of the theorem is proved in Lemma 6.

**Lemma 1** *Let  $Y = (x, r; z, 1 - r)$  with mean  $E[Y] = \mu$ ,  $x < z$ , and  $r \leq \frac{1}{2}$ . Then there is a sequence of lotteries  $X_i$  with expected value  $\mu$  such that  $X_1 = (\mu, 1)$ ,  $X_i \rightarrow Y$ , and  $X_{i+1}$  is obtained from  $X_i$  by a left symmetric split. Moreover, if  $r_i$  and  $r'_i$  are the probabilities of  $x$  and  $z$  in  $X_i$ , then  $r_i \uparrow r$  and  $r'_i \uparrow 1 - r$ .*



**Proof:** The main idea of the proof is to have at each step at most five outcomes:  $x, \mu, z$ , and up to two outcomes between  $x$  and  $\mu$ . In a typical move either  $\mu$  or one of the outcomes between  $x$  and  $\mu$ , denote it  $w$ , is split “as far as possible,” which means:

1. If  $w \in (x, \frac{x+\mu}{2}]$ , then split its probability between  $x$  and  $w + (w - x) = 2w - x$ . Observe that  $x < 2w - x \leq \mu$ .
2. If  $w \in [\frac{x+z}{2}, \mu]$ , then split its probability between  $z$  and  $w - (z - w) = 2w - z$ . Observe that  $x \leq 2w - z < \mu$ .
3. If  $w \in (\frac{x+\mu}{2}, \frac{x+z}{2})$ , then split its probability between  $\mu$  and  $w - (\mu - w) = 2w - \mu$ . Observe that  $x < 2w - \mu < \mu$ .

If  $r = \frac{1}{2}$ , that is, if  $\mu = \frac{x+z}{2}$  then the sequence terminates after the first split. We will therefore assume that  $r < \frac{1}{2}$ . Observe that the this procedures never split the probabilities of  $x$  and  $z$  hence these probabilities form increasing sequences. We identify and analyze three cases: *a.* For every  $i$  the support of  $X_i$  is  $\{x, y_i, z\}$ . *b.* There is  $k > 1$  such that the support of  $X_k$  is  $\{x, \mu, z\}$ . *c.* Case *b* does not happen, but there is  $k > 1$  such that the support of  $X_k$  is  $\{x, w_k, \mu, z\}$ . We also show that if for all  $i > 1$ ,  $\mu$  is not in the support of  $X_i$ , then case *a* prevails.

*a.* The simplest case is when for every  $i$  the support of  $X_i$  has three outcomes at most,  $x < y_i < z$ . By construction, the probability of  $y_i$  is  $\frac{1}{2^i}$ , hence  $X_i$  puts  $1 - \frac{1}{2^i}$  probability on  $x$  and  $z$ . In the limit these converge to a lottery over  $x$  and  $z$  only, and since for every  $i$ ,  $E[X_i] = \mu$ , this limit must be  $Y$ . The two examples of Section 2.2 show that this procedure may or may not terminate after a finite number of steps.

*b.* Suppose now that even though at a certain step the obtained lottery has more than three outcomes, it is nevertheless the case that after  $k$  splits we reach a lottery of the form  $X_k = (x, p_k; \mu, q_k; z, 1 - p_k - q_k)$ . For example, let  $X = (17, 1)$  and  $Y = (24, \frac{17}{24}; 0, \frac{7}{24})$ . The first five splits are

$$\begin{aligned}
X &= (17, 1) \rightarrow (10, \frac{1}{2}; 24, \frac{1}{2}) \rightarrow (3, \frac{1}{4}; 17, \frac{1}{4}; 24, \frac{1}{2}) \rightarrow \\
&\quad (0, \frac{1}{8}; 6, \frac{1}{8}; 17, \frac{1}{4}; 24, \frac{1}{2}) \rightarrow (0, \frac{3}{16}; 12, \frac{1}{16}; 17, \frac{1}{4}; 24, \frac{1}{2}) \rightarrow \\
&\quad (0, \frac{7}{32}; 17, \frac{1}{4}; 24, \frac{17}{32})
\end{aligned} \tag{6}$$

By construction  $k \geq 2$  and  $q_k \leq \frac{1}{4}$ . Repeating these  $k$  steps  $j$  times will yield the lottery  $X_{jk} = (x, p_{jk}; \mu, q_k^j; z, 1 - p_{jk} - q_k^j) \rightarrow Y$  as  $q_k^j \rightarrow 0$  and as the expected value of all lotteries is  $\mu$ ,  $p_{jk} \uparrow r$  and  $1 - p_{jk} - q_k^j \uparrow 1 - r$ .

*c.* If at each stage  $X_i$  puts no probability on  $\mu$  then we are in case *a*. The reason is that as splits of type 3 do not happen, in each stage the probability of the outcome between  $x$  and  $z$  is split between a new such outcome and either  $x$  or  $z$ , and the number of different outcomes is still no more than three. Suppose therefore that at each stage  $X_i$  puts positive probability on at least one outcome  $w$  strictly between  $x$  and  $\mu$  (although these outcomes  $w$  may change from one lottery  $X_i$  to another) and at some stage  $X_i$  puts (again) positive probability on  $\mu$ . Let  $k \geq 2$  be the first split that puts positive probability on  $\mu$ . We consider two cases.

*c*<sub>1</sub>.  $k = 2$ : In the first step, the probability of  $\mu$  is divided between  $z$  and  $2\mu - z$  and in the second step the probability of  $2\mu - z$  is split and half of it is shifted back to  $\mu$  (see for example the second split in eq. (6) above). In other words, the first split is of type 2 while the second is of type 3. By the description of the latter,

$$\frac{x + \mu}{2} < 2\mu - z < \frac{x + z}{2} \iff \frac{2}{3} < \frac{\mu - x}{z - x} < \frac{3}{4} \quad (7)$$

The other one quarter of the original probability of  $\mu$  is shifted from  $2\mu - z$  to

$$2\mu - z - (\mu - [2\mu - z]) = 3\mu - 2z \leq \frac{x + \mu}{2} \iff 4(z - x) \geq 5(\mu - x)$$

Which is satisfied by eq. (7). Therefore, in the next step a split of type 1 will be used, and one eighth of the original probability of  $\mu$  will be shifted away from  $2\mu - z$  to  $x$ . In other words, in three steps  $\frac{5}{8}$  of the original probability of  $\mu$  is shifted to  $x$  and  $z$ , one quarter of it is back at  $\mu$ , and one eighth of it is now on an outcome  $w_1 < \mu$ .  $\diamond$

*c*<sub>2</sub>.  $k \geq 3$ : For example,  $X = (29, 1)$  and  $Y = (48, \frac{29}{48}; 0, \frac{19}{48})$ . Then

$$\begin{aligned} X &= (29, 1) \rightarrow (10, \frac{1}{2}; 48, \frac{1}{2}) \rightarrow (0, \frac{1}{4}; 20, \frac{1}{4}; 48, \frac{1}{2}) \rightarrow \\ &(0, \frac{1}{4}; 11, \frac{1}{8}; 29, \frac{1}{8}; 48, \frac{1}{2}) \rightarrow \dots \end{aligned} \quad (8)$$

After  $k$  splits  $\frac{1}{2^k}$  of the original probability of  $\mu$  is shifted back to  $\mu$  and  $\frac{1}{2^k}$  is shifted to another outcome  $w_1 < \mu$ . The rest of the original probability is split (not necessarily equally) between  $x$  and  $z$ .  $\diamond$

Let  $\ell = \max\{k, 3\}$ . We now construct inductively a sequence of cycles, where the length of cycle  $j$  is  $\ell + j - 1$ . Such a cycle will end with the probability distributed over  $x < w_j < \mu < z$ . Denote the probability of  $\mu$  by  $p_j$  and that of  $w_j$  by  $q_j$ . We show that  $p_j + q_j \rightarrow 0$ . The probabilities of  $x$  and  $z$  are such that the expected value is kept at  $\mu$ , and as  $p_j + q_j \rightarrow 0$ , it will follow that the probabilities of  $x$  and  $z$  go up to  $r$  and  $1 - r$ , respectively. In the example of eq. (8),  $\ell = 3$ , the length of the first cycle (where  $j = 1$ ) is 3, and  $w_1 = 11$ .

Suppose that we've finished the first  $j$  cycles. Cycle  $j + 1$  starts with splitting the  $p_j$  probability of  $\mu$  to  $\{x, w_1, \mu, z\}$  as in the first cycle. One of the outcomes along this sequence may be  $w_j$ , but we will continue to split only the "new" probability of this outcome (and will not yet touch the probability  $q_j$  of  $w_j$ ). At the end of this part of the new cycle, the probability is distributed over  $x, w_1, w_j, \mu$ , and  $z$ . At least half of  $p_j$ , the earlier probability of  $\mu$ , is shifted to  $\{x, z\}$ , and the probabilities of both these outcomes did not decrease. Continuing the example of eq. (8), the first part of the second cycle (where  $j = 1$ ) is

$$(0, \frac{1}{4}; 11, \frac{1}{8}; 29, \frac{1}{8}; 48, \frac{1}{2}) \rightarrow (0, \frac{1}{4}; 10, \frac{1}{16}; 11, \frac{1}{8}; 48, \frac{9}{16}) \rightarrow \\ (0, \frac{9}{32}; 11, \frac{1}{8}; 20, \frac{1}{32}; 48, \frac{9}{16}) \rightarrow (0, \frac{9}{32}; 11, \frac{9}{64}; 29, \frac{1}{64}; 48, \frac{9}{16})$$

The second part of cycle  $j + 1$  begins with  $j - 1$  splits starting with  $w_1$ . At the end of these steps, the probability is spread over  $x, w_j, \mu$ , and  $z$ . Split the probability of  $w_j$  between an element of  $\{x, \mu, z\}$  and  $w_{j+1}$  which is not in this set to get  $p_{j+1}$  and  $q_{j+1}$ . In the above example, as  $j = 1$  there is only one split at this stage to

$$(0, \frac{45}{128}; 22, \frac{9}{128}; 29, \frac{1}{64}; 48, \frac{9}{16})$$

And  $w_2 = 22$ . The first part of the third cycle ( $j = 2$ ) leads to

$$(0, \frac{91}{256}; 11, \frac{1}{512}; 22, \frac{9}{128}; 29, \frac{1}{512}; 48, \frac{73}{128})$$

The second part of this cycle has two splits. Of  $w_1 = 11$  into 0 and 22, and then of  $w_2 = 22$  into  $\mu = 29$  and  $w_3 = 15$ .

$$\rightarrow (0, \frac{365}{1024}; 22, \frac{73}{1024}; 29, \frac{1}{512}; 48, \frac{73}{128}) \rightarrow (0, \frac{365}{1024}; 15, \frac{73}{2048}; 29, \frac{77}{2048}; 48, \frac{73}{128})$$

We now show that for every  $j$ ,

$$p_{j+2} + q_{j+2} \leq \frac{3}{4}(p_j + q_j) \tag{9}$$

We first observe that for every  $j$ ,  $p_{j+1} + q_{j+1} < p_j + q_j$ . This is due to the fact that the rest of the probability is spread over  $x$  and  $z$ , the probability of  $z$  must increase (because of the initial split in the probability of  $\mu$ ), and the probabilities of  $x$  and  $z$  cannot go down.

When moving from  $(p_j, q_j)$  to  $(p_{j+2}, q_{j+2})$ , half of  $p_j$  is switched to  $z$ . Later on, half of  $q_j$  is switched either to  $x$  or  $z$ , or to  $\mu$ , in which case half of it (that is, one quarter of  $q_j$ ) will be switched to  $z$  on the move from  $p_{j+1}$  to  $p_{j+2}$ . This proves inequality (9), hence the lemma.  $\square$

**Lemma 2** *Let  $X = (x_1, p_1; \dots; x_n, p_n)$  and  $Y = (y_1, q_1; \dots; y_m, q_m)$  where  $x_1 \leq \dots \leq x_n$  and  $y_1 \leq \dots \leq y_m$  be two lotteries such that  $X$  dominates  $Y$  by second-order stochastic dominance. Then there is a sequence of lotteries  $X_i$  such that  $X_1 = X$ ,  $X_i \rightarrow Y$ ,  $X_{i+1}$  is obtained from  $X_i$  by a symmetric (not necessarily always left or always right) split of one of the outcomes of  $X_i$ , all the outcomes of  $X_i$  are between  $y_1$  and  $y_m$ , and the probabilities the lotteries  $X_i$  put on  $y_1$  and  $y_m$  go up to  $q_1$  and  $q_m$ , respectively.*

**Proof:** From Rothschild and Stiglitz [35, p. 236] we know that we can present  $Y$  as  $(y_{11}, q_{11}; \dots; y_{nn}, q_{nn})$  such that  $\sum_j q_{kj} = p_k$  and  $\sum_j q_{kj} y_{kj} / p_k = x_k$ ,  $k = 1, \dots, n$ .

Let  $Z = (z_1, r_1; \dots; z_\ell, r_\ell)$  such that  $z_1 < \dots < z_\ell$  and  $E[Z] = z$ . Let  $Z_0 = (z, 1)$ . One can move from  $Z_0$  to  $Z$  in at most  $\ell$  steps, where at each step some of the probability of  $z$  is split into two outcomes of  $Z$  without affecting the expected value of the lottery, in the following way. If

$$\frac{r_1 z_1 + r_\ell z_\ell}{r_1 + r_\ell} \geq z \quad (10)$$

then move  $r_1$  probability to  $z_1$  and  $r'_\ell \leq r_\ell$  to  $z_\ell$  such that  $r_1 z_1 + r'_\ell z_\ell = z(r_1 + r'_\ell)$ . However, if the sign of the inequality in (10) is reversed, then move  $r_\ell$  probability to  $z_\ell$  and  $r'_1 \leq r_1$  probability to  $z_1$  such that  $r'_1 z_1 + r_\ell z_\ell = z(r'_1 + r_\ell)$ . Either way the move shifted all the required probability from  $z$  to one of the outcomes of  $Z$  without changing the expected value of the lottery.

Consequently, one can move from  $X$  to  $Y$  in  $\ell^2$  steps, where at each step some probability of an outcomes of  $X$  is split between two outcomes of  $Y$ . By Lemma 1, each such split can be obtained as the limit of symmetric splits (recall that we do not require in the current lemma that the symmetric splits will be left or right splits). That all the outcomes of the obtained lotteries are between  $y_1$  and  $y_m$ , and that the probabilities these put on  $y_1$  and  $y_m$  go up to  $q_1$  and  $q_m$  follow by Lemma 1.  $\square$

**Lemma 3** Let  $Y = (y_1, p_1; \dots; y_n, p_n)$ ,  $y_1 \leq \dots \leq y_n$ , with expected value  $\mu$  be skewed to the left such that  $F_Y(\mu) \geq \frac{1}{2}$ . Then there is a sequence of lotteries  $X_i$  with expected value  $\mu$  such that  $X_1 = (\mu, 1)$ ,  $X_i \rightarrow Y$ , and  $X_{i+1}$  is obtained from  $X_i$  by a left symmetric split. Moreover, if  $r_i$  and  $r'_i$  are the probabilities of  $y_1$  and  $y_n$  in  $X_i$ , then  $r_i \uparrow p_1$  and  $r'_i \uparrow p_n$ .

**Proof:** Suppose wlg that  $y_{j^*} = \mu$  (of course, it may be that  $p_{j^*} = 0$ ). Since  $F_Y(y_{j^*}) \geq \frac{1}{2}$ , it follows that  $t := \sum_{j=j^*+1}^n p_j \leq \frac{1}{2}$ . As  $Y$  is skewed to the left,  $y_n - \mu \leq \mu - y_1$ , hence  $2\mu - y_n \geq y_1$ . Let  $m = n - j^*$  be the number of outcomes of  $Y$  that are strictly above the expected value  $\mu$ . Move from  $X_1$  to  $X_m = (2\mu - y_n, p_n; \dots; 2\mu - y_{j^*+1}, p_{j^*+1}; y_{j^*}, 1 - 2t; y_{j^*+1}, p_{j^*+1}; \dots; y_n, p_n)$  by repeatedly splitting probabilities away from  $\mu$ . All these splits are symmetric, hence left symmetric splits.

Next we show that  $Y$  is a mean preserving spread of  $X_m$ . Obviously,  $E[X_m] = E[Y] = \mu$ . Integrating by parts, we have for  $x \geq \mu$

$$\begin{aligned} y_n - \mu &= \int_{y_1}^{y_n} F_Y(z) dz = \int_{y_1}^x F_Y(z) dz + \int_x^{y_n} F_Y(z) dz \\ y_n - \mu &= \int_{y_1}^{y_n} F_{X_m}(z) dz = \int_{y_1}^x F_{X_m}(z) dz + \int_x^{y_n} F_{X_m}(z) dz \end{aligned}$$

Since  $F_Y$  and  $F_{X_m}$  coincide for  $z \geq \mu$ , we have, for  $x \geq \mu$ ,  $\int_{y_1}^x F_{X_m}(z) dz = \int_{y_1}^x F_Y(z) dz$  and in particular,  $\int_{y_1}^x F_{X_m}(z) dz \leq \int_{y_1}^x F_Y(z) dz$ .

For  $x < \mu$  it follows by the assumption that  $Y$  is skewed to the left and by the construction of  $X_m$  as a symmetric lottery around  $\mu$  that

$$\begin{aligned} \int_{y_1}^x F_{X_m}(z) dz &= \int_{2\mu-x}^{2\mu-y_1} [1 - F_{X_m}(z)] dz = \\ &= \int_{2\mu-x}^{2\mu-y_1} [1 - F_Y(z)] dz \leq \int_{y_1}^x F_Y(z) dz \end{aligned}$$

Since to the right of  $\mu$ ,  $X_m$  and  $Y$  coincide, we can view the left side of  $Y$  as a mean preserving spread of the left side of  $X_m$ . By Lemma 2 the left side of  $Y$  is the limit of symmetric mean preserving spreads of the left side of  $X_m$ . Moreover, all these splits take place between  $y_1$  and  $\mu$  and are therefore left symmetric splits. By Lemma 2 it also follows that  $r_i \uparrow p_1$  and  $r'_i \uparrow p_n$ .  $\square$

We now show that Lemma 3 holds without the restriction  $F_Y(\mu) \geq \frac{1}{2}$ .

**Lemma 4** *Let  $Y$  with expected value  $\mu$  be skewed to the left. Then there is a sequence of lotteries  $X_i$  with expected value  $\mu$  such that  $X_1 = (\mu, 1)$ ,  $X_i \rightarrow Y$ , and  $X_{i+1}$  is obtained from  $X_i$  by a left symmetric split.*

**Proof:** The first step in the proof of Lemma 3 was to create a symmetric distribution around  $\mu$  such that its upper tail (above  $\mu$ ) agrees with  $F_Y$ . Obviously this can be done only if  $F_Y(\mu) \geq \frac{1}{2}$ , which is no longer assumed. Instead, we apply the proof of Lemma 3 successively to mixtures of  $F_Y$  and  $\delta_\mu$ , the distribution that yields  $\mu$  with probability one.

Suppose that  $F_Y(\mu) = \lambda < \frac{1}{2}$ . Let  $\gamma = 1/2(1 - \lambda)$  and define  $Z$  to be the lottery obtained from the distribution  $\gamma F_Y + (1 - \gamma)\delta_\mu$ . Observe that

$$F_Z(\mu) = \gamma F_Y(\mu) + (1 - \gamma)\delta_\mu(\mu) = \frac{\lambda}{2(1 - \lambda)} + \frac{1 - 2\lambda}{2(1 - \lambda)} = \frac{1}{2}$$

It follows that the lotteries  $Z$  and  $(\mu, 1)$  satisfy the conditions of Lemma 3, and therefore there is sequence of lotteries  $X_i$  with expected value  $\mu$  such that  $X_1 = (\mu, 1)$ ,  $X_i \rightarrow Z$ , and  $X_{i+1}$  is obtained from  $X_i$  by a left symmetric split. This is done in two stages. First we create a symmetric distribution around  $\mu$  that agrees with  $Z$  above  $\mu$  (denote the number of splits needed in this stage by  $t$ ), and then we manipulate the part of the distribution which is weakly to the left of  $\mu$  by taking successive symmetric splits (which are all left symmetric splits when related to  $\mu$ ) to get nearer and nearer to the second-order stochastically dominated left side of  $Z$  as in Lemma 2. Observe that the highest outcome of this part of the distribution  $Z$  is  $\mu$ , and its probability is  $1 - \gamma$ . By Lemma 2, for every  $k \geq 1$  there is  $\ell_k$  such that after  $\ell_k$  splits of this second phase the probability of  $\mu$  will be at least  $r_k = (1 - \gamma)(1 - \frac{1}{k+1})$  and  $\|X_{t+\ell_k} - Z\| < \frac{1}{k}$ .

The first cycle will end after  $t + \ell_1$  splits with the distribution  $F_{Z_1}$ . Observe that the probabilities of the outcomes to the right of  $\mu$  in  $Z_1$  are those of the lottery  $Y$  multiplied by  $\gamma$ . The first part of second cycle will be the same as the first cycle, applied to the  $r_k$  conditional probability of  $\mu$ . At the end of this part we'll get the lottery  $Z'_1$  which is the same as  $Z_1$ , conditional on the probability of  $\mu$ . We now continue the second cycle by splitting the combination of  $Z_1$  and  $Z'_1$  for the total of  $t + \ell_1 + \ell_2$  steps. As we continue to add such cycles inductively we get closer and closer to  $Y$ , hence the lemma.  $\square$

Next we show that part 1 of the theorem can be achieved by using bounded spreads. The first steps in the proof of Lemma 3 involve shifting probabilities from  $\mu$  to all the outcomes of  $Y$  to the right of  $\mu$ , and these

outcomes are not more than  $\max y_i - \mu$  away from  $\mu$ . All other shifts are symmetric shifts involving only outcomes to the left of  $\mu$ . The next lemma shows that such shifts can be achieved as the limit of symmetric bounded shifts.

**Lemma 5** *Let  $Z = (z - \alpha, \frac{1}{2}; z + \alpha, \frac{1}{2})$  and let  $\varepsilon > 0$ . Then there is a sequence of lotteries  $Z_i$  such that  $Z_0 = (z, 1)$ ,  $Z_i \rightarrow Z$ , and  $Z_{i+1}$  is obtained from  $Z_i$  by a symmetric (not necessarily left or right) split of size smaller than  $\varepsilon$ .*

**Proof:** The claim is interesting only when  $\varepsilon < \alpha$ . Fix  $n$  such that  $\varepsilon > \frac{\alpha}{n}$ . We show that the lemma can be proved by choosing the size of the splits to be  $\frac{\alpha}{n}$ . Consider the  $2n + 1$  points  $z_k = z + \frac{k}{n}$ ,  $k = -n, \dots, n$  and construct the sequence  $\{Z_i\}$  where  $Z_i = (z - \alpha, p_{i,-n}; z - \frac{n-1}{n}\alpha, p_{i,-n+1}; \dots; z + \alpha, p_{i,n})$  as follows.

THE INDEX  $i$  IS ODD: Let  $z_j$  be the highest outcome in  $\{z, \dots, z + \frac{n-1}{n}\alpha\}$  with the highest probability in  $Z_{i-1}$ . Formally,  $j$  satisfies:

- $0 \leq j \leq n - 1$
- $p_{i-1,j} \geq p_{i-1,k}$  for all  $k$
- If for some  $j' \in \{0, \dots, n - 1\}$ ,  $p_{i-1,j'} \geq p_{i-1,k}$  for all  $k$ , then  $j \geq j'$ .

Split the probability of  $z_j$  between  $z_j - \frac{\alpha}{n}$  and  $z_j + \frac{\alpha}{n}$  (i.e., between  $z_{j-1}$  and  $z_{j+1}$ ). That is,  $p_{i,j-1} = p_{i-1,j-1} + \frac{1}{2}p_{i-1,j}$ ,  $p_{i,j+1} = p_{i-1,j+1} + \frac{1}{2}p_{i-1,j}$ ,  $p_{i,j} = 0$ , and for all  $k \neq j - 1, j, j + 1$ ,  $p_{i,k} = p_{i-1,k}$ .

THE INDEX  $i$  IS EVEN: In this step we create the mirror split of the one done in the previous step. Formally, If  $j$  of the previous stage is zero, do nothing. Otherwise, split the probability of  $z_{-j}$  between  $z_{-j} - \frac{\alpha}{n}$  and  $z_{-j} + \frac{\alpha}{n}$ . That is,  $p_{i,-j-1} = p_{i-1,-j-1} + \frac{1}{2}p_{i-1,-j}$ ,  $p_{i,-j+1} = p_{i-1,-j+1} + \frac{1}{2}p_{i-1,-j}$ ,  $p_{i,-j} = 0$ , and for all  $k \neq -j - 1, -j, -j + 1$ ,  $p_{i,k} = p_{i-1,k}$ .

After each pair of these steps, the probability distribution is symmetric around  $z$ . Also, the sequences  $\{p_{i,-n}\}_i$  and  $\{p_{i,n}\}$  are non decreasing. Being bounded by  $\frac{1}{2}$ , they converge to a limit  $L$ . Our aim is to show that  $L = \frac{1}{2}$ . Suppose not. Then at each step the highest probability of  $\{p_{i-1,-n+1}, \dots, p_{i-1,n-1}\}$  must be at least  $\ell := (1 - 2L)/(2n - 1) > 0$ . The variance of  $Z_i$  is bounded from above by the variance of  $(\mu - \alpha, \frac{1}{2}; \mu + \alpha, \frac{1}{2})$ , which is  $\alpha^2$ . Splitting  $p$  probability from  $z$  to  $z - \frac{\alpha}{n}$  and  $z + \frac{\alpha}{n}$  will increase

the variance by  $p\left(\frac{\alpha}{n}\right)^2$ . Likewise, for  $k \neq -n, 0, n$ , splitting  $p$  probability from  $z + \frac{k\alpha}{n}$  to  $z + \frac{(k+1)\alpha}{n}$  and  $z - \frac{(k-1)\alpha}{n}$  will increase the variance by  $\frac{p}{2}\left(\frac{\alpha}{n}\right)^2$ . Therefore, for positive even  $i$  we have

$$\sigma^2(Z_i) - \sigma^2(Z_{i-2}) \geq \frac{1 - 2L}{2n - 1} \left(\frac{\alpha}{n}\right)^2$$

If  $L < \frac{1}{2}$ , then after enough steps the variance of  $Z_i$  will exceed  $\alpha^2$ , a contradiction.  $\square$

That we can do part 1 of the theorem for all lotteries  $Y$  follows by the fact that a countable set of countable sequences is countable.

Finally, the following lemma proves part 2 of the theorem.

**Lemma 6** *Any sequence of left symmetric split starting at  $\delta_\mu$  converges (in the  $L^1$  topology) to a skewed to the left distribution with expected value  $\mu$ .*

**Proof:** That such sequences converge follows from the fact that a symmetric split will increase the variance of the distribution, but as all distributions are over the bounded  $[\underline{x}, \bar{x}]$  segment of  $\mathfrak{R}$ , the variances of the distributions increase to a limit. Replacing  $(x, p)$  with  $(x - \alpha, \frac{p}{2}; x + \alpha, \frac{p}{2})$  increases the variance of the distribution by

$$\frac{p}{2}(x - \alpha - \mu)^2 + \frac{p}{2}(x + \alpha - \mu)^2 - p(x - \mu)^2 = p\alpha^2$$

and therefore the distance between two successive distributions in the sequences is bounded by  $\bar{x} - \underline{x}$  times the change in the variance. The sum of the changes in the variances is bounded, as is therefore the sum of distances between successive distributions, hence Cauchy criterion is satisfied and the sequence converges.

Next we prove that the limit is a skewed to the left distribution with expected value  $\mu$ . Let  $F$  be the distribution of  $X = (x_1, p_1; \dots; x_n, p_n)$  with expected value  $\mu$  be skewed to the left. Suppose wlg that  $x_1 \leq \mu$ , and break it symmetrically to obtain  $X' = (x_1 - \alpha, \frac{p_1}{2}; x_1 + \alpha, \frac{p_1}{2}; x_2, p_2; \dots; x_n, p_n)$  with the distribution  $F'$ . Note that  $E[X'] = \mu$ . Consider the following two cases.

*Case 1:*  $x_1 + \alpha \leq \mu$ . Then for all  $\delta$ ,  $\eta_2(F, \delta) = \eta_2(F', \delta)$ . For  $\delta$  such that  $\mu - \delta \leq x_1 - \alpha$  or such that  $x_1 + \alpha \leq \mu - \delta$ ,  $\eta_1(F', \delta) = \eta_1(F, \delta) \geq \eta_2(F, \delta) = \eta_2(F', \delta)$ . For  $\delta$  such that  $x_1 - \alpha < \mu - \delta \leq x_1$ ,  $\eta_1(F', \delta) = \eta_1(F, \delta) + [(\mu -$



$\delta) - (x_1 - \alpha)] \frac{p_1}{2} > \eta_1(F, \delta) \geq \eta_2(F, \delta) = \eta_2(F, \delta)$ . Finally, for  $\delta$  such that  $x_1 < \mu - \delta < x_1 + \alpha (\leq \mu)$ ,  $\eta_1(F', \delta) = \eta_1(F, \delta) + [(x_1 + \alpha) - (\mu - \delta)] \frac{p_1}{2} > \eta_1(F, \delta) \geq \eta_2(F, \delta) = \eta_2(F', \delta)$ .

*Case 2:*  $x_1 + \alpha > \mu$ . Then for all  $\delta$  such that  $\mu + \delta \geq x_1 + \alpha$ ,  $\eta_2(F, \delta) = \eta_2(F', \delta)$ . For  $\delta$  such that  $\mu - \delta \leq x_1 - \alpha$ ,  $\eta_1(F', \delta) = \eta_1(F, \delta) \geq \eta_2(F, \delta) = \eta_2(F', \delta)$ . For  $\delta$  such that  $x_1 - \alpha < \mu - \delta \leq x_1$ ,  $\eta_1(F', \delta) = \eta_1(F, \delta) + [(\mu - \delta) - (x_1 - \alpha)] \frac{p_1}{2} \geq \eta_2(F, \delta) + [(\mu - \delta) - (x_1 - \alpha)] \frac{p_1}{2} \geq \eta_2(F, \delta) + \max\{0, (x_1 + \alpha) - (\mu + \delta)\} \frac{p_1}{2} = \eta_2(F', \delta)$ . Finally, for  $\delta$  such that  $\mu - \delta > x_1$ ,  $\eta_1(F', \delta) = \eta_1(F, \delta) + [(x_1 + \alpha) - (\mu - \delta)] \frac{p_1}{2} \geq \eta_2(F, \delta) + \max\{0, (x_1 + \alpha) - (\mu + \delta)\} \frac{p_1}{2} = \eta_2(F', \delta)$ .

If  $X_n \rightarrow Y$ , all have the same expected value and for all  $n$ ,  $X_n$  is skewed to the left, then so is  $Y$ . ■

**Proof of Theorem 3:** We first note the following immediate corollary of Theorem 1.

**Corollary 1** *Under the assumptions of Theorem 1, the decision maker rejects skewed-to-the-left splits of outcomes below the mean.*

In proving Theorem 3, we will use the following rules:

- a. Rejection of left symmetric splits (Theorem 1). For example,  $(8, \frac{1}{2}; 4, \frac{1}{2}) \succ_a (8, \frac{1}{2}; 6, \frac{1}{4}; 2, \frac{1}{4})$ .
- b. Rejection of larger left-symmetric splits (Definition 8). For example,  $(8, \frac{1}{2}; 6, \frac{1}{4}; 2, \frac{1}{4}) \succ_b (8, \frac{1}{2}; 7, \frac{1}{4}; 1, \frac{1}{4})$ .
- c. Rejection of skewed-to-the-left splits of outcomes below the mean (Corollary 1). For example,  $(10, \frac{1}{2}; 6, \frac{1}{2}) \succ_c (10, \frac{1}{2}; 7, \frac{1}{3}; 4, \frac{1}{6})$ .

The proof is divided into two parts.

**Part 1.** Suppose first that  $q > p$ . Since  $p \geq \frac{1}{2}$ ,

$$1 - (1 - q + p) \leq 1 - q + p - \frac{1 - q}{1 - p}$$

Hence

$$(1 - q + p, 1) \succeq_c \left(1, p; \frac{1 - q}{1 - p}, 1 - p\right) = X_1$$

**Case 1.1** If  $q \leq \frac{2p}{1+p}$ , then

$$1 - \frac{1-q}{1-p} \leq \frac{1-q}{1-p} - (1-q)$$

In which case

$$X_1 = \left(1, p; \frac{1-q}{1-p}, 1-p\right) \succeq_c \left(1, \frac{p}{q}; 1-q, \frac{q-p}{q}\right) = Y_1$$

Fig. 3 depicts this case for  $p = \frac{2}{3}$  and  $q = \frac{3}{4}$ .

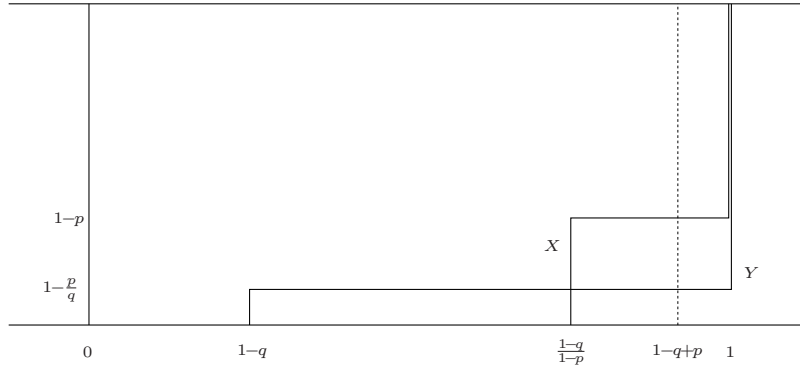


Figure 3: Case 1.1

**Case 1.2** If  $q > \frac{2p}{1+p}$  but  $q \leq 2p - p^2$ , then

$$1 - (1+p-q) \leq \frac{1-q}{1-p} - (1-q)$$

And

$$(1-p) - \left(1 - \frac{p}{q}\right) < 1 - \frac{p}{q}$$

Now

$$X_1 = \left(1, p; \frac{1-q}{1-p}, 1-p\right) \succeq_a$$

$$Z_1 := \left(1, p; 1-q+p, \frac{p}{q} - p; 1-p, \frac{p}{q} - p; 1-q, 1+p - \frac{2p}{q}\right)$$

Explanation:  $Z_1$  is a mean preserving spread of  $X_1$  and can therefore be obtained from the latter by a sequence of symmetric splits. Since  $X_1$  and  $Z_1$  differ only for outcomes below the average  $1 + p - q$ , all these splits are left symmetric splits. Finally,

$$\left(1, p; 1 - \frac{q}{2}, \frac{2p}{q} - 2p; 1 - q, 1 + p - \frac{2p}{q}\right) \succeq_a Z_1 \succeq_b Y_1$$

Fig. 4 depicts this case for  $p = \frac{2}{3}$  and  $q = \frac{5}{6}$ . Lottery  $Z_1$  is depicted by the heavy line.

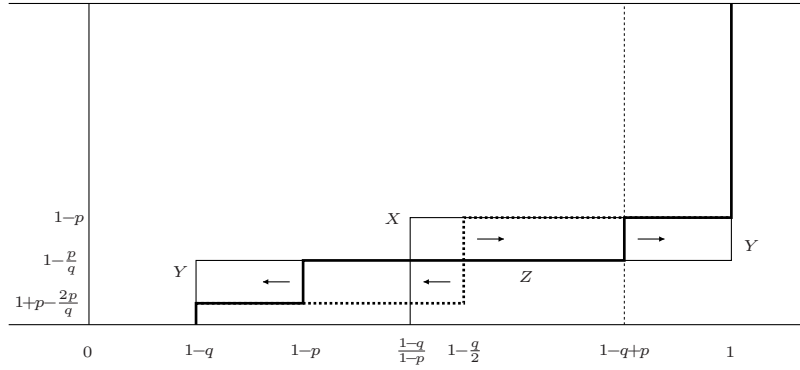


Figure 4: Case 1.2

**Case 1.3** Suppose now that  $q > \frac{2p}{1+p}$  and  $q > 2p - p^2$ . Then

$$1 - (1 + p - q) > \frac{1 - q}{1 - p} - (1 - q)$$

And

$$(1 - p) - \left(1 - \frac{p}{q}\right) < 1 - \frac{p}{q}$$

Similarly to the analysis of case 2,

$$X_1 \succeq_a Z'_1 := \left(1, p; 1 - \frac{q}{2}, \frac{2p}{q} - 2p; 1 - q, 1 + p - \frac{2p}{q}\right)$$

Observe that  $Z'_1$  is is mean preserving spread of  $X_1$  and that the two differ only below the average  $1 - q + p$ . The case follows by  $Z'_1 \succeq_a Y_1$ . Fig. 5 depicts this case for  $p = \frac{2}{3}$  and  $q = \frac{11}{12}$ . Lottery  $Z'_1$  is depicted by the heavy line.

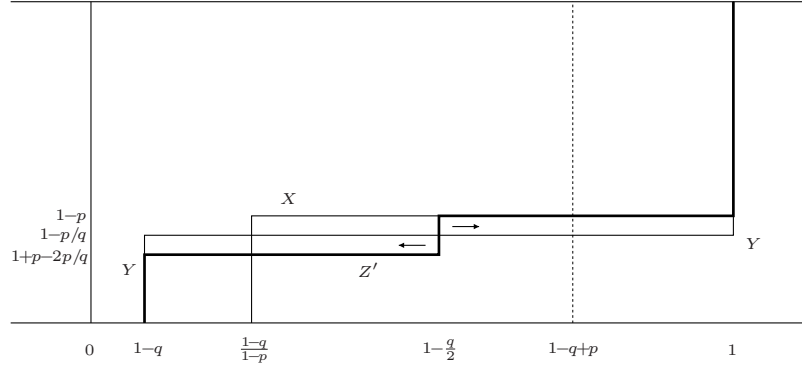


Figure 5: Case 1.3

**Part 2:** Here we deal with the case  $q < p$  and show that  $X_2 \succeq Y_2$ .

**Case 2.1** If  $q \geq \frac{p}{2-p}$  then  $1 - \frac{q}{p} \leq \frac{q}{p} - q$  hence  $X_2 \succeq_c Y_2$ .

**Case 2.2** Suppose that  $p^2 \leq q < \frac{p}{2-p}$ . Then

$$1 - (1 - p + q) \leq \frac{q}{p} - q \quad \text{and} \quad p \leq \frac{2(p - q)}{1 - q}$$

Therefore

$$X_2 \succeq_a \left( 1, 1 - p; p, p - \frac{p - q}{1 - q}; q, \frac{p - 2q + pq}{1 - q} \right) \succeq_a Y_2$$

**Case 2.3** Suppose that  $2p - 1 \leq q \leq p^2$ . Then

$$1 - (1 - p + q) \geq \frac{q}{p} - q \tag{11}$$

And the outcome  $1 - (\frac{q}{p} - q)$  is to the right of the average  $1 - p + q$ . Also,

$$p \leq \frac{2(p - q)}{1 - q} \iff p - 2q + pq \geq 0$$

The expression  $p - 2q + pq$  is decreasing in  $q$ , and at  $q = p^2$  it is  $p(1 - 2p + p^2) = p(1 - p)^2 > 0$ . Therefore

$$\frac{p - q}{1 - q} = (m + s) \left( p - \frac{p - q}{1 - q} \right) = \frac{(m + s)q(1 - p)}{1 - q}$$

Where  $m$  is a positive integer and  $0 \leq s < 1$ . We now move from  $X_2$  to  $Y_2$  in  $m + 1$  utility-reducing steps (If  $s = 0$ , then there are only  $m$  steps).

Step 0: Move from  $X_2$  to

$$Z_2^0 = \left( q, \frac{sq(1-p)}{1-q}; \frac{q}{p}, \frac{mq(1-p)}{1-q}; \frac{q}{p} + s \left( \frac{q}{p} - q \right), \frac{q(1-p)}{1-q}; 1, 1-p \right)$$

This is a left split to the left of the average, hence by Theorem 1  $X_2 \succeq_c Z_2^0$ . If  $s = 0$  denote  $Z_2^0 = X_2$ .

Steps 1,  $\dots$ ,  $m$ : For  $i = 1, \dots, m$ , let

$$Z_2^i = \left( q, \frac{(i+s)q(1-p)}{1-q}; \frac{q}{p}, \frac{(m-i)q(1-p)}{1-q}; \frac{q}{p} + (i+s) \left( \frac{q}{p} - q \right), \frac{q(1-p)}{1-q}; 1, 1-p \right)$$

Observe that  $Z_2^i$  is obtained from  $Z_2^{i-1}$  by a continuation of a symmetric split around

$$\begin{aligned} d_i : &= \frac{1}{2} \left( \frac{q}{p} + (i+s) \left( \frac{q}{p} - q \right) \right) \leq \\ &\frac{1}{2} \left( \frac{q}{p} + \left[ 1 - \frac{q}{p} + q \right] \right) = \frac{1+q}{2} \leq 1-p+q \iff \\ &2p \leq 1+q \end{aligned}$$

We assumed  $q \geq 2p-1$ , hence these splits are all symmetric around outcomes (weakly) below the average  $1-p+q$ . Therefore

$$X_2 \succeq_c Z_2^0 \succeq_b Z_2^1 \succeq_b \dots \succeq_b Z_2^m = Y_2$$

Lottery  $Z_2^{m-1}$  is depicted by the heavy line in Fig. 6 which is drawn for the case  $p = \frac{2}{3}$  and  $q = \frac{1}{3}$ . Observe that in this case  $s = 0$ . ■

**Proof of Proposition 2:** Suppose that for  $q$  sufficiently close to zero,  $\langle 1, q; 0, 1-q \rangle \succ \langle q, 1 \rangle$ . These preferences imply that the derivative of  $h(q) := V(1, q; 0, 1-q) - V(q, 1)$  at  $q = 0$  is positive. We obtain:

$$\begin{aligned} h'(q) &= V_2(1, q; 0, 1-q) - V_4(1, q; 0, 1-q) - V_1(q, 1; 0, 0) \implies \\ h'(0) &= V_2(1, 0; 0, 1) - V_4(1, 0; 0, 1) - V_1(0, 1; 0, 0) > 0 \end{aligned} \quad (12)$$

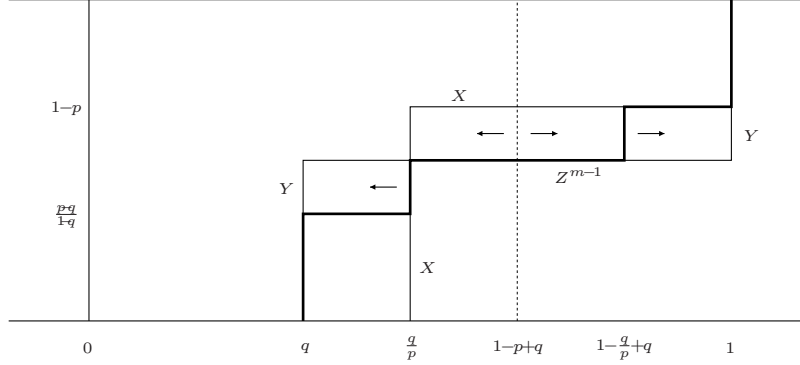


Figure 6: Case 2.3

Consider now the lotteries  $X_2 = \langle 1, 1-p; \frac{q}{p}, p \rangle$  and  $Y_2 = \langle 1, \frac{1-p}{1-q}; q, \frac{p-q}{1-q} \rangle$ , and let  $p = 1 - kq$ . We want to check conditions under which for sufficiently small  $q$ ,  $g(q) := V(Y_2) - V(X_2) > 0$ . In other words, we want to show that the first non-zero derivative with respect to  $q$  of  $g$  at  $q = 0$  is positive. Let  $A = (1, \frac{kq}{1-q}; q, \frac{1-(k+1)q}{1-q})$  and  $B = (1, kq; \frac{q}{1-kq}, 1 - kq)$ . We get

$$\begin{aligned}
g(q) &= V(A) - V(B) \\
g'(q) &= \frac{k}{(1-q)^2} [V_2(A) - V_4(A)] + V_3(A) - \\
&\quad k[V_2(B) - V_4(B)] - \frac{1}{(1-kq)^2} V_3(B) \implies \\
g'(0) &= k[V_2(A) - V_4(A)] + V_3(A) - k[V_2(B) - V_4(B)] - V_3(B)
\end{aligned}$$

At  $q = 0$ ,  $V_2(A) - V_4(A)$  and  $V_2(B) - V_4(B)$  describe the same effect: the change in the value of the lottery  $(1, r; 0, 1-r)$  as the probability  $r$  starts to go up from zero. Likewise, at  $q = 0$ ,  $V_3(A)$  and  $V_3(B)$  describe the change in the value of the lottery  $(1, 0; s, 1)$  as the outcome  $s$  starts to go up from zero. It thus follows that  $g'(0) = 0$ .

$$\begin{aligned}
g''(q) &= \frac{2k}{(1-q)^3} [V_2(A) - V_4(A)] + \frac{k^2}{(1-q)^4} [V_{22}(A) - 2V_{24}(A) + V_{44}(A)] + \\
&\quad \frac{2k}{(1-q)^2} [V_{23}(A) - V_{43}(A)] + V_{33}(A) - \\
&\quad k^2 [V_{22}(B) - 2V_{24}(B) + V_{44}(B)] - \frac{2k}{(1-kq)^2} [V_{23}(B) - V_{43}(B)] - \\
&\quad \frac{2k}{(1-kq)^3} V_3(B) - \frac{1}{(1-kq)^4} V_{33}(B) \implies \\
g''(0) &= 2k[V_2(A) - V_4(A)] + k^2 [V_{22}(A) - 2V_{24}(A) + V_{44}(A)] + \\
&\quad 2k[V_{23}(A) - V_{43}(A)] + V_{33}(A) -
\end{aligned}$$

$$k^2[V_{22}(B) - 2V_{24}(B) + V_{44}(B)] - 2k[V_{23}(B) - V_{43}(B)] - 2kV_3(B) - V_{33}(B)$$

Similarly to the previous analysis, at  $q = 0$ ,  $V_{ij}(A) = V_{ij}(B)$  for all  $i$  and  $j$ , hence

$$g''(0) = 2k[V_2(A) - V_4(A) - V_3(B)]$$

At  $q = 0$ ,  $A = B = (1, 0; 0, 1) \sim (0, 1; 1, 0)$ . Moreover, for all  $a$ ,  $(a, 1; 0, 0) \sim (a, 1; 1, 0)$ . Therefore, at  $q = 0$ ,  $V_3(B) = V_1(0, 1; 0, 0)$ . It thus follows that  $g''(0) = 2kh'(0) > 0$ . ■

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## Appendix A: Example 1

A quadratic utility (Chew, Epstein, and Segal [10]) functional is given by  $V(p) = \sum_x \sum_y p_x p_y \theta(x, y)$ , where  $\theta$  is symmetric. Following [10, Example 5 (p. 145)], If  $\theta(x, y) = \frac{v(x)w(y)+v(y)w(x)}{2}$ , where  $v$  and  $w$  are positive functions, then  $V(p) = E[v(p)] \times E[w(p)]$ . This is the form of  $V$  we analyze below.

The function  $V$  is the product of two positive linear functions of the probabilities, hence quasi concave. To see why, observe that  $\ln V(p) = \ln E[v(p)] + \ln E[w(p)]$ . The sum of concave functions is concave, hence quasi concave, and any monotone nondecreasing transformation of a quasi concave function is quasi concave.

Direct calculations show that the local utility function of any quadratic utility is given by  $u_F(x) = 2 \int \theta(x, y) dF(y)$ . Since we are only interested in the behavior of the function in lotteries of the form  $\delta_y := (y, 1)$ , we have

$$u_{\delta_y}(x) = 2\theta(x, y) = v(x)w(y) + v(y)w(x)$$

Take  $v(x) = x$  and let  $w$  be any increasing, concave, and differential function such that  $w(0) = 0$ . We now show that  $V$  satisfies Weak Hypothesis II. That is, we show that

$$RA := -\frac{u''_{\delta_y}(x)}{u'_{\delta_y}(x)} = -\frac{yw''(x)}{w(y) + yw'(x)}$$

is an increasing function of  $y$ . We have

$$\begin{aligned} -\frac{\partial}{\partial y} \left( \frac{yw''(x)}{w(y) + yw'(x)} \right) &> 0 \iff \\ w''(x)(w(y) + yw'(x)) &< (w'(y) + w'(x))yw''(x) \iff \\ w(y) &> w'(y)y \iff \\ w(y)/y &> w'(y) \end{aligned}$$

which holds since  $w$  is concave.

Next we analyze the functional form  $V(\langle p_1, q_1; \dots; p_n, q_n \rangle) = E[w(c_p)] \times E[c_p]$  where  $w(x) = \frac{\zeta x - x^\zeta}{\zeta - 1}$ ,  $c_p = \beta p + (1 - \beta)p^\kappa$ ,  $\zeta = 1.024$ ,<sup>16</sup>  $\kappa = 1.1$ , and

---

<sup>16</sup>since  $\zeta > 1$ , we have that  $w'(x) = \frac{\zeta - \zeta x^{\zeta-1}}{\zeta - 1} > 0$  and  $w''(x) = \frac{(\zeta - 1)\zeta x^{\zeta-2}}{\zeta - 1} < 0$ , hence  $w$  is increasing and concave.

$\beta = 0.15$ . Since all the inequalities below are strict, there is an open set of parameters for which they are satisfied as well. Observe that

$$w(c_p) = \frac{\zeta [\beta p + (1 - \beta)p^\kappa] - [\beta p + (1 - \beta)p^\kappa]^\zeta}{\zeta - 1}$$

We show first that this functional rejects all symmetric noise. For any  $0 < p < 1$  and  $\varepsilon \leq \min\{p, 1 - p\}$ , let

$$f(\varepsilon, p) := [w(c_{p+\varepsilon}) + w(c_{p-\varepsilon})] \times [c_{p+\varepsilon} + c_{p-\varepsilon}]$$

Rejection of symmetric noise requires that  $f(0, p) - f(\varepsilon, p) > 0$  for all  $p \in (0, 1)$  and  $\varepsilon \in (0, \min\{p, 1 - p\})$ . Numerical calculations show that this is indeed the case. See graph below.

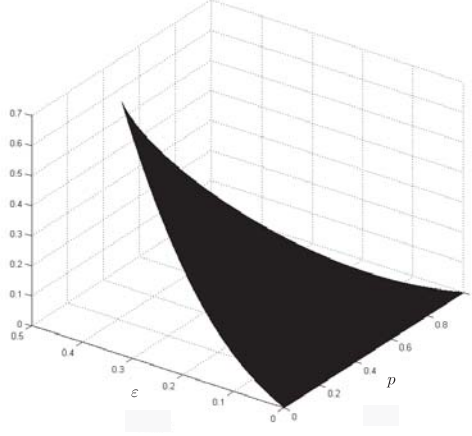


Figure A1: Rejection of symmetric noise

Using the same functional as above, we now show that for every  $p > 0$  there exists a sufficiently small  $q > 0$  such that  $\langle p, q; 0, 1 - q \rangle \succeq \langle pq, 1 \rangle$ , that is, the decision maker always accepts some positively skewed noise.

For  $q = 0$ ,  $V(c_{pq}, 1) - V(c_p, q; 0, 1 - q) = 0$ . We show that for every  $p < 1$ , the first non-zero derivative of this expression with respect to  $q$  at  $q = 0$  is negative. We get

$$\begin{aligned} (\zeta - 1)V(c_{pq}, 1) &= (\zeta - 1)w(c_{pq})c_{pq} = \\ &\left( \zeta [\beta pq + (1 - \beta)p^\kappa q^\kappa] - [\beta pq + (1 - \beta)p^\kappa q^\kappa]^\zeta \right) \times [\beta pq + (1 - \beta)p^\kappa q^\kappa] \end{aligned}$$

Differentiate with respect to  $q$  to obtain

$$\begin{aligned} & \left( \zeta [\beta p + \kappa(1 - \beta)p^\kappa q^{\kappa-1}] - \zeta [\beta pq + (1 - \beta)p^\kappa q^\kappa]^{\zeta-1} [\beta p + \kappa(1 - \beta)p^\kappa q^{\kappa-1}] \right) \times \\ & \quad [\beta pq + (1 - \beta)p^\kappa q^\kappa] + \\ & \left( \zeta [\beta pq + (1 - \beta)p^\kappa q^\kappa] - [\beta pq + (1 - \beta)p^\kappa q^\kappa]^\zeta \right) \times [\beta p + \kappa(1 - \beta)p^\kappa q^{\kappa-1}] \end{aligned}$$

At  $q = 0$ , this expression equals 0. Differentiate again with respect to  $q$  to obtain

$$\begin{aligned} & \zeta \left( \kappa(\kappa - 1)(1 - \beta)p^\kappa q^{\kappa-2} - (\zeta - 1) [\beta pq + (1 - \beta)p^\kappa q^\kappa]^{\zeta-2} [\beta p + \kappa(1 - \beta)p^\kappa q^{\kappa-1}]^2 - \right. \\ & \quad \left. [\beta pq + (1 - \beta)p^\kappa q^\kappa]^{\zeta-1} \kappa(\kappa - 1)(1 - \beta)p^\kappa q^{\kappa-2} \right) \times [\beta pq + (1 - \beta)p^\kappa q^\kappa] + \\ & 2\zeta \left( [\beta p + \kappa(1 - \beta)p^\kappa q^{\kappa-1}] - [\beta pq + (1 - \beta)p^\kappa q^\kappa]^{\zeta-1} [\beta p + \kappa(1 - \beta)p^\kappa q^{\kappa-1}] \right) \times \\ & \quad [\beta p + \kappa(1 - \beta)p^\kappa q^{\kappa-1}] + \\ & \left( \zeta [\beta pq + (1 - \beta)p^\kappa q^\kappa] - [\beta pq + (1 - \beta)p^\kappa q^\kappa]^\zeta \right) \times \kappa(\kappa - 1)(1 - \beta)p^\kappa q^{\kappa-2} \end{aligned}$$

Observe that

$$\begin{aligned} & \zeta \left( \kappa(\kappa - 1)(1 - \beta)p^\kappa q^{\kappa-2} - (\zeta - 1) [\beta pq + (1 - \beta)p^\kappa q^\kappa]^{\zeta-2} [\beta p + \kappa(1 - \beta)p^\kappa q^{\kappa-1}]^2 - \right. \\ & \quad \left. [\beta pq + (1 - \beta)p^\kappa q^\kappa]^{\zeta-1} \kappa(\kappa - 1)(1 - \beta)p^\kappa q^{\kappa-2} \right) \times [\beta pq + (1 - \beta)p^\kappa q^\kappa] = \\ & \zeta \left( \kappa(\kappa - 1)(1 - \beta)p^\kappa q^{\kappa-1} - (\zeta - 1)q [\beta pq + (1 - \beta)p^\kappa q^\kappa]^{\zeta-2} [\beta p + \kappa(1 - \beta)p^\kappa q^{\kappa-1}]^2 - \right. \\ & \quad \left. [\beta pq + (1 - \beta)p^\kappa q^\kappa]^{\zeta-1} \kappa(\kappa - 1)(1 - \beta)p^\kappa q^{\kappa-1} \right) \times [\beta p + (1 - \beta)p^\kappa q^{\kappa-1}] \end{aligned}$$

This expression converges to zero with  $q$ . This is obvious for  $\zeta \geq 2$ . If  $2 > \zeta > 1$ , then notice that by l'Hospital's rule

$$\lim_{q \rightarrow 0} \frac{q}{[\beta pq + (1 - \beta)p^\kappa q^\kappa]^{2-\zeta}} = \lim_{q \rightarrow 0} \frac{[\beta pq + (1 - \beta)p^\kappa q^\kappa]^{\zeta-1}}{(2 - \zeta) [\beta p + \kappa(1 - \beta)p^\kappa q^{\kappa-1}]} = 0$$

Also, as  $q \rightarrow 0$ , the limit of the expression

$$2\zeta \left( [\beta p + \kappa(1 - \beta)p^\kappa q^{\kappa-1}] - [\beta pq + (1 - \beta)p^\kappa q^\kappa]^{\zeta-1} [\beta p + \kappa(1 - \beta)p^\kappa q^{\kappa-1}] \right) \times [\beta p + \kappa(1 - \beta)p^\kappa q^{\kappa-1}]$$

is  $2\zeta\beta^2p^2$ . Finally,

$$\begin{aligned} & \left( \zeta [\beta pq + (1 - \beta)p^\kappa q^\kappa] - [\beta pq + (1 - \beta)p^\kappa q^\kappa]^\zeta \right) \times \kappa(\kappa - 1)(1 - \beta)p^\kappa q^{\kappa-2} = \\ & \left( \zeta [\beta p + (1 - \beta)p^\kappa q^{\kappa-1}] - [\beta pq^{1-\frac{1}{\zeta}} + (1 - \beta)p^\kappa q^{\kappa-\frac{1}{\zeta}}]^\zeta \right) \times \kappa(\kappa - 1)(1 - \beta)p^\kappa q^{\kappa-1} \end{aligned}$$

As  $\zeta, \kappa > 1$ , this expression goes to zero with  $q$ .

On the other hand,  $(\zeta - 1)V(c_p, q; 0, 1 - q)$  equals

$$q^2 \left( \zeta [\beta p + (1 - \beta)p^\kappa] - [\beta p + (1 - \beta)p^\kappa]^\zeta \right) \times [\beta p + (1 - \beta)p^\kappa]$$

Its first order derivative with respect to  $q$  at  $q = 0$  is zero, while the second derivative at this point equals

$$2 \left( \zeta [\beta p + (1 - \beta)p^\kappa] - [\beta p + (1 - \beta)p^\kappa]^\zeta \right) \times [\beta p + (1 - \beta)p^\kappa]$$

We therefore get that the first order derivative of  $V(c_{pq}, 1) - V(c_p, q; 0, 1 - q)$  at  $q = 0$  is zero, and that

$$\begin{aligned} & (\zeta - 1) \lim_{q \rightarrow 0} \frac{\partial^2}{\partial q^2} [V(c_{pq}, 1) - V(c_p, q; 0, 1 - q)] = g(p; \beta, \zeta, \kappa) := \\ & 2\zeta\beta^2p^2 - 2 \left( \zeta [\beta p + (1 - \beta)p^\kappa] - [\beta p + (1 - \beta)p^\kappa]^\zeta \right) \times [\beta p + (1 - \beta)p^\kappa] \end{aligned}$$

The graph below shows  $g(p; \beta, \zeta, \kappa)$  for  $\beta = 0.15$ ,  $\kappa = 1.1$ , and  $\zeta = 1.024$ . Note that for these values  $g(p; \beta, \zeta, \kappa) < 0$  for all  $p \in (0, 1)$ , which means that for  $q > 0$  small enough, the positively skewed noise  $\langle p, q; 0, 1 - q \rangle$  is accepted.

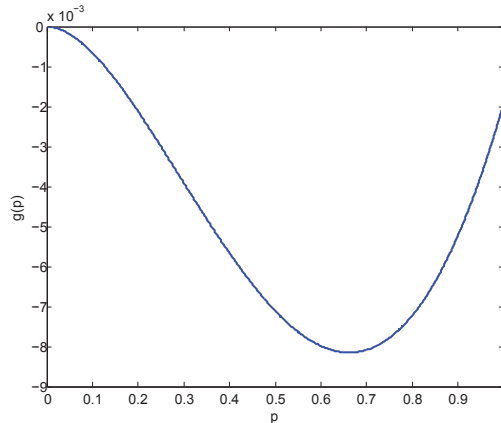


Figure A2:  $g(p; 0.15, 1.024, 1.1)$

Finally, to illustrate why Proposition 2 and the predictions of Theorem 3 are not contradictory, we provide numerical results showing that TC is always preferred to SD if  $q \geq 2p - 1$ . Recall that  $w$  and  $c_p$  are bounded in  $[0, 1]$ . The following graphs show the combinations of  $p$  and  $q$  for which the value of TC exceeds the value of SD by 0.01 and 0.001. In both pictures, the left panel is for the case  $q > p$  and the right panel is for the case  $q < p$ . In both cases, all pairs  $(p, q)$  such that  $q \geq 2p - 1$  satisfy the requirements.

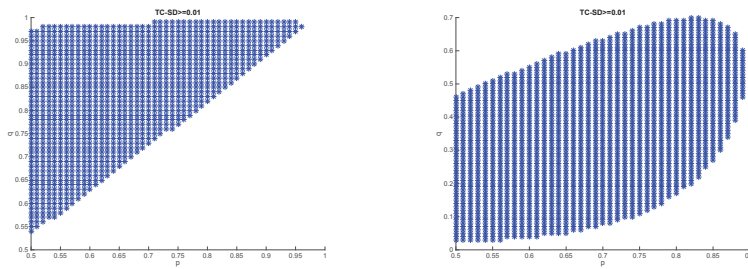


Figure A3:  $V(TC) - V(SD) > 0.01$

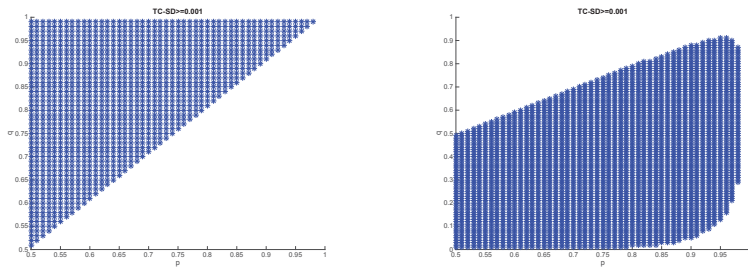


Figure A4:  $V(TC) - V(SD) > 0.001$