

COSTLY CONCESSIONS: AN EMPIRICAL FRAMEWORK FOR MATCHING WITH IMPERFECTLY TRANSFERABLE UTILITY

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ABSTRACT. We introduce an empirical framework for models of matching with imperfectly transferable utility and unobserved heterogeneity in tastes. Our framework allows us to characterize matching equilibrium in a flexible way that includes as special cases the classical fully- and non-transferable utility models, collective models, and settings with taxes on transfers, deadweight losses, and risk aversion. We allow for the introduction of a very general class of unobserved heterogeneity on agents' preferences. Under minimal assumptions, we show existence and uniqueness of equilibrium. We provide two algorithms to compute the equilibria in our model. The first algorithm operates under any structure of heterogeneity in preferences. The second algorithm is more efficient, but applies only in the case when random utilities are logit. We show that the log-likelihood of the model has a particularly simple expression and we compute its derivatives. As an application, we build a model of marriage with preferences over the partner type and private consumption. We estimate our model using the 2013 "Living Costs and Food Survey" database.

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1. INTRODUCTION

The field of Family Economics has two principal approaches to modeling and understanding marriage patterns: *matching models* emphasize market-level forces and take heterogeneous tastes over partners as primitives; *collective models*, by contrast, focus on the impact of intra-household bargaining. Economists’ two approaches to marriage have not themselves been “married,” however, because collective models necessarily include frictions of a form absent from classical matching frameworks. In this paper, we develop an *Imperfectly Transferable Utility* (ITU) matching model that allows us to unify marriage matching with the collective framework. Our setting moreover allows for the introduction of a very general class of unobserved heterogeneity on agents’ preferences, under which existence and uniqueness of equilibrium is typically guaranteed. We provide two algorithms to compute the equilibria in our model, and then demonstrate a simple application in which we use the 2013 “Living Costs and Food Survey” database to estimate the extent to which frictions appear to affect intra-household transfers.

Naturally, matching models have been extensively used to model the marriage market, in which men and women with heterogeneous tastes may form pairs; this approach, pioneered in Economics by Becker (1973, 1991) and Shapley and Shubik (1972), focuses mainly on matching patterns and the sharing of the surplus in a Transferable Utility (TU) setting. While appealing from a theoretical point of view, TU matching models suffer from several limitations. Indeed, TU models rely on the assumption that there is a numéraire good which is freely transferable across partners. Consequently, a man and a woman who match and generate a joint surplus Φ may decide on splitting this surplus between the utility of the man u and the utility of the woman v in any way such that $u + v \leq \Phi$. In this case, the efficient frontier in the (u, v) -space is simply a straight line of slope -1 . However, assuming this particular shape for the efficient frontier may be inappropriate—one can think of many cases in which there are frictions that partially impede the transfer of utility between matched partners. This possibility arises naturally in marriage markets, where the transfers between partners might take the form of favor exchange (rather than cash), and the cost of a favor to one partner may not exactly equal the benefit to the other. An

extreme alternative would be to assume Nontransferable Utility (NTU) (Gale and Shapley, 1962), in which there is no possibility of compensating transfer between partners. However, although NTU matching seems well-suited to specific examples such as school choice or organ exchanges, where transfers are often explicitly ruled out, it is not, in general, the most realistic assumption.

The collective model approach of Chiappori (1992), which focuses on intra-household bargaining over a potentially complex feasible utility set, cannot generally be expressed in terms of TU matching models, because inefficiencies in the bargaining process create transfer frictions.¹ Consequently, the matching and collective approaches to modeling marriage have not yet been combined into a single empirical framework. Choo and Siow (2006) observed this issue explicitly, stating that “[their] model of marriage matching should be integrated with models of intrahousehold allocations”—an integration which, in the ten years since Choo and Siow (2006) were writing, was not been achieved prior to our work.

In our Imperfectly Transferable Utility framework, partners match as in a classical marriage model, but utility transfers within matches are not necessarily additive. As already noted, this allows us to embed TU, NTU, and collective approaches to the marriage market. Our framework also makes sense for thinking about labor markets—because of taxation, an employer must pay more in wages than its employees actually receive (Jaffe and Kominers, 2014). In contrast with prior ITU matching models, our setting allows for a compact characterization of equilibrium as a set of nonlinear equations, as well as efficient computational approaches and clean comparative statics.

We prove existence and uniqueness of the equilibrium outcome in our ITU model with general heterogeneity in tastes. In the case when the heterogeneity are logit, we show how maximum likelihood estimation of our model can be performed in a very straightforward manner, which we illustrate by estimating a simple collective model of matching in a market with marital preferences and private consumption.

¹There are exceptions—see, e.g., the model described in Bowring et al. (2014), pp. 83 and 118, in which one private good is assumed to provide the same marginal utility to both members of the household, and thus can be used to transfer utility without friction.

Relation to the Literature. The theory of ITU matching has been studied by a number of authors: Crawford and Knoer (1981), Kelso and Crawford (1982), and Hatfield and Milgrom (2005) find conditions and algorithms for determining competitive equilibrium outcomes in ITU matching markets, and analyze the structure of the sets of equilibria. Kaneko (1982), Quinzii (1984), Alkan (1989), Alkan and Gale (1990), Gale (1984), and Demange and Gale (1985) provide results on the existence of equilibrium and study properties of the core. Pycia (2012) considers a general many-to-one matching setting with imperfectly transferable utility and characterizes the sharing rules that lead to pairwise alignment of preferences and existence of an equilibrium. Jaffe and Kominers (2014) study the problem of matching with linear and lump-sum taxes and provide comparative statics results. Legros and Newman (2007) find conditions under which positive assortativeness arises in ITU models; they apply these findings to problems of matching under uncertainty with risk aversion. Chade and Eeckhout (2014) extended the work of Legros and Newman (2007) to the case when agents have different risky endowments. Nöldeke and Samuelson (2015) connect ITU matching with abstract notions of convexity. Chiappori (2012) provides an illustrative example of how collective models naturally embed into ITU matching models.

However, the literature on the structural estimation of matching models has so far been restricted to the TU and NTU cases only. In the wake of the seminal work by Choo and Siow (2006), many papers have exploited heterogeneity in preferences for identification in the TU case (see Fox (2010), Chiappori, Orefice and Quintana-Domeque (2012), Galichon and Salanié (2014), Chiappori, Salanié, and Weiss (2014), and Dupuy and Galichon (2015)). Choo and Seitz (2013) present one of the first attempts to reconcile the matching and the collective approaches, albeit still in the TU case. Other research in the collective model literature have endogenize the sharing rule, but mostly in a TU framework (see Chapter 8 and 9 in the textbook by Browning and al. (2014) for a review, and references therein, e.g. Chiappori and al. (2009) and Iyigun and Walsh (2007)). Cherchye et al. (2015) derive Afriat-style inequalities that result from ITU stability in collective models. Similar strategies have been successfully applied in the NTU case (see Dagsvik (2000), Menzel (2015), Hitsch, Hortaçsu, and Ariely (2010), and Agarwal (2015)). To the best of our knowledge, our work is the first to provide an empirical framework for general ITU models with random utility.

Organization of the Paper. The remainder of the paper is organized as follows. Section 2 provides an introduction to the ITU framework building off the classic TU case. Section 3 formally describes the model we consider, introduces important technical machinery used throughout, and provides a number of examples. Section 4 introduces heterogeneity in tastes, defines the notion of aggregate equilibrium, and relates it with the classical notion of individual stability. Then, Section 5 determines the equations characterizing the aggregate equilibrium, shows existence and uniqueness results, and provides an algorithm to find equilibria in our framework. Section 6 deals with the important special case of logit heterogeneity, providing a more efficient algorithm for find equilibria in that case, and discussing maximum likelihood estimation. Section 7 uses our tools to estimate a matching model of marriage with consumption and assortativeness in education. Section 8 concludes. All proofs are presented in the Appendix.

2. PRELUDE: FROM TU MATCHING TO ITU MATCHING

We start with a brief overview of the structure of our model, which we hope will be particularly useful for readers who have already some degree of familiarity with TU matching models. To guide intuition, we start with the classical TU model, and show how it extends to the more general ITU model. Although less popular than the more restrictive TU and NTU models, ITU models have been studied in various forms in the literature (see, e.g., Alkan (1989), Chapter 9 in Roth and Sotomayor (1990), and Hatfield and Milgrom (2005)). However, unlike prior work, our presentation will introduce ITU matching in a form that is general enough to embed the TU and the NTU models, while still being amenable to the introduction of unobserved heterogeneity in preferences.

2.1. The TU matching model. We first recall the basics of the Transferable Utility model. In this model, it is assumed that there are sets \mathcal{I} and \mathcal{J} of men and women. If a man $i \in \mathcal{I}$ and a woman $j \in \mathcal{J}$ decide to match, they respectively enjoy utilities α_{ij} and γ_{ij} , where α and γ are primitives of the model.

If i and j match, the man and the woman also may agree on a transfer w_{ij} (determined at equilibrium) from the woman to the man (positive or negative), so that their utilities

after transfer are respectively $\alpha_{ij} + w_{ij}$ and $\gamma_{ij} - w_{ij}$. If i or j decide to remain unmatched, they enjoy respective payoffs \mathcal{U}_{i0} and \mathcal{V}_{0j} , which are exogenous reservation utilities.

Let μ_{ij} be the “matching” (also determined at equilibrium), which is equal to 1 if i and j are matched, and 0 otherwise. Hence, a matching should satisfy the *feasibility conditions*

$$(\mathbf{F}) \quad \begin{cases} \mu_{ij} \in \{0, 1\} \\ \sum_{j \in \mathcal{J}} \mu_{ij} \leq 1 \\ \sum_{i \in \mathcal{I}} \mu_{ij} \leq 1, \end{cases}$$

Let u_i and v_j be the indirect payoffs of man i and woman j , respectively. These quantities are determined at equilibrium, and one has $u_i = \max_{j \in \mathcal{J}} \{\alpha_{ij} + w_{ij}, \mathcal{U}_{i0}\}$ and $v_j = \max_{i \in \mathcal{I}} \{\gamma_{ij} - w_{ij}, \mathcal{V}_{0j}\}$, which implies in particular that for any i and j , the inequalities $u_i \geq \alpha_{ij} + w_{ij}$ and $v_j \geq \gamma_{ij} - w_{ij}$ jointly hold, implying that $u_i + v_j \geq \alpha_{ij} + \gamma_{ij}$ should hold for every $i \in \mathcal{I}$ and $j \in \mathcal{J}$. Likewise, $u_i \geq \mathcal{U}_{i0}$ and $v_j \geq \mathcal{V}_{0j}$ should hold for all i and j . Thus, the equilibrium payoffs should satisfy the *stability conditions*

$$(\mathbf{S}) \quad \begin{cases} u_i + v_j \geq \alpha_{ij} + \gamma_{ij} \\ u_i \geq \mathcal{U}_{i0} \\ v_j \geq \mathcal{V}_{0j}. \end{cases}$$

Finally, we relate the equilibrium matching μ and the equilibrium payoffs (u, v) . If $\mu_{ij} > 0$, then $\mu_{ij} = 1$ and i and j are matched, so the first line of **(S)** should hold as an equality. On the contrary, if $\sum_j \mu_{ij} < 1$, then $\sum_j \mu_{ij} = 0$, so i is unmatched and $u_i = \mathcal{U}_{i0}$. Similar conditions hold for j . To summarize, the equilibrium quantities are related by the following set of *complementary slackness* conditions:

$$(\mathbf{CS}) \quad \begin{cases} \mu_{ij} > 0 \implies u_i + v_j = \alpha_{ij} + \gamma_{ij} \\ \sum_j \mu_{ij} < 1 \implies u_i = \mathcal{U}_{i0} \\ \sum_i \mu_{ij} < 1 \implies v_j = \mathcal{V}_{0j} \end{cases} .$$

Following the classical definition, (μ, u, v) is an *equilibrium outcome* in the TU matching model if the feasibility conditions **(F)**, stability conditions **(S)**, and complementary slackness conditions **(CS)** are met. The characterization of the solutions to that problem in terms of Linear Programming are well known (see, e.g., Roth and Sotomayor, Chapter 8).

The equilibrium outcomes (μ, u, v) are such that μ maximizes the utilitarian social welfare $\sum_{ij} \mu_{ij} (\alpha_{ij} + \gamma_{ij} - \mathcal{U}_{i0} - \mathcal{V}_{0j})$ with respect to $\mu \geq 0$ subject to $\sum_j \mu_{ij} \leq 1$ and $\sum_i \mu_{ij} \leq 1$, which is the primal problem; and (u, v) are the solution of the corresponding dual problem, hence they minimize $\sum_i u_i + \sum_j v_j$ subject to $u_i + v_j \geq \alpha_{ij} + \gamma_{ij}$, and $u_i \geq \mathcal{U}_{i0}$, $v_j \geq \mathcal{V}_{0j}$. However, as we see in Section 3.4 below, this interpretation in terms of optimality is very specific to the present TU case.

2.2. The ITU matching model. The ITU matching model is a natural generalization of the TU model. If man $i \in \mathcal{I}$ and woman $j \in \mathcal{J}$ decide to match with transfer w_{ij} , their utilities after transfer are respectively $\mathcal{U}_{ij}(w_{ij})$ and $\mathcal{V}_{ij}(w_{ij})$, where $\mathcal{U}_{ij}(\cdot)$ is a continuous and nondecreasing function and $\mathcal{V}_{ij}(\cdot)$ is a continuous and nonincreasing function. (Note that in the specialization to the TU case, $\mathcal{U}_{ij}(w_{ij}) = \alpha_{ij} + w_{ij}$ and $\mathcal{V}_{ij}(w_{ij}) = \gamma_{ij} - w_{ij}$). If i or j decide to remain unmatched, they enjoy respective payoffs $\alpha_{i0} \in \mathbb{R}$ and $\gamma_{0j} \in \mathbb{R}$, which are exogenous reservation utilities

As before, the matching μ has term μ_{ij} equal to 1 if i and j are matched, 0 otherwise; clearly, the set of conditions **(F)** defining feasible matchings is unchanged.

At equilibrium, the indirect payoffs are now given by $u_i = \max_{j \in \mathcal{J}} \{\mathcal{U}_{ij}(w_{ij}), \mathcal{U}_{i0}\}$ and $v_j = \max_{i \in \mathcal{I}} \{\mathcal{V}_{ij}(w_{ij}), \mathcal{V}_{0j}\}$, which implies in particular that for any i and j , the inequalities $u_i \geq \mathcal{U}_{ij}(w_{ij})$ and $v_j \geq \mathcal{V}_{ij}(w_{ij})$ jointly hold. However, in contrast to the TU case, adding up the utility inequalities does not cancel out the w_{ij} term. As a way out of this problem, we show in Section 3 that there exists a function $D_{ij}(u, v)$, called *distance function*, which is increasing in u and v and has $D_{ij}(\mathcal{U}_{ij}(w), \mathcal{V}_{ij}(w)) = 0$ for all w . Then $u_i \geq \mathcal{U}_{ij}(w_{ij})$ and $v_j \geq \mathcal{V}_{ij}(w_{ij})$ jointly imply that $D_{ij}(u_i, v_j) \geq D_{ij}(\mathcal{U}_{ij}(w), \mathcal{V}_{ij}(w)) = 0$. Hence the equilibrium payoffs in an ITU model must satisfy the *nonlinear stability conditions*

$$(\mathbf{S}') \quad \begin{cases} D_{ij}(u_i, v_j) \geq 0 \\ u_i \geq \mathcal{U}_{i0} \\ v_j \geq \mathcal{V}_{0j}, \end{cases}$$

and the *nonlinear complementary slackness conditions*

$$(\mathbf{CS}') \quad \begin{cases} \mu_{ij} > 0 \implies D_{ij}(u_i, v_j) = 0 \\ \sum \mu_{ij} < 1 \implies u_i = \mathcal{U}_{i0} \\ \sum \mu_{ij} < 1 \implies v_j = \mathcal{V}_{0j}. \end{cases}$$

A triple (μ, u, v) is an equilibrium outcome in the matching model with Imperfectly Transferable Utility whenever conditions (\mathbf{F}) , (\mathbf{S}') and (\mathbf{CS}') are met.

3. FRAMEWORK

We now give a rigorous description of the framework introduced in the previous section. We consider a population of men indexed by $i \in \mathcal{I}$ and women indexed by $j \in \mathcal{J}$ who may decide either to remain single or to form heterosexual pairs. It will be assumed that if i and j match, then they bargain over utility outcomes (u_i, v_j) lying within a *feasible set* \mathcal{F}_{ij} , the structure of which is described in Section 3.1. If i and j decide to remain single, then they receive their respective reservation utilities \mathcal{U}_{i0} and \mathcal{V}_{0j} .

An *outcome* (formally defined in Section 3.2) is comprised of

- a *matching* $\mu_{ij} \in \{0, 1\}$, which is a binary variable equal to 1 if and only if i and j are matched, and
- the *payoffs* u_i and v_j , which are in \mathcal{F}_{ij} if i and j are matched, and are equal to the reservation utilities when i and j are unmatched.

Our concept of equilibrium, which we formalize in Definition 4 of Section 3.2, is based on *pairwise stability*: an outcome (μ, u, v) is an equilibrium outcome if there is no blocking coalition, i.e., if all the payoffs are above reservation value, and if there is no pair (i, j) of individuals who would be able to reach a feasible pair of utilities dominating u_i and v_j .

We give a number of examples cases of our model in Section 3.3, including the classic TU and NTU models, as well as several intermediate cases of interest.

3.1. The feasible bargaining sets. If man $i \in \mathcal{I}$ and a woman $j \in \mathcal{J}$ are matched, then they bargain over a set of feasible utilities $(u_i, v_j) \in \mathcal{F}_{ij}$. We begin by describing the pairwise bargaining sets \mathcal{F}_{ij} ; then, we provide two different—but equivalent—useful descriptions.

First, we represent the feasible sets “implicitly,” by describing the efficient frontier as the set of zeros of a function, $\{(u_i, v_j) \in \mathbb{R}^2 : D_{ij}(u_i, v_j) = 0\}$. Next, we represent the feasible sets “explicitly,” by their frontiers as the range of a map: $\{(\mathcal{U}_{ij}(w_{ij}), \mathcal{V}_{ij}(w_{ij})) : w_{ij} \in \mathbb{R}\}$.

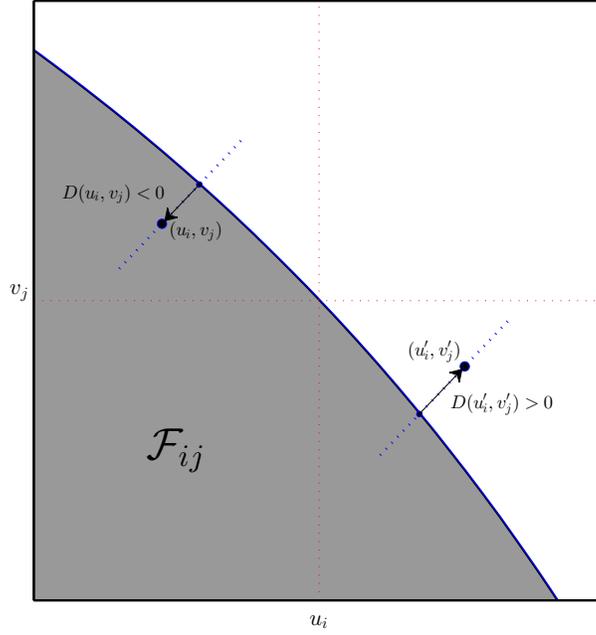
3.1.1. *Assumptions on the feasible sets.* The following natural assumptions on the geometry of the sets \mathcal{F}_{ij} will be extensively employed throughout the paper.

Definition 1. The set \mathcal{F}_{ij} is a *proper bargaining set* if the four following conditions are met:

- (i) \mathcal{F}_{ij} is closed.
- (ii) \mathcal{F}_{ij} is *lower comprehensive*: if $(u, v) \in \mathcal{F}_{ij}$, then $(u', v') \in \mathcal{F}_{ij}$ provided $u' \leq u$ and $v' \leq v$.
- (iii) \mathcal{F}_{ij} is *scarce*: Assume $u_n \rightarrow +\infty$ and v_n bounded by below then for N large enough (u_n, v_n) does not belong in \mathcal{F} for $n \geq N$; similarly u_n bounded by below and $v_n \rightarrow +\infty$
- (iv) \mathcal{F}_{ij} is *provisive*: Assume $u_n \rightarrow -\infty$ and $v_n \rightarrow -\infty$, then for N large enough (u_n, v_n) belongs in \mathcal{F}_{ij} for $n \geq N$.

Some comments on the preceding requirements are useful at this stage. The closedness of \mathcal{F}_{ij} is classically needed for efficient allocations to exist. The fact that \mathcal{F}_{ij} is lower comprehensive is equivalent to free disposal; in particular, it rules out the case in which \mathcal{F}_{ij} has finite cardinality. The scarcity property rules out the possibility that both partners can obtain arbitrarily large payoffs. The fact that \mathcal{F}_{ij} is provisive means that if both partner’s demands are low enough, they can always be fulfilled; in particular, it implies that \mathcal{F}_{ij} is nonempty.

3.1.2. *Implicit representation of the efficient frontier.* We now provide an equivalent representation of the sets \mathcal{F}_{ij} which is very useful for the later part of the analysis. This consists in representing \mathcal{F}_{ij} as the lower level set of a function D_{ij} , which we call *distance function* because $D_{ij}(u, v)$ measures the (signed) distance of (u, v) from the efficient frontier of \mathcal{F}_{ij} , when running along the diagonal. See Figure 1. $D_{ij}(u, v)$ is positive if (u, v) is outside of

FIGURE 1. Bargaining set \mathcal{F}_{ij} and the transfer function.

the feasible set, and negative if (u, v) is in the interior of the feasible set; its value is 0 at the frontier. Formally:

Definition 2. The distance function $D_{\mathcal{F}_{ij}} : \mathbb{R}^2 \rightarrow \mathbb{R}$ of a proper bargaining set \mathcal{F}_{ij} is defined by

$$D_{\mathcal{F}_{ij}}(u, v) = \min \{z \in \mathbb{R} : (u - z, v - z) \in \mathcal{F}_{ij}\}. \quad (3.1)$$

The function $D_{\mathcal{F}_{ij}}$ defined by (3.1) exists: indeed, the set $\{z \in \mathbb{R} : (u - z, v - z) \in \mathcal{F}_{ij}\}$ is closed because \mathcal{F}_{ij} is closed, bounded above because \mathcal{F}_{ij} is scarce, and nonempty because \mathcal{F}_{ij} is provisory; hence the minimum in (3.1) exists. By the definition of $D_{\mathcal{F}_{ij}}$, one has $\mathcal{F}_{ij} = \{(u, v) \in \mathbb{R}^2 : D_{\mathcal{F}_{ij}}(u, v) \leq 0\}$, and $D_{\mathcal{F}_{ij}}(u, v) = 0$ if and only if (u, v) lies on the frontier of \mathcal{F}_{ij} . The quantity $D_{\mathcal{F}_{ij}}(u, v)$ is thus interpreted as the distance (positive or negative) between (u, v) and the frontier of \mathcal{F}_{ij} along the diagonal. In particular, $D_{\mathcal{F}_{ij}}(a + u, a + v) = a + D_{\mathcal{F}_{ij}}(u, v)$ for any reals a, u and v . By the same token, if $D_{\mathcal{F}_{ij}}$ is differentiable at (u, v) , then $\partial_u D_{\mathcal{F}_{ij}} + \partial_v D_{\mathcal{F}_{ij}} = 1$. The following lemma summarizes important properties of $D_{\mathcal{F}_{ij}}$.

Lemma 1. *Let \mathcal{F}_{ij} be a proper bargaining set. Then:*

(i) $\mathcal{F}_{ij} = \{(u, v) \in \mathbb{R}^2 : D_{\mathcal{F}_{ij}}(u, v) \leq 0\}$.

(ii) For every $(u, v) \in \mathbb{R}^2$, $D_{\mathcal{F}_{ij}}(u, v) \in (-\infty, +\infty)$.

(iii) $D_{\mathcal{F}_{ij}}$ is \gg -isotone:

$$(u, v) \leq (u', v') \text{ implies } D_{\mathcal{F}_{ij}}(u, v) \leq D_{\mathcal{F}_{ij}}(u', v'), \text{ and}$$

$$u < u' \text{ and } v < v' \text{ implies } D_{\mathcal{F}_{ij}}(u, v) < D_{\mathcal{F}_{ij}}(u', v').$$

(iv) $D_{\mathcal{F}_{ij}}$ is continuous.

(v) $D_{\mathcal{F}_{ij}}(a + u, a + v) = a + D_{\mathcal{F}_{ij}}(u, v)$.

3.1.3. *Explicit representation of the efficient frontier.* We now give an explicit parametrization of the efficient frontier.

Given two utilities (u, v) such that $D_{\mathcal{F}_{ij}}(u, v) = 0$, let us introduce the *wedge* w as the difference $w = u - v$.

Definition 3. Define $\mathcal{U}_{\mathcal{F}_{ij}}(w)$ and $\mathcal{V}_{\mathcal{F}_{ij}}(w)$ as the values of u and v such that

$$D_{\mathcal{F}_{ij}}(u, v) = 0 \text{ and } w = u - v. \tag{3.2}$$

See figure 2. This definition (and the existence of the functions $\mathcal{U}_{\mathcal{F}_{ij}}$ and $\mathcal{V}_{\mathcal{F}_{ij}}$) is motivated by the following result.

Lemma 2. *Let \mathcal{F}_{ij} be a proper bargaining set. There are two 1-Lipschitz functions $\mathcal{U}_{\mathcal{F}_{ij}}$ and $\mathcal{V}_{\mathcal{F}_{ij}}$ defined on a nonempty open interval $(\underline{w}_{ij}, \bar{w}_{ij})$ such that $\mathcal{U}_{\mathcal{F}_{ij}}$ is nondecreasing and $\mathcal{V}_{\mathcal{F}_{ij}}$ is nonincreasing, and such that the set of (u, v) such that $D_{\mathcal{F}_{ij}}(u, v) = 0$ is given by $\{(\mathcal{U}_{\mathcal{F}_{ij}}(w), \mathcal{V}_{\mathcal{F}_{ij}}(w)) : w \in (\underline{w}_{ij}, \bar{w}_{ij})\}$. Further, $\mathcal{U}_{\mathcal{F}_{ij}}(w)$ and $\mathcal{V}_{\mathcal{F}_{ij}}(w)$ are the unique values of u and v solving of (3.2), and they are given by*

$$\mathcal{U}_{\mathcal{F}_{ij}}(w) = -D_{\mathcal{F}_{ij}}(0, -w), \text{ and } \mathcal{V}_{\mathcal{F}_{ij}}(w) = -D_{\mathcal{F}_{ij}}(w, 0). \tag{3.3}$$

Note that whenever u and v exist, we have $\mathcal{U}'_{\mathcal{F}_{ij}}(w) = \partial_v D_{\mathcal{F}_{ij}}(0, -w)$ and $\mathcal{V}'_{\mathcal{F}_{ij}}(w) = -\partial_u D_{\mathcal{F}_{ij}}(w, 0)$. Further, as $\mathcal{U}_{\mathcal{F}_{ij}}(w)$ is increasing and 1-Lipschitz, \bar{w}_{ij} is finite if and only if

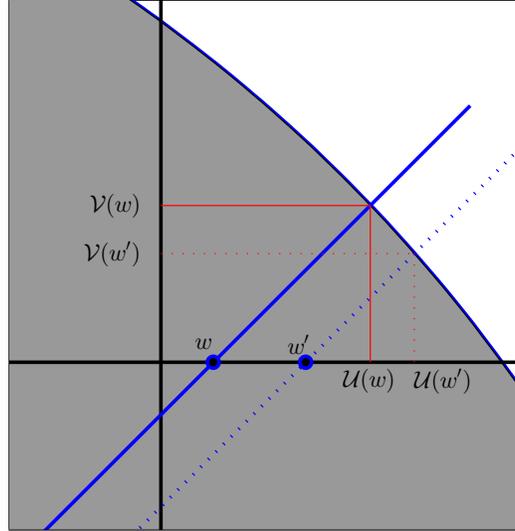


FIGURE 2. Explicit representation of the efficient frontier

the maximal utility u obtainable by the man for some feasible $(u, v) \in \mathcal{F}_{ij}$ is finite. Similarly, \underline{w}_{ij} is finite if and only if the maximal utility v obtainable by the woman for some feasible $(u, v) \in \mathcal{F}_{ij}$ is finite.

3.2. Basic model. Having established the structure of the feasible bargains among matched couples, we describe the matching process. Men and women may form (heterosexual) pairs or decide to remain unmatched. If i (resp. j) decides to remain unmatched, he (resp. she) gets reservation utility \mathcal{U}_{i0} (resp. \mathcal{V}_{0j}). If i and j decide to match, they bargain over a set \mathcal{F}_{ij} of feasible payoffs (u, v) , where \mathcal{F}_{ij} is a proper bargaining set, whose associated distance function is denoted $D_{ij} := D_{\mathcal{F}_{ij}}$ and whose functions $\mathcal{U}_{\mathcal{F}_{ij}}$ and $\mathcal{V}_{\mathcal{F}_{ij}}$ are respectively denoted \mathcal{U}_{ij} and \mathcal{V}_{ij} .

We denote by u_i (resp. v_j) the (equilibrium) outcome utility of man i (resp. woman j). At equilibrium, we must have $u_i \geq \mathcal{U}_{i0}$ and $v_j \geq \mathcal{V}_{0j}$ as it is always possible to leave an arrangement which yields less than the reservation utility. Similarly, at equilibrium, $D_{\mathcal{F}}(u_i, v_j) \geq 0$ must hold for every i and j ; indeed, if this were not the case, there would be a pair (i, j) such that (u_i, v_j) is in the strict interior of the feasible set \mathcal{F}_{ij} , so that there would exist payoffs $u' \geq u_i$ and $v' \geq v_j$ with one strict inequality and $(u', v') \in \mathcal{F}_{ij}$, which would imply that i and j can be better off by matching together. Let μ_{ij} be indicator

variable which is equal to 1 if i and j are matched, 0 otherwise. If $\mu_{ij} = 1$, we require that (u_i, v_j) be feasible, that is $D_{ij}(u_i, v_j) \leq 0$, hence equality should hold.

Combining the conditions just described, we are ready to define equilibrium in our ITU matching model. We call this equilibrium “individual” as opposed to the concept of “aggregate” equilibrium defined later in Section 4.

Definition 4 (Individual Equilibrium). The triple $(\mu_{ij}, u_i, v_j)_{i \in \mathcal{I}, j \in \mathcal{J}}$ is an *individual equilibrium outcome* if the following three conditions are met:

- (i) $\mu_{ij} \in \{0, 1\}$, $\sum_j \mu_{ij} \leq 1$ and $\sum_i \mu_{ij} \leq 1$;
- (ii) for all i and j , $D_{ij}(u_i, v_j) \geq 0$, with equality if $\mu_{ij} = 1$;
- (iii) $u_i \geq \mathcal{U}_{i0}$ and $v_j \geq \mathcal{V}_{0j}$, with equality respectively if $\sum_j \mu_{ij} = 0$, and if $\sum_i \mu_{ij} = 0$.

The vector $(\mu_{ij})_{i \in \mathcal{I}, j \in \mathcal{J}}$ an *individual equilibrium matching* if and only if there exists a pair of vectors $(u_i, v_j)_{i \in \mathcal{I}, j \in \mathcal{J}}$ such that (μ, u, v) is an individual equilibrium outcome.

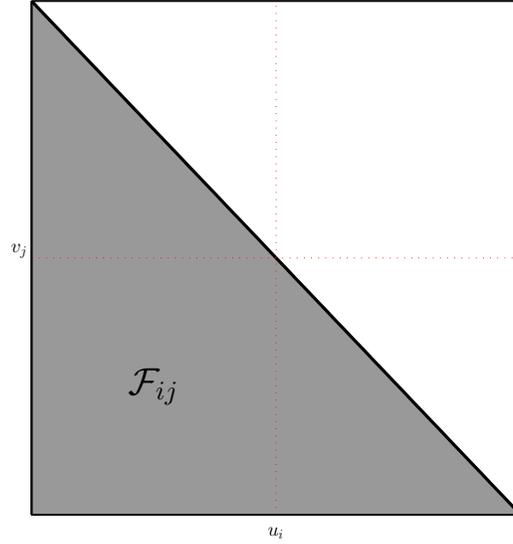
As we detail in the next section, our setting embeds both the standard TU matching model, the standard NTU model, and many others.

3.3. Example Specifications. Now, we provide examples of specifications of frontiers \mathcal{F} (or equivalently, distance functions D) that illustrate the wide array of applications encompassed by our framework.

3.3.1. Matching with Transferable Utility (TU). The classical TU matching model has been widely used in Economics—it is the cornerstone of Becker’s marriage model, which has found a vast array of applications in labor markets, marriage markets, and housing markets (Koopmans and Beckmann, 1957; Shapley and Shubik, 1971; Becker, 1973). To recover the TU model in our framework, we take

$$\mathcal{F}_{ij} = \{(u, v) \in \mathbb{R}^2 : u + v \leq \Phi_{ij}\},$$

that is, the partners can additively share the quantity Φ_{ij} , which is interpreted as a *joint surplus* (see Figure 3). The Pareto efficient payoffs will be such that $u + v = \Phi_{ij}$. In this setting, utility is perfectly transferable: if one partner gives up one unit of utility, the other

FIGURE 3. Bargaining set \mathcal{F}_{ij} in the TU case.

partner fully appropriates it. It is easily verified that in this case,

$$D_{ij}(u, v) = \frac{u + v - \Phi_{ij}}{2}, \quad (3.4)$$

$$\mathcal{U}_{ij}(w) = \frac{\Phi_{ij} + w}{2}, \text{ and } \mathcal{V}_{ij}(w) = \frac{\Phi_{ij} - w}{2}.$$

3.3.2. *Matching with Non-Transferable Utility (NTU)*. Equally important is the NTU matching model of Gale and Shapley (1962), which has frequently been used to model school choice markets, organ exchange matching, and centralized job assignment. In this case, utility is not transferable at all, and the maximum utility that each partner can obtain is fixed and does not depend on what the other partner gets. Like the TU model, we can embed the NTU model in our ITU framework: in this case,

$$\mathcal{F}_{ij} = \{(u, v) \in \mathbb{R}^2 : u \leq \alpha_{ij}, v \leq \gamma_{ij}\}.$$

which means that the only efficient pair of payoffs has $u = \alpha_{ij}$ and $v = \gamma_{ij}$ (see Figure 4).

It is easily checked that

$$D_{ij}(u, v) = \max\{u - \alpha_{ij}, v - \gamma_{ij}\}. \quad (3.5)$$

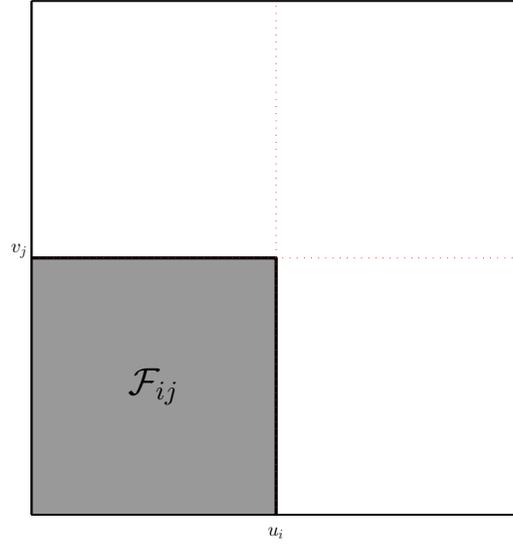


FIGURE 4. Bargaining set \mathcal{F}_{ij} in the NTU case.

In this case, the Pareto efficient matchings in the sense of Definition 4 coincide with NTU-stable matchings under the classical definition. Indeed, condition (3) of that definition implies that $\max \{u_i - \alpha_{ij}, v_j - \gamma_{ij}\} < 0$ cannot hold, which is equivalent to the absence of a blocking pair in the classical definition, while condition (2) expresses that there is no blocking individual. Further, condition (4) and Pareto efficiency imply that if i is matched to j , then $u_i = \alpha_{ij}$ and $v_j = \gamma_{ij}$. We have $\mathcal{U}_{ij}(w) = \min \{\alpha_{ij}, w + \gamma_{ij}\}$ and $\mathcal{V}_{ij}(w) = \min \{\alpha_{ij} - w, \gamma_{ij}\}$.

3.3.3. Matching with marital utility and consumption. We consider a matching model of the marriage market in which each partner's utility depends both on his or her private consumption, and his or her marital utility. More specifically, assume that if man i and woman j are matched, their respective utilities combine marital utilities and private consumptions in a Cobb-Douglas form, i.e. $U_i = A_{ij}c_i^\tau$ and $V_j = \Gamma_{ij}c_j^\tau$, where A_{ij} and Γ_{ij} are respectively i and j 's marital utilities, and c_i and c_j are their respective private consumptions, and τ is the elasticity of substitution between marital utility and consumption. Assume the budget of an (i, j) pair is $c_i + c_j = I_i + I_j$, where I_i and I_j are i and j 's incomes, respectively. Then, letting $u_i = \log U_i$ and $v_j = \log V_j$ be the log utilities, and $\alpha_{ij} = \log A_{ij} + \tau \log (I_i + I_j) / 2$

and $\gamma_{ij} = \log \Gamma_{ij} + \tau \log (I_i + I_j) / 2$, one is led to the model of matching with *Exponentially Transferable Utility* (ETU), where α and γ play the role of “premuneration values,” as defined in Liu et al. (2014) and Mailath et al. (2013). A particular case of the ETU model can be found in Legros and Newman (2007, p. 1086). In our setting, the feasible set is

$$\mathcal{F}_{ij} = \left\{ (u, v) \in \mathbb{R}^2 : \exp\left(\frac{u - \alpha_{ij}}{\tau_{ij}}\right) + \exp\left(\frac{v - \gamma_{ij}}{\tau_{ij}}\right) \leq 2 \right\}.$$

See Figure 5. The expression of D_{ij} follows easily as

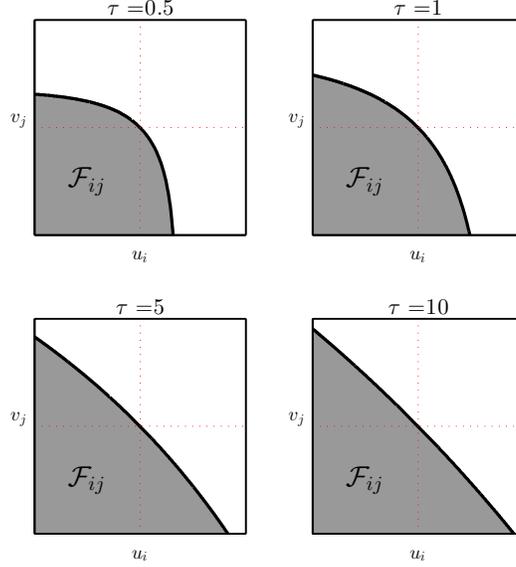
$$D_{ij}(u, v) = \tau_{ij} \log \left(\frac{\exp\left(\frac{u - \alpha_{ij}}{\tau_{ij}}\right) + \exp\left(\frac{v - \gamma_{ij}}{\tau_{ij}}\right)}{2} \right). \quad (3.6)$$

As $\tau_{ij} \rightarrow 0$, we recover the NTU model (3.5), and when $\tau_{ij} \rightarrow +\infty$, we obtain the TU model (3.4). Hence, the ETU model interpolates between the nontransferable and fully transferable utility models. Here, the parameter τ_{ij} , which captures the elasticity of substitution between marital well-being and consumption, equivalently parameterizes the *degree of transferability*. We have

$$\mathcal{U}_{ij}(w) = -\tau_{ij} \log \left(\frac{e^{\frac{-\alpha_{ij}}{\tau_{ij}}} + e^{\frac{-w - \gamma_{ij}}{\tau_{ij}}}}{2} \right) \text{ and } \mathcal{V}_{ij}(w) = -\tau_{ij} \log \left(\frac{e^{\frac{w - \alpha_{ij}}{\tau_{ij}}} + e^{\frac{-\gamma_{ij}}{\tau_{ij}}}}{2} \right).$$

3.3.4. Matching with a Linear Tax Schedule. Our framework can also model a labor market with linear tax: Assume the nominal wage W_{ij} is taxed at rate $1 - R_{ij}$ on the employee’s side (income tax) and at rate $1 + C_{ij}$ on the firm’s side (social contributions). The tax rates are allowed to depend on both employer and employee characteristics. Assume that if employee i and employer j match and decide on a wage W_{ij} , they respectively have (post-transfer) utilities $u_i = \alpha_{ij} + R_{ij}W_{ij}$ and $v_j = \gamma_{ij} - C_{ij}W_{ij}$, where α_{ij} is job j ’s amenity to worker i , and γ_{ij} is the productivity of worker i on job j . This specification is called the *Linearly Transferable Utility (LTU)* model, and the feasible set is given by

$$\mathcal{F}_{ij} = \{ (u, v) \in \mathbb{R}^2 : \lambda_{ij}u + \zeta_{ij}v \leq \Phi_{ij} \},$$


 FIGURE 5. Bargaining set \mathcal{F}_{ij} in the ETU case.

where $\lambda_{ij} = 1/R_{ij} > 0$, and $\zeta_{ij} = 1/C_{ij} > 0$, and $\Phi_{ij} = \lambda_{ij}\alpha_{ij} + \zeta_{ij}\gamma_{ij}$. Note that the TU case is recovered when $\lambda_{ij} = 1$ and $\zeta_{ij} = 1$. A simple calculation yields

$$D_{ij}(u, v) = \frac{\lambda_{ij}u + \zeta_{ij}v - \Phi_{ij}}{\lambda_{ij} + \zeta_{ij}}. \quad (3.7)$$

The LTU model (3.7) extends that of Jaffe and Kominers (2014), in which R_{ij} only depends on i and ζ_{ij} only depends on j . We have

$$\mathcal{U}_{ij}(w) = \frac{\Phi_{ij} + \zeta_{ij}w}{\lambda_{ij} + \zeta_{ij}} \text{ and } \mathcal{V}(w) = \frac{\Phi_{ij} - \lambda_{ij}w}{\lambda_{ij} + \zeta_{ij}}.$$

3.3.5. Matching with a Nonlinear Tax Schedule. Our framework is general enough to extend to a nonlinear tax schedule, well beyond linear taxes. Assume that if the nominal wage is W_{ij} , the after-tax income of employee is $R_{ij}(W_{ij})$, and the after-tax cost of employment for the firm is $C_{ij}(W_{ij})$. The nonlinear tax schedules $W \mapsto R_{ij}(W)$ and $W \mapsto C_{ij}(W)$ are usually assumed convex and increasing, and R_{ij} and C_{ij} satisfy $0 < R_{ij}(W) < W$ and $0 < C_{ij}(W) < W$. Note that they are allowed to depend on both employer and employee characteristics. Assume that if employer i and employee j match, they enjoy respective (post-transfer) utilities $u_i = \alpha_{ij} + R_{ij}(W_{ij})$ and $v_j = \gamma_{ij} - C_{ij}(W_{ij})$, so that the feasible

set is given by

$$\mathcal{F}_{ij} = \{(u, v) \in \mathbb{R}^2 : \lambda_{ij}(u) + \zeta_{ij}(v) \leq 0\},$$

where $\lambda_{ij}(u) = R_{ij}^{-1}(u - \alpha_{ij})$ and $\zeta_{ij}(v) = C_{ij}^{-1}(v - \gamma_{ij})$. This model was introduced by Demange and Gale (1985) (see also Roth and Sotomayor (1990), Chapter 9).

Here, $D_{ij}(u, v)$ is defined as the value of $t \in \mathbb{R}$ such that there exists $W \in \mathbb{R}$ for which $u - t = \alpha_{ij} + R_{ij}(W)$ and $v - t = \gamma_{ij} - C_{ij}(W)$. Thus, $u - \alpha_{ij} - v + \gamma_{ij} = R_{ij}(W) + C_{ij}(W)$, thus $W = (R_{ij} + C_{ij})^{-1}(u - \alpha_{ij} - v + \gamma_{ij})$, from which we deduce $t = u - \alpha_{ij} - R_{ij}(W)$. Hence, this specification gives rise to:

$$D_{ij}(u, v) = u - \alpha_{ij} - R_{ij}\left((R_{ij} + C_{ij})^{-1}(u - \alpha_{ij} - v + \gamma_{ij})\right). \quad (3.8)$$

This model therefore allows for a nonlinear tax schedule that depends on the types of the employers and the employees. In this case, $\mathcal{U}_{ij}(w) = \alpha_{ij} + R_{ij}((R_{ij} + C_{ij})^{-1}(w - \alpha_{ij} + \gamma_{ij}))$, and $\mathcal{V}_{ij}(w) = \mathcal{U}_{ij}(w) - w$.

3.3.6. Matching with Uncertainty. Now, we consider a model of the labor market with uncertainty regarding match quality; such a model is considered by Legros and Newman (2007) and Chade and Eeckhout (2014), who focus on characterizing positive assortativeness.

We assume that a worker of type i decides to match with a firm of type j , and decide on a wage W_{ij} . The job amenity is \tilde{e}_{ij} , where \tilde{e}_{ij} is a stochastic term learned only after the match is formed; the distribution of \tilde{e}_{ij} may depend on i and j . The employee is risk-averse and has an increasing and concave utility function $\mathfrak{U}(\cdot)$. Then the employees' and employers' systematic utilities are respectively $u_i = \mathbb{E}[\mathfrak{U}_i(\tilde{e}_{ij} + W_{ij})]$ and $v_j = \gamma_{ij} - W_{ij}$. This model therefore has

$$\mathcal{F}_{ij} = \{(u, v) \in \mathbb{R}^2 : u \leq \mathbb{E}[\mathfrak{U}_i(\tilde{e}_{ij} + \gamma_{ij} - v)]\},$$

which is a particular case of the previous model with $\alpha_{ij} = 0$, $R_{ij}(W) = \mathbb{E}[\mathfrak{U}_i(\tilde{e}_{ij} + W)]$, and $C_{ij}(W) = W$, so that D_{ij} can be deduced from these expressions of R_{ij} and C_{ij} by relation (3.8).

3.3.7. *Collective Models.* Finally, we consider a situation in which a man i and a woman j have respective utilities $\mathbf{u}_i(c_i, l_i, g)$ and $\mathbf{v}_j(c_j, l_j, g)$ which depend on private consumptions c_i and c_j , private leisure l_i and l_j , and a public good g . The wages of man i and of woman j are respectively denoted W_i and W_j , and the price of the public good is denoted p . The budget constraint of the household is therefore $c_i + c_j + W_i l_i + W_j l_j + pg = (W_i + W_j)T$, where T is the total amount of time available to each partner. The feasible set is therefore given by

$$\mathcal{F}_{ij} = \left\{ \begin{array}{l} (u, v) \in \mathbb{R}^2 : u \leq \mathbf{u}_i(c_i, l_i, g), v \leq \mathbf{v}_j(c_j, l_j, g), \\ \text{and } c_i + c_j + W_i l_i + W_j l_j + pg = (W_i + W_j)T \\ c_i, c_j \geq 0, g \geq 0, \text{ and } 0 \leq l_i, l_j \leq T. \end{array} \right\}.$$

The “collective” approach initiated by Chiappori (1992) assumes that the outcome (u, v) lies on the Pareto frontier of the feasible set of achievable utilities, given some exogenous sharing rule. Assuming that i and j are exogenously matched, it assumes a pair of Pareto weights θ_i and θ_j and such that the outcome (u_i, v_j) maximizes $\theta_i u_i + \theta_j v_j$ subject to $(u_i, v_j) \in \mathcal{F}_{ij}$. In contrast, in the present context, both the existence of a match between i and j and the sharing rule are determined endogenously by the stability conditions.

3.4. Remarks.

Remark 3.1 (Equilibrium vs. Optimality). As argued in Example 3.3.1 above, the TU matching model (also sometimes called the *Optimal Assignment Model*), is recovered in the case $D_{ij}(u, v) = u + v - \Phi_{ij}$ for some vector of joint surplus Φ_{ij} , shared additively between partners. It is well known in this case that the equilibrium conditions are the Complementary Slackness conditions for optimality in a Linear Programming problem, so in this case, equilibrium and optimality coincide. However, outside of this case, these conditions are *not* the first-order conditions associated to an optimization problem, and equilibrium does not have an interpretation as the maximizer of some welfare function.

Remark 3.2 (Galois connections). When D_{ij} is strictly increasing in each of its arguments (or equivalently, when the upper frontier of \mathcal{F}_{ij} is strictly downward sloping), one may define

$$\mathbb{U}_{ij}(v) = \max \{u : D_{ij}(u, v) \leq 0\} \text{ and } \mathbb{V}_{ij}(u) = \max \{v : D_{ij}(u, v) \leq 0\}$$

and it can be verified that \mathbb{U}_{ij} and \mathbb{V}_{ij} are continuous, strictly decreasing, and inverse of one another. In this case, if u and v are equilibrium payoff vectors, then

$$v_j = \max_{i \in \mathcal{I}} \{\mathbb{V}_{ij}(u_i), \mathcal{V}_{0j}\} \quad \text{and} \quad u_i = \max_{j \in \mathcal{J}} \{\mathbb{U}_{ij}(v_j), \mathcal{U}_{i0}\}.$$

In particular, in the TU case studied in Example 3.3.1 above, $\mathbb{U}_{ij}(v) = \Phi_{ij} - v$ and $\mathbb{V}_{ij}(u) = \Phi_{ij} - u$. The maps \mathbb{U}_{ij} and \mathbb{V}_{ij} are called *Galois connections*, and are investigated by Noldeke and Samuelson (2015). Our setting is more general, in the sense that it allows D to be only weakly increasing in each of its arguments, as in the NTU case studied in Example 3.3.2 above, where \mathbb{U}_{ij} and \mathbb{V}_{ij} are not defined and continuous. Unfortunately, to our knowledge, there is no obvious way to describe the NTU case using Galois connections.

4. AGGREGATE EQUILIBRIUM: MOTIVATION AND DEFINITION

In this section we shall add structure to our previous model by assuming that agents can be grouped into a finite number of types, which are observable to the econometrician, and vary according to an unobserved taste parameter. Section 4.1 precisely describes this setting. The individual, or “microscopic” equilibrium defined in Section 3 above has a “macroscopic” analog: the *aggregate equilibrium*, which describes the equilibrium matching patterns and systematic payoffs across observable types; we define this concept in Section 4.2.

4.1. Unobserved heterogeneity. We assume that individuals may be gathered in groups of agents of similar observable characteristics, or types, but heterogeneous tastes. We let \mathcal{X} and \mathcal{Y} be the sets of *types* of men and women, respectively; we assume that \mathcal{X} and \mathcal{Y} are finite. Let $x_i \in \mathcal{X}$ (resp. $y_j \in \mathcal{Y}$) be the type of individual man i (resp. woman j). We let n_x be the mass of men of type x , and let m_y be the mass of women of type y . In the sequel, we denote by $\mathcal{X}_0 \equiv \mathcal{X} \cup \{0\}$ the set of marital options available to women (either type of male partner or singlehood, denoted 0); analogously, $\mathcal{Y}_0 \equiv \mathcal{Y} \cup \{0\}$ denotes the set of marital options available to men (either type of female partner or singlehood, again denoted 0). We assume in the sequel that $D_{ij}(\cdot, \cdot)$ depends only on agent types—that is, $D_{ij}(\cdot, \cdot) \equiv D_{x_i y_j}(\cdot, \cdot)$.

Consider a market in which men and women either decide to match or to remain single. Let u_i and v_j be the utilities that man i and woman j obtain respectively on this market.

Assumption 1. *Assume that if i and j are matched, then $u_i = U_i + \varepsilon_{iy_j}$ and $v_j = V_j + \eta_{x_ij}$, while if they remain single, then $u_i = \varepsilon_{i0}$ and $v_j = \eta_{0j}$, where:*

(i) *the “systematic” parts of their utilities belong to the feasible set, that is $(U_i, V_j) \in \mathcal{F}_{ij}$, and*

(ii) *the “idiosyncratic” parts of their utilities are the entries of random vectors $(\varepsilon_{iy})_{y \in \mathcal{Y}_0}$ and $(\eta_{xj})_{x \in \mathcal{X}_0}$ which are i.i.d. draws from distributions \mathbf{P}_x and \mathbf{Q}_y , respectively.*

Assumption 1 implies that there is unobserved heterogeneity only on preferences. In other words, while two different women with the same observable type may have different rankings of partners, a given woman will be indifferent between any men within a given observable type. This assumption was made in Choo and Siow (2006), who were the first to realize its analytical convenience, and it has played a central role in the subsequent literature.²

We now introduce the following restrictions on the bargaining sets \mathcal{F}_{ij} :

Assumption 2. *The sets \mathcal{F}_{ij} are such that:*

(i) *For $i \in \mathcal{I}$ and $j \in \mathcal{J}$, \mathcal{F}_{ij} only depends on the types of i and j , hence $\mathcal{F}_{ij} = \mathcal{F}_{x_i y_j}$, where x_i is the type of i and y_j is the type of j .*

(ii) *For $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, \mathcal{F}_{xy} is a proper bargaining set in the sense of Definition 1.*

(iii) *For each man type $x \in \mathcal{X}$, either all the \bar{w}_{xy} , $y \in \mathcal{Y}$ are finite, or all the \bar{w}_{xy} , $y \in \mathcal{Y}$ coincide with $+\infty$ (where \bar{w}_{xy} and \underline{w}_{xy} are as defined in Section 3.1.3). For each woman type $y \in \mathcal{Y}$, either all the \underline{w}_{xy} , $x \in \mathcal{X}$ are finite, or all the \underline{w}_{xy} , $x \in \mathcal{X}$ coincide with $-\infty$.*

The more substantial restriction introduced in Assumption 2 is part (i), which extends the “additive separability assumption” highlighted by Chiappori, Salanié, and Weiss (2014), building on the work of Choo and Siow (2006). In the case of TU models (see Example 3.3.1 above), our restriction simply states that the joint surplus Φ_{ij} can be decomposed in the form $\Phi_{ij} = \Phi_{x_i y_j} + \varepsilon_{iy} + \eta_{jx}$. Note that, while the transfers U_i and V_j are allowed to vary in

²In contrast, Dagsvik (2000) and Menzel (2015) assume that the heterogeneity in tastes is of the form ε_{ij} and η_{ij} , where the utility shocks are i.i.d. across partners, and hence is individual-specific.

an idiosyncratic manner within observable types, it will be a fundamental property of the equilibrium (stated in Theorem 5 below) that U_i is the same for all men i of type x matched with a woman of type y , while V_j is the same for all the women j of type y matched with a man of type x . As a result of Assumption 2, part (i), the notations D_{xy} , \mathcal{U}_{xy} , and \mathcal{V}_{xy} will naturally substitute for $D_{\mathcal{F}_{xy}}$, $\mathcal{U}_{\mathcal{F}_{xy}}$, and $\mathcal{V}_{\mathcal{F}_{xy}}$. Part (ii) simply assumes that \mathcal{F}_{xy} has the properties described in Section 3.1.1. Part (iii) is a technical assumption expressing that given any agent (man or woman), the maximum utility that this agent can obtain with any partner is either always finite, or always infinite; this is needed to ensure existence of an equilibrium, and it is satisfied in all the examples we have.

We finally impose assumptions on \mathbf{P}_x and \mathbf{Q}_y , the distributions of the idiosyncratic terms $(\varepsilon_{iy})_{y \in \mathcal{Y}_0}$ and $(\eta_{xj})_{x \in \mathcal{X}_0}$, which are i.i.d. random vectors respectively valued in $\mathbb{R}^{\mathcal{Y}_0}$ and $\mathbb{R}^{\mathcal{X}_0}$.

Assumption 3. \mathbf{P}_x and \mathbf{Q}_y have non-vanishing densities on $\mathbb{R}^{\mathcal{Y}_0}$ and $\mathbb{R}^{\mathcal{X}_0}$.

There are two components to Assumption 3: the requirement that \mathbf{P}_x and \mathbf{Q}_y have full support, and the requirement that they are absolutely continuous. The full-support requirement implies that given any pair of types x and y , there are individuals of these types with arbitrarily large valuations for each other; this implies that at equilibrium, any matching between observable pairs of types will be observed. The absolute continuity requirement ensures that with probability 1 the men and the women's choice problems have a unique solution.

Transposing Definition 4 to the framework with parameterized heterogeneity, we see that (μ_{ij}, u_i, v_j) is an individual equilibrium outcome when:

- (i) $\mu_{ij} \in \{0, 1\}$, $\sum_j \mu_{ij} \leq 1$ and $\sum_i \mu_{ij} \leq 1$;
- (ii) for all i and j , $D_{x_i y_j}(u_i - \varepsilon_{iy_j}, v_j - \eta_{x_i j}) \geq 0$, with equality if $\mu_{ij} = 1$;
- (iii) $u_i \geq \varepsilon_{i0}$ and $v_j \geq \eta_{0j}$ with equality if respectively $\sum_j \mu_{ij} = 0$ and $\sum_i \mu_{ij} = 0$.

Remark 4.1 (An informal preview of the next steps). To provide some intuition on the definition of aggregate equilibrium to follow, we summarize the next steps. We start with an equivalent condition to point (ii) in the definition of an individual equilibrium above

(Definition 4): for any pair of types $x \in \mathcal{X}$ and $y \in \mathcal{Y}$,

$$\min_{\substack{i:x_i=x \\ j:y_j=y}} D_{xy}(u_i - \varepsilon_{iy}, v_j - \eta_{xj}) \geq 0,$$

with equality if there is a matching between a man of type x and a woman of type y . Thus, defining $U_{xy} = \min_{i:x_i=x} \{u_i - \varepsilon_{iy}\}$ and $V_{xy} = \min_{j:y_j=y} \{v_j - \eta_{xj}\}$ yields $D_{xy}(U_{xy}, V_{xy}) \geq 0$. We will show that under weak conditions, this is actually an equality, hence:

$$D_{xy}(U_{xy}, V_{xy}) = 0. \tag{4.1}$$

Further, one sees from the definition of U_{xy} and V_{xy} that $u_i \geq \max_{y \in \mathcal{Y}} \{U_{xy} + \varepsilon_{iy}, \varepsilon_{i0}\}$ and $v_j \geq \max_{x \in \mathcal{X}} \{V_{xy} + \eta_{xj}, \eta_{0j}\}$. Again under rather weak conditions (stated in Appendix A), this actually will hold as an equality, so that $u_i = \max_{y \in \mathcal{Y}} \{U_{xy} + \varepsilon_{iy}, \varepsilon_{i0}\}$ and $v_j = \max_{x \in \mathcal{X}} \{V_{xy} + \eta_{xj}, \eta_{0j}\}$. Hence, agents face discrete choice problems when choosing the type of their partner. At equilibrium, the mass of men of type x choosing type y women should coincide with the mass of women of type y choosing men of type x . Thus, we need to relate this common quantity μ_{xy} to the vector of systematic utilities (U_{xy}) and (V_{xy}). This is done in the next paragraph using results from the literature on Conditional Choice Probability (CCP) inversion, which will allow us to state a definition of aggregate equilibrium.

4.2. Aggregate Equilibrium. An *aggregate matching* (or just a *matching*, when no confusion is possible), is specified by a vector $(\mu_{xy})_{x \in \mathcal{X}, y \in \mathcal{Y}}$ measuring the mass of matches between men of type x and women of type y . Let \mathcal{M} be the set of matchings, that is, the set of $\mu_{xy} \geq 0$ such that $\sum_{y \in \mathcal{Y}} \mu_{xy} \leq n_x$ and $\sum_{x \in \mathcal{X}} \mu_{xy} \leq m_y$. For later purposes, we shall need to consider the strict interior of \mathcal{M} , denoted \mathcal{M}^0 , i.e. the set of $\mu_{xy} > 0$ such that $\sum_{y \in \mathcal{Y}} \mu_{xy} < n_x$ and $\sum_{x \in \mathcal{X}} \mu_{xy} < m_y$. The elements of \mathcal{M}^0 are called *interior matchings*.

We will look for an individual equilibrium (μ_{ij}, u_i, v_j) with the property that there exist two vectors (U_{xy}) and (V_{xy}) such that if i is matched with j , then $u_i = U_{x_i y_j} + \varepsilon_{i y_j}$, and $v_j = V_{x_i y_j} + \eta_{x_i j}$.³

Under such an equilibrium, each agent is faced with a choice between the observable types of his or her potential partners, and man i and woman j solve respectively the following discrete choice problems

$$u_i = \max_{y \in \mathcal{Y}} \{U_{x_i y} + \varepsilon_{i y}, \varepsilon_{i 0}\} \quad \text{and} \quad v_j = \max_{x \in \mathcal{X}} \{V_{x y_j} + \eta_{x j}, \eta_{0 j}\}.$$

This yields an important extension of Choo and Siow's (2006) original insight that the matching problem with heterogeneity in tastes is equivalent to a pair of discrete choice problems on both sides of the market. This will allow us to relate the vector of utilities (U_{xy}) and (V_{xy}) to the equilibrium matching μ such that μ_{xy} is the mass of men of type x and women of type y mutually preferring each other. In order to establish this relation, we make use of the convex analytic apparatus of Galichon and Salanié (2015). We define the total indirect surplus of men and women by respectively

$$G(U) = \sum_{x \in \mathcal{X}} n_x \mathbb{E} \left[\max_{y \in \mathcal{Y}} \{U_{xy} + \varepsilon_{iy}, \varepsilon_{i0}\} \right] \quad \text{and} \quad \sum_{y \in \mathcal{Y}} m_y \mathbb{E} \left[\max_{x \in \mathcal{X}} \{V_{xy} + \eta_{xj}, \eta_{0j}\} \right]. \quad (4.2)$$

By the Daly-Zachary-Williams theorem, the mass of men of type x demanding a partner of type y is a quantity $\mu_{xy} = \partial G(U) / \partial U_{xy}$, which we denote in vector notation by $\mu \equiv \nabla G(U)$. Similarly, the mass of women of type y demanding a partner of type x is given by ν_{xy} , where $\nu \equiv \nabla H(V)$. At equilibrium, the mass of men of type x demanding women of type y should coincide with the mass of women of type y demanding men of type x , thus $\mu_{xy} = \nu_{xy}$ should hold for any pair, so reexpresses as $\nabla G(U) = \nabla H(V)$. Of course, U and V are related by the feasibility equation $D_{xy}(U_{xy}, V_{xy}) = 0$ for each $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. This leads to the following definition:

Definition 5 (Aggregate Equilibrium). The triple $(\mu_{xy}, U_{xy}, V_{xy})_{x \in \mathcal{X}, y \in \mathcal{Y}}$ is an *aggregate equilibrium outcome* if the following three conditions are met:

³While this may look like a restriction, we show in Appendix A that: (i) there always exists an individual equilibrium of this form, and (ii) under a very mild additional assumption on the feasible sets (namely, Assumption 2' in Appendix A), *any* individual equilibrium is of this form.

- (i) μ is an interior matching, i.e. $\mu \in \mathcal{M}^0$;
- (ii) (U, V) is feasible, i.e.

$$D_{xy}(U_{xy}, V_{xy}) = 0, \quad \forall x \in \mathcal{X}, y \in \mathcal{Y}; \quad (4.3)$$

- (iii) $\mu, U,$ and V are related by the market clearing condition

$$\mu = \nabla G(U) = \nabla H(V). \quad (4.4)$$

The vector $(\mu_{xy})_{x \in \mathcal{X}, y \in \mathcal{Y}}$ an *aggregate equilibrium matching* if and only if there exists a pair of vectors $(U_{xy}, V_{xy})_{x \in \mathcal{X}, y \in \mathcal{Y}}$ such that (μ, U, V) is an aggregate equilibrium outcome.

Remark 4.2. A simple count of variables shows that the vector of unknowns (μ, U, V) is of dimension $3 \times |\mathcal{X}| \times |\mathcal{Y}|$. This number coincides with the number of equations provided by (4.3) and (4.4), which is consistent with the fact, proven below, that a solution exists and is unique.

Remark 4.3. We discuss the equivalence of individual and aggregate equilibrium in Theorem 5 of Appendix A.

4.3. Aggregate matching equation. Before ending this section, we rewrite the system of equations in Definition 5 as a simpler system of equations which involves the matching vector μ only. To do this, we need to invert $\mu = \nabla G(U)$ and $\mu = \nabla H(V)$ in order to express U and V as a function of μ . For this purpose, we introduce the Legendre-Fenchel transform (a.k.a. convex conjugate) of G and H :

$$G^*(\mu) = \sup_U \left\{ \sum_{xy} \mu_{xy} U_{xy} - G(U) \right\} \quad \text{and} \quad H^*(\nu) = \sup_V \left\{ \sum_{xy} \nu_{xy} V_{xy} - H(V) \right\}. \quad (4.5)$$

It is a well-known fact from convex analysis (cf. Rockafellar 1970) that, under smoothness assumptions that hold here given Assumption 3,

$$\mu = \nabla G(U) \iff U = \nabla G^*(\mu) \quad \text{and} \quad \nu = \nabla H(V) \iff V = \nabla H^*(\nu),$$

so we may substitute out U and V as an expression of μ in the system of equations in Definition 5, so that equilibrium is characterized by a set of $|\mathcal{X}| \times |\mathcal{Y}|$ equations expressed only in terms of μ .

Proposition 1. *Matching $\mu \in \mathcal{M}^0$ is an aggregate equilibrium matching if and only if*

$$D_{xy} \left(\frac{\partial G^*(\mu)}{\partial \mu_{xy}}, \frac{\partial H^*(\mu)}{\partial \mu_{xy}} \right) = 0 \text{ for all } x \in \mathcal{X}, y \in \mathcal{Y}. \quad (4.6)$$

Although the reformulation in Proposition 1 is not used to obtain existence and uniqueness of an equilibrium in Section 5, it is extremely useful in Section 6 when the particular case of logit heterogeneity is considered; in that case, equation (4.6) can be inverted easily.

Remark 4.4. In the TU setting, $D_{xy}(u, v) = (u + v - \Phi_{xy})/2$; thus, the fundamental matching equation (4.6) can be rewritten as $\nabla G^*(\mu) + \nabla H^*(\mu) = \Phi_{xy}$. In this case, Galichon and Salanié (2015) have shown the existence and uniqueness of a solution to (4.6) by showing that this equation coincides with the first-order conditions associated to the utilitarian welfare maximization problem, namely

$$\max_{\mu} \left\{ \sum_{xy} \mu_{xy} \Phi_{xy} - \mathcal{E}(\mu) \right\},$$

where $\Phi = \alpha + \gamma$ is the systematic part of the joint affinity, and $\mathcal{E} := G^* + H^*$ is an entropy penalization that trades-off against the maximization of the observable part of the joint affinity. However, besides the particular case of Transferable Utility, Equation (4.6) cannot be interpreted in general as the first-order conditions of an optimization problem. Hence, the methods developed in the present paper, which are based on Gross Substitutes are very different than those in Galichon and Salanié (2015), which rely on convex optimization.

5. AGGREGATE EQUILIBRIUM: EXISTENCE, UNIQUENESS AND COMPUTATION

In this section, we study aggregate equilibria by reformulating the ITU matching market in terms of a demand system. The couple types xy will be treated as goods; men as producers, and women as consumers. Each man of type x chooses to produce one of the goods of type xy , where $y \in \mathcal{Y}_0$; similarly, each woman of type y chooses to consume one of the goods of type xy , where $x \in \mathcal{X}_0$. The wedges W_{xy} are interpreted as prices, and $\partial G(\mathcal{U}(W))/\partial U_{xy}$ is interpreted as the supply of the xy good, and $\partial H(\mathcal{V}(W))/\partial V_{xy}$ is interpreted as the demand for that good if the price vector is W . An increase in W_{xy} raises the supply of the xy good and decreases the demand for it. We can define the excess

demand function as

$$Z(W) := \nabla H(\mathcal{V}(W)) - \nabla G(\mathcal{U}(W)), \quad (5.1)$$

so that $Z_{xy}(W)$ is the mass of women of type y willing to match with men of type x minus the mass of men of type x willing to match with women of type y , if the vector of market wedges is W . At equilibrium, the market wedges are such that $Z(W) = 0$. In Section 5.1, we show that our demand system satisfies the gross substitutability property; this observation is the basis of our existence and uniqueness proofs in Section 5.2.

5.1. Reformulation as a demand system. Thanks to the explicit representation of the feasible sets, we obtain an alternative description of our matching model as a demand system, in the spirit of Azevedo and Leshno's (2015) approach to NTU models without unobserved heterogeneity. As we recall, $D_{xy}(U_{xy}, V_{xy}) = 0$ is equivalent to the existence of W_{xy} such that $U = \mathcal{U}(W)$ and $V = \mathcal{V}(W)$, where the xy -entries of $\mathcal{U}(W)$ and $\mathcal{V}(W)$ are $\mathcal{U}_{xy}(W_{xy})$ and $\mathcal{V}_{xy}(W_{xy})$, as introduced in Definition 3.

Proposition 2. *Outcome (μ, U, V) is an aggregate equilibrium outcome if and only if $\mu = \nabla G(U) = \nabla H(V)$, and there exists a vector (W_{xy}) such that $U = \mathcal{U}(W)$, $V = \mathcal{V}(W)$, and*

$$Z(W) = 0. \quad (5.2)$$

As we recall, $Z(\cdot)$ is to be interpreted as an excess demand function, and (W_{xy}) as a vector of market prices: if W_{xy} increases and all the other entries of W remain constant, the systematic utility V_{xy} of women in the xy category decreases and the utility U_{xy} of men in that category increases, hence Z_{xy} , the excess demand for category xy , decreases. It is possible to express that in this demand interpretation various categories of goods xy are gross substitutes, in the following sense:

Proposition 3 (Gross Substitutes). *(a) If W_{xy} increases and all other entries of W remain constant, then:*

- (a.1) $Z_{xy}(W)$ decreases,
- (a.2) $Z_{x'y'}(W)$ increases if either $x = x'$ or $y = y'$ (but both equalities do not hold),
- (a.3) $Z_{x'y'}(W)$ remains constant if $x \neq x'$ and $y \neq y'$.

(b) the sum $\sum_{x' \in \mathcal{X}, y' \in \mathcal{Y}} Z_{x'y'}(W)$ decreases.

The result implies that the excess demand function Z satisfies the *gross substitutability* condition. Point (a.1) means that when W_{xy} increases, one moves along the Pareto frontier of the feasible set \mathcal{F}_{xy} towards a direction which is more favorable to the men (U_{xy} increases, V_{xy} decreases), and thus there is *ceteris paribus* less demand from women and more from men for the category xy , and excess demand Z_{xy} decreases. Point (a.2) expresses that when the price of some category, say W_{xy} increases, and all the other entries of W remain constant, then the prospects of women in the category xy deteriorates, thus some of these women will switch to category $x'y$, and hence the excess demand Z_{xy} for category xy increases. Point (a.3) simply means that an agent (man or woman) does not respond to the price change of a category which does not involve his or her type. Finally, point (b) expresses that when the price of category xy increases, then singlehood becomes weakly less attractive for all men, and strongly less so for men of category x ; while singlehood becomes more attractive for women, which explains that the sum of $Z_{x'y'}$ over all categories, decreases.

5.2. The result. We will now state and prove a theorem that ensures the existence and uniqueness of an equilibrium using the characterization of aggregate equilibrium as a demand system introduced in Proposition 2. We show that there is a unique vector of prices (W_{xy}) at which the value of excess demand is 0. This is stated in the following result:

Theorem 1 (Existence and uniqueness of a Walrasian equilibrium). *Under Assumptions 1, 2, and 3, there exists a unique vector W such that*

$$Z(W) = 0. \tag{5.3}$$

5.2.1. Existence. The proof of existence is constructive, and W is obtained as the outcome of the following algorithm. It is shown in the proof of Theorem 1 that one can find an initial vector of prices (W_{xy}^0) for which excess demand is negative, that is $Z(W^0) \leq 0$. This suggests that prices (W_{xy}^0) are too high. Our algorithm consists of lowering these prices such that at each step, the excess demand at current price $Z(W^t)$ remains negative. More precisely, we set W_{xy}^t , the price of category xy at time t , to be such that $Z(W_{xy}^t, W_{-xy}^{t-1}) =$

0, where $(W_{xy}^t, W_{-xy}^{t-1})$ denotes the price vector which coincides with W^{t-1} on all entries except on the xy entry and which sets price W_{xy}^t to the xy entry. In other words, the prices of each category are updated in order to cancel the corresponding excess demand, holding the prices of other categories constant. More formally, it is possible to define a map $\mathcal{W} : \mathbb{R}^{\mathcal{X} \times \mathcal{Y}} \rightarrow \mathbb{R}^{\mathcal{X} \times \mathcal{Y}}$ such that $W' = \mathcal{W}(W)$ if and only if for all $xy \in \mathcal{X} \times \mathcal{Y}$

$$Z_{xy}(W'_{xy}, W_{-xy}) = 0,$$

and the proposed algorithm simply consists in setting $W^t = \mathcal{W}(W^{t-1})$. By Proposition 3, point (a.1), it follows that that $W_{xy}^t \leq W_{xy}^{t-1}$ for each xy . Because of the gross substitutability property (Proposition 3, point (a.2)), $Z(W_{xy}^t, W_{-xy}^t) \leq Z(W_{xy}^t, W_{-xy}^{t-1}) = 0$, so that excess demand is still negative at step t . Finally, it is possible to show that W_{xy}^t remains bounded by below; thus, it converges monotonically. The limit is therefore a fixed point of \mathcal{W} , hence a zero of Z . This leads us to the following algorithm:

Algorithm 1.

- Step 0 | Start with $w^0 = \bar{W}$.
 - Step t | For each $x \in \mathcal{X}, y \in \mathcal{Y}$, define $W^{t+1} = \mathcal{W}(W^t)$.
- The algorithm terminates when $\sup_{xy \in \mathcal{X} \times \mathcal{Y}} |W_{xy}^{t+1} - W_{xy}^t| < \epsilon$.

5.2.2. *Uniqueness.* The proof of uniqueness is based on a result of Berry, Gandhi, and Haile (2013), that implies that Z is inverse isotone. Hence, if there are two vectors W and \tilde{W} such that $Z(W) = Z(\tilde{W}) = 0$, it would follow that $W \leq \tilde{W}$ and $\tilde{W} \leq W$ altogether, hence $W = \tilde{W}$.

Combining Theorem 1 and Proposition 2, it follows that there exists a unique equilibrium outcome (μ, U, V) , where μ , U , and V are related to W by $U_{xy} = \mathcal{U}_{xy}(W_{xy})$, $V_{xy} = \mathcal{V}_{xy}(W_{xy})$, and $\mu = \nabla G(U) = \nabla H(V)$.

Corollary 1 (Existence and uniqueness of an equilibrium outcome). *Under Assumptions 1, 2, and 3, there exists a unique equilibrium outcome (μ, U, V) , and μ , U , and V are related to the solution W to system (5.3) by $U_{xy} = \mathcal{U}_{xy}(W_{xy})$, $V_{xy} = \mathcal{V}_{xy}(W_{xy})$, and $\mu = \nabla G(U) = \nabla H(V)$.*

6. THE ITU-LOGIT MODEL

In this section, and for the rest of the paper, we consider the model of matching with Imperfectly Transferable Utility and *logit* heterogeneity. We therefore replace Assumption 3 by:

Assumption 3'. \mathbf{P}_x and \mathbf{Q}_y are the distributions of *i.i.d.* Gumbel (standard type I extreme value) random variables.

Of course, Assumption 3' is a specialization of Assumption 3, as the Gumbel distribution has a positive density on the real line. As we show in this section, the logit assumption carries strong implications. We shall show in Section 6.1 that the equilibrium matching equations (4.6) can be drastically simplified, and that an algorithm more efficient than Algorithm 1 can be used to solve them. Next, in Section 6.2, we provide a number of illustrative applications of the logit assumption in the various example instances introduced in Section 3.3. Finally, we will show in Section 6.3 that maximum likelihood estimation is particularly straightforward in the logit context.

6.1. Equilibrium and computation, logit case. With logit random utilities, it is well-known that the systematic part of the utility U_{xy} can be identified by the log the ratio of the odds of choosing alternative y , relative to choosing the default option, and a similar formula applies to V_{xy} , hence $U_{xy} = \log(\mu_{xy}/\mu_{x0})$ and $V_{xy} = \log(\mu_{xy}/\mu_{0y})$, where $\mu_{x0} = n_x - \sum_{y \in \mathcal{Y}} \mu_{xy}$, and $\mu_{0y} = m_y - \sum_{x \in \mathcal{X}} \mu_{xy}$. Hence, the feasibility equation $D_{xy}(U_{xy}, V_{xy}) = 0$ in expression (4.6) becomes $D_{xy}(\log \mu_{xy} - \log \mu_{x0}, \log \mu_{xy} - \log \mu_{0y}) = 0$, which, given the translation invariance property (v) of Lemma 1, yields

$$\log \mu_{xy} = -D_{xy}(-\log \mu_{x0}, -\log \mu_{0y}),$$

which explicitly defines μ_{xy} as a function of μ_{x0} and μ_{0y} :

$$\mu_{xy} = M_{xy}(\mu_{x0}, \mu_{0y}), \text{ where } M_{xy}(\mu_{x0}, \mu_{0y}) = \exp(-D_{xy}(-\log \mu_{x0}, -\log \mu_{0y})). \quad (6.1)$$

Remark 6.1. In the search and matching literature, maps such as M_{xy} that relate the mass of matches of type (x, y) to the mass of unmatched agents of type x and y are called

aggregate matching functions (see, e.g., Petrongolo and Pissarides (2001) and Siow (2008)). However, there is an important difference between our aggregate matching function and much of the prior work. Here, μ_{x0} and μ_{0y} are the masses of men and women selected into singlehood, which is endogenous, and determined by equilibrium equations (6.2). In the demography literature (up to the important exception of Choo and Siow (2006) and the subsequent literature), μ_{x0} and μ_{0y} are usually the masses of *available* men and women, assumed to be exogenous.

The expression of μ_{xy} as a function of μ_{x0} and μ_{0y} , combined with the requirement that $\mu \in \mathcal{M}^0$, provides a set of equations that fully characterize the aggregate matching equilibrium, as argued in the following result:

Theorem 2. *Under Assumptions 1, 2, and 3', the equilibrium outcome (μ, U, V) in the ITU-logit model is given by*

$$\mu_{xy} = M_{xy}(\mu_{x0}, \mu_{0y}), \quad U_{xy} = \log \frac{\mu_{xy}}{\mu_{x0}}, \quad V_{xy} = \log \frac{\mu_{xy}}{\mu_{0y}},$$

where the pair of vectors $(\mu_{x0})_{x \in \mathcal{X}}$ and $(\mu_{0y})_{y \in \mathcal{Y}}$ is the unique solution to the system of equations

$$\begin{cases} \sum_y M_{xy}(\mu_{x0}, \mu_{0y}) + \mu_{x0} = n_x \\ \sum_x M_{xy}(\mu_{x0}, \mu_{0y}) + \mu_{0y} = m_y. \end{cases} \quad (6.2)$$

Theorem 2 implies that computing aggregate equilibria in the logit case is equivalent to solving the system of nonlinear equations (6.2)—a system of $|\mathcal{X}| + |\mathcal{Y}|$ equations in the same number of unknowns. It turns out that a very simple iterative procedure provides a very efficient means of solving these equations. The basic idea is each equation in the first set of equations is an equation in the full set of (μ_{0y}) , but in the single unknown μ_{x0} . Hence, these can be inverted to obtain the values of (μ_{x0}) from the values of (μ_{0y}) . A similar logic applies to the second set of equations, where the values of (μ_{0y}) can be obtained from the values of (μ_{x0}) . The proposed algorithm operates by iterating the expression of (μ_{x0}) from (μ_{0y}) and vice-versa. Provided the initial choice of (μ_{0y}) is high enough, the procedure converges isototonically, as argued in the theorem below.

Algorithm 2.

Step 0	Fix the initial value of μ_{0y} at $\mu_{0y}^0 = m_y$.
Step $2t + 1$	Keep the values μ_{0y}^{2t} fixed. For each $x \in \mathcal{X}$, solve for the value, μ_{x0}^{2t+1} , of μ_{x0} such that equality $\sum_{y \in \mathcal{Y}} M_{xy}(\mu_{x0}, \mu_{0y}^{2t}) + \mu_{x0} = n_x$ holds.
Step $2t + 2$	Keep the values μ_{x0}^{2t+1} fixed. For each $y \in \mathcal{Y}$, solve for which is the value, μ_{0y}^{2t+2} , of μ_{0y} such that equality $\sum_{x \in \mathcal{X}} M_{xy}(\mu_{x0}^{2t+1}, \mu_{0y}) + \mu_{0y} = m_y$ holds.

The algorithm terminates when $\sup_y |\mu_{0y}^{2t+2} - \mu_{0y}^{2t}| < \epsilon$.

Theorem 3. *Under Assumptions 1, 2, and 3', Algorithm 2 converges toward the solution (6.2), in such a way that (μ_{0y}^t) is nonincreasing with t , and (μ_{x0}^t) is nondecreasing with t .*

6.2. Example Specifications, logit case.

6.2.1. *TU-logit Specification.* In the logit case of the TU specification introduced in Section 3.3.1, the matching function becomes

$$\mu_{xy} = \mu_{x0}^{1/2} \mu_{0y}^{1/2} \exp \frac{\Phi_{xy}}{2}, \quad (6.3)$$

which is Choo and Siow's (2006) formula.

6.2.2. *NTU-logit Specification.* In the logit case of the NTU specification introduced in Section 3.3.2, the matching function becomes

$$\mu_{xy} = \min(\mu_{x0} e^{\alpha_{xy}}, \mu_{0y} e^{\gamma_{xy}}). \quad (6.4)$$

When $\mu_{x0} e^{\alpha_{xy}} \leq \mu_{0y} e^{\gamma_{xy}}$, $\mu_{xy} = \mu_{x0} e^{\alpha_{xy}}$ is constrained by the choice problem of men; we say that, relative to pair xy , men are on the *short side (of the market)* and women are on the *long side (of the market)*, and visa versa. Galichon and Hsieh (2016) study this model in detail. In particular, they show that existence and computation of a more general version of this model can be provided via an aggregate version of the Gale–Shapley (1962) algorithm.⁴

⁴Note that Dagsvik (2000) and Menzel (2015) obtain $\mu_{xy} = \mu_{x0} \mu_{0y} e^{\alpha_{xy} + \gamma_{xy}}$, in contrast with our formula (6.4). The reason for this difference is that Dagsvik (2000) and Menzel (2015) assume that the stochastic matching affinities are given by $\alpha_{ij} = \alpha_{xy} + \varepsilon_{ij}$ and $\gamma_{ij} = \gamma_{xy} + \eta_{ij}$, where the ε_{ij} and η_{ij} terms are i.i.d. type I extreme value distributions. In contrast, in our setting, $\alpha_{ij} = \alpha_{xy} + \varepsilon_{iyj}$ and $\gamma_{ij} = \gamma_{xy} + \eta_{xij}$.

6.2.3. *ETU-logit Specification.* In the logit case of the the Exponentially Transferable Utility specification introduced in Section 3.3.3, the feasibility frontier takes the form

$$\exp\left(\frac{U_{xy} - \alpha_{xy}}{\tau_{xy}}\right) + \exp\left(\frac{V_{xy} - \gamma_{xy}}{\tau_{xy}}\right) = 2,$$

which, when combined with the log-odds ratio formulae identifying U and V , yields the following expression for the matching function:

$$\mu_{xy} = \left(\frac{e^{-\alpha_{xy}/\tau_{xy}} \mu_{x0}^{-1/\tau_{xy}} + e^{-\gamma_{xy}/\tau_{xy}} \mu_{0y}^{-1/\tau_{xy}}}{2} \right)^{-\tau_{xy}}. \quad (6.5)$$

As expected, when $\tau_{xy} \rightarrow 0$, formula (6.5) converges to the NTU-logit formula, (6.4). Likewise, when $\tau_{xy} \rightarrow +\infty$, (6.5) converges to the TU-logit formula, (6.3). But when $\tau_{xy} = 1$, then (up to multiplicative constants) μ_{xy} becomes the harmonic mean between μ_{x0} and μ_{0y} . We thus recover a classical matching function form—the “Harmonic Marriage Matching Function” that has been used by demographers for decades (see, e.g. Schoen (1981)). To our knowledge, our framework gives the first behavioral/microfounded justification of harmonic marriage matching function. (Indeed, as Siow (2008, p. 5) argued, this choice of matching function heretofore had “no coherent behavioral foundation.”)

6.2.4. *LTU-logit Specification.* In the logit case of the the Linearly Transferable Utility specification introduced in Section 3.3.4, the matching function becomes

$$\mu_{xy} = e^{(\lambda_{xy}\alpha_{xy} + \zeta_{xy}\gamma_{xy})/(\lambda_{xy} + \zeta_{xy})} \mu_{x0}^{\lambda_{xy}/(\lambda_{xy} + \zeta_{xy})} \mu_{0y}^{\zeta_{xy}/(\lambda_{xy} + \zeta_{xy})}. \quad (6.6)$$

In particular, when $\lambda_{xy} = 1$ and $\zeta_{xy} = 1$, we again recover the Choo and Siow (2006) identification formula.

6.3. **Maximum likelihood estimation, logit case.** In this section, we assume that (D_{xy}) belongs to a parametric family (D_{xy}^θ) and we estimate θ by maximum likelihood. Letting N be the total number of households in our sample, the log-likelihood of observation $\hat{\mu}_{xy}$ is given (up to a $N \log N$ factor) by

$$l(\theta) = \sum_{xy \in \mathcal{X}\mathcal{Y}} \hat{\mu}_{xy} \log \mu_{xy}^\theta + \sum_{x \in \mathcal{X}} \hat{\mu}_{x0} \log \mu_{x0}^\theta + \sum_{y \in \mathcal{Y}} \hat{\mu}_{0y} \log \mu_{0y}^\theta, \quad (6.7)$$

given the expression of μ_{xy}^θ as a function of μ_{x0}^θ and μ_{0y}^θ provided in (6.1). Denoting $u_x^\theta = -\log \mu_{x0}^\theta$ and $v_y^\theta = -\log \mu_{0y}^\theta$, the expression of the log-likelihood reformulates as:

Theorem 4. *The log-likelihood $l(\theta)$ is expressed using*

$$-l(\theta) = \sum_{xy \in \mathcal{X} \times \mathcal{Y}} \hat{\mu}_{xy} D_{xy}^\theta(u_x^\theta, v_y^\theta) + \sum_{x \in \mathcal{X}} \hat{\mu}_{x0} u_x^\theta + \sum_{y \in \mathcal{Y}} \hat{\mu}_{0y} v_y^\theta, \quad (6.8)$$

where the expected utilities u_x^θ and v_y^θ are the unique pair of vectors (u, v) solution to

$$\begin{cases} e^{-u_x} + \sum_{y \in \mathcal{Y}} e^{-D_{xy}^\theta(u_x, v_y)} = n_x \\ e^{-v_y} + \sum_{x \in \mathcal{X}} e^{-D_{xy}^\theta(u_x, v_y)} = m_y, \end{cases} \quad (6.9)$$

Expression (6.8) has a striking interpretation. For a matched pair (x, y) , $D_{xy}(u_x, v_y)$ is the signed distance to the efficient frontier. For a single individual man or woman of type x or y , u_x or v_y is also the signed distance to the efficient frontier, which is 0. Hence, the value of the likelihood is the opposite of the sum of the distances to the efficient frontier. Therefore, we arrive at a very important re-interpretation of the maximum likelihood procedure: *the maximum likelihood procedure finds the value of the parameter vector such that the sum of the distances of $u_x^\theta = -\log \mu_{x0}^\theta$ and $v_y^\theta = -\log \mu_{0y}^\theta$ to the efficient frontier is minimized.*

In order to compute the Maximum Likelihood Estimator of θ by a descent method, one typically needs the gradient of the likelihood. As $\mu_{xy} = M_{xy}^\theta(\mu_{x0}^\theta, \mu_{0y}^\theta)$, the derivative $\partial_{\theta^k} \mu_{xy}$ is the sum of three contributions: $\partial_{\theta^k} M_{xy}$, and the contributions of $\partial_{\theta^k} \mu_{x0}$ and $\partial_{\theta^k} \mu_{0y}$ which can be computed by the Implicit Function Theorem given that (μ_{x0}, μ_{0y}) is a solution to system (6.2). The computational details are provided a companion note (Galichon and Weber, 2016).

7. APPLICATION

In this section, we bring to the data a richer version of the simple model described in Section 3.3.3—a model with marital complementarities and private consumption. It is very hard in practice to select among models with different transferability structures simply based on matching patterns. However, this difficulty is overcome if the data set is supplemented

with information on transfers or information on personal expenditures. We use observations on demographic characteristics and personal expenditures from the British Living Costs and Food Expenditures survey (2013) to structurally estimate the British marriage market. Our estimation procedure is carried out using the maximum likelihood procedure described in the preceding section.

7.1. The model for estimation. We consider a simple model of matching with marital utility and consumption, similar to the one introduced in Section 3.3.3, and logit heterogeneity, as introduced in Section 6. The systematic utilities of a married man of type x and a married woman of type y paired together are specified as follows

$$\alpha_{xy} + \tau \log c^m \text{ and } \gamma_{xy} + \tau \log c^w,$$

where c^m and c^w are the private consumptions of the man and the woman, respectively. Private consumption should satisfy the budget constraint $c^m + c^w = I(x) + I(y)$, where $I(x)$ and $I(y)$ are the income of men of type x and women of type y , respectively. At equilibrium, $c^m = c^m(x, y)$ and $c^w = c^w(x, y)$ only depend on the man and the woman's observable types. If a man x or a woman y chooses to remain single, they respectively receive

$$\alpha_{x0} + \tau \log I(x) \text{ and } \gamma_{0y} + \tau \log I(y).$$

The systematic parts of the matching surpluses, relative to singlehood, for a married pair x, y are given by

$$U_{xy} = \alpha_{xy} - \alpha_{x0} + \tau \log \left(\frac{c^m(x, y)}{I(x)} \right) \text{ and } V_{xy} = \gamma_{xy} - \gamma_{0y} + \tau \log \left(\frac{c^w(x, y)}{I(y)} \right).$$

Without loss of generality, we assume in the sequel that $\alpha_{x0} = 0$ and $\gamma_{0y} = 0$. The budget constraint $c^m(x, y) + c^w(x, y) = I(x) + I(y)$ implies an expression for the feasible set \mathcal{F} and the distance function D_{xy} . A simple calculation similar to the one in Section 3.3.3 shows that

$$D_{xy}(U_{xy}, V_{xy}) = \tau \log \left(\rho(x, y) \exp\left(\frac{U_{xy} - \alpha_{xy}}{\tau}\right) + (1 - \rho(x, y)) \exp\left(\frac{V_{xy} - \gamma_{xy}}{\tau}\right) \right),$$

where $\rho(x, y) = \frac{I(x)}{I(x) + I(y)}$ denotes the man's share of contribution to total income of the household.

7.2. Estimation. Assuming logit heterogeneity, estimation follows the steps described in Section 6. We assume that the frequency of each type of man and woman is identical, so that $n_x = m_y = 1$ for all $x, y \in \mathcal{X} \times \mathcal{Y}$. We will not worry here about the fact that the types are sampled from a continuous distribution of types; the theoretical model would incorporate this feature by appealing to a continuous logit model as used in Dupuy and Galichon (2014) or Menzel (2015), but the estimation would be identical.

The likelihood function is similar to expression (6.7), with a notable difference. Indeed, in general, data sets do not allow the researcher to observe private consumption, but often some proxy of personal expenditures. Indeed, our model allows us to recover the expenditure levels, say of man i , as

$$c^m(x_i, y_j) = I(x_i) \exp\left(\frac{u_{x_i} - D_{x_i y_j}(u_{x_i}, v_{y_j}) - \alpha_{x_i y_j}}{\tau}\right),$$

with the notation $u_x = -\log \mu_{x0}$ and $v_y = -\log \mu_{0y}$. (Note that women's private consumptions can be deduced by $\hat{c}_j^w = I(x_i) + I(y_j) - \hat{c}_i^m$). Assume further that we measure private expenditures with some measurement error, that is we observe men's private consumptions as $\hat{c}_i^m = c^m(x_i, y_j) + \epsilon_{ij}$, where ϵ is a Gaussian measurement error with variance s_ϵ^2 , and independently distributed across the (x, y) pairs.

Letting θ be a parameterizations of (α, γ, τ) , we can now compute the log-likelihood (up to constants):

$$\begin{aligned} \log \mathcal{L}(\theta, s_\epsilon, s_\eta) = & - \sum_{(i,j) \in \mathcal{C}} D^\theta(u_{x_i}^\theta, v_{y_j}^\theta) - \sum_{i \in \mathcal{S}_M} u_{x_i}^\theta - \sum_{j \in \mathcal{S}_F} v_{y_j}^\theta \\ & - 2 \sum_{(i,j) \in \mathcal{C}} \frac{(\hat{c}_i^m - c^m(x_i, y_j))^2}{2s_\epsilon^2} - 2|\mathcal{C}| \log s_\epsilon, \end{aligned}$$

where \mathcal{C} denotes the set of matched pairs (i, j) observed in the data, \mathcal{S}_M and \mathcal{S}_F respectively denote the set of single men and the set of single women observed in the data, and where u_x^θ and v_y^θ satisfy equilibrium equations (6.2).

The parameters $\xi = (\theta, s_\epsilon)$ are estimated by maximum likelihood. Estimation is performed in R using the NLOPT package and the BFGS algorithm, with bound constraints on τ and on s_ϵ (these parameters are restricted to be positive). We compute analytically

the gradient to improve performance. At each step of the estimation process, the following computations are performed:

- (i) The parameters ξ are updated using the gradient computed in the previous step.
- (ii) The updated values of α , γ and τ are deduced.
- (iii) The equilibrium quantities u and v are computed using Algorithm 2, and the predicted consumption levels are constructed.
- (iv) The log-likelihood is updated.

The gradient computation procedure is part of a R package, called TraME (Transportation Methods for Econometrics; Galichon and Weber (2015)). This package simplifies the computation and the estimation of a wide range of discrete choice and matching problems, as it relies on a flexible formulation of these models in terms of transferability or heterogeneity structure. Under TraME, user-defined models can be solved using core equilibrium algorithms (mainly via Linear Programming, Convex Optimization, Jacobi iterations, Deferred Acceptance, or Iterative Fitting of which Algorithm 2 is an instance of) and estimated by maximum likelihood.

7.3. Data. To estimate our model, we use the British Living costs and Food Survey data set (which replaced the Family Expenditure Survey in 2008) for the year 2013. The data allows us to construct a representative sample of the British population that includes raw estimates of personal expenditures. We focus on married heterosexual pairs, in which case we gather information on both partners, as well as singles (never married, divorced, separated or widowed)⁵ who are heads of their households. We only keep couples in which both members have positive income, and singles with positive income. Additionally, we restrict our attention to households of size 1 for singles and size 2 for couples (hence excluding households with children or relatives and non-relatives), as we focus our attention on the sharing of resources between the married partners. Another advantage of such restriction is that we exclude from the analysis a major public good, namely, investment in children and their education. Finally, we select households in which the head is between 25 and 40 years old, and drop singles or couples with missing information.

⁵Ideally, it would be preferable to focus on first-time married couples and never-married singles, but such detailed information on marital history is usually missing in expenditures data sets.

The total income of a matched pair is the sum of the partners’ personal incomes. Ideally, our application would combine income data with data on private consumption. Of course, private consumption variables are rarely available, and researchers must instead use a proxy of personal expenditures. The data offers a variable called “Total Personal Expenditures”; this is an imperfect measure of consumption, however, it excludes major public goods such as rent, heating or car purchases, while aggregating individual-level expenditures on food, household equipment, leisure goods and services, and clothing. For singles, the variable is set equal to total personal income. For couples, personal expenditure is taken by breaking down the total income proportionally according to each partner’s share of personal expenditures. This ensures that the sum of personal expenditures across partners coincides with the couple’s total income.

TABLE 1. Summary statistics, full sample

	Married				Single			
	Male		Female		Male		Female	
	mean	sd	mean	sd	mean	sd	mean	sd
Age	32.37	4.60	30.33	4.94	33.43	4.61	32.58	4.35
White	0.92	0.27	0.90	0.30	0.87	0.34	0.89	0.31
Black	0.01	0.11	0.02	0.14	0.04	0.20	0.05	0.21
Education	19.84	2.99	20.14	3.08	18.80	2.97	19.17	2.65
Personal Income	638.05	325.88	485.36	264.95	544.27	330.86	478.32	275.12
Share Expenditures	0.47	0.23	0.53	0.23
Observations	161		161		76		66	

To ease estimation, we use a subsample of households, randomly drawn from the full data, with summary statistics displayed in Table (1). Our sample is mostly composed of White individuals. Married men appears to be older than married women (with an average age difference of two years, a fairly standard fact in marriage markets), but somewhat less educated. The data displays large variations in personal income, and shows that women accounts for a slightly larger share of personal expenditures than men. However, this may be

a consequence of measurement error on private consumption, as the latter is only imperfectly observed.

7.4. Specification and results. We use a simple parametrization of couples' pre-transfer utilities:

$$\alpha_{xy} = \lambda_1 |\text{educ}_x - \text{educ}_y| \text{ and } \gamma_{xy} = \lambda_2 |\text{educ}_x - \text{educ}_y|$$

where educ_x and educ_y are the (standardized) ages at which the members of the couple left the schooling system—a proxy for years of education. The parameter vector to be estimated is $\xi = (\lambda_1, \lambda_2, s_\epsilon, \tau) \in \mathbb{R}^4$.

The results from our maximum likelihood estimation are displayed in Table (2) below.

TABLE 2. Estimates

Parameters	λ_1	λ_2	s_ϵ	τ	LL
Estimates	-0.87	-1.09	308.38	3.32	2955.93

Note : These estimates are obtained using the TraME package (Galichon and Weber (2015)) and the NLOPT optimization routine. Parameters λ_1 and λ_2 measure education assortativeness, while τ is the transferability parameter. The standard deviation of our measurement error is estimated as s_ϵ . Finally LL provides the value of the log-likelihood at the optimal point.

Several remarks are in order. The coefficients corresponding to education assortativeness are in line with the prior literature on marriage—they indicate that as utility decreases as distance between the education level of the partners increases. Hence, our results suggest positive assortative mating in education. Additionally, it appears that the penalty in utility for a higher distance between education levels is higher for women than it is for men. This may be reconciled with the observation that women account for a larger share of personal expenditures. Indeed, for a given level of utility and a given education difference, our results suggest that the woman must receive a higher compensation in terms of consumption than the man.

An interesting interpretation of these results can be made in terms of substitutability between education assortativeness and consumption. Indeed, one can use λ and τ to compute marginal rates of substitution (see, for example Chiappori and al. (2012)): for a man, an

increase of one standard deviation in the education difference must be compensated by a 26% ($= 0.87/3.32$) increase in consumption to keep constant the level of utility. For women, a similar computation shows that an increase of one standard deviation in the education difference must be compensated by a 33% increase in consumption.

The parameter τ weights the gains from consumption in this particular specification of the utility functions. However, it has also a deeper meaning: it represents the degree of transferability. (Recall from Section 3.3.3 that we recover the TU case when $\tau \rightarrow +\infty$ and recover the NTU case as $\tau \rightarrow 0$.) Hence, our results suggest that the marriage market considered here is best described by an intermediate model, that is, neither by the classical TU or NTU frameworks. Although this application is modest as it focus on a relatively simple model of marriage with education assortativeness and consumption and makes use of crude expenditure data, it highlights the benefits of our general ITU approach, in terms of flexibility and implementation.

8. DISCUSSION AND PERSPECTIVES

We have introduced an empirical framework for ITU matching models with unobserved heterogeneity in tastes. Our framework includes as special cases the classical fully- and non-transferable utility models, collective models, and settings with taxes on transfers, deadweight losses, and risk aversion. We have characterized the equilibrium conditions, provided results on the existence and uniqueness of an equilibrium, described algorithms to compute the equilibrium, and worked out the maximum likelihood estimation of these models.

The present contribution brings together a number of approaches. In terms of the techniques used, it builds on concepts from Game Theory, General Equilibrium, and Econometrics. In terms of models allowed, it embeds models with and without Transferable Utility. It also provides an integrated approach for both matching models and collective models. Lastly, it can also be used in conjunction with reduced-form methodologies, as it allows to compute the equilibrium outcome's response to a shock in the matching primitives, e.g. a demographic shock, and to regress the former on the latter.

Beyond the class of problems investigated in the present paper, the methods developed here, based on fixed point theorems for isotone functions, may be more broadly applicable. In particular, they may be a useful tool for the investigation of matching problems with peer effects put forward by Mourifié and Siow (2014). They may also prove useful for studying certain commodity flow problems in trade networks, and may also extend to one-to-many matching problems. We leave these last two extensions for further work.

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APPENDIX A. RELATING INDIVIDUAL AND AGGREGATE EQUILIBRIA

In this Appendix, we establish a precise connection between individual equilibria (defined in Section 3), and aggregate equilibria (defined in Section 4). It will be useful at some point to introduce a slightly stronger assumption than Assumption 2, to handle the case when the frontiers of the bargaining sets are strictly downward sloping. This leads us to formulate:

Assumption 2’. *The sets \mathcal{F}_{ij} satisfy Assumption 2, and in addition, D_{ij} is strictly increasing in both its arguments for all i and j .*

Note that in the NTU case, the frontier of the feasible set is not strictly downward-sloping, and therefore Assumption 2’ is not satisfied, while it is satisfied for all the other examples in Section 3.3.

The following result relates the individual and aggregate equilibria.

Theorem 5. (i) Under Assumptions 1, 2, and 3, let (μ, U, V) be an aggregate equilibrium outcome. Then, defining

$$u_i = \max_{y \in \mathcal{Y}} \{U_{x_i y} + \varepsilon_{iy}, \varepsilon_{i0}\} \quad \text{and} \quad v_j = \max_{x \in \mathcal{X}} \{V_{xy_j} + \eta_{xj}, \eta_{0j}\}, \quad (\text{A.1})$$

there is an individual matching μ_{ij} such that (μ_{ij}, u_i, v_j) is an individual equilibrium outcome, which is such that $\mu_{ij} > 0$ implies $u_i = U_{x_i y} + \varepsilon_{iy}$ and $v_j = V_{xy_j} + \eta_{xj}$.

(ii) Under Assumptions 1, 2', and 3, let (μ_{ij}, u_i, v_j) be an individual equilibrium outcome. Then, defining

$$U_{xy} = \min_{i: x_i = x} \{u_i - \varepsilon_{iy}\} \quad \text{and} \quad V_{xy} = \min_{j: y_j = y} \{v_j - \eta_{jx}\}, \quad (\text{A.2})$$

and $\mu_{xy} = \sum_{ij \in \mathcal{I} \times \mathcal{J}} \mu_{ij} \mathbb{1}\{x_i = x\} \mathbb{1}\{y_j = y\}$, it follows that (μ, U, V) is an aggregate equilibrium outcome.

Note that deducing an aggregate equilibrium based on an individual equilibrium (part ii) requires a slightly stronger assumption than deducing an individual equilibrium based on an aggregate equilibrium (part i). The NTU case (not covered under Assumption 2') thus deserves further investigations, which are carried in Galichon and Hsieh (2016).

Theorem 5 implies that agents keep their entire utility shocks at equilibrium, even when they could transfer them fully or partially.

Corollary 2. Under Assumptions 1, 2', and 3, consider a pair of matched individuals i and j of types x and y respectively. Then the equilibrium payoffs of i and j are respectively given by $u_i = U_{xy} + \varepsilon_{iy}$ and $v_j = V_{xy} + \eta_{xj}$, where U and V are aggregate equilibrium payoffs. Therefore, individuals keep their idiosyncratic utility shocks at equilibrium.

This finding, which carries strong testable implications, was known in the TU case (see Chiappori, Salanié, and Weiss (2014)). Our theorem clarifies the deep mechanism that drives this result: the crucial assumption is that the distance function D_{ij} should only depend on i and j through the observable types x_i and y_j , and that some transfers are possible.

APPENDIX B. PROOFS

B.1. Proof of Lemma 1.

Proof. (i) directly follows from the definition of D . (ii) is straightforward given requirements (iii) and (iv) of Definition 1. Let us show (iii). Assume $(u, v) \leq (u', v')$. Then by requirement (ii) of Definition 1, for any $z \in \mathbb{R}$, $(u' - z, v' - z) \in \mathcal{F}_{ij}$ implies $(u - z, v - z) \in \mathcal{F}_{ij}$. Thus $D_{\mathcal{F}}(u, v) \leq D_{\mathcal{F}}(u', v')$, which is the first part of the claim.

Now assume $u < u'$ and $v < v'$ and $D_{\mathcal{F}}(u, v) = D_{\mathcal{F}}(u', v')$. Then $u - D_{\mathcal{F}}(u, v) < u' - D_{\mathcal{F}}(u', v')$ and $v - D_{\mathcal{F}}(u, v) < v' - D_{\mathcal{F}}(u', v')$. But this implies that there exists $\epsilon > 0$ such that $u - D_{\mathcal{F}}(u, v) + \epsilon \leq u' - D_{\mathcal{F}}(u', v')$ and $v - D_{\mathcal{F}}(u, v) + \epsilon \leq v' - D_{\mathcal{F}}(u', v')$; however, as $(u' - D_{\mathcal{F}}(u', v'), v' - D_{\mathcal{F}}(u', v')) \in \mathcal{F}_{ij}$, this implies, still by requirement (ii) of Definition 1, that $(u - D_{\mathcal{F}}(u, v) + \epsilon, v - D_{\mathcal{F}}(u, v) + \epsilon) \in \mathcal{F}_{ij}$, a contradiction. Thus $D_{\mathcal{F}}(u, v) < D_{\mathcal{F}}(u', v')$, which completes the proof that $D_{\mathcal{F}}$ is \gg -isotone.

To show point (iv) ($D_{\mathcal{F}}$ is continuous), consider (u, v) and (u', v') , and assume that $v - u \geq v' - u'$. Then $u - D(u, v) \leq u' - D_{\mathcal{F}}(u', v')$; indeed, assume by contradiction that $u - D(u, v) > u' - D_{\mathcal{F}}(u', v')$, then by summation $v - D(u, v) > v' - D_{\mathcal{F}}(u', v')$. By the same argument as above, this leads to a contradiction; hence, $u - D(u, v) \leq u' - D_{\mathcal{F}}(u', v')$. It is easy to check that $v - u \leq v' - u'$ implies $u - D(u, v) \geq u' - D_{\mathcal{F}}(u', v')$. Hence, in general

$$\min(v' - v, u' - u) \leq D_{\mathcal{F}}(u', v') - D_{\mathcal{F}}(u, v) \leq \max(u' - u, v' - v)$$

which shows continuity of $D_{\mathcal{F}}$.

(v) One has $D_{\mathcal{F}}(u + a, v + a) = \min\{z \in \mathbb{R} : (u + a - z, v + a - z) \in \mathcal{F}\}$ by the very definition of $D_{\mathcal{F}}$, which immediately shows that $D_{\mathcal{F}}(u + a, v + a) = a + D_{\mathcal{F}}(u, v)$. ■

B.2. Proof of Lemma 2.

Proof. The proof is divided in several parts.

First part: let us show that the set of wedges w that can be expressed as $w = u - v$ for u and v such that $D_{\mathcal{F}}(u, v) = 0$ is an open interval. Consider u, u', v and v' such that $D(u, v) = 0$ and $D(u', v') = 0$, and $(u, v) \neq (u', v')$. Let $w = u - v$ and $w' = u' - v'$. Assume

w.l.o.g $u' > u$, then one has necessarily $v \geq v'$, hence $u' - v' > u - v$. In this case, let $u_t = t(u' - u) + u$ and $v_t = t(v' - v) + v$. Let $\tilde{u}_t = u_t - D(u_t, v_t)$ and $\tilde{v}_t = v_t - D(u_t, v_t)$, so that $D(\tilde{u}_t, \tilde{v}_t) = 0$. One has $\tilde{u}_t - \tilde{v}_t = u_t - v_t = t(u' - v') + (1 - t)(u - v) = tw' + (1 - t)w$, which shows that the set of wedges is an interval, denoted I . Let us now show that this interval is open. Call \underline{w} the infimum of the interval, and assume it is finite. Then there is a sequence (u_n, v_n) such that u_n is decreasing, v_n is increasing, $D(u_n, v_n) = 0$ and $u_n - v_n \rightarrow \underline{w}$. Then by the scarcity of \mathcal{F} , u_n and v_n need to remain bounded, hence they converge in \mathcal{F} . Let (u^*, v^*) be their limit; one has $D(u^*, v^*) = 0$ and $u^* - v^* = \underline{w}$. For any $u' < u^*$, one has $D(u', v^*) \leq 0$; hence, by scarcity of \mathcal{F} , there is some $v' \geq v^*$ such that $D(u', v') = 0$. $u' < u^*$ and $v' \geq v^*$, thus $u' - v' < u^* - v^* = \underline{w}$, a contradiction. Thus $\underline{w} \in I$. A symmetric argument shows that if the supremum of I is finite, then it belongs in I . Thus, I is an open interval.

Second part: let us show that \mathcal{U} and \mathcal{V} are well defined on I . For $w \in I$, there exists by definition (u, v) such that $D(u, v) = 0$ and $u - v = w$. The argument at the beginning of part (i) implies that (u, v) is unique. Hence \mathcal{U} and \mathcal{V} are well defined.

Third part: let us show that \mathcal{U} is increasing and \mathcal{V} is decreasing. Suppose $w < w'$ and $\mathcal{U}(w) \geq \mathcal{U}(w')$. Then $w - \mathcal{U}(w) < w' - \mathcal{U}(w')$, hence $\mathcal{V}(w) > \mathcal{V}(w')$, a contradiction. Thus $\mathcal{U}(w) < \mathcal{U}(w')$, which shows that \mathcal{U} is increasing. By a similar logic, $\mathcal{V}(w) > \mathcal{V}(w')$, and \mathcal{V} is decreasing.

Fourth part: let us show that \mathcal{U} is 1-Lipschitz. Take $\epsilon > 0$ and assume by contradiction $u' > u + \epsilon$ where $u = \mathcal{U}(w)$ and $u' = \mathcal{U}(w + \epsilon)$. Then $D(u, u - w) = 0$ with and $D(u', u' - w - \epsilon) = 0$. Then because $u' > u$ and $u' - \epsilon > u$, it follows that $D(u', u' - w - \epsilon) > D(u, u - w) = 0$, a contradiction. Hence, $0 \leq \mathcal{U}(w + \epsilon) - \mathcal{U}(w) \leq \epsilon$, and thus \mathcal{U} is 1-Lipschitz. A similar argument for \mathcal{V} completes the proof.

Fifth part: let us show that expression (3.3) holds. By applying point (v) of Lemma 1 twice, once with $a = -u$ and once $a = -v$, it follows respectively that $D_{\mathcal{F}}(0, v - u) = D_{\mathcal{F}}(u, v) - u$ and that $D_{\mathcal{F}}(u - v, 0) = D_{\mathcal{F}}(u, v) - v$. Hence, if (u, v, w) are solutions to (3.2), it follows that $u = -D_{\mathcal{F}}(0, -w)$, and thus $v = -D_{\mathcal{F}}(w, 0)$. Hence (3.3) holds. ■

B.3. Proof of Proposition 1.

Proof. Assume μ is an aggregate equilibrium matching. Then, by definition, there exists a pair of vectors U and V such that (μ, U, V) is an aggregate equilibrium outcome. Thus $D_{xy}(U_{xy}, V_{xy}) = 0$ for every $x \in \mathcal{X}$, $y \in \mathcal{Y}$, and $\mu_{xy} = \partial G(U) / \partial U_{xy}$ and $\mu_{xy} = \partial H(V) / \partial V_{xy}$, which inverts into

$$U_{xy} = \partial G^*(\mu) / \partial \mu_{xy} \text{ and } V_{xy} = \partial H^*(\mu) / \partial \mu_{xy}, \quad (\text{B.1})$$

and thus by substitution,

$$D_{xy}(\partial G^*(\mu) / \partial \mu_{xy}, \partial H^*(\mu) / \partial \mu_{xy}) = 0 \quad (\text{B.2})$$

holds for every $x \in \mathcal{X}$, $y \in \mathcal{Y}$. Conversely, assume (B.2) holds. Then, defining U and V by (B.1), one sees that (μ, U, V) is an aggregate equilibrium outcome. ■

B.4. Proof of Proposition 2.

Proof. Assume (μ, U, V) is an aggregate equilibrium outcome. Then $D_{xy}(U_{xy}, V_{xy}) = 0$ for every $x \in \mathcal{X}$, $y \in \mathcal{Y}$, and

$$\mu_{xy} = \partial G(U) / \partial U_{xy} = \partial H(V) / \partial V_{xy} \quad (\text{B.3})$$

thus, by Lemma 2, there exists a vector (W_{xy}) such that for every $x \in \mathcal{X}$, $y \in \mathcal{Y}$,

$$U_{xy} = \mathcal{U}_{xy}(W_{xy}) \text{ and } V_{xy} = \mathcal{V}_{xy}(W_{xy}), \quad (\text{B.4})$$

where \mathcal{U}_{xy} and \mathcal{V}_{xy} are defined in (3.3). Thus it follows that $Z(W) = 0$. Conversely, assume that $Z(W) = 0$. Then letting U and V as in (B.4), and μ such that $\mu_{xy} = \partial G(U) / \partial U_{xy}$, it follows that (μ, U, V) is an aggregate equilibrium outcome. ■

B.5. Proof of Proposition 3.

Proof. Recall that

$$Z_{x'y'}(W) = \frac{\partial H}{\partial V_{x'y'}}(\mathcal{V}(W)) - \frac{\partial G}{\partial U_{x'y'}}(\mathcal{U}(W))$$

and, because of Assumption 3, $\partial G / \partial U_{x'y'}(U)$ is increasing in $U_{x'y'}$, decreasing in U_{xy} if either of the conditions $x = x'$ or $y = y'$ holds (but not both), a similar conditions holds for H , and $W_{xy} \rightarrow \mathcal{V}_{xy}(W_{xy})$ is nonincreasing, while $W_{xy} \rightarrow \mathcal{U}_{xy}(W_{xy})$ is nondecreasing. At

the same time, $\mathcal{U}_{xy}(W_{xy}) - \mathcal{V}_{xy}(W_{xy}) = W_{xy}$, so \mathcal{U}_{xy} and \mathcal{V}_{xy} cannot be stationary at the same point W_{xy} .

Proof of (a.1): One has $Z_{x'y'}(W) = \partial H / \partial V_{x'y'}(\mathcal{V}(W)) - \partial G / \partial U_{x'y'}(\mathcal{U}(W))$, thus the map $W_{x'y'} \rightarrow Z_{x'y'}(W)$ is nonincreasing. At the same time, as $\partial G / \partial U_{x'y'}(U)$ is increasing in $U_{x'y'}$ and $\partial H / \partial V_{x'y'}(V)$ is increasing in $V_{x'y'}$ and as \mathcal{U}_{xy} and \mathcal{V}_{xy} cannot be stationary at the same point W_{xy} , it follows that $W_{x'y'} \rightarrow Z_{x'y'}(W)$ is decreasing.

Proof of (a.2): The proof is based on the same logic as above.

Proof of (a.3): When $x \neq x'$ and $y \neq y'$, then the quantity $\partial H / \partial V_{x'y'}(\mathcal{V}(W))$ does not depend on W_{xy} and nor does $\partial G / \partial U_{x'y'}(\mathcal{U}(W))$. Thus $Z_{x'y'}(W)$ does not depend on W_{xy} .

Proof of (b): One has

$$\sum_{x' \in \mathcal{X}, y' \in \mathcal{Y}} Z_{x'y'}(W) = \sum_{y'} m_{y'} - \sum_{x'} n_{x'} + \sum_{x'} \mu_{x'0}(\mathcal{U}(W)) - \sum_{y'} \mu_{0y'}(\mathcal{V}(W))$$

where $\mu_{x'0}(U)$ is defined as $n_{x'} - \sum_{y' \in \mathcal{Y}} \partial G(U) / \partial U_{x'y'}$, and $\mu_{0y'}(V)$ is defined as $m_{y'} - \sum_{x' \in \mathcal{X}} \partial H(V) / \partial V_{x'y'}$. But it is easy to check that $\mu_{x'0}(U) = n_{x'} \Pr(\varepsilon_{i0} > \max_{y' \in \mathcal{Y}} \{U_{x'y'} + \varepsilon_{iy'}\})$, thus $\mu_{x'0}(U)$ is decreasing with respect to all the entries of vector $U_{x'y'}$, $y' \in \mathcal{Y}$. A similar logic applies to show that $\mu_{0y'}(V)$ is decreasing with respect to all the entries of vector $V_{x'y'}$, $x' \in \mathcal{X}$. Hence, $\sum_{x' \in \mathcal{X}, y' \in \mathcal{Y}} Z_{x'y'}(W)$ is decreasing with respect to any entry of the vector W .

Remark: Conditions (a.1)–(a.3) express that $-Z$ is a Z-function, while conditions (a) and (b) together imply that $-Z$ is a M-function. See [48]. ■

B.6. Proof of Theorem 1. The existence part of Theorem 1 is constructive, and consists in showing that Algorithm 1 converges to a solution of equations (4.6); this convergence in turns follows from two claims, which are rather classical but included here for completeness. The uniqueness part relies on the fact that, by a result of Berry et al. (2013), the Gross Substitute property established in Proposition 3 implies that the excess demand function Z is inverse antitone, thus injective.

We show that:

Claim 1. *There exist two vectors w^l and w^u such that $w^l \leq w^u$ and*

$$Z(w^u) \leq 0 \leq Z(w^l).$$

Proof. By Assumption 2 (iii), for each $x \in \mathcal{X}$, either all the men's payoffs U_{xy} are bounded above or they all converge to $+\infty$. Let \mathcal{X}_1 be the set of $x \in \mathcal{X}$ such that for each $y \in \mathcal{Y}$, $\mathcal{U}_{xy}(w_{xy})$ all converge to $+\infty$ as $w_{xy} \rightarrow \bar{w}_{xy}$. For $x \in \mathcal{X}_1$, let $p_y = n_x(1 - 1/k) / |\mathcal{Y}|$, and let $U_{xy}^k = \partial G^* / \partial \mu_{xy}(p)$. It is easy to see that $U_{xy}^k \rightarrow +\infty$, thus $V_{xy}^k \rightarrow -\infty$. Hence there exists w_{xy} such that $\mathcal{U}_{xy}(w_{xy}) = U_{xy}^k$ and $V_{xy}^k = \mathcal{V}_{xy}(w^k)$. Now for $x \notin \mathcal{X}_1$, then for each $y \in \mathcal{Y}$, \bar{w}_{xy} is finite, and $\mathcal{U}_{xy}(w_{xy})$ all converge to a finite value $\bar{U}_{xy} \in \mathbb{R}$. Then, let $U_{xy}^k = \mathcal{U}_{xy}(\bar{w}_{xy} - 1/k)$ and $V_{xy}^k = \mathcal{V}_{xy}(\bar{w}_{xy} - 1/k)$, so that $V_{xy}^k \rightarrow -\infty$ and $U_{xy}^k \rightarrow \bar{U}_{xy} \in \mathbb{R}$. We have thus constructed vectors w^k such that $w_{xy}^k \rightarrow \bar{w}_{xy}$ for all x and y , and $\mathcal{V}_{xy}(w^k) \rightarrow -\infty$, while $\mathcal{U}_{xy}(w^k)$ converges to a vector of positive numbers. Thus, for k large enough, setting $w^u = w^k$ implies $Z(w^u) \leq 0$. A similar logic implies that there exists w^l such that $Z(w^l) \geq 0$. ■

Claim 2. *Z is inverse antitone: if $Z(w) \leq Z(w')$ for some two vectors w and w' , then $w \geq w'$.*

Proof. We show that $-Z$ satisfies the assumptions in Berry et al. (2013), Theorem 1, see also related results in Moré (1972), Theorem 3.3. We verify the three assumptions in Berry et al. (2013). Assumption 1 in that paper is met because Z is defined on the Cartesian product of the intervals (w_{xy}, \bar{w}_{xy}) . Next, by part (a.2) of Proposition 3 above, $-Z_{xy}(w)$ is weakly decreasing in $w_{x'y'}$ for $x'y' \neq xy$, and letting

$$Z_0(w) = \sum_{y'} m_{y'} - \sum_{x'} n_{x'} - \sum_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} Z_{xy}(w),$$

it follows from part (b) of Proposition 3 above that $-Z_0$ is strictly decreasing in all the w_{xy} . Thus Assumption 2 and 3 in Berry et al. (2013) are also satisfied, hence $-Z$ is inverse isotone, Z is inverse antitone. ■

With these preparations, a proof of Theorem 1 can be provided.

Proof of Theorem 1. We prove existence first, then uniqueness.

Proof of existence: It is easy to see that Z is continuous, and by the results of Proposition 3, it is strictly diagonally antitone, and off-diagonally isotone. Existence follows from Theorem 3.1 in Rheinboldt (1970) jointly with Proposition 3 and Claim 1. The proof there is based on a constructive argument based on nonlinear Gauss-Seidel iterations, as discussed in Section 5.2.1.

Proof of uniqueness: As noted in Berry et al. (2013), uniqueness follows from Claim 2 as in Corollary 1. Indeed, assume $Z(w) = Z(w')$. Then, by Claim 2, both inequalities $w \geq w'$ and $w' \geq w$ hold, and thus $w = w'$. ■

B.7. Proof of Corollary 1.

Proof. This corollary directly follows from a combination of Proposition 2 and Theorem 1. ■

B.8. Proof of Theorem 2.

Proof. By combining Theorem 1 with Proposition 1, it follows that Equation (4.6), namely

$$D_{xy} \left(\frac{\partial G^*}{\partial \mu_{xy}}(\mu), \frac{\partial H^*}{\partial \mu_{xy}}(\mu) \right) = 0$$

has a unique solution. But when Assumption 3 is strengthened into Assumption 3', then

$$\partial G^* / \partial \mu_{xy}(\mu) = \log(\mu_{xy} / \mu_{x0}) \quad \text{and} \quad \partial H^* / \partial \mu_{xy}(\mu) = \log(\mu_{xy} / \mu_{0y})$$

where $\mu_{x0} = n_x - \sum_{y \in \mathcal{Y}} \mu_{xy}$ and $\mu_{0y} = m_y - \sum_{x \in \mathcal{X}} \mu_{xy}$. Hence Equation (4.6) rewrites as

$$\begin{cases} D_{xy}(\log \mu_{xy} - \log \mu_{x0}, \log \mu_{xy} - \log \mu_{0y}) = 0 \\ \mu_{x0} + \sum_{y \in \mathcal{Y}} \mu_{xy} = n_x \\ \mu_{0y} + \sum_{x \in \mathcal{X}} \mu_{xy} = m_y \end{cases} \quad (\text{B.5})$$

but $D_{xy}(\log \mu_{xy} - \log \mu_{x0}, \log \mu_{xy} - \log \mu_{0y}) = \log \mu_{xy} + D_{xy}(-\log \mu_{x0}, -\log \mu_{0y})$, thus system (B.5) rewrites as system (6.2). Conversely, assume (μ_{x0}, μ_{0y}) satisfy the system (6.2). Then, letting $\mu_{xy} = M_{xy}(\mu_{x0}, \mu_{0y})$, $U_{xy} = \log(\mu_{xy} / \mu_{x0})$ and $V_{xy} = \log(\mu_{xy} / \mu_{0y})$, one has \mathcal{M}^{int} , $D_{xy}(U_{xy}, V_{xy}) = 0$ and $U_{xy} = \log(\mu_{xy} / \mu_{x0})$ and $V_{xy} = \log(\mu_{xy} / \mu_{0y})$, thus (μ, U, V) is an aggregate equilibrium outcome. ■

B.9. Proof of Theorem 3.

Proof. The proof of Theorem 2 is based on the following set of properties of $M_{xy}(\mu_{x0}, \mu_{0y}) = \exp(-D_{xy}(-\log \mu_{x0}, -\log \mu_{0y}))$, which are direct consequences of Definition 1 and of Lemma 1. For every pair $x \in \mathcal{X}$, $y \in \mathcal{Y}$:

(i) Map $M_{xy} : (a, b) \mapsto M_{xy}(a, b)$ is continuous.

(ii) Map $M_{xy} : (a, b) \mapsto M_{xy}(a, b)$ is weakly isotone, i.e. if $a \leq a'$ and $b \leq b'$, then $M_{xy}(a, b) \leq M_{xy}(a', b')$.

(iii) For each $a > 0$, $\lim_{b \rightarrow 0^+} M_{xy}(a, b) = 0$, and for each $b > 0$, $\lim_{a \rightarrow 0^+} M_{xy}(a, b) = 0$.

Given these properties, the existence of a solution (μ_{x0}, μ_{0y}) is essentially an application of Tarski's fixed point theorem; we provide an explicit proof for concreteness. We show that the construction of μ_{x0}^{2t+1} and μ_{0y}^{2t+2} at each step is well defined. Consider step $2t + 1$. For each $x \in \mathcal{X}$, the equation to solve is

$$\sum_{y \in \mathcal{Y}} M_{xy}(\mu_{x0}, \mu_{0y}) + \mu_{x0} = n_x$$

but the right-hand side is a continuous and increasing function of μ_{x0} , tends to 0 when $\mu_{x0} \rightarrow 0$ and tends to $+\infty$ when $\mu_{x0} \rightarrow +\infty$. Hence μ_{x0}^{2t+1} is well defined and belongs in $(0, +\infty)$. Denoting

$$\mu_{x0}^{2t+1} = \mathfrak{F}_x(\mu_{0.}^{2t}),$$

we see that \mathfrak{F} is antitone, meaning that $\mu_{0y}^{2t} \leq \tilde{\mu}_{0y}^{2t}$ for all $y \in \mathcal{Y}$ implies $\mathfrak{F}_x(\tilde{\mu}_{0.}^{2t}) \leq \mathfrak{F}_x(\mu_{0.}^{2t})$ for all $x \in \mathcal{X}$. By the same token, at step $2t + 2$, μ_{0y}^{2t+2} is well defined in $(0, +\infty)$, and we can denote

$$\mu_{0y}^{2t+2} = \mathfrak{G}_y(\mu_{0.}^{2t+1})$$

where, similarly, \mathfrak{G} is antitone. Thus, $\mu_{0.}^{2t+2} = \mathfrak{G} \circ \mathfrak{F}(\mu_{0.}^{2t})$, where $\mathfrak{G} \circ \mathfrak{F}$ is isotone. But $\mu_{0y}^2 \leq m_y = \mu_{0y}^0$ implies that $\mu_{0.}^{2t+2} \leq \mathfrak{G} \circ \mathfrak{F}(\mu_{0.}^{2t})$. Hence $(\mu_{0.}^{2t+2})_{t \in \mathbb{N}}$ is a decreasing sequence, bounded from below by 0. As a result $\mu_{0.}^{2t+2}$ converges. Letting $\bar{\mu}_{0.}$ be its limit, and letting $\bar{\mu}_{0.} = \mathfrak{F}(\bar{\mu}_{0.})$, it is not hard to see that $(\bar{\mu}_{0x}, \bar{\mu}_{0y})$ is a solution to (6.2). ■

B.10. Proof of Theorem 4.

Proof. Rearranging Expression (6.7) yields:

$$l(\theta) = 2 \sum_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} \hat{\mu}_{xy} \log \mu_{xy}^\theta + \sum_{x \in \mathcal{X}} \hat{\mu}_{x0} \log \mu_{x0}^\theta + \sum_{y \in \mathcal{Y}} \hat{\mu}_{0y} \log \mu_{0y}^\theta,$$

but $\log \mu_{xy}^\theta = -D_{xy}(-\log \mu_{x0}^\theta, -\log \mu_{0y}^\theta)$, thus letting $u_x^\theta = -\log \mu_{x0}^\theta$ and $v_y^\theta = -\log \mu_{0y}^\theta$ yields

$$-l(\theta) = 2 \sum_{\substack{x \in \mathcal{X} \\ y \in \mathcal{Y}}} \hat{\mu}_{xy} D_{xy}(u_x^\theta, v_y^\theta) + \sum_{x \in \mathcal{X}} \hat{\mu}_{x0} u_x^\theta + \sum_{y \in \mathcal{Y}} \hat{\mu}_{0y} v_y^\theta.$$

■

B.11. Proof of Theorem 5.

Proof. Proof of part (i): Let (μ, U, V) be an aggregate equilibrium matching, and let u_i and v_j as in (A.1). By definition of these quantities, one has $u_i - \varepsilon_{iy} \geq U_{xy}$ and $v_j - \eta_{xj} \geq V_{xy}$, thus $D_{xy}(u_i - \varepsilon_{iy}, v_j - \eta_{xj}) \geq D_{xy}(U_{xy}, V_{xy}) = 0$. Further, $u_i \geq \varepsilon_{i0}$ and $v_j \geq \eta_{0j}$, hence the stability condition holds. Let us show that one can construct μ_{ij} so that (μ_{ij}, u_i, v_j) is feasible. For $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, let \mathcal{I}_{xy} be the set of $i \in \mathcal{I}$ such that $x_i = x$ and $y = \arg \max_{y \in \mathcal{Y}_0} \{U_{xy} + \varepsilon_{iy}, \varepsilon_{i0}\}$. Similarly, let \mathcal{J}_{xy} be the set of $j \in \mathcal{J}$ such that $y_j = y$ and $x = \arg \max_{x \in \mathcal{X}_0} \{V_{xy} + \eta_{xj}, \eta_{0j}\}$. The mass of \mathcal{I}_{xy} is $\partial G(U) / \partial U_{xy}$ and the mass of \mathcal{J}_{xy} is $\partial H(V) / \partial V_{xy}$. The equilibrium condition $\mu = \nabla G(U) = \nabla H(V)$ implies therefore that the mass of \mathcal{I}_{xy} and the mass of \mathcal{J}_{xy} coincide. One can therefore take any assignment of men in \mathcal{I}_{xy} to women in \mathcal{J}_{xy} . Let μ_{ij} be the resulting individual assignment. If $\mu_{ij} > 0$, then $i \in \mathcal{I}_{x_i y_j}$ and $j \in \mathcal{J}_{x_i y_j}$, therefore $u_i = U_{xy} + \varepsilon_{iy}$ and $v_j = V_{xy} + \eta_{xj}$, thus $D_{xy}(u_i - \varepsilon_{iy}, v_j - \eta_{xj}) = D_{xy}(U_{xy}, V_{xy}) = 0$. Assume i is unassigned under (μ_{ij}) ; then for all $y \in \mathcal{Y}$, $u_i > U_{xy} + \varepsilon_{iy}$, and thus $u_i = \varepsilon_{i0}$. Similarly, if j is unassigned under (μ_{ij}) , then $v_j = \eta_{0j}$. Hence, (μ_{ij}, u_i, v_j) is an individual equilibrium.

Proof of part (ii): Now assume (μ_{ij}, u_i, v_j) is an individual equilibrium. Then for all i and j , the stability condition

$$D_{x_i y_j}(u_i - \varepsilon_{iy}, v_j - \eta_{jx}) \geq 0,$$

holds, and holds with equality if $\mu_{ij} > 0$. Hence, for all pairs x and y , we have the inequality

$$\min_{i:x_i=x} \min_{j:y_j=y} \{D_{x_i y_j}(u_i - \varepsilon_{iy}, v_j - \eta_{jx})\} \geq 0,$$

with equality if $\mu_{xy} > 0$, that is, if there is at least one marriage between a man of type x and a woman of type y . Taking U and V as (A.2), and making use of the strict monotonicity of D_{xy} in both its arguments, matching $\mu \in \mathcal{M}$ is an equilibrium matching if inequality $D_{xy}(U_{xy}, V_{xy}) \geq 0$ holds for any x and y , with equality if $\mu_{xy} > 0$. By definition of U and V , one has

$$u_i \geq \max_y \{U_{xy} + \varepsilon_{iy}, \varepsilon_{i0}\} \text{ and } v_j \geq \max_{x \in \mathcal{X}} \{V_{xyj} + \eta_{xj}, \eta_{0j}\}.$$

Assume one of these inequalities holds strict, for instance $u_i > \max_y \{U_{xy} + \varepsilon_{iy}, \varepsilon_{i0}\}$. Then $u_i - \varepsilon_{iy} > U_{xy}$. Because D was assumed strictly increasing, this implies that for all j

$$D_{xy}(u_i - \varepsilon_{iy}, v_j - \eta_{yj}) > D_{xy}(U_{xy}, v_j - \eta_{yj}) \geq D_{xy}(U_{xy}, V_{xy}) \geq 0$$

thus for all j , $\mu_{ij} = 0$. Therefore i is single, but $u_i > \varepsilon_{i0}$ yields a contradiction. Now Assumption 3 implies $\mu_{xy} > 0$ for all x and y , thus $D_{xy}(U_{xy}, V_{xy}) = 0$. ■

B.12. Proof of Corollary 2.

Proof. Let (μ_{ij}, u_i, v_j) be an individual outcome. By part (ii) of Theorem 5, the aggregate outcome (μ, U, V) is such that

$$U_{xy} = \min_{i:x_i=x} \{u_i - \varepsilon_{iy}\} \text{ and } V_{xy} = \min_{j:y_j=y} \{v_j - \eta_{jx}\},$$

hence $U_{xy} \geq u_i - \varepsilon_{iy}$ and $V_{xy} \geq v_j - \eta_{jx}$, but $D_{xy}(U_{xy}, V_{xy}) = 0$ and $D_{xy}(u_i - \varepsilon_{iy}, v_j - \eta_{jx}) = 0$, thus, by Assumption 2', $U_{xy} = u_i - \varepsilon_{iy}$ and $V_{xy} = v_j - \eta_{jx}$. Hence $u_i = U_{xy} + \varepsilon_{iy}$ and $v_j = V_{xy} + \eta_{jx}$. ■

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