Written Portion of the Candidacy Examination for

the Degree of Doctor of Philosophy

MICROECONOMIC THEORY

Instructions: This examination contains two sections, each of which contains three questions. You must answer two questions from each section. You will not receive additional credit, and may receive less credit, if you answer more than four questions. You have 3½ hours to complete this exam.
Section I

1. Let $d : \mathbb{R}_+^L \times \mathbb{R}_+ \to \mathbb{R}_+^L$ satisfy $pd(p, m) \leq m$ for all $(p, m) \in \mathbb{R}_+^L \times \mathbb{R}_+$.  
   a) State the strong axiom of revealed preference (SARP).
   b) Prove that SARP is necessary and sufficient for $d$ to be the demand function for a regular (total, reflexive and transitive) preference relation on $\mathbb{R}_+^L$.

2. 
   a) Define a competitive quasi-equilibrium for an economy with production.
   b) State assumptions sufficient to ensure that a competitive quasi-equilibrium allocation is Pareto efficient. Be as general as you can.
   c) Prove that your assumptions are sufficient. Your proof should be self-contained. For example, if you believe that a duality result is relevant, instead of invoking the duality result your proof should contain the relevant elements of the proof of the duality result.

3. Let $((\succeq_i, \omega_i)_i, R)$ be an asset exchange economy with two traders and three states. The traders have identical preferences on $\mathbb{R}_+^3$, which are represented by the utility function

\[ u_i(x_i) = \frac{1}{2} \sqrt{x_{1i}} + \frac{1}{3} \sqrt{x_{2i}} + \frac{1}{6} \sqrt{x_{3i}}. \]

Their endowments are

\[ w_1 = (2, 0, 0), \quad w_2 = (0, 2, 2). \]

a) Suppose there are three assets, with the return vectors (columns of $R$),

\[ r_1 = (1, 1, 0), \quad r_2 = (1, 0, 1), \quad r_3 = (0, 1, 1). \]

Find a competitive equilibrium $(q; x_1, x_2)$, where $q \in \mathbb{R}_+^3$ is the vector of asset prices and $x_i \in \mathbb{R}_+^3$ is trader $i$'s asset portfolio (short sales are permitted).

b) Suppose there are only two assets, with the return vectors

\[ r_1 = (1, 0, 0), \quad r_2 = (0, 1, 1). \]

Find a competitive equilibrium.

c) Suppose there are only two assets, with the return vectors

\[ r_1 = (1, 1, 0), \quad r_2 = (0, 0, 1). \]

Does a competitive equilibrium exist? Justify your answer.
Section II

1. A single object is for sale to two bidders. Each bidder’s value $X_i$ is a random variable that is independently distributed on $[0, 1]$ according to the distribution function $F$ with density $f$. Bidder $i$ knows the realization $x_i$ of $X_i$ and only that other bidders’ values are independently and uniformly distributed. The object is sold via an all-pay auction—each bidder $i$ submits a sealed-bid of $b_i$, the object is awarded to the highest bidder and all bidders pay what they bid.

(a) Find a symmetric equilibrium of the all-pay auction. What is the expected revenue of the seller?

(b) (Due to E. Akyol). Let $H$ denote the bid distribution of a player in the symmetric equilibrium derived in part (a). Thus $H(b)$ is the probability that a bidder submits a bid not exceeding $b$. Show that $H$ is a concave function.

(Note: You may assume that $F$ is twice-continuously differentiable).

2. Consider a dynamic pricing problem facing a monopoly seller of goods that last at most two periods. There are four potential customers, $H_1, H_2$ and $L_1$ and $L_2$, each of whom wants at most one good. The respective (per period utilities (in money units) they get are in the table below:

<table>
<thead>
<tr>
<th></th>
<th>Period 1</th>
<th>Period 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_1, H_2$</td>
<td>1200</td>
<td>500</td>
</tr>
<tr>
<td>$L_1, L_2$</td>
<td>500</td>
<td>200</td>
</tr>
</tbody>
</table>

So if $H_1$, say, buys in the first period at a price of $p^1$, she gets a total utility of $1200 + 500) - p^1$. If she waits until the second period and buys at a price of $p^2$, she gets $500 - p^2$. There is no discounting by any of the parties and the monopolist’s costs are zero.

(a) First, suppose that the monopolist can irrevocably commit to a pricing policy; that is, in period 1 the monopolist announces both prices $p^1$ and $p^2$ which are then fixed. What is the optimal pricing policy under commitment? What are the monopolist’s profits?

(b) Now suppose that the monopolist cannot commit to a pricing policy. The monopolist announces a first-period price $p^1$; all buyers $H_i, L_i$ simultaneously and independently, decide whether to buy or not. The monopolist observes who has bought and who is left to buy and then chooses a second-period price $p^2$. What is the subgame perfect Nash equilibrium pricing policy for the monopolist? What are his profits?
(c) What would happen if the monopolist could lease (rent) the product in each period instead of making a once-and-for-all sale? What are his profits from an optimal leasing policy?

3. (Due to Chae and Yang) Let $G_t(\pi; P_t, P_{t+1})$ denote the two-player alternating offer Rubinstein bargaining game in which players $P_t$ and $P_{t+1}$ bargain over a pie of size $\pi$ with $P_t$ making the first offer in period $t$. Now let $G_t(\pi; P_t, P_{t+1}, P_k)$ denote a bargaining game in which three players $P_t, P_{t+1}, P_k$ bargain over a pie of size $\pi$ beginning in period $t$ as follows: player $P_t$ makes an offer $x_t \in [0, \pi]$ to $P_{t+1}$ (the next player in the sequence). If $P_{t+1}$ says yes, then his share is fixed at $x_t$ and then the game $G_t(\pi - x_t; P_{t+1}, P_k)$ is played in which $P_{t+1}$ is no longer active and the two remaining players bargain over the remaining pie of size $1 - x_t$. If $P_{t+1}$ says no, then the game $G_t(\pi - x_t; P_{t+1}, P_k)$ is played, beginning in period $t + 1$. All players discount future payoffs using a discount factor $\delta$ and payoffs are obtained only when all players have agreed to a division of the pie. So, for instance, if player 2 agrees to $x_2$ in period 1 but players 1 and 3 agree to a division $(x_1, x_3)$ of the remaining pie $1 - x_2$ only in period $t > 1$, then the payoffs of the three players, discounted back to the beginning of period 1, are $(\delta^{t-1}x_1, \delta^{t-1}x_2, \delta^{t-1}x_3)$.

(a) Draw a schematic tree to depict the game $G_1(1; P_1, P_2, P_3)$. Find a stationary subgame perfect equilibrium of the game. What is the equilibrium division of the pie?

(b) Now consider $G_t^*(\pi; P_t, P_{t+1}, P_k)$, a variant of the game above, defined as follows: in period $t$, player $P_t$ makes an offer $x_t \in [0, \pi]$ to $P_{t+1}$ (the last player in the sequence). Again, if $P_{t+1}$ says yes, then his share is fixed at $x_t$ and then the game $G_t(\pi - x_t; P_{t+1}, P_k)$ is played in which $P_{t+1}$ is no longer active and the two remaining players bargain over the remaining pie of size $1 - x_t$. If $P_{t+1}$ says no, then the game $G_t(\pi - x_t; P_{t+1}, P_k)$ is played, beginning in period $t + 1$. As before, players discount future payoffs using a discount factor $\delta$ and payoffs are obtained only when all players have agreed to a division of the pie.

Draw a schematic tree to depict the game $G_1^*(1; P_1, P_2, P_3)$. Find a stationary subgame perfect equilibrium of the game. What is the equilibrium division of the pie?