Nonparametric Identification of k-Double Auctions using Price Data*

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Abstract

This paper studies the model identification problem of k-double auctions between one buyer and one seller when the transaction price, rather than the traders' bids, can be observed. Given only the price data is available, I explore an identification strategy that utilizes the double auctions with extreme pricing weight ($k = 1$ or 0) and exclusive covariates that shift only one trader's value distribution to identify both the buyer's and the seller's value distributions nonparametrically. First, I show that because each exclusive covariate can take at least two values, the buyer's and the seller's value distributions are partially identified from the price distribution for $k = 1$ or $k = 0$. The the identified set is sharp and can be easily computed. I provide a set of sufficient conditions under which the traders' value distributions are point identified. Second, when the exclusive covariates are continuous, I show that the buyer's and the seller's value distributions will be uniquely determined by a partial differential equation that only depends on the price distribution, provided that the value distributions are known for at least one value of the exclusive covariates.

Keywords: Double auctions, bargaining, nonparametric identification.

JEL Classification: C14, C57, C78, D44, D82

1 Introduction

Double auctions are one of the most common exchange institutions. They permit both offers to buy and offers to sell and usually set the transaction price according to traders' offers from both sides.

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They are extensively used in many field markets such as stock markets and commodity markets.

Recent theoretical studies on double auctions within a game theoretic framework, e.g. Chatterjee and Samuelson (1983), Myerson and Satterthwaite (1983), Leininger, Linhart, and Radner (1989), Satterthwaite and Williams (1989, 2002), Kadan (2007), provide researchers and practitioners with insights about the behavioral theory of double auction markets. Meanwhile, there has been a large body of work which examines identification and estimation of one-sided auctions. Examples includes Laffont, Ossard, and Vuong (1995), Laffont and Vuong (1996), Donald and Paarsch (1996), Guerre, Perrigne, and Vuong (2000, 2009), Li, Perrigne, and Vuong (2000, 2002), Athey and Haile (2002), Haile, Hong, and Shum (2003), Haile and Tamer (2003), Hendricks, Pinkse, and Porter (2003), McAdams (2008), Li and Zheng (2009), An, Hu, and Shum (2010), Krasnokutskaya (2011), Tang (2011), Marmer and Shneyerov (2012), Hubbard, Li, and Paarsch (2012), Hu, McAdams, and Shum (2013), Gentry and Li (2014). However, in contrast to the intensive investigation of one-sided auctions, the empirical analysis of double auction models is still in its infancy.

Motivated by the lack of identification and estimation results for double auction model, I study a simple yet important type of double auctions called k-double auctions with a single buyer and a single seller, which employs a pricing rule that takes the weighted average of the two traders' offers as the transaction price. Such a double auction model is closely related to the structural analysis of noncooperative bargaining models with incomplete information (see, e.g. Sieg, 2000; Watanabe, 2006; Merlo, Ortalo-Magne, and Rust, 2015) and has a wide range of applications including negotiations, dispute settlements, and real estate sales. Previous work (Li and Liu, 2015), within the independent private value paradigm, obtained theoretical results for nonparametric identification and estimation of the buyer's and the seller's value distributions if their bids are observed, both in the case in which bids are observed for all double auctions and when they are observed only in double auctions where transactions take place. In this paper, I pursue the problem of nonparametrically identifying both traders' value distributions if I only observe the transaction price in each double auction with a transaction. It is common in many applications that the researchers can only access limited sets of observables for reasons such as the design of the trading mechanism. For markets governed by the double auction institution, the transaction prices rather than the traders' offers are usually more readily available. Limiting the observables to the transaction price makes identification more appealing but also more difficult. A similar problem has been addressed for one-sided auctions, see, e.g. Athey and Haile (2002), Adams (2007). However, the transaction price in double auctions depends on both the buyer's and the seller's strategic offers at the same time. This creates greater challenges to identify the traders' value distributions from the price data alone.

As part of my research, I focus on the case in which the pricing weight $k = 1$ or $k = 0$, noting that my results can be readily extended to the situations in which k is observed and equals one or zero some of the time. Based on the distribution of the transaction price in double auctions with $k = 1$

or $k = 0$, under mild assumptions, I give the sharp bound for the identified value distributions of the buyer and the seller on part of their supports, provided that there exist exogenous value distribution shifters Y for the buyer and Z for the seller, while point identification can be reached under stronger conditions. I further establish identification of the two value distributions when the value distribution shifters are continuously distributed.

The rest of this paper proceeds as follows. Section 2 presents the k-double auction model. In Section 3, I exploit exclusion restrictions to achieve nonparametric identification of the buyer's and the seller's private value distributions, mainly in the case where the pricing weight on the buyer's bid is equal to 1. Section 4 concludes with a discussion of possible estimation approaches. The supplementary results and the proofs are collected in the appendix.

2 The Sealed-Bid k-Double Auction Model

Consider a sealed-bid k-double auction where a single indivisible good is traded between a buyer and a seller. The value of the good to the buyer is V and the reservation value to the seller is C. Both traders are risk neutral expected utility maximizers. In the auction, both the buyer and the seller simultaneously submit sealed bids B and S, respectively. If $B \geqslant S$, the transaction is struck at price $P = kB + (1-k)S$ where $0 \leqslant k \leqslant 1$. The seller's utility is $P - C$ and the buyer's utility is $V - P$. If $B < S$, there is no transaction occurring and each gets zero utility. Each trader knows his own private value and observes some auction-specific covariates X. However, he only knows his adversary's value is drawn from a certain distribution. The joint distribution of these random variables and the pricing rule (including the pricing weight k) are all common knowledge between the buyer and the seller.

I impose the following assumption on the traders' value distributions.

Assumption A (Independent Private Value)**.**

(i) V and C are independent conditional on X;

(ii) The conditional distributions of V and C given $X = x$, $F_V(\cdot \mid x)$ and $F_C(\cdot \mid x)$, are absolutely continuous with densities $f_V(\cdot \mid x)$ and $f_C(\cdot \mid x)$ on the same support $[\underline{c}(x), \overline{v}(x)] \subset \mathbb{R}_+$.

Assumption A requires that both traders' values are conditionally independent and drawn from absolutely continuous distributions which share the same bounded supports. It allows unconditional correlation between V and C as long as the auction-specific covariates X account for all the dependence structure.

Denote by $\beta_B(\cdot, x) : [\underline{c}(x), \overline{v}(x)] \to \mathbb{R}_+$ and $\beta_S(\cdot, x) : [\underline{c}(x), \overline{v}(x)] \to \mathbb{R}_+$ the respective strategies of the buyer and the seller. The Bayesian Nash equilibrium (BNE) concept is adopted throughout. However, Leininger, Linhart, and Radner (1989) and Satterthwaite and Williams (1989) showed that there can exist multiple BNE's in a given k-double auction. To exclude some irregular cases and focus on a certain class of equilibria which are well-behaved as described in Chatterjee and Samuelson (1983), the following restrictions are imposed on the equilibrium under consideration here:

Assumption B (Regular equilibrium)**.** *The equilibrium strategy profile (β_B, β_S) satisfies, for any x,*

(i) $\beta_B(\cdot, x)$ and $\beta_S(\cdot, x)$ are continuous and strictly increasing;

(ii) for traders who have positive probability of trade under the strategy profile, $\beta_B(\cdot, x)$ and $\beta_S(\cdot, x)$ are differentiable;

(iii) for traders who have zero probability of trade under the strategy profile, $\beta_B(v, x) = v$ and $\beta_S(c, x) = c$.

An equilibrium is called "regular" if it satisfies Assumption B. Assumption B restricts me to the equilibria with strictly monotonic and (piecewise) differentiable strategies. Here the equilibrium strategies can depend on covariates X in two ways. First, the equilibrium strategies will change as the model primitives such as the value distributions or the pricing weight k vary. Second, when the k-double auction with given model primitives has multiple regular equilibria, the equilibrium strategies can differ as the covariates X affect the equilibrium selection.

Let $G_{Bk}(\cdot \mid x)$ and $G_{Sk}(\cdot \mid x)$ denote the respective distributions of buyer's and seller's equilibrium bids conditional on $X = x$, which are induced by the value distributions and some equilibrium strategy profile (β_{Bk}, β_{Sk}). The k in the subscript is used to indicate the dependence of these functions on the pricing weight k. Since the regular equilibrium strategies are strictly increasing, it follows that $F_V(v \mid x) = G_{Bk}(\beta_{Bk}(v, x) \mid x)$, $F_C(c \mid x) = G_{Sk}(\beta_{Sk}(c, x) \mid x)$, and the respective supports of G_{Bk} and G_{Sk} are given by $[\underline{b}_k(x), \overline{b}_k(x)] = [\beta_{Bk}(\underline{c}(x), x), \beta_{Bk}(\overline{v}(x), x)]$ and $[\underline{s}_k(x), \overline{s}_k(x)] = [\beta_{Sk}(\underline{c}(x), x), \beta_{Sk}(\overline{v}(x), x)]$. As shown in Li and Liu (2015), the regular equilibrium bids of the buyer and of the seller are independent conditional on X and their supports should satisfy $\underline{b}_k(x) \leqslant \underline{s}_k(x) < \overline{b}_k(x) \leqslant \overline{s}_k(x)$. Because the transaction price P is defined only when the buyer's bid is greater than the seller's bid, the support of P is $[\underline{s}_k(x), \overline{b}_k(x)]$. Therefore, by the conditional independence of B and S, the density function of the transaction price is

$$h_k(p \mid x) = a_k(x) \int_0^{T_k(p,x)} g_{Bk}(p + (1-k)t \mid x) g_{Sk}(p - kt \mid x)\, \mathrm{d}t, \tag{2.1}$$

where $g_{Bk}(\cdot \mid x)$ and $g_{Sk}(\cdot \mid x)$ are the corresponding densities of $G_{Bk}(\cdot \mid x)$ and $G_{Bk}(\cdot \mid x)$, $a_k(x)$ is a constant which makes $\int_{\underline{s}_k(x)}^{\overline{b}_k(x)} h_k(p \mid x)\, \mathrm{d}p = 1$, and the upper limit of the integral is $T_k(p, x) = \min\left(\frac{\overline{b}_k(x) - p}{1-k}, \frac{p - \underline{s}_k(x)}{k}\right)$.

3 Nonparametric Identification

In this section, I study the identification of the buyer's and the seller's conditional private value distributions, $F_V(\cdot \mid x)$ and $F_C(\cdot \mid x)$. In contrast to Li and Liu (2015), here I assume the econometricians have less information about the traders' behavior—rather than the bids of the buyer and the seller, they can only observe the final transaction price P, auction-specific covariates X and the pricing rule parameter k.

Li and Liu (2015) showed that given that the traders' private values are independent, the value distributions are nonparametrically identified as long as the econometricians can observe the bids of the buyer and the seller (at least those bids with a successful transaction). So a way of identifying the value distributions is using (2.1) to recover the bid distributions from the price distribution then apply the conclusion of Li and Liu (2015). However, this approach faces challenges.

First, the difficulty comes from the fact that the price distribution is obtained by projecting the joint bid distribution in a certain direction so the price distribution compresses the information of both the buyer's and the seller's bid distributions. It is usually impossible to recover a two-dimensional bid distribution from a one-dimensional price distribution, even if both traders' bids are independent (in this case, it aims to recover two one-dimensional bid distributions).

Second, although the price distribution can be treated as a weighted mixture of the two bid distributions because $P = kB + (1-k)S$, the methods that are typically used to identify the component distributions in finite mixture models will not be applicable due to special features of double auction models. In double auctions, the transaction price is defined only when the buyer's bid is greater than the seller's bid. This means that the price distribution is obtained from a truncated bid distribution. The original independence between the buyer's and the seller's bids breaks down because of the truncation. Therefore, the deconvolution method that is usually used to decompose the mixture distribution does no longer work without the independence condition. In addition, the parameter k does not only play a role as a weight used to calculate the price in double auction transaction and thereby a mixture weight, but it also directly determines the buyer's and the seller's equilibrium bidding strategies and therefore the equilibrium bid distributions. So any changes in the value of k will inevitably change the two bid distributions. As a result, the method employed in many studies about finite mixture models, which rely on the existence of a variable that shifts the mixture weight without affecting the component distributions, cannot be used in the double auction case.

In view of these difficulties, I explore a different identification strategy based on extreme values of pricing weight and exclusion restrictions to identify the buyer's and the seller's conditional private value distributions nonparametrically.

To start, I posit the following assumption on the observed pricing rule parameter k.

Assumption C. *The pricing weight $k = 0$ or $k = 1$ with positive probability.*

This assumption requires that the econometricians are able to observe the double auctions in which one of the two traders has full bargaining power to unilaterally set the final transaction price once a successful transaction takes place. This condition is satisfied in many applications, for example, either the buyer or the seller can propose a take-it-or-leave-it offer.

The pricing weight k taking the extreme values brings several benefits. As shown by Satterthwaite and Williams (1989), when $k = 0$ or $k = 1$, the k-double auction game has a unique regular equilibrium, so specifying an equilibrium selection mechanism can be avoided. Meanwhile, in these two extreme cases, the strategies in that equilibrium have closed-form expressions, which allows me to interpret the corresponding equilibrium bid distributions as well as the distribution of transaction price in terms of the value distributions and then establish a direct connection between the observables and the model primitives of interest. Precisely, according to Satterthwaite and Williams (1989): When $k = 1$, the seller will choose the weakly dominant strategy of bidding his private value truthfully, therefore the seller's equilibrium inverse bidding function is $\beta_{S1}^{-1}(s, x) = s$ and the buyer's equilibrium inverse bidding function is given by

$$\beta_{B1}^{-1}(b, x) = b + \mathbb{1}(b \geqslant \underline{s}_1(x)) \cdot \lambda(b, x),$$

where $\lambda(\cdot, x) = F_C(\cdot \mid x)/f_C(\cdot \mid x)$. Then by (2.1), the density function of the transaction price when $k = 1$ is

$$\begin{aligned} h_1(p \mid x) &= a_1(x) \int_0^{p-\underline{s}_1(x)} g_{B1}(p \mid x) g_{S1}(p - t \mid x) \, \mathrm{d}t \\ &= a_1(x) g_{B1}(p \mid x) G_{S1}(p \mid x) \\ &= a_1(x) F_C(p \mid x) f_V(p + \lambda(p, x) \mid x) \, [1 + \partial_1 \lambda(p, x)] \end{aligned} \tag{3.1}$$

for $\underline{s}_1(x) \leqslant p \leqslant \overline{b}_1(x)$, where $\partial_1 \lambda$ denotes the partial derivative of λ with respect to the first argument. When $k = 0$, the buyer plays the truth-telling strategy in the unique regular equilibrium, so the buyer's equilibrium inverse bidding function is $\beta_{B0}^{-1}(b, x) = b$ and the seller's equilibrium inverse bidding function is

$$\beta_{S0}^{-1}(s, x) = s - \mathbb{1}(s \leqslant \overline{b}_0(x)) \cdot \delta(s, x),$$

where $\delta(\cdot, x) = [1 - F_V(\cdot \mid x)]/f_C(\cdot \mid x)$. As a result, the price density when $k = 0$ is

$$\begin{aligned} h_0(p \mid x) &= a_0(x) g_{S0}(p \mid x) \, [1 - G_{B0}(p \mid x)] \\ &= a_0(x) [1 - F_V(p \mid x)] f_C(p - \delta(p, x) \mid x) \, [1 - \partial_1 \delta(p, x)] \end{aligned} \tag{3.2}$$

for $\underline{s}_0(x) \leqslant p \leqslant \overline{b}_0(x)$. Here $\partial_1 \delta$ is the partial derivative of δ with respect to the first argument.

Another key identification restriction is a source of variation in one trader's value distribution that leaves the other trader's value distribution unchanged. Suppose that the observed covariates can be partitioned into three parts $X = (Y, Z, W)$ where $Y \in \mathbb{R}^{d_y}$, $Z \in \mathbb{R}^{d_z}$ and $W \in \mathbb{R}^{d_w}$, and suppose Y and Z only affect the value distribution of one trader. I assume the following exclusion restriction holds.

Assumption D (Exclusion Restriction)**.** *For any realization* $x = (y, z, w)$*:*

(i) $\underline{c}(y, z, w) = \underline{c}(w)$, $\overline{v}(y, z, w) = \overline{v}(w)$*;*

(ii) $F_V(v \mid y, z, w) = F_V(v \mid y, w)$ *and* $F_C(c \mid y, z, w) = F_C(c \mid z, w)$ *for any* $v, c \in [\underline{c}(w), \overline{v}(w)]$*.*

According to Assumption D, Y only affects the buyer's value distribution while Z only affects the seller's. However, the exclusive covariates Y and Z only change the shape but not the supports of the private value distributions.

For simplicity, w is dropped from the notation unless specifically stated otherwise; all quantities considered are implicitly functions of w. To illustrate the idea of the identification strategy, I will focus on identification for the $k = 1$ case. Symmetric results for the $k = 0$ case can be obtained by using similar assumptions and arguments (see Appendix A.1).

3.1 Identification with Binary-valued Z

Let $\mathcal{Y}$ and $\mathcal{Z}$ denote the respective supports of Y and Z. To guarantee the existence of a regular equilibrium when $k = 1$, i.e. to ensure the buyer has a strictly increasing and differentiable bidding strategy, I assume that the seller's conditional value distribution $F_C(\cdot \mid z)$ satisfies the following assumption.

Assumption E. *For any* $z \in \mathcal{Z}$, $\lambda(\underline{c}, z) = 0$*, and* $\lambda(\cdot, z)$ *is continuously differentiable with* $0 < \partial_1 \lambda(c, z) < \infty$ *for all* $c \in [\underline{c}, \overline{v}]$*.*

Such an admissibility condition, which requires that the seller's conditional value distribution admits a continuously differentiable and strictly decreasing reverse hazard rate, is usually imposed in the literature about one-sided auctions and double auctions (see Satterthwaite and Williams, 1989). Then it can be shown first that:

Lemma 1. *Under Assumptions A to E, if* $h_1(\cdot \mid y, z)$ *is the density of the transaction price in the regular equilibrium of a sealed-bid k-double auction with* $k = 1$ *for* $F_V(\cdot \mid y)$, $y \in \mathcal{Y}$ *and* $F_C(\cdot \mid z)$, $z \in \mathcal{Z}$*, then for any* $y \in \mathcal{Y}$ *and any* $z \in \mathcal{Z}$*,*

(i) $\underline{s}_1(y, z) = \underline{c}$*, and* $h_1(\underline{c} \mid y, z) = 0$*;*

(ii) $\overline{b}_1(y, z) = \overline{b}_1(z)$ *which solves* $\overline{b}_1(z) + \lambda(\overline{b}_1(z), z) = \overline{v}$*.*

Proof. See Appendix A.2. □

According to this lemma, when $k = 1$, all price distributions have identical lower endpoints of their support at $\underline{c}$, where the price distribution has zero density. It is also implied that the upper endpoint of the price distribution support only depends on the value of Z which affects the seller's value distribution. These properties are mainly attributed to the exclusion restrictions assumed in Assumption D.

Given this result, for $y^*, y^{**} \in \mathcal{Y}$ and $z \in \mathcal{Z}$, if I define[1]

$$\Gamma_1(p,z) \equiv \lim_{q \to \underline{c}} \left[\frac{h_1(p \,|\, y^*, z)}{h_1(q \,|\, y^*, z)} \Big/ \frac{h_1(p \,|\, y^{**}, z)}{h_1(q \,|\, y^{**}, z)} \right] \tag{3.3}$$

for $\underline{c} < p < \bar{b}_1(z)$ and let $\Gamma_1(\underline{c}, z) \equiv \lim_{p \to \underline{c}} \Gamma_1(p,z)$, $\Gamma_1(\bar{b}_1(z), z) \equiv \lim_{p \to \bar{b}_1(z)} \Gamma_1(p,z)$, then it follows from (3.1) that

$$\Gamma_1(p,z) = \frac{f_V(p + \lambda(p,z) \,|\, y^*)}{f_V(p + \lambda(p,z) \,|\, y^{**})} \cdot \frac{f_V(\underline{c} \,|\, y^{**})}{f_V(\underline{c} \,|\, y^*)}. \tag{3.4}$$

Because the function Γ_1 only depends on the price density h_1 by definition, (3.4) means that the likelihood ratio of the buyer's value distributions in the k-double auction model as specified in Section 2 is identified up to scale. Note that the buyer's unobservable private value can be inferred from the identified $\Gamma_1(\cdot, z)$ function if it is invertible. So in order to have invertibility, I assume that for some values of Y, the buyer's conditional value distribution possesses the monotone likelihood ratio property. Specifically,

Assumption F. *There exist $y^* \neq y^{**}$ in $\mathcal{Y}$ such that $f_V(\cdot \,|\, y^*)/f_V(\cdot \,|\, y^{**})$ is continuously differentiable with negative derivative on $[\underline{c}, \bar{v}]$.*

Therefore, for the values y^* and y^{**} of covariate Y such that Assumption F holds, I have:

Lemma 2. *Under Assumptions A to F, if $h_1(\cdot \,|\, y, z)$ is the density of the transaction price in the regular equilibrium of a sealed-bid k-double auction with $k = 1$ for $F_V(\cdot \,|\, y)$, $y \in \{y^*, y^{**}\}$ and $F_C(\cdot \,|\, z)$, $z \in \mathcal{Z}$, then*

(i) $\Gamma_1(\cdot, z)$ is continuously differentiable with negative derivative on $[\underline{c}, \bar{b}_1(z)]$ for any $z \in \mathcal{Z}$;

(ii) $\Gamma_1(\underline{c}, z^*) = \Gamma_1(\underline{c}, z^{**}) = 1$ and $\Gamma_1(\bar{b}_1(z^*), z^*) = \Gamma_1(\bar{b}_1(z^{**}), z^{**})$ for any $z^*, z^{**} \in \mathcal{Z}$.

Proof. See Appendix A.3. □

Lemma 2 shows some restrictions that the specified nonparametric double auction model imposes on the observable conditional distribution of the transaction price in terms of Γ_1. First, condition (i) mainly points out that the model implies a strictly decreasing $\Gamma_1(\cdot, z)$. This is mainly due to the monotonicity of the reverse hazard rate of the seller's value distribution and the monotonicity of the likelihood ratio of the buyer's value distribution. Together with the monotonicity, condition (ii) implies that the range of $\Gamma_1(\cdot, z)$, whose domain coincides with the support of transaction price, should keep constant as the value of the exclusive covariate for seller's value distribution changes.

[1] I take the limit as $q \to \underline{c}$ rather than directly take $q = \underline{c}$ because the price has zero density at $\underline{c}$ if the price is generated from the k-double auction specified in Section 2. Here y^*, y^{**} are regarded as fixed in order to simplify the notation.

I will start the discussion about identifying the traders' value distributions by showing a few important properties of the traders' conditional value distributions that can rationalize a given price data by the k-double auction model.

Suppose that for $y^* \neq y^{**}$ in $\mathcal{Y}$ and $z^* \neq z^{**}$ in $\mathcal{Z}$, the functions $\Gamma_1(\cdot, z^*)$ and $\Gamma_1(\cdot, z^{**})$ induced by the price densities $h_1(\cdot \mid y, z)$ with $y \in \{y^*, y^{**}\}$ and $z \in \{z^*, z^{**}\}$, satisfy the conditions of Lemma 2; that is, both $\Gamma_1(\cdot, z^*)$ and $\Gamma_1(\cdot, z^{**})$ are strictly decreasing and have the same range. Then, for any p in $\Gamma_1(\cdot, z^*)$'s domain $[\underline{c}, \bar{b}_1(z)]$, there is a unique $\psi(p) \in [\underline{c}, \bar{b}_1(z^{**})]$ such that[2]

$$\Gamma_1(p, z^*) = \Gamma_1(\psi(p), z^{**}). \tag{3.5}$$

By the implicit function theorem, $\psi(\cdot)$ defined above is continuously differentiable and strictly increasing and it satisfies $\psi(\underline{c}) = \underline{c}$. Suppose that the price data is generated by the k-double auction model, then under Assumptions A to E, by (3.4), equation (3.5) is equivalent to

$$p + \lambda(p, z^*) = \psi(p) + \lambda(\psi(p), z^{**}). \tag{3.6}$$

Because in a k-double auction with $k = 1$, when the seller has private value distribution $F_V(\cdot \mid z)$, the equilibrium inverse bidding function for the buyer is $b + \lambda(b, z)$ where b is the bid, (3.6) means that a buyer who bids $\psi(p)$ when facing a seller with value distribution $F_C(\cdot \mid z^{**})$, will have the same private value as a buyer who bids p when the seller's value distribution is $F_C(\cdot \mid z^*)$.

Moreover, (3.6) directly implies

$$f_V(p + \lambda(p, z^*) \mid y^*) = f_V(\psi(p) + \lambda(\psi(p), z^{**}) \mid y^*),$$

and differentiating both sides of (3.6) with respect to p yields

$$1 + \partial_1 \lambda(p, z^*) = [1 + \partial_1 \lambda(\psi(p), z^{**})] \cdot \psi'(p).$$

Then, by combining these two equations and (3.1), I have for any $p \in [\underline{c}, \bar{b}_1(z^*)]$,

$$\frac{F_C(\psi(p) \mid z^{**})}{F_C(p \mid z^*)} = \frac{a_1(y^*, z^*)}{a_1(y^*, z^{**})} \cdot \frac{h_1(\psi(p) \mid y^*, z^{**})}{h_1(p \mid y^*, z^*)} \psi'(p) = \frac{a_1(y^*, z^*)}{a_1(y^*, z^{**})} m(p), \tag{3.7}$$

where

$$m(p) \equiv \frac{h_1(\psi(p) \mid y^*, z^{**})}{h_1(p \mid y^*, z^*)} \psi'(p).$$

Equation (3.7) requires the ratio $F_C(\psi(\cdot) \mid z^{**}) / F_C(\cdot \mid z^*)$ to be proportional to the function $m(\cdot)$, which is determined by the price distributions, on the interval $(\underline{c}, \bar{b}_1(z^*)]$. It indeed imposes another

[2] In fact, $\psi(p) = \Gamma_1^{-1}(\Gamma_1(p, z^*), z^{**})$ where $\Gamma_1^{-1}(\cdot, z^{**})$ is the inverse function of $\Gamma_1(\cdot, z^{**})$ provided that $\Gamma_1(\cdot, z^{**})$ is strictly monotone.

restriction besides (3.6) on the seller's conditional value distribution that can rationalizes the observed price distribution. This is because by the exclusion restrictions, the same conditional value distribution for the buyer generates the price densities $h_1(\cdot \mid y^*, z^*)$ and $h_1(\cdot \mid y^*, z^{**})$ with the seller's conditional value distributions $F_C(\cdot \mid z^*)$ and $F_C(\cdot \mid z^{**})$, respectively. However, it follows from (3.1) that

$$F_V(v \mid y) = \frac{1}{a_1(y,z)} \int_{\underline{c}}^{b} \frac{h_1(u \mid y,z)}{F_C(u \mid z)} \, \mathrm{d}u,$$

where b solves $b + \lambda(b, z) = v$. It suggests that any conditional value distribution for the seller that satisfies Assumption E will automatically induce a conditional value distribution for the buyer,[3] namely

$$\tilde{F}_V(v \mid y, z) = \left[\int_{\underline{c}}^{\bar{b}_1(z)} \frac{h_1(u \mid y,z)}{F_C(u \mid z)} \, \mathrm{d}u \right]^{-1} \int_{\underline{c}}^{b} \frac{h_1(u \mid y,z)}{F_C(u \mid z)} \, \mathrm{d}u, \quad v \in [\underline{c}, \overline{v}], \tag{3.8}$$

and $\tilde{F}_V(v \mid y, z)$ rationalizes a given price density by a k-double auction with $k = 1$. So the implication of condition (3.7) is to make sure that, the buyer's conditional value distribution induced by $F_C(\cdot \mid z^*)$ and $h_1(\cdot \mid y^*, z^*)$ is the same as the one induced by $F_C(\cdot \mid z^{**})$ and $h_1(\cdot \mid y^*, z^{**})$, and therefore it does not depend on Z. Meanwhile, because any given seller's conditional value distribution induces an associated conditional value distribution for the buyer in the above way, when I try to identify both traders' value distributions, it suffices to consider only the one for the seller.

These restrictions are summarized by the following theorem.

Theorem 1. *Under Assumptions A to F, suppose $h_1(\cdot \mid y, z)$, $y \in \{y^*, y^{**}\}$, $z \in \{z^*, z^{**}\}$ satisfies the conditions of Lemmas 1 and 2, then $h_1(\cdot \mid y, z)$ is the density of the transaction price in the regular equilibrium of a sealed-bid k-double auction with $k = 1$ for some $F_V(\cdot \mid y)$, $y \in \{y^*, y^{**}\}$ and $F_C(\cdot \mid z)$, $z \in \{z^*, z^{**}\}$ if and only if $F_C(\cdot \mid z^*)$ and $F_C(\cdot \mid z^{**})$ satisfy equations (3.6) and (3.7).*

Proof. See Appendix A.4. □

Theorem 1 is important. On one hand, it gives necessary conditions for the private value distributions that can rationalize the given distribution of transaction price by showing that only those seller's conditional value distributions satisfying (3.6) and (3.7) are consistent with price data. These conditions characterize a bound for the identified set of the model primitives. On the other hand, it also shows that any seller's conditional value distributions such that Assumption E holds can rationalize the given price distribution and therefore are observationally equivalent, as long as they satisfy (3.6) and (3.7). This indicates that such a bound for the identified set is actually sharp.

The identified set with the bound imposed by (3.6) and (3.7) can be easily computed. To see

[3] By definition, such a conditional value distribution for the buyer which is induced by the price density and a given seller's conditional value distribution will depend not only the covariate Y but also the covariate Z.

that, first rewrite (3.7) as

$$F_C(\psi(p)\,|\,z^{**}) = \frac{a_1(y^*,z^*)}{a_1(y^*,z^{**})} \cdot F_C(p\,|\,z^*)m(p),$$

and then taking the derivative with respect to p yields

$$f_C(\psi(p)\,|\,z^{**}) = \frac{a_1(y^*,z^*)}{a_1(y^*,z^{**})} \cdot \frac{f_C(p\,|\,z^*)m(p) + F_C(p\,|\,z^*)m'(p)}{\psi'(p)}.$$

Plugging these two into (3.6) will give

$$m'(p)\,[\lambda(p,z^*)]^2 + [(p-\psi(p))m(p)]'\,\lambda(p,z^*) + (p-\psi(p))m(p) = 0 \tag{3.9}$$

for $p \in [\underline{c}, \overline{b}_1(z^*)]$. By the construction of (3.9), $F_C(\cdot\,|\,z^*)$ and $F_C(\cdot\,|\,z^{**})$ satisfy (3.6) and (3.7) if and only if $F_C(\cdot\,|\,z^*)$ satisfies (3.9) (and induces $F_C(\cdot\,|\,z^{**})$ according to (3.6) or (3.7)). This allows me to obtain the identified set by looking for the solution to (3.9). Since $\psi(\cdot)$ and $m(\cdot)$ are identified from the price distribution, (3.9) only serves to determine $\lambda(\cdot, z^*)$. Interestingly, for any fixed $p \in (\underline{c}, \overline{b}_1(z^*)]$, (3.9) is a quadratic equation whenever $m'(p) \neq 0$,[4] so it will have two real solutions

$$\lambda(p,z^*) = \frac{-[(p-\psi(p))m(p)]' \pm \sqrt{\{[(p-\psi(p))m(p)]'\}^2 - 4(p-\psi(p))m(p)m'(p)}}{2m'(p)} \tag{3.10}$$

provided that $\Delta_1(p) \equiv \{[(p-\psi(p))m(p)]'\}^2 - 4(p-\psi(p))m(p)m'(p) \geqslant 0$.Then, the identified set for $\lambda(\cdot, z^*)$, or equivalently for $F_C(\cdot\,|\,z^*)$, can be obtained by screening all functions that conform to the form of (3.10) with the conditions of Assumption E.

Furthermore, utilizing the quadratic feature of (3.9), I can find conditions on the observables, under which the identified set for the seller's conditional value distributions collapses to a singleton so that the model is point identified. The following theorem provides an example.

Theorem 2. *Under Assumptions A to F, suppose $h_1(\cdot\,|\,y,z)$, $y \in \{y^*, y^{**}\}$, $z \in \{z^*, z^{**}\}$ satisfies the conditions of Lemma 1 and Lemma 2, if in addition for all $p \in (\underline{c}, \overline{b}_1(z^*)]$,*

(i) $\Delta_1(p) \geqslant 0$,

(ii) $\psi(p) < p$, *and*

(iii) $m'(p) < 0$,

then $F_C(\cdot\,|\,z^*)$ is identified on $[\underline{c}, \overline{b}_1(z^*)]$ and $F_C(\cdot\,|\,z^{**})$ is identified on $[\underline{c}, \overline{b}_1(z^{**})]$.

Proof. See Appendix A.5. □

[4] If $m'(p) = 0$ but $[(p-\psi(p))m(p)]' \neq 0$, (3.9) will become a linear equation and has a unique solution $\lambda(p,z^*) = -(p-\psi(p))m(p)/[(p-\psi(p))m(p)]'$ which coincides with the limit of original quadratic equation's solution $\left\{-[(p-\psi(p))m(p)]' + \mathrm{sgn}\left([(p-\psi(p))m(p)]'\right)\sqrt{\Delta_1(p)}\right\}/[2m'(p)]$ as p approaches the zeros of $m'(\cdot)$.

Condition (i) in Theorem 2 is imposed to ensure the existence of a function $\lambda(\cdot, z^*)$ that satisfies equation (3.9). This is because if the observed prices come from the regular equilibrium of a k-double auction with $k = 1$ and conditional value distributions determined by Assumptions A, D, E and F, then $\lambda(\cdot, z^*)$ corresponding to the true conditional value distribution for the seller given $Z = z^*$ will be a solution to equation (3.9), so it must be true that $\Delta_1(p) \geqslant 0$. Meanwhile, the uniqueness of the identified value distribution for the seller is mainly due to conditions (ii) and (iii) in Theorem 2. The economic implication behind these two conditions can reveal some properties of the identified value distributions. First, recall that p and $\psi(p)$ stand for the equilibrium bids of a buyer with private value $v = p + \lambda(p, z^*) = \psi(p) + \lambda(\psi(p), z^{**})$ in a k-double auction with $k = 1$ against seller with value distributions $F_C(\cdot \mid z^*)$ and $F_C(\cdot \mid z^{**})$, respectively. So $\psi(p) < p$ means that, the seller's conditional value distributions $F_C(\cdot \mid z^*)$ and $F_C(\cdot \mid z^{**})$ are such that the same buyer, regardless of the realization of his own private value, will always bid a higher price when the seller's conditional value distribution is $F_C(\cdot \mid z^*)$ than when it is $F_C(\cdot \mid z^{**})$. This is the case if and only if $\lambda(c, z^{**}) > \lambda(c, z^*)$ for all $c \in [\underline{c}, \overline{b}_1(z^*)]$, or equivalently, the ratio $F_C(\cdot \mid z^{**})/F_C(\cdot \mid z^*)$ is strictly decreasing on $[\underline{c}, \overline{b}_1(z^*)]$.[5] Second, since in a k-double auction with $k = 1$, the seller will bid his true private value in regular equilibrium and the transaction takes place only when the buyer's bid is no less than the seller's bid, so given that p and $\psi(p)$ are the same buyer's respective bids when he faces two possible seller's value distributions, $F_C(p \mid z^*)$ and $F_C(\psi(p) \mid z^{**})$ happen to be the probabilities of trade in these two cases, respectively. According to (3.7), $m(\cdot)$ is proportional to the ratio $F_C(\psi(\cdot) \mid z^{**})/F_C(\cdot \mid z^*)$, therefore, a strictly decreasing $m(\cdot)$ in condition (iii) requires such a ratio to be strictly decreasing, too. This means that the seller's conditional value distributions should be such that the ratio of trade probabilities for the same buyer decreases as the buyer gets higher valuation for the good.

There are a few points regarding Theorem 1 and Theorem 2 that need further clarification. First, the conclusions of there two theorems only need two different values for the exclusive covariate Z. This is useful in establishing nonparametric identification no matter whether the covariate Z is discrete or continuous. Second, these two theorems aim to recover the buyer's and the seller's conditional value distributions from the price densities $h_1(\cdot \mid y, z)$ only for $y \in \{y^*, y^{**}\}$ and $z \in \{z^*, z^{**}\}$. When the supports of the covariates, espcially $\mathcal{Z}$, are richer and allow for more variation in the covariates, it is possible to shrink the identified set that Theorem 1 gives by choosing other values for Y and Z. Finally, it should be pointed out that those conditions on the function $\psi(\cdot)$ and $m(\cdot)$ in Theorem 2 except the discriminant one, are sufficient but not necessary for point identifying the seller's conditional value distributions (see the example in Appendix A.7 where the conditions of Theorem 2 are violated but $\lambda(\cdot, z^*)$ is still point identified).

[5] Refer to Shaked and Shanthikumar (2007) about the latter equivalence.

3.2 Identification with Continuous Z

When the covariate Z that exclusively shifts the seller's value distribution is continuous, it is feasible in theory but practically inefficient to establish the identification of the seller's value distribution, by repeatedly applying the results from the previous subsection to investigate all pairs of Z's values in its support. But it will be shown next that the property of Z varying continuously provides a shortcut to identify the seller's conditional value distribution for all $z \in \mathcal{Z}$.

For ease of discussion, I will assume that Z is scalar (i.e. $d_z = 1$) and the support $\mathcal{Z}$ is an interval $[\underline{z}, \overline{z}]$ in $\mathbb{R}$ for the time being.

First, as a supplement to Assumption E, I assume that:

Assumption G. *When Z is continuous, $\lambda(c, z)$ is continuously differentiable in z.*

With this assumption, Lemma 2 is augmented to include the following conclusion.

Lemma 3. *When Z is continuous, under Assumptions A to G, if $h_1(\cdot \mid y, z)$ is the density of transaction price in the regular equilibrium of a sealed-bid k-double auction with $k = 1$ for $F_V(\cdot \mid y)$, $y \in \{y^*, y^{**}\}$ and $F_C(\cdot \mid z)$, $z \in \mathcal{Z}$, then $\Gamma_1(p, z)$ is continuously differentiable in z.*

Now suppose the function $\Gamma_1(p, z)$ defined for $z \in \mathcal{Z}$ and $p \in [\underline{c}, \overline{b}_1(z)]$ satisfies the conditions in Lemmas 2 and 3, then define[6]

$$\ell_1(p, z) = \frac{\partial_2 \Gamma_1(p, z)}{\partial_1 \Gamma_1(p, z)}, \quad z \in \mathcal{Z}, \quad p \in [\underline{c}, \overline{b}_1(z)], \tag{3.11}$$

where $\partial_1 \Gamma_1$ and $\partial_2 \Gamma_1$ are the partial derivatives of Γ_1 with respect to the first and the second arguments, respectively. By (3.4), I have

$$\ell_1(p, z) = \frac{\partial_2 \lambda(p, z)}{1 + \partial_1 \lambda(p, z)}$$

which can be written as the following form

$$\partial_2 \lambda(p, z) - \ell_1(p, z) \cdot \partial_1 \lambda(p, z) = \ell_1(p, z), \tag{3.12}$$

where $\partial_2 \lambda$ denotes the partial derivative of λ with respect to the second argument. Equation (3.12) turns out to be a first-order linear inhomogeneous partial differential equation about λ, which should be satisfied by all the conditional value distributions for the seller that can rationalize the price distribution. Thus, applying the theory of partial differential equations will give the following identification result when Z is continuous.

[6] $\ell_1(p, z)$ is always well-defined because $\partial_1 \Gamma_1(p, z) < 0$ by Lemma 2.

Theorem 3. *Under Assumptions A to G, suppose $h_1(\cdot \mid y, z)$, $y \in \{y^*, y^{**}\}$, $z \in \mathcal{Z}$ satisfies the conditions of Lemmas 1 to 3, if $\lambda(\cdot, z^*)$ is known for some $z^* \in \mathcal{Z}$, then $\lambda(\cdot, \cdot)$ is identified on the set $\{(p, z) : z \in \mathcal{Z},\ p \in [\underline{c}, \bar{b}_1(z)]\}$.*

Proof. See Appendix A.6. □

According to Theorem 3, I can pin down the value of $\lambda(\cdot, z)$ for all other values of z in the support $\mathcal{Z}$ by solving the partial differential equation (3.12), as long as $\lambda(\cdot, z)$ or $F_C(\cdot \mid z)$ is identified for just one realization of covariate Z, which can be done by applying Theorem 1 or Theorem 2. As a result, the seller's conditional value distribution is identified as

$$\frac{F_C(c \mid z)}{F_C(\bar{b}_1(z) \mid z)} = \exp\left(-\int_c^{\bar{b}_1(z)} \frac{1}{\lambda(u, z)}\,\mathrm{d}u\right), \quad z \in \mathcal{Z}, \quad c \in [\underline{c}, \bar{b}_1(z)],$$

and the buyer's conditional value distribution is given by (3.8) for $y \in \{y^*, y^{**}\}$.

Theorem 3 holds for the case where covariate Z is vector-valued (i.e. $d_z > 1$) and the support $\mathcal{Z}$ takes the form of $[\underline{z}_1, \bar{z}_1] \times \cdots \times [\underline{z}_{d_z}, \bar{z}_{d_z}] \subset \mathbb{R}^{d_z}$. This is because, by taking the partial derivatives of $\Gamma_1(\cdot, \cdot)$ and $\lambda(\cdot, \cdot)$ with respect to each component of Z instead, I can define a series of functions similar to (3.11) and construct a system of partial differential equations similar to (3.12), and then it can be shown that the solution to the system of partial differential equations is still uniquely determined by a boundary value condition such as a known $\lambda(\cdot, z^*)$.

4 Conclusion

This paper addresses the problem of nonparametric identification of the buyer's and the seller's value distributions in k-double auctions given only the transaction price is observed. I use exclusion restrictions which take the form of two exogenous covariates that respectively shift the buyer's and the seller's value distributions. I show that in the k-double auctions with either $k = 1$ or $k = 0$, both traders' value distributions can be partially identified in general from the distribution of transaction price, as long as both exclusive value distribution shifters can take at least two distinct values. Besides showing my bound for the identified set is sharp, I provide some sufficient conditions under which the traders' value distributions are point identified. When the value distribution shifters are continuous, I also show that the traders' value distributions can be recovered by solving a partial differential equation that only depends on the observed price distribution.

A nonparametric estimation method for the buyer's and the seller's conditional value distributions, $(F_V(\cdot \mid y), F_C(\cdot \mid z))$, can be developed given they are point identified. A strategy could rely on the previous identification strategy. Specifically, for example, when $k = 1$, first let $\hat{\Gamma}_1(p, z)$ be the estimate of $\Gamma_1(p, z)$ in (3.3), where $h_1(p \mid y, z)$ is replaced by its nonparametric estimate $\hat{h}_1(p \mid y, z)$.

Then define the estimates for $\psi(p)$ and $m(p)$ by

$$\hat{\Gamma}_1(p, z^*) = \hat{\Gamma}_1(\hat{\psi}(p), z^{**}) \quad \text{and} \quad \hat{m}(p) = \frac{\hat{h}_1(\hat{\psi}(p), z^{**})}{\hat{h}_1(p, z^*)} \hat{\psi}'(p).$$

Finally, an estimator for $\lambda(\,\cdot\,, z^*)$ is obtained by solving the counterpart of equation (3.9), that is,

$$\hat{m}'(p) \left[\hat{\lambda}(p, z^*)\right]^2 + \left[(p - \hat{\psi}(p))\hat{m}(p)\right]' \hat{\lambda}(p, z^*) + (p - \hat{\psi}(p))\hat{m}(p)\hat{m}'(p) = 0.$$

However, because it requires estimating the functions $\Gamma_1(\,\cdot\,,\,\cdot\,)$, $\psi(\,\cdot\,)$ and $m(\,\cdot\,)$ as intermediate steps which involves operations such as taking limits or derivatives, this strategy could be computationally demanding and complicated in implementation.

Another possible strategy replies on the feature that when $k = 1$ or 0, the conditional price density $h_1(\,\cdot \mid y, z)$ or $h_0(\,\cdot \mid y, z)$ can be explicitly expressed as a function of the traders' conditional value distributions. This feature allows the conditional value distributions to be estimated by directly looking for $F_V(\,\cdot \mid y)$ and $F_C(\,\cdot \mid z)$ in the parameter space to match the observed and the predicted distributions of the transaction price. Consider a simple example of n double auctions with $k = 1$, for each of which the observables consist of the transaction price P_i and the associated covariates (Y_i, Z_i). First, estimate $\underline{c}$ by the lowest observed price as $\hat{\underline{c}} = \min_i P_i$ and assume the rest unknown parameters $\theta = (\overline{v}, F_V(\,\cdot \mid \cdot\,), F_C(\,\cdot \mid \cdot\,)) \in \Theta = \mathcal{V} \times \mathcal{F}_V \times \mathcal{F}_C$, where $\mathcal{V}$ is a compact subset of $\mathbb{R}_+$ and $\mathcal{F}_V$, $\mathcal{F}_C$ are the respective sets of the buyer's and the seller's conditional value distributions that satisfy all the relevant assumptions on traders' value distributions. Next, take $\Theta_n = \mathcal{V} \times \mathcal{F}_{V,n} \times \mathcal{F}_{C,n}$ as the sieve approximation of Θ such that the sieve preserves the shape and smoothness restrictions on the unknown functions, and then an estimator for θ is given by the minimizer of a criterion function Q_n, i.e. $\hat{\theta} = \arg\min_{\theta_n = (\overline{v}_n, F_{Vn}, F_{Cn}) \in \Theta_n} Q_n(\theta_n)$. A candidate for the criterion function is the negative likelihood function which, by (3.1), takes the form of

$$Q_n(\theta_n) = -\frac{1}{n} \sum_{i=1}^{n} \frac{H_{1n}(P_i, Y_i, Z_i)}{\int_{\hat{\underline{c}}}^{\overline{b}_{1n}(Z_i)} H_{1n}(u, Y_i, Z_i)\, \mathrm{d}u},$$

where

$$H_{1n}(p, y, z) = F'_{Vn}\left(p + \frac{F_{Cn}(p \mid z)}{F'_{Cn}(p \mid z)} \,\middle|\, y\right) F_{Cn}(p \mid z) \left\{2 - \frac{F_{Cn}(p \mid z) F''_{Cn}(p \mid z)}{[F'_{Cn}(p \mid z)]^2}\right\}$$

and $\overline{b}_{1n}(Z_i)$ is determined by $\overline{b}_{1n}(Z_i) + F_{Cn}(\overline{b}_{1n}(Z_i) \mid Z_i) / F'_{Cn}(\overline{b}_{1n}(Z_i) \mid Z_i) = \overline{v}_n$. Alternatively, inspired by Bierens and Song (2012, 2014), the criterion function can be chosen as

$$Q_n(\theta_n) = \int_{[-t,t]^{d_y + d_z + 1}} \left| \frac{1}{n} \sum_{i=1}^{n} \exp\left[\mathrm{i} \cdot (P_i, Y_i, Z_i)\tau\right] - \frac{1}{n} \sum_{i=1}^{n} \exp\left[\mathrm{i} \cdot (\tilde{P}_i(\theta_n), Y_i, Z_i)\tau\right] \right|^2 \mathrm{d}\tau$$

with some $t > 0$, where $\tau \in \mathbb{R}^{d_y+d_z+1}$, $\mathrm{i} = \sqrt{-1}$, and $\tilde{P}_i(\theta_n)$ is the simulated transaction price in a double auction with $k = 1$ for the traders' value distributions specified by $F_{Vn}(\cdot \mid Y_i)$ and $F_{Cn}(\cdot \mid Z_i)$.

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A Supplementary Results and Proofs

A.1 Results for the Case with $k = 0$

The identification results based on the price distributions of k-double auctions with $k = 0$ can be established in a similar way by using symmetric assumptions. Here I will list the relevant assumptions and provide the statements of the conclusions without proof.

When $k = 0$, I assume that the buyer's conditional value distribution $F_V(\cdot \mid y)$ satisfies:

Assumption H. *For any $y \in \mathcal{Y}$, $\delta(\overline{v}, y) = 0$, and $\delta(\cdot, y)$ is continuously differentiable with $-\infty < \partial_1 \delta(v, z) < 0$ for all $v \in [\underline{c}, \overline{v}]$.*

Just a reminder, here $\delta(\cdot, y) = [1 - F_V(\cdot \mid y)]/f_V(\cdot \mid y)$ and $\partial_1 \delta(v, y)$ denotes the partial derivative of $\delta(\cdot, \cdot)$ with respect to the first argument evaluated at (v, y). I also assumed that the seller's conditional value distribution $F_C(\cdot \mid z)$ satisfies:

Assumption I. *There exist $z^* \neq z^{**}$ in $\mathcal{Z}$ such that $f_C(\cdot \mid z^*)/f_C(\cdot \mid z**)$ is continuously differentiable with positive derivative on $[\underline{c}, \overline{v}]$.*

Assumption H and Assumption I are parallel to Assumptions E and F, respectively.

With these assumptions, I can show the following parallel results for the $k = 0$ case.

Lemma 4. *Under Assumptions A to D and Assumption H, if $h_0(\cdot \mid y, z)$ is the density of the transaction price in the regular equilibrium of a sealed-bid k-double auction with $k = 0$ for $F_V(\cdot \mid y)$, $y \in \mathcal{Y}$ and $F_C(\cdot \mid z)$, $z \in \mathcal{Z}$, then for any $y \in \mathcal{Y}$ and any $z \in \mathcal{Z}$,*

(i) $\overline{b}_0(y, z) = \overline{v}$, and $h_0(\overline{v} \mid y, z) = 0$;

(ii) $\underline{s}_0(y, z) = \underline{s}_0(y)$ which solves $\underline{s}_0(y) - \delta(\underline{s}_0(y), y) = \underline{c}$.

By defining

$$\Gamma_0(p, y) \equiv \lim_{q \to \overline{v}} \left[\frac{h_0(p \mid y, z^*)}{h_0(q \mid y, z^*)} \middle/ \frac{h_0(p \mid y, z^{**})}{h_0(q \mid y, z^{**})} \right]$$

for $\underline{s}_0(y) < p < \overline{v}$ and $\Gamma_0(\underline{s}_0(y), y) \equiv \lim_{p \to \underline{s}_0(y)} \Gamma_0(p, y)$, $\Gamma_0(\overline{v}, y) \equiv \lim_{p \to \overline{v}} \Gamma_0(p, y)$, it follows that

Lemma 5. *Under Assumptions A to D and Assumptions H and I, if $h_0(\cdot \mid y, z)$ is the density of the transaction price in the regular equilibrium of a sealed-bid k-double auction with $k = 0$ for $F_V(\cdot \mid y)$, $y \in \mathcal{Y}$ and $F_C(\cdot \mid z)$, $z \in \{z^*, z^{**}\}$, then* *(i)* $\Gamma_0(\cdot, y)$ *is continuously differentiable with positive derivative on* $[\underline{s}_0(y), \overline{v}]$ *for any* $y \in \mathcal{Y}$*;*

(ii) $\Gamma_0(\overline{v}, y^*) = \Gamma_0(\overline{v}, y^{**}) = 1$ *and* $\Gamma_0(\underline{s}_0(y^*), y^*) = \Gamma_0(\underline{s}_0(y^{**}), y^{**})$ *for any* $y^*, y^{**} \in \mathcal{Y}$*.*

For $y^* \neq y^{**}$ and for $p \in [\underline{s}_0(y^*), \overline{v}]$, define $\varphi(p)$ be such that

$$\Gamma_0(p, y^*) = \Gamma_0(\varphi(p), y^{**}),$$

and let

$$r(p) \equiv \frac{h_0(\varphi(p) \,|\, y^{**}, z^*)}{h_0(p \,|\, y^*, z^*)} \varphi'(p),$$

then

Theorem 4. *Under Assumptions A to D and Assumptions H and I, suppose* $h_0(\cdot \,|\, y, z)$*,* $y \in \{y^*, y^{**}\}$*,* $z \in \{z^*, z^{**}\}$ *satisfies the conditions of Lemmas 4 and 5, then* $h_0(\cdot \,|\, y, z)$ *is the density of the transaction price in the regular equilibrium of a sealed-bid k-double auction with* $k = 0$ *for some* $F_V(\cdot \,|\, y)$*,* $y \in \{y^*, y^{**}\}$ *and* $F_C(\cdot \,|\, z)$*,* $z \in \{z^*, z^{**}\}$ *if and only if* $F_V(\cdot \,|\, y^*)$ *and* $F_V(\cdot \,|\, y^{**})$ *satisfy*

$$p + \delta(p, y^*) = \varphi(p) + \delta(\varphi(p), y^{**}),$$

and

$$\frac{F_V(\varphi(p) \,|\, y^{**})}{F_V(p \,|\, y^*)} = \frac{a_0(y^*, z^*)}{a_0(y^{**}, z^*)} \cdot r(p),$$

for all $p \in [\underline{s}_0(y^*), \overline{v}]$*.*

Theorem 5. *Under Assumptions A to D and Assumptions H and I, suppose* $h_0(\cdot \,|\, y, z)$*,* $y \in \{y^*, y^{**}\}$*,* $z \in \{z^*, z^{**}\}$ *satisfies conditions of Lemmas 4 and 5, if in addition for all* $p \in [\underline{s}_0(y^*), \overline{v})$*,*

(i) $\Delta_0(p) \equiv \{[(p - \varphi(p))r(p)]'\}^2 - 4(p - \varphi(p))r(p)r'(p) \geqslant 0$*,*

(ii) $\varphi(p) < p$*, and*

(iii) $r'(p) < 0$*,*

then $F_V(\cdot \,|\, y^*)$ *is identified on* $[\underline{s}_0(y^*), \overline{v}]$ *and* $F_V(\cdot \,|\, y^*)$ *is identified on* $[\underline{s}_0(y^{**}), \overline{v}]$*.*

When the covariate Y is continuous random variable and the support $\mathcal{Y} = [\underline{y}, \overline{y}] \subset \mathbb{R}$, I assume in addition that:

Assumption J. *When* Y *is continuous,* $\delta(v, y)$ *is continuously differentiable in* y*.*

then,

Lemma 6. *Under Assumptions A to D and H to J, if* $h_0(\cdot \,|\, y, z)$ *is the density of the transaction price in the regular equilibrium of a sealed-bid k-double auction with* $k = 0$ *for* $F_V(\cdot \,|\, y)$*,* $y \in \mathcal{Y}$ *and* $F_C(\cdot \,|\, z)$*,* $z \in \{z^*, z^{**}\}$*, then* $\Gamma_0(p, y)$ *is continuously differentiable in* y*.*

Therefore, similarly define $\ell_0(p,y) = \partial_1\Gamma_0(p,y)/\partial_2\Gamma_0(p,y)$ for $y \in \mathcal{Y}$ and $p \in [\underline{s}_0(y), \overline{v}]$, and the following theorem parallel to Theorem 3 holds.

Theorem 6. *Under Assumptions A to D and H to J, suppose $h_0(\,\cdot\,|\,y,z)$, $y \in \mathcal{Y}$, $z \in \{z^*, z^{**}\}$ satisfies the conditions of Lemmas 4 to 6, if $\delta(\,\cdot\,,y^*)$ is known for some $y^* \in \mathcal{Y}$, then $\delta(\,\cdot\,,\,\cdot\,)$ is identified on the set $\{(p,y) : y \in \mathcal{Y},\ p \in [\underline{s}_0(y), \overline{v}]\}$ through the following partial differential equation*

$$\ell_0(p,y) \cdot \partial_1\delta(p,y) - \partial_2\delta(p,y) = \ell_0(p,y).$$

A.2 Proof of Lemma 1

By Assumption D, the lowest possible private values of the buyer and the seller are $\underline{c}$. When $k = 1$, because the equilibrium bidding strategy for the seller is submitting his true valuation, the seller with the lowest private value will bid $\underline{c}$. Also, the buyer with the lowest private value will also bid $\underline{c}$ because $\underline{c} + \lambda(\underline{c}, z) = \underline{c} + 0 = \underline{c}$. Therefore, the lower endpoint of the price support is $\underline{s}_1(y,z) = \underline{c}$, which is independent from Y and Z; and it follows from (3.1) directly that $h_1(\underline{c}\,|\,y,z) = 0$ because $F_C(\underline{c}\,|\,z) = 0$.

For conclusion (ii), because the seller's value distribution does not depend on covariate Y by condition (i) of Assumption D, the buyer's equilibrium bidding strategy, whose inverse function is given by $b + \lambda(b,z)$, does not depend on Y, either. Also, because the highest possible private value of the buyer, $\overline{v}$, does not depend on Y or Z, the buyer's highest bid $\overline{b}_1(y,z)$ will just depend on covariate Z. Since the buyer's inverse bidding function is strictly increasing as $\lambda(\,\cdot\,,z)$ is assumed to be strictly increasing, the buyer's highest bid, which is also the upper endpoint of the price support, is the unique solution to equation $\overline{b}_1(z) + \lambda(\overline{b}_1(z), z) = \overline{v}$. □

A.3 Proof of Lemma 2

For (i), given the differentiability of $f_V(\,\cdot\,|\,y^*)/f_V(\,\cdot\,|\,y^{**})$ and $\lambda(\,\cdot\,,z)$ assumed in Assumptions E and F, differentiating $\Gamma_1(p,z)$ as (3.4) with respect to p yields

$$\partial_1\Gamma_1(p,z) = \frac{f_V(\underline{c}\,|\,y^{**})}{f_V(\underline{c}\,|\,y^*)} \cdot \frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{f_V(u\,|\,y^*)}{f_V(u\,|\,y^{**})}\right]\Bigg|_{u=p+\lambda(p,z)} \cdot [1 + \partial_1\lambda(p,z)]\,.$$

Since $\frac{\mathrm{d}}{\mathrm{d}u}\left[f_V(u\,|\,y^*)/f_V(u\,|\,y^{**})\right]$ is continuous and both $p + \lambda(p,z)$ and $1 + \partial_1\lambda(p,z)$ are continuous in p, $\partial_1\Gamma_1(p,z)$ is continuous in p and therefore $\Gamma_1(\,\cdot\,,z)$ is continuously differentiable. In the mean time, $\partial_1\Gamma_1(p,z) < 0$ because $\frac{\mathrm{d}}{\mathrm{d}u}\left[f_V(u\,|\,y^*)/f_V(u\,|\,y^{**})\right] < 0$ by Assumption F and $1 + \partial_1\lambda(p,z) > 1$ by Assumption E.

To show (ii), first note that by Assumption E, $\lambda(\underline{c}, z^*) = \lambda(\underline{c}, z^{**}) = 0$, so by (3.4),

$$\Gamma_1(\underline{c}, z^*) = \frac{f_V(\underline{c} + \lambda(\underline{c}, z^*)\,|\,y^*)}{f_V(\underline{c} + \lambda(\underline{c}, z^*)\,|\,y^{**})} \cdot \frac{f_V(\underline{c}\,|\,y^{**})}{f_V(\underline{c}\,|\,y^*)} = \frac{f_V(\underline{c}\,|\,y^*)}{f_V(\underline{c}\,|\,y^{**})} \cdot \frac{f_V(\underline{c}\,|\,y^{**})}{f_V(\underline{c}\,|\,y^*)} = 1$$

and

$$\Gamma_1(\underline{c}, z^{**}) = \frac{f_V(\underline{c} + \lambda(\underline{c}, z^{**}) \,|\, y^*)}{f_V(\underline{c} + \lambda(\underline{c}, z^{**}) \,|\, y^{**})} \cdot \frac{f_V(\underline{c} \,|\, y^{**})}{f_V(\underline{c} \,|\, y^*)} = \frac{f_V(\underline{c} \,|\, y^*)}{f_V(\underline{c} \,|\, y^{**})} \cdot \frac{f_V(\underline{c} \,|\, y^{**})}{f_V(\underline{c} \,|\, y^*)} = 1.$$

Next, it follows from conclusion (ii) of Lemma 1 that $\overline{b}_1(z^*)$ and $\overline{b}_1(z^{**})$ satisfy that $\overline{b}_1(z^*) + \lambda(\overline{b}_1(z^*), z^*) = \overline{b}_1(z^{**}) + \lambda(\overline{b}_1(z^{**}), z^{**}) = \overline{v}$. Therefore,

$$\Gamma_1(\overline{b}_1(z^*), z^*) = \frac{f_V(\overline{b}_1(z^*) + \lambda(\overline{b}_1(z^*), z^*) \,|\, y^*)}{f_V(\overline{b}_1(z^*) + \lambda(\overline{b}_1(z^*), z^*) \,|\, y^{**})} \cdot \frac{f_V(\underline{c} \,|\, y^{**})}{f_V(\underline{c} \,|\, y^*)} = \frac{f_V(\overline{v} \,|\, y^*)}{f_V(\overline{v} \,|\, y^{**})} \cdot \frac{f_V(\underline{c} \,|\, y^{**})}{f_V(\underline{c} \,|\, y^*)},$$

$$\Gamma_1(\overline{b}_1(z^{**}), z^{**}) = \frac{f_V(\overline{b}_1(z^{**}) + \lambda(\overline{b}_1(z^{**}), z^{**}) \,|\, y^*)}{f_V(\overline{b}_1(z^{**}) + \lambda(\overline{b}_1(z^{**}), z^{**}) \,|\, y^{**})} \cdot \frac{f_V(\underline{c} \,|\, y^{**})}{f_V(\underline{c} \,|\, y^*)} = \frac{f_V(\overline{v} \,|\, y^*)}{f_V(\overline{v} \,|\, y^{**})} \cdot \frac{f_V(\underline{c} \,|\, y^{**})}{f_V(\underline{c} \,|\, y^*)},$$

and then $\Gamma_1(\overline{b}_1(z^*), z^*) = \Gamma_1(\overline{b}_1(z^{**}), z^{**})$. □

A.4 Proof of Theorem 1

First, for the "only if" part, I shall show that under Assumptions A to F, for any buyer's conditional value distributions $F_V(\cdot \,|\, y)$, $y \in \{y^*, y^{**}\}$ and seller's conditional value distributions $F_C(\cdot \,|\, z)$, $z \in \{z^*, z^{**}\}$ that can rationalize the price distributions with densities $h_1(\cdot \,|\, y, z)$, $y \in \{y^*, y^{**}\}$, $z \in \{z^*, z^{**}\}$, the seller's conditional value distributions $F_C(\cdot \,|\, z^*)$ and $F_C(\cdot \,|\, z^{**})$ must satisfy (3.6) and (3.7). This is straightforward based on the derivation of (3.6) and (3.7).

Since $F_V(\cdot \,|\, y)$, $y \in \{y^*, y^{**}\}$ and $F_C(\cdot \,|\, z)$, $z \in \{z^*, z^{**}\}$ can rationalize the given price distribution, then by (3.1) it must be true that for $y \in \{y^*, y^{**}\}$ and $z \in \{z^*, z^{**}\}$, the price density can be written as

$$h_1(p \,|\, y, z) = a_1(y, z) F_C(p \,|\, z) f_V(p + \lambda(p, z) \,|\, y) \left[1 + \partial_1 \lambda(p, z)\right], \quad p \in [\underline{c}, \overline{b}_1(z)],$$

for some $a_1(y, z) > 0$. Then by the definition of $\Gamma_1(\cdot, z)$ (i.e. (3.3)), it follows that for $z \in \{z^*, z^{**}\}$,

$$\begin{aligned}
\Gamma_1(p, z) &= \lim_{q \to \underline{c}} \left[\frac{h_1(p \,|\, y^*, z)}{h_1(q \,|\, y^*, z)} \Big/ \frac{h_1(p \,|\, y^{**}, z)}{h_1(q \,|\, y^{**}, z)} \right] \\
&= \lim_{q \to \underline{c}} \left[\frac{f_V(p + \lambda(p, z) \,|\, y^*)}{f_V(p + \lambda(p, z) \,|\, y^{**})} \cdot \frac{f_V(q + \lambda(q, z) \,|\, y^{**})}{f_V(q + \lambda(q, z) \,|\, y^*)} \right] \\
&= \frac{f_V(p + \lambda(p, z) \,|\, y^*)}{f_V(p + \lambda(p, z) \,|\, y^{**})} \cdot \frac{f_V(\underline{c} \,|\, y^{**})}{f_V(\underline{c} \,|\, y^*)},
\end{aligned}$$

which is equation (3.4). The last equality is due to $\lambda(\underline{c}, z) = 0$ and the continuity of $\lambda(\cdot, z)$ by Assumption E as well as the continuity of $f_V(\cdot \,|\, y^{**}) / f_V(\cdot \,|\, y^*)$ by Assumption F. Because the function $\psi(\cdot)$ is defined as such that $\Gamma_1(p, z^*) = \Gamma_1(\psi(p), z^{**})$ for $p \in [\underline{c}, \overline{b}_1(z^*)]$, and because $f_V(\cdot \,|\, y^*) / f_V(\cdot \,|\, y^{**})$ is strictly monotone by Assumption F, $F_C(\cdot \,|\, z^*)$ and $F_C(\cdot \,|\, z^{**})$, or $\lambda(\cdot \,|\, z^*)$

and $\lambda(\cdot\,|\,z^{**})$, must satisfy

$$\frac{f_V(p+\lambda(p,z^*)\,|\,y^*)}{f_V(p+\lambda(p,z^*)\,|\,y^{**})}=\frac{f_V(\psi(p)+\lambda(\psi(p),z^{**})\,|\,y^*)}{f_V(\psi(p)+\lambda(\psi(p),z^{**})\,|\,y^{**})}\;\Rightarrow\; p+\lambda(p,z^*)=\psi(p)+\lambda(\psi(p),z^{**}),$$

which is equation (3.6).

Given $F_C(\cdot\,|\,z^*)$ and $F_C(\cdot\,|\,z^{**})$ satisfy (3.6), differentiating both sides of (3.6) yields

$$1+\partial_1\lambda(p,z^*)=\psi'(p)+\partial_1\lambda(\psi(p),z^{**})\cdot\psi'(p)=\psi'(p)\left[1+\partial_1\lambda(\psi(p),z^{**})\right].$$

Since for $y\in\{y^*,y^{**}\}$ and $p\in[\underline{c},\overline{b}_1(z^*)]$,

$$h_1(p\,|\,y,z^*)=a_1(y,z^*)F_C(p\,|\,z^*)f_V(p+\lambda(p,z^*)\,|\,y)\left[1+\partial_1\lambda(p,z^*)\right],$$
$$h_1(\psi(p)\,|\,y,z^{**})=a_1(y,z^{**})F_C(\psi(p)\,|\,z^{**})f_V(\psi(p)+\lambda(\psi(p),z^{**})\,|\,y)\left[1+\partial_1\lambda(\psi(p),z^{**})\right],$$

so taking the ratio of these two and using $f_V(p+\lambda(p,z^*)\,|\,y)=f_V(\psi(p)+\lambda(\psi(p),z^{**})\,|\,y)$ implied by (3.6), I have that $F_C(\cdot\,|\,z^*)$ and $F_C(\cdot\,|\,z^{**})$ should also satisfy

$$\frac{h_1(p\,|\,y,z^*)}{h_1(\psi(p)\,|\,y,z^{**})}=\frac{a_1(y,z^*)}{a_1(y,z^{**})}\cdot\frac{F_C(p\,|\,z^*)}{F_C(\psi(p)\,|\,z^{**})}\psi'(p),$$

which can be written into the form of (3.7).

Now I shall show the "if" part, i.e. under Assumptions A to D, for any seller's conditional value distributions $F_C(\cdot\,|\,z^*)$ and $F_C(\cdot\,|\,z^{**})$ that satisfy Assumption E, (3.6) and (3.7), there exist buyer's conditional value distributions $F_V(\cdot\,|\,y^*)$ and $F_V(\cdot\,|\,y^{**})$ satisfying Assumption F such that $h_1(\cdot\,|\,y,z)$, $y\in\{y^*,y^{**}\}$, $z\in\{z^*,z^{**}\}$ are the price densities generated from the regular equilibrium of a k-double auction with $k=1$.

I will start off by claiming that given a price density $h_1(\cdot\,|\,y,z)$ and a seller's conditional value distribution $F_C(\cdot\,|\,z)$ that satisfies Assumption E, a distribution given by (3.8) with $\overline{v}=\overline{b}_1(z)+\lambda(\overline{b}_1(z),z)$ is the conditional value distribution for the buyer with which the seller's conditional value distribution rationalizes the price distribution. To show this, first, because b solves $b+\lambda(b,z)=v$, it follows that

$$\frac{\mathrm{d}b}{\mathrm{d}v}+\partial_1\lambda(b,z)\cdot\frac{\mathrm{d}b}{\mathrm{d}v}=1\;\Rightarrow\;\frac{\mathrm{d}b}{\mathrm{d}v}=\frac{1}{1+\partial_1\lambda(b,z)}.$$

Note that $\left\{\int_{\underline{c}}^{\overline{b}_1(z)}[h_1(u\,|\,y,z)/F_C(u\,|\,z)]\,\mathrm{d}u\right\}^{-1}$ in (3.8) is a constant only depending on y and z so denote it by $\sigma_1(y,z)$. Therefore, differentiating (3.8) with respect to v gives the corresponding density of $\tilde{F}_V(\cdot \mid y, z)$ as

$$\tilde{f}_V(v \mid y, z) = \sigma_1(y, z) \cdot \frac{h_1(b \mid y, z)}{F_C(b \mid z)} \cdot \frac{\mathrm{d}b}{\mathrm{d}v} = \sigma_1(y, z) \cdot \frac{h_1(b \mid y, z)}{F_C(b \mid z)} \cdot \frac{1}{1 + \partial_1 \lambda(b, z)}. \tag{A.1}$$

Since $v = b + \lambda(b, z)$, so (A.1) can be written as

$$\tilde{f}_V(b + \lambda(b, z) \mid y, z) = \sigma_1(y, z) \cdot \frac{h_1(b \mid y, z)}{F_C(b \mid z)} \cdot \frac{1}{1 + \partial_1 \lambda(b, z)} \tag{A.2}$$

for $b \in [\underline{c}, \overline{b}_1(z)]$. Then by (3.1), the density of the price distribution generated by $F_C(\cdot \mid z)$ and $\tilde{F}_V(\cdot \mid y, z)$ is

$$\tilde{h}_1(p \mid y, z) = \tilde{a}_1(y, z) F_C(p \mid z) \tilde{f}_V(p + \lambda(p, z) \mid y, z) \left[1 + \partial_1 \lambda(p, z)\right],$$

where $\tilde{a}_1(y, z)$ is the normalization constant. It immediately follows from (A.2) that

$$\tilde{h}_1(p \mid y, z) = \tilde{a}_1(y, z) \sigma_1(y, z) h_1(p \mid y, z).$$

Since both $\tilde{h}_1(\cdot \mid y, z)$ and $h_1(\cdot \mid y, z)$ are density functions on support $[\underline{c}, \overline{b}_1(z)]$,

$$1 = \int_{\underline{c}}^{\overline{b}_1(z)} \tilde{h}_1(p \mid y, z) \, \mathrm{d}p = \tilde{a}_1(y, z) \sigma_1(y, z) \int_{\underline{c}}^{\overline{b}_1(z)} h_1(p \mid y, z) \, \mathrm{d}p = \tilde{a}_1(y, z) \sigma_1(y, z)$$

and hence $\tilde{h}_1(p \mid y, z) = h_1(p \mid y, z)$ for all $p \in [\underline{c}, \overline{b}_1(z)]$.

Let $\tilde{F}_V(\cdot \mid y, z)$, $y \in \{y^*, y^{**}\}$, $z \in \{z^*, z^{**}\}$ be the respective buyer's conditional value distributions induced by the corresponding $h_1(\cdot \mid y, z)$ and the seller's conditional value distributions $F_C(\cdot \mid z^*)$, $F_C(\cdot \mid z^{**})$ that satisfy (3.6) and (3.7). Next I will show that for $y \in \{y^*, y^{**}\}$, $\tilde{F}_V(\cdot \mid y, z^*)$ and $\tilde{F}_V(\cdot \mid y, z^{**})$ are in fact the same distribution.

First, by condition (ii) of Lemma 2, I have $\psi(\underline{c}) = \underline{c}$ and $\psi(\overline{b}_1(z^*)) = \overline{b}_1(z^{**})$, then (3.6) implies

$$\overline{b}_1(z^*) + \lambda(\overline{b}_1(z^*), z^*) = \psi(\overline{b}_1(z^*)) + \lambda(\psi(\overline{b}_1(z^*)), z^{**}) = \overline{b}_1(z^{**}) + \lambda(\overline{b}_1(z^{**}), z^{**}).$$

Since the induce buyer's conditional value distribution has support $[\underline{c}, \overline{v}]$ with $\overline{v} = \overline{b}_1(z) + \lambda(\overline{b}_1(z), z)$, this means that both $F_V(\cdot \mid y, z^*)$ and $F_V(\cdot \mid y, z^{**})$ have the same support. Second, for any $p \in [\underline{c}, \overline{b}_1(z^*)$, let $v = p + \lambda(p, z^*)$. Given $F_C(\cdot \mid z^*)$ and $F_C(\cdot \mid z^{**})$ satisfy (3.6), $v = \psi(p) + \lambda(\psi(p), z^{**})$, too. So by taking $b = p$ and $b = \psi(p)$ in (A.2) for $z = z^*$ and $z = z^{**}$ respectively, I get

$$\tilde{f}_V(v \mid y, z^*) = \sigma_1(y, z^*) \cdot \frac{h_1(p \mid y, z^*)}{F_C(p \mid z^*)} \cdot \frac{1}{1 + \partial_1 \lambda(p, z^*)},$$

$$\tilde{f}_V(v \,|\, y, z^{**}) = \sigma_1(y, z^{**}) \cdot \frac{h_1(\psi(p) \,|\, y, z^{**})}{F_C(\psi(p) \,|\, z^{**})} \cdot \frac{1}{1 + \partial_1 \lambda(\psi(p), z^{**})}.$$

Then by (3.7),

$$\begin{aligned}
\tilde{f}_V(v \,|\, y, z^{**}) &= \sigma_1(y, z^{**}) \frac{a_1(y, z^{**})}{a_1(y, z^{*})} \cdot \frac{h_1(p \,|\, y, z^{*})}{F_C(p \,|\, z^{*})} \cdot \frac{1}{[1 + \partial_1 \lambda(\psi(p), z^{**})]\, \psi'(p)} \\
&= \sigma_1(y, z^{**}) \frac{a_1(y, z^{**})}{a_1(y, z^{*})} \left[\frac{1 + \partial_1 \lambda(p, z^{*})}{\sigma_1(y, z^{*})} \cdot \tilde{f}_V(v \,|\, y, z^{*}) \right] \frac{1}{[1 + \partial_1 \lambda(\psi(p), z^{**})]\, \psi'(p)} \\
&= \frac{\sigma_1(y, z^{**}) a_1(y, z^{**})}{\sigma_1(y, z^{*}) a_1(y, z^{*})} \cdot \frac{1 + \partial_1 \lambda(p, z^{*})}{[1 + \partial_1 \lambda(\psi(p), z^{**})]\, \psi'(p)} \cdot \tilde{f}_V(v \,|\, y, z^{*}).
\end{aligned}$$

Equation (3.6) implies $1 + \partial_1 \lambda(p, z^*) = \psi'(p) + \partial_1 \lambda(\psi(p), z^{**}) \psi'(p)$, so $\frac{1 + \partial_1 \lambda(p, z^*)}{[1 + \partial_1 \lambda(\psi(p), z^{**})] \psi'(p)} = 1$. Consequently, $\tilde{f}_V(v \,|\, y, z^*) = \tilde{f}_V(v \,|\, y, z^{**})$ because

$$\frac{\sigma_1(y, z^{**}) a_1(y, z^{**})}{\sigma_1(y, z^{*}) a_1(y, z^{*})} = \frac{\sigma_1(y, z^{**}) a_1(y, z^{**})}{\sigma_1(y, z^{*}) a_1(y, z^{*})} \int_{\underline{c}}^{\overline{v}} \tilde{f}_V(v \,|\, y, z^{*}) \, \mathrm{d}v = \int_{\underline{c}}^{\overline{v}} \tilde{f}_V(v \,|\, y, z^{**}) \, \mathrm{d}v = 1.$$

Thus, the induced $\tilde{F}_V(\cdot \,|\, y, z), y \in \{y^*, y^{**}\}$ do not depend on covariate Z, so $\tilde{F}_V(\cdot \,|\, y, z) = \tilde{F}_V(\cdot \,|\, y)$ and they are valid conditional value distributions for the buyer that satisfies Assumption D.

As a final point, it is remained to show that $\tilde{F}_V(\cdot \,|\, y^*)$ and $\tilde{F}_V(\cdot \,|\, y^{**})$ satisfy Assumption F. Since $\tilde{F}_V(\cdot \,|\, y^*)$ and $\tilde{F}_V(\cdot \,|\, y^{**})$ can rationalize the observed price distributions, straightforwardly, for either $z = z^*$ or $z = z^{**}$,

$$\Gamma_1(p, z) = \frac{\tilde{f}_V(p + \lambda(p, z) \,|\, y^*)}{\tilde{f}_V(p + \lambda(p, z) \,|\, y^{**})} \cdot \frac{\tilde{f}_V(\underline{c} \,|\, y^{**})}{\tilde{f}_V(\underline{c} \,|\, y^{*})}, \quad p \in [\underline{c}, \overline{b}_1(z)].$$

Given that $\lambda(\cdot, z)$ is continuously differentiable and strictly increasing by Assumption E, the likelihood ratio $\tilde{f}_V(\cdot \,|\, y^*)/\tilde{f}_V(\cdot \,|\, y^{**})$ is continuously differentiable and strictly decreasing as $\Gamma_1(\cdot, z)$ is continuously differentiable and strictly decreasing due to Lemma 2. □

A.5 Proof of Theorem 2

Because for any $p \in (\underline{c}, \overline{b}_1(z^*)]$, $m'(p) < 0$ and $\Delta_1(p) \geqslant 0$, (3.9) has two real solutions, namely $\lambda_{(1)}(\cdot, z^*)$ and $\lambda_{(2)}(\cdot, z_2)$.

Since $m(p) > 0$, so when $\psi(p) < p$ and $m'(p) < 0$, the product of these two solution

$$\lambda_{(1)}(p, z^*) \cdot \lambda_{(2)}(p, z^*) = \frac{(p - \psi(p)) m(p)}{m'(p)} < 0.$$

It implies that either one of the two solutions is positive and the other is negative. Because it is assumed that $\lambda(p, z^*) > 0$ for all $p \in (\underline{c}, \overline{b}_1(z^*)]$, the negative solution should be ruled out.

Therefore, $\lambda(\cdot, z^*)$ is identified as the positive solution, i.e.

$$\lambda(p,z^*) = \frac{-[(p-\psi(p))m(p)]' + \operatorname{sgn}(p-\psi(p))\sqrt{\{[(p-\psi(p))m(p)]'\}^2 - 4(p-\psi(p))m(p)m'(p)}}{2m'(p)},$$

where $\operatorname{sgn}(\cdot)$ is the sign function.

Then by (3.6), $\lambda(\cdot, z^{**})$ will be identified as

$$\lambda(p,z^{**}) = \psi^{-1}(p) + \lambda(\psi^{-1}(p), z^*) - p, \quad p \in [\underline{c}, \overline{b}_1(z^{**})],$$

where $\psi^{-1}(\cdot)$ is the inverse function of $\psi(\cdot)$. It follows from the definition of $\lambda(\cdot,\cdot)$ that the corresponding seller's conditional value distributions are identified as

$$\frac{F_C(c \mid z)}{F_C(\overline{b}_1(z) \mid z)} = \exp\left(-\int_c^{\overline{b}_1(z)} \frac{1}{\lambda(u,z)}\, \mathrm{d}u\right), \quad c \in [\underline{c}, \overline{b}_1(z)],$$

for $z \in \{z^*, z^{**}\}$. □

A.6 Proof of Theorem 3

By applying the method of characteristics (see Rhee, Aris, and Amundson, 1986 or Vvedensky, 1993), the first-order linear differential equation in the form of (3.12) can be solved by solving its system of characteristic differential equations

$$\frac{\mathrm{d}z}{1} = \frac{\mathrm{d}p}{-\ell_1(p,z)} = \frac{\mathrm{d}\lambda}{\ell_1(p,z)} \tag{A.3}$$

with the boundary condition $\lambda(\cdot, z^*) = \xi(\cdot)$ where z^* is arbitrary point in $\mathcal{Z}$ and $\xi(\cdot)$ is a continuously differentiable function satisfying $\xi(\underline{c}) = 0$ and $\xi'(\cdot) > 0$.

Rewrite the boundary condition curve into its parametric form:

$$z(u) = z^*, \quad p(u) = u, \quad \lambda(u) = \xi(u), \quad u \in [\underline{c}, \overline{b}_1(z^*)].$$

Because

$$1 \cdot [-p'(u)] - [-\ell_1(p(u), z(u))] \cdot z'(u) = -1 \neq 0$$

for all $t \in [\underline{c}, \overline{b}_1(z^*)]$, the boundary condition prescribe above is non-characteristic; that is, geometrically, the projection of the boundary condition curve onto the (p,z)-plane does not coincide with the projections of any integral curves that satisfy (A.3). Therefore, the solution to the partial differential equation (3.12) subject to the boundary condition $\lambda(\cdot, z^*) = \xi(\cdot)$ is unique and the desired result follows. □

Remark. As a matter of fact, given the partial differential equation under review is in fact strictly linear, it can be shown that how the solution is uniquely obtained. First, it follows from (A.3) that

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\ell_1(p,z).$$

Since $\ell_1(\cdot,\cdot)$ is continuous by the fact that $\Gamma_1(p,z)$ is continuously differentiable in both p and z, this ordinary differential equation can be solved and denote the solution by $p = \eta(z, A)$ where A is a constant of integration which is determined by boundary condition $\eta(z^*, A) = p^*$ where $p^* \in [\underline{c}, \overline{b}_2(z^*)]$. Next, it also follows from (A.3) that

$$\frac{\mathrm{d}\lambda}{\mathrm{d}z} = \ell_1(p,z).$$

Then for any value of A, since $\lambda(\eta(z^*, A), z^*) = \lambda(p^*, z^*) = \xi(p^*)$ by the boundary condition specified, then taking the integral of the above ordinary differential equation along the curve $\{(p,z) : z \in \mathcal{Z},\ p = \eta(z, A)\}$, I can get for any $z \in \mathcal{Z}$,

$$\lambda(\eta(z,A),z) = \xi(\eta(z^*,A)) + \int_{z^*}^{z} \ell_1(\eta(t,A),t)\,\mathrm{d}t,$$

which is the desired solution of the partial differential equation (3.12).

A.7 Example that violates conditions of Theorem 2 but has $\lambda(\cdot, z^*)$ identified

Here is a numerical example in which $m(p)$ is not monotone and $p - \psi(p)$ changes sign as p varies within $[\underline{c}, \overline{b}_1(z^*)]$, however, $\Delta_1(p) = \left\{[(p-\psi(p))m(p)]'\right\}^2 - 4(p-\psi(p))m(p)m'(p) \geqslant 0$ for all p and $\lambda(p, z^*)$ is point identified.

Suppose the private value support is $[\underline{c}, \overline{v}] = [0, 1]$. The (true) seller's conditional value distributions are specified as such that

$$\lambda(p,z^*) = \frac{F_C(p \mid z^*)}{f_C(p \mid z^*)} = p, \quad 0 \leqslant p \leqslant \frac{1}{2}$$

and

$$\lambda(p,z^{**}) = \frac{F_C(p \mid z^{**})}{f_C(p \mid z^{**})} = \frac{2\sqrt{9p+1}-2}{3} - p, \quad 0 \leqslant p \leqslant \frac{7}{12}.$$

Let the (true) buyer's conditional value densities be $f_V(v \mid y^*) = 1$ and $f_V(v \mid y^{**}) = 2v$. Given such specification, the price densities $h_1(\cdot \mid y^*, z^*)$ and $h_1(\cdot \mid y^{**}, z^*)$ will have support $[\underline{c}, \overline{b}_1(z^*)] = \left[0, \frac{1}{2}\right]$; the price densities $h_1(\cdot \mid y^*, z^{**})$ and $h_1(\cdot \mid y^{**}, z^{**})$ will have support $[\underline{c}, \overline{b}_1(z^{**})] = \left[0, \frac{7}{12}\right]$.

It can be verified that the price distributions imply

$$\psi(p) = p^2 + \frac{2p}{3}, \quad 0 \leqslant p \leqslant \frac{1}{2},$$

which features $\psi(p) < p$ if $0 < p < \frac{1}{3}$ and $\psi(p) > p$ if $\frac{1}{3} < p \leqslant \frac{1}{2}$. Meanwhile, it can also be verified that with some $A > 0$,

$$m(p) = A \cdot (4-3p)^{-\frac{5}{2}} p^{-\frac{1}{2}}, \quad m'(p) = A \cdot (9p-2)(4-3p)^{-\frac{7}{2}} p^{-\frac{3}{2}}, \quad 0 \leqslant p \leqslant \frac{1}{2}.$$

So $m(p)$ is strictly decreasing when $0 \leqslant p \leqslant \frac{2}{9}$ and strictly increasing when $\frac{2}{9} \leqslant p \leqslant \frac{1}{2}$. These mean that the conditions in Theorem 2 are not satisfied, except

$$\Delta_1(p) = A^2 \cdot \frac{(9p^2 - 42p + 10)^2}{9p(4-3p)^7} \geqslant 0$$

for all $0 \leqslant p \leqslant \frac{1}{2}$.

However, note that for $p \in \left[0, \frac{1}{2}\right] \setminus \left\{\frac{2}{9}\right\}$, $m'(p) \neq 0$, so by the quadratic formula, the two real solutions are

$$\lambda_{(1)}(p, z^*) = \frac{-\left[(p - \psi(p))m(p)\right]' + \sqrt{\Delta_1(p)}}{2m'(p)} = \begin{cases} \dfrac{9p^3 - 15p^2 + 4p}{27p - 6} & \text{if } p < \dfrac{7 - \sqrt{39}}{3}, \\ p & \text{otherwise;} \end{cases}$$

$$\lambda_{(2)}(p, z^*) = \frac{-\left[(p - \psi(p))m(p)\right]' + \sqrt{\Delta_1(p)}}{2m'(p)} = \begin{cases} p & \text{if } p < \dfrac{7 - \sqrt{39}}{3}, \\ \dfrac{9p^3 - 15p^2 + 4p}{27p - 6} & \text{otherwise.} \end{cases}$$

When $p = \frac{2}{9}$, (3.9) has only one real solution

$$-\left.\frac{(p - \psi(p))m(p)}{\left[(p - \psi(p))m(p)\right]'}\right|_{p = \frac{2}{9}} = \frac{2}{9}.$$

The definition of solution $\lambda_{(2)}(p, z^*)$ above can be modified to accommodate this case.

However, there is only one solution of (3.9) satisfying Assumption E, i.e. $\lambda(p, z^*)$ is continuously differentiable and strictly increasing with $\lambda(\underline{c}, z^*) = 0$ (see Figure 1):

$$\lambda(p, z^*) = \mathbb{1}\left(0 \leqslant p < \frac{7 - \sqrt{39}}{3}\right) \cdot \lambda_{(2)}(p, z^*) + \mathbb{1}\left(\frac{7 - \sqrt{39}}{3} \leqslant p \leqslant \frac{1}{2}\right) \cdot \lambda_{(1)}(p, z^*) = p.$$

Therefore, the seller's conditional value distribution $F_C(\cdot \mid z^*)$ on interval $[\underline{c}, \bar{b}_1(z^*)] = \left[0, \frac{1}{2}\right]$ is still identified, though conditions (ii) and (iii) required by Theorem 2 are not fulfilled.

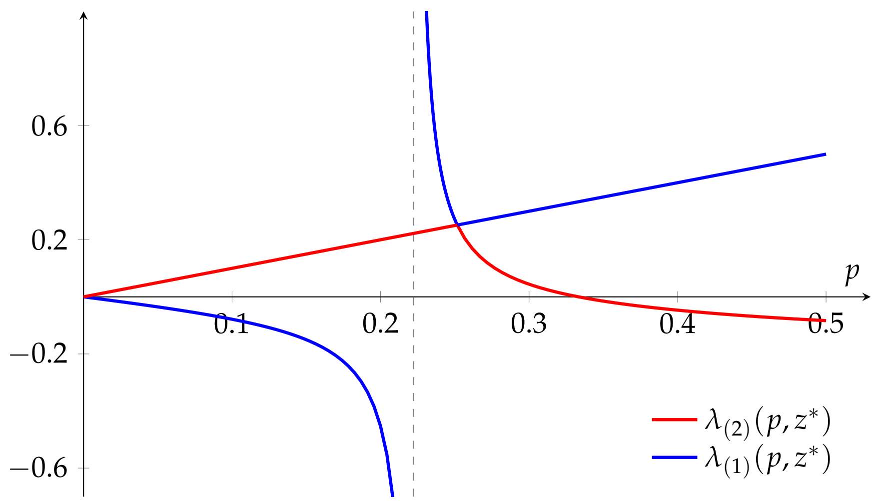

Figure 1: Examine the solutions of equation (3.9)