Econometrics prelim 2020

1. Suppose that (X, Y) has a joint probability density function $f_{XY}(x, y) = 8xy\mathbb{I}(0 \le x \le y \le 1)$.

   (a) Obtain the function $g : (0, 1) \to \mathbb{R}$ such that for any measurable function $h : (0, 1) \to \mathbb{R}$, we have $\mathbb{E}\big[\{Y - g(X)\}^2\big] \le \mathbb{E}\big[\{Y - h(X)\}^2\big]$.

   (b) Let $Z = X^2$. Obtain the probability density function of Z.

   (c) Compute the correlation coefficient between $Y - g(X)$ and Z, where g and Z are defined in parts (a) and (b), respectively.

2. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables. Suppose that $\mathbb{P}(X_1 \le 0) = 0$ and $\lim_{h \downarrow 0} \mathbb{P}(X_1 \le x + h \mid X_1 > x)/h = 1$ for all $x > 0$. Further, let $X_{(j)}$ be the jth smallest variable among $X_1, X_2, \ldots, X_n$.

   (a) Please compute $\mathbb{E}(X_1)$ and $\mathbb{E}(X_{(1)})$.

   (b) Let F be the cumulative distribution function of X_1. Please derive the cumulative distribution function and the probability density function of $Y_{(j)} := F(X_{(j)})$ for $j = 1, 2, \ldots, n$.

3. Suppose that X and Y are random variables with $\mathbb{E}\{\exp(tX + sY)\} = \exp\{(t^2 + s^2)/2\}$ for all $t, s \in \mathbb{R}$. Please compute $\mathbb{E}(2X + Y \mid X + 2Y > 0)$.

4. Suppose that (Y_1, Y_0) and T are independent, where T is Bernoulli. Define $Y := Y_1 T + Y_0(1 - T)$.

   (a) Can we identify the (marginal) distribution functions of Y_0 and Y_1 from the joint distribution of (Y, T)? Prove or disprove.

   (b) Suppose that Y_1 and Y_0 are also Bernoulli and that they are dependent such that $Y_1 \ge Y_0$ with probability one. Can we identify $\mathbb{P}(Y_1 = 1 \mid Y_0 = 0)$ from the joint distribution of (Y, T)? Prove or disprove.

5. Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables with $\mathbb{E}(X_1^4) < \infty$. Let $\hat{\mu} = n^{-1}\sum_{i=1}^n X_i$ and $\hat{\sigma}^2 = n^{-1}\sum_{i=1}^n X_i^2 - \hat{\mu}^2$. You may assume that $\sigma^2 > 0$ and $\mathbb{E}\{(X_1 - \mu)^4\} > \sigma^4$, where $\mu = \mathbb{E}(X_1)$ and $\sigma^2 = \mathbb{V}(X_1)$.

   (a) Please derive the (non-degenerate) limit distribution of $n^r(\hat{\sigma}^2 - \sigma^2)$ for some appropriate number $r > 0$.

   (b) Please derive the (non-degenerate) limit distribution of $n^r(\hat{\mu}^2 - \mu^2)$ for some appropriate number $r > 0$.

6. Prove or disprove each of the following statements. You may assume that all the random variables are defined on the same probability space.

   (a) If $X_n \xrightarrow{\text{a.s.}} 0$, then we must have $\mathbb{E}(X_n) \to 0$.

   (b) If $\mathbb{E}(|X_n|^3) = O(n)$, then we must have $X_n = O_p(n^{1/3})$.

(c) If $X_n \xrightarrow{d} X$, then we must have $h(X_n) \xrightarrow{d} h(X)$ whenever h is a continuous function.

(d) If $X_n \xrightarrow{d} 0$, then we must have $X_n \xrightarrow{p} 0$ as well.

(e) If $\mathbb{E}\{(X_n - Y)^2\} \to 0$, where $\mathbb{E}(Y^2) < \infty$ and $\mathbb{E}(X_n^2) < \infty$ for all n, then we must have $\mathbb{E}(X_n^2) \to \mathbb{E}(Y^2)$.

7. Create a joint distribution of two random variables x, y with finite variances for each of the following three situations:

   (a) x, y are independent;

   (b) y is conditionally mean independent of x but x, y are *not* independent;

   (c) y and x are uncorrelated but y is *not* conditionally mean independent of x.

   You can choose whether x, y are discrete, continuous, or mixed and whether you provide distribution functions or probability mass/density functions.

8. Consider $y_i^* = x_i^\top \beta_0 + u_i$, where x_i, u_i are mutually independent, data are i.i.d., and $\mathbb{E}(x_1 x_1^\top) > 0$. Further, $u_i \sim N(0, 1)$. The variable y_i^* is unobserved. Instead, you observe $y_i = y_i^*$ if and only if $|u_i| \leq 1$.

   (a) Show that the OLS estimator $\hat{\beta}$ of β_0 is consistent.

   (b) Figure out what the asymptotic distribution of $\hat{\beta}$ is in this specific case. *Note:* you only need to figure out what it is in this specific case: no additional points for writing down or deriving the general formula.

   (c) Suppose that one would instead use ML. Write down the loglikelihood function for this specific case. *Note:* again, no additional points for writing down extraneous information, although showing the steps and arriving at an incorrect answer may yield partial credit.

   (d) Would the MLE in this case be more or less efficient than absent the truncation? Explain. *Hint:* think this through carefully; things are not as clear–cut as they may seem. A thought–through incorrect answer may still yield points.

9. Assume i.i.d. data and consider $y_i^* = x_i^\top \beta_0 + u_i$, where $u_i | x_i = x \sim N(0, \sigma_0^2)$ for some $0 < \sigma_0^2 < \infty$ and $0 < \mathbb{E}(x_1 x_1^\top) < \infty$ where the support of x_i includes zero. The variable y_i^* is latent. Instead, you observe

$$y_i = \begin{cases} -1, & y_i^* \leq -1, \\ 0, & |y_i^*| \leq 1, \\ 1, & y_i^* \geq 1. \end{cases} \tag{1}$$

   (a) Argue whether or not β_0 is identified.

   If β_0 is not identified then assume that $\sigma_0 = 1$ below.

   (b) What is the limit distribution of the MLE of β_0? *Note:* again, no additional points for writing down extraneous information or general formulas, although showing the steps and arriving at an incorrect answer may yield partial credit.

10. Assume i.i.d. data and consider the model

$$\begin{cases} y_i^* = x_i^\intercal \beta_0 + q_i \gamma_0 + u_i, \\ q_i = w_i^\intercal \delta_0 + v_i, \end{cases} \tag{2}$$

where y_i^* is latent, u_i, v_i are mean zero jointly normal and independent of $z_i = [x_i^\intercal, w_i^\intercal]^\intercal$ and u_i has variance one. Assume that everything is well–behaved in terms of collinearity and the existence of moments and that w_i, x_i are vectors with multiple elements that have no elements in common. Instead of y_i^* you observe $y_i = \mathbb{1}(y_i^* \geq 0)$. It can be shown that

$$\mathbb{E}(y_i \mid z_i = z) = \Phi\left(\frac{x^\intercal \beta_0 + w^\intercal \delta_0 \gamma_0}{\sqrt{1 + \gamma_0^2 \sigma_0^2 + 2\gamma_0 \sigma_0 \rho_0}}\right), \tag{3}$$

where σ_0^2, ρ_0 are respectively the variance of v_i and the correlation of u_i, v_i. Now suppose that you form moment conditions

$$\begin{cases} \mathbb{E}\left[z_i \left\{y_i - \Phi\left(\frac{x_i^\intercal \beta_0 + w_i^\intercal \delta_0 \gamma_0}{\sqrt{1 + \gamma_0^2 \sigma_0^2 + 2\gamma_0 \sigma_0 \rho_0}}\right)\right\}\right] = 0, \\ \mathbb{E}\{z_i (q_i - w_i^\intercal \delta_0)\} = 0, \\ \mathbb{E}\{(q_i - w_i^\intercal \delta_0)^2 - \sigma_0^2\} = 0. \end{cases} \tag{4}$$

(a) Do the moment conditions provide underidentification, exact identification, or overidentification? Explain briefly.

(b) Name one advantage of using GMM with an overidentified system compared to using an asymptotically equally efficient GMM estimator based on an exactly identified system.

(c) Finish the sentence: "Exactly identified GMM using a given fixed vector of instruments z_i is to OLS what GMM with optimal instruments is to ...."

11. Suppose that $y_i = x_i^\intercal \beta_0 + u_i$, where

$$\begin{cases} x_i = \Pi_0^\intercal z_i + v_i, \\ u_i = \exp(z_i^\intercal \gamma_0) e_i, \end{cases} \tag{5}$$

where e_i is mean zero unit variance and e_i, v_i are independent of z_i and all variables have finite fourth moments.

(a) Write down some moment conditions that provide identification under reasonable additional conditions and state what those conditions are.

(b) Derive the optimal instruments.

12. True or false? Please explain.

(a) This exam is hard.

....