Income Anonymity*

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Abstract

How does income inequality affect social welfare? This paper characterizes a normative criterion for ranking income distributions. We consider two axioms: Pareto and Income Anonymity. Pareto requires that, if everyone supports a simultaneous change in the distribution of income and in prices, then that change is socially desirable. Income Anonymity requires that social welfare can be evaluated based on the anonymized distribution of income. When individuals have heterogeneous preferences, there exists at most one social preference relation that satisfies both axioms.

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Economic analysis often aims to identify the optimal policy in various contexts, such as determining the best trade agreements or designing a fair taxation system. The success of this analysis depends on a clear understanding of the social objective. The overarching goal of welfare economics is to narrow down the set of plausible social welfare criteria based on ethical principles.

One such principle is the Pareto principle, which states that a change in policy is socially desirable if it benefits at least one person without harming anyone else. However, the Pareto principle does not get us very far, as too many social preference relations satisfy it. For instance, it is satisfied by a dictatorial social preference relation which elevates the preferences of one individual over everybody else.

This suggests that we can further narrow down the set of plausible welfare criteria by imposing some kind of fairness requirement. The question, then, is which. The standard approach postulates that each individual's "utility" should matter equally for social welfare. A well-known shortcoming of this approach is that it is not clear what is the appropriate notion of utility to use for interpersonal comparisons, or how to measure it. Even setting these issues aside, it is not clear that evaluating fairness based on utilities is the right approach.

To illustrate, consider a simple example in which there are two people and one divisible consumption good. Utility is defined as some interpersonally-comparable measure of pleasure that is derived from consumption. The two people differ in how much they enjoy consumption. If person 1 consumes c units of consumption, then his utility is $A_1 c^\alpha$ (where $\alpha \in (0,1)$). If person 2 consumes c units of consumption, then is utility is $A_2 c^\alpha$. According to the utilitarian criterion, an equitable allocation is optimal only if $A_1 = A_2$. Otherwise, it is better to allocate more consumption to the person who is more "productive" at transforming consumption into "utils". When the difference between A_1 and A_2 is large, then the optimal decision rule approximates the dictatorial one, and allocates most of the consumption good to one person.

Dworkin proposes an alternative approach, according to which an equitable allocation is always optimal, regardless of the productivity parameters, A_1 and A_2. According to Dworkin, what matters for social welfare is the distribution of incomes, and not the utilities that individuals derive from those incomes. This captures an idea of fairness that is based on the quantity of resources that is devoted to the life of each individual.

This paper studies the broader implications of this approach for social welfare analysis. We introduce a new axiom, Income Anonymity, which states that each person's income should matter equally for social welfare. Although we are the first to formally study this axiom (to the best of our knowledge), it is implicit in any welfare criterion that is based on the income distribution. An obvious advantage of this approach is that it does not rely on the existence or measurement of an interpersonally-comparable utility; the only things that matter are the distribution of income and the prices that people face.

Our main result is that, combined with the Pareto condition, Income Anonymity implies concrete guidelines on how to do social welfare analysis. Whenever preferences are heterogeneous, then there exists at most one social preference relation that satisfies Pareto and Income Anonymity. This result is powerful as it means that if we agree on Pareto and on Income Anonymity, then we should also agree on everything else, from optimal trade agreements to optimal redistributive policy.

We obtain analytical characterizations of the social welfare function for three classes of individual preferences: homothetic preferences, Stone-Geary preferences, and Gorman form preferences with a common slope. However, for certain individual preference profiles, there can be no social preference relation that satisfies both Pareto and Income Anonymity. Loosely speaking, existence requires that non-homotheticities are common to all individuals. The possibility that the two axioms are inconsistent reveals a tension between the Pareto condition and the idea that fairness should be judged based on the distribution of incomes.

This paper contributes to a rich literature on the axiomatic characterization of social preferences. The approach here is closely related to the money-metric utility approach (Deaton and Muellbauer [1980], Fleurbaey and Maniquet [2011], Fleurbaey and Maniquet [2018]). The money-metric utility approach proposes a social preference relation that is an aggregation of individuals' equivalent incomes at common reference prices ("money-metric utilities"). There are many social welfare functions that satisfy anonymity with respect to the distribution of money-metric utilities. The anonymity condition that we consider is more restrictive, as it requires anonymity with respect to the distribution of income at any prices – not just at the reference price. The social preferences must therefore be symmetric in money-metric utilities for any reference price.

This paper is also related to the literature on price-independent welfare prescriptions (Roberts [1980], Slesnick [1991], Blackorby et al. [1993] and Fleurbaey and Blanchet [2013]). Roberts [1980] studies the conditions under which income distributions can be ranked irrespective of prices, and finds that they are "highly restrictive".[1] This sparked a debate about whether it is appropriate to require welfare prescriptions to be independent of prevailing prices (Fleurbaey and Blanchet [2013]). The analysis here suggests that it is not. The social preference relation that we derive generates price-independent welfare prescriptions only in special cases. This leaves open the possibility that income inequality is more tolerable when necessities are cheaper, consistent with the intuition that one person's necessities take

[1] He finds that, unless the welfare criterion is dictatorial, both individual and social preferences must be homothetic.

precedent over another person's luxuries.

Finally, this paper is related to the literature on the inconsistency of Pareto and other normative principles. Sen [1970a] and Kaplow and Shavell [2001] show that the Pareto principle implies a social ranking that is welfarist. This leaves limited room for expressing a concern for non-welfarist normative principles such as freedom, justice or procedural fairness (see also Sher [2021]). In the context of resource allocation problems, Sen [1970b], Suzumura [1981a], Suzumura [1981b], Suzumura [1983], Tadenuma [2002] and Fleurbaey and Trannoy [2003] uncover tensions between Pareto and various egalitarian principles pertaining to the fair allocation of resources. Fleurbaey and Trannoy [2003] show that, whenever preferences are heterogeneous, Pareto is inconsistent with a social preference for redistributing resources from rich to poor (the Pigou-Dalton principle).[2] We add to these impossibility results by showing that, for some preference profiles, Pareto is also inconsistent with Income Anonymity, which is a procedural fairness requirement. However, in light of the negative results in this literature, our positive results are perhaps more surprising: we show that, for various preference profiles (including profiles of heterogeneous preferences), there is no conflict between Income Anonymity and Pareto.

1 Preliminaries

There are $2 \leq I < \infty$ individuals indexed $i = 1, ..., I$, and $2 \leq J < \infty$ goods. Individual i's indirect preferences over combinations of income, $m \in \mathbb{R}_{++}$, and prices $p = (p^1, ..., p^J) \in \mathbb{R}^J_{++}$, are represented by $v_i(m,p)$, where v_i is homogeneous of degree 0, strictly increasing in m and strictly decreasing in each p^j. Let $\preceq_i$ denote the binary preference relation represented by v_i:

$$(m,p) \preceq_i (m',p') \Leftrightarrow v_i(m,p) \leq v_i(m',p')$$

Throughout, we use bold letters to denote vectors of length I (the number of individuals). The social preference ranking, $\preceq$ (with no subscript), is defined over elements of the form $(\mathbf{m}, \mathbf{p})$, where $\mathbf{m} = (m_1, ..., m_I) \in \mathbb{R}^I_{++}$ is the distribution of income and $\mathbf{p} = (p_1, ..., p_I) \in \mathbb{R}^J_{++}$ are individual price vectors (individual i faces the prices $p_i = (p_i^1, ..., p_i^J)$). For a price vector p, the notation $(\mathbf{m}, p)$ is used as a shorthand for the allocation $(\mathbf{m}, (p, ..., p))$.

We consider two axioms on the social preference relation. Like all axioms, their normative appeal is not universal; it is always possible to come up with examples in which the axioms contradict basic moral intuitions. The axioms below are not meant to apply to all possible

[2]More precisely, the Pigou-Dalton principle states that if one person has more of every good than another person, then any transfer of goods that maintains this ordering but reduces inequality is a socially-desirable transfer.

situations; instead, they characterize the environments in which the social preference relation that we derive is applicable.

The first axiom is the standard Pareto condition, which is stated as follows.

Axiom (Pareto)**.** *For each* $\mathbf{m}, \mathbf{m}', \mathbf{p}, \mathbf{p}'$*, (a) if* $(m_i, p_i) \preceq_i (m_i', p_i')$ *for all* i*, then* $(\mathbf{m}, \mathbf{p}) \preceq (\mathbf{m}', \mathbf{p}')$*; and (b) if, in addition,* $(m_i, p_i) \prec_i (m_i', p_i')$ *for some* i*, then* $(\mathbf{m}, \mathbf{p}) \prec (\mathbf{m}', \mathbf{p}')$*.*

This axiom is sometimes referred to as "unanimity". It states that, if all individuals support a certain change in prices and incomes, then it should be considered socially desirable. This axiom rules out situations in which people's decisions are not in their best interests, such as drug addiction. But it is broadly thought to be an appealing condition in most circumstances.

The second axiom is Income Anonymity:

Axiom (Income Anonymity)**.** *For every common price vector* $p \in \mathbb{R}^J_{++}$ *and income distribution* $\mathbf{m} \in \mathbb{R}^I_{++}$*, it holds that*

$$(\mathbf{m}, p) \sim ((m_{\sigma(1)}, ..., m_{\sigma(I)}), p)$$

for any permutation $\sigma : \{1, ..., I\} \mapsto \{1, ..., I\}$*.*

Income Anonymity states that the normative ranking of income distributions should not depend on which person receives which income. Instead, income distributions can be ranked anonymously.

This anonymity condition is in the spirit of Dworkin [1981].[3] Dworkin [1981] argues that fairness should be judged based on the distribution of income, regardless of individuals' preferences, and regardless of prevailing prices.[4] According to Dworkin, what matters is the value of the resources devoted to each person's life – and not how that individual chooses to use those resources, or the utility that he derives from their use.

Dworkin writes, "the true measure of the social resources devoted to the life of one person is fixed by asking how important, in fact, that resource is for others." This implies a measure of value that is based on common equilibrium prices. To reflect this, Income Anonymity requires anonymity with respect to income only when all people face the same price vectors. In this case, the value of each person's consumption bundle is given by that person's income.

[3] Varian [1976] also considers the question of how to define a fair allocation when people have heterogeneous preferences. He discusses the sense in which an equal distribution of income results in a better market allocation than an unequal distribution of income.

[4] See Keller [2002] for a discussion. A similar notion of equality is also reflected in the Laisser-Faire axiom in Fleurbaey and Maniquet [2006], which postulates that there is no scope for redistribution between two people who face the same earning opportunities, even when they choose to work different amounts.

As with all axioms, the appeal of this axiom is not universal. As Sen [1980] illustrates, Income Anonymity is particularly controversial when people have heterogeneous needs. Sen [1980] considers an example of two people, a disabled person and an able-bodied person. The disabled person needs to spend a certain amount on accessibility in order to achieve a level of mobility that the able-bodied person obtains for free. Because of this, at a given income level, he will have less money to spend on other things, and more limited consumption possibilities.[5]

Dworkin himself concedes that "certain kinds of preferences, which people wish they did not have, may call for compensation as handicaps".[6] However, Dworkin contends that in many circumstances, resource equality is the appropriate criterion for evaluating fairness. We return to the question of heterogeneous needs in section 5.

2 Uniqueness

On their own, both Pareto and Income Anonymity are consistent with various degrees of inequality aversion. For example, the Pareto condition is consistent with any social welfare function of the form

$$W(\mathbf{m}, \mathbf{p}) = \sum_{i=1}^{I} \phi_i(v_i(m_i, p_i))$$

where ϕ_i is strictly increasing. In this class of social welfare functions, the concavities of the functions $\{\phi_i\}_{i=1}^{I}$ determine aversion to inequalities in utilities, and hence, indirectly, aversion to income inequality.

Similarly, Income Anonymity it is consistent with any social welfare function of the form

$$W(\mathbf{m}, \mathbf{p}) = \sum_{i=1}^{I} \frac{m_i^{1-\eta}}{1-\eta}$$

here, η determines the degree of inequality aversion. This functional form is often identified as the Atkinson index (Atkinson [1970]). In this class of social welfare functions, Income Anonymity is satisfied because the social ranking is symmetric with respect to all incomes.

When all individuals have the same preferences, there are many social preference relations that are consistent with both axioms. For example, when $v_i = v$ for all i, then both axioms

[5] In light of this concern, Roemer [2009], Fleurbaey and Maniquet [2012] and Piacquadio [2017] argue for fairness conditions based on consumption opportunity sets, rather than income.

[6] Dworkin [1981], Page 288, footnote 4.

are satisfied when the social preference relation is represented by

$$W(\mathbf{m}, \mathbf{p}) = \sum_{i=1}^{I} \phi(v(m_i, p_i))$$

here, the concavity of ϕ determines the amount of inequality aversion.

The following theorem establishes that, when preferences are heterogeneous, the combination of Pareto and Income Anonymity uniquely characterizes the entire social preference relation. In particular, only one level of inequality aversion is consistent with both axioms.

Theorem 1. *Assume that individual preferences are heterogeneous. If there exists a social preference relation that satisfies Pareto and Income Anonymity, then it is unique.*

The complete proof is in the appendix, together with other omitted proofs. Below we sketch its key steps. Consider a simple case in which there are two individuals and two goods ($I = J = 2$). Normalize the price of good 1 to be $p^1 = 1$, so that both individuals face the price vector $p = (1, p^2)$.

Figure 1 presents the two individuals' indifference curves over combinations of income and prices. Their indifference curves are upward sloping, because higher prices can be compensated with higher incomes. In this figure, the two individuals have different preferences over the two goods, so their indifference curves are not the same.

Let $p' = (1, p'^2)$ be such that $p'^2 < p^2$, as in Figure 1. Assume that individual 1 is indifferent between (m, p) and (m'_1, p'), and individual 2 is indifferent between (m, p) and (m'_2, p'). Pareto requires that, when all individuals are indifferent, then the social preference relation is indifferent as well. Consequently, Pareto implies

$$((m, m), p) \sim ((m'_1, m'_2), p') \tag{1}$$

Income Anonymity requires that, at any given prices, income distributions can be ranked anonymously. In particular, social preferences must be indifferent with respect to switching the incomes of the two individuals at the prices p':

$$((m'_1, m'_2), p') \sim ((m'_2, m'_1), p') \tag{2}$$

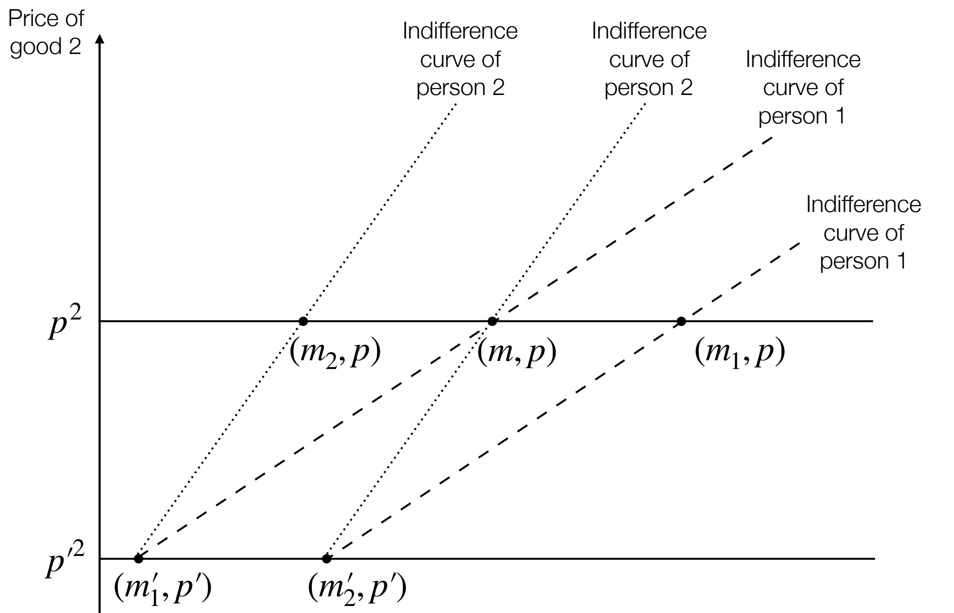
Let m_1 and m_2 be such that the following indifference conditions are satisfied:

$$(m'_1, p') \sim_2 (m_2, p) \text{ and } (m'_2, p') \sim_1 (m_1, p) \tag{3}$$

Figure 1 illustrates this construction. By Pareto, it follows that

$$((m'_2, m'_1), p') \sim ((m_1, m_2), p) \tag{4}$$

Figure 1: Sketch of the proof of uniqueness



The transitivity of the indifference relation implies that

$$((m, m), p) \sim ((m'_1, m'_2), p') \sim ((m'_2, m'_1), p') \sim ((m_1, m_2), p) \tag{5}$$

Consequently, given a price level of p, any social preferences that satisfy Pareto and Income Anonymity must be indifferent between the equal allocation of income, (m, m), and the unequal allocation, (m_1, m_2). This is a restriction on the amount of inequality aversion. A sufficiently inequality-averse social preference relation would strictly prefer the equal allocation over the unequal one; a sufficiently inequality-tolerant relation would prefer the unequal allocation instead. Neither of these preferences would be consistent with the combination of Pareto and Income Anonymity. The uniqueness of the social preference relation follows from the uniqueness of the level of inequality aversion.

3 Existence

To characterize the conditions under which the two axioms are consistent, it is useful to introduce the following definitions. Let $H : \mathbb{R}^J_{++} \mapsto \mathbb{R}_{++}$ be a strictly increasing function, and let $d : \mathbb{R}_{++} \times \mathbb{R}^J_{++} \mapsto \mathbb{R}_{++}$ be a function that is strictly increasing in the first argument and homogeneous of degree 1. Define the function $V(\cdot,\cdot|H,d)$ as

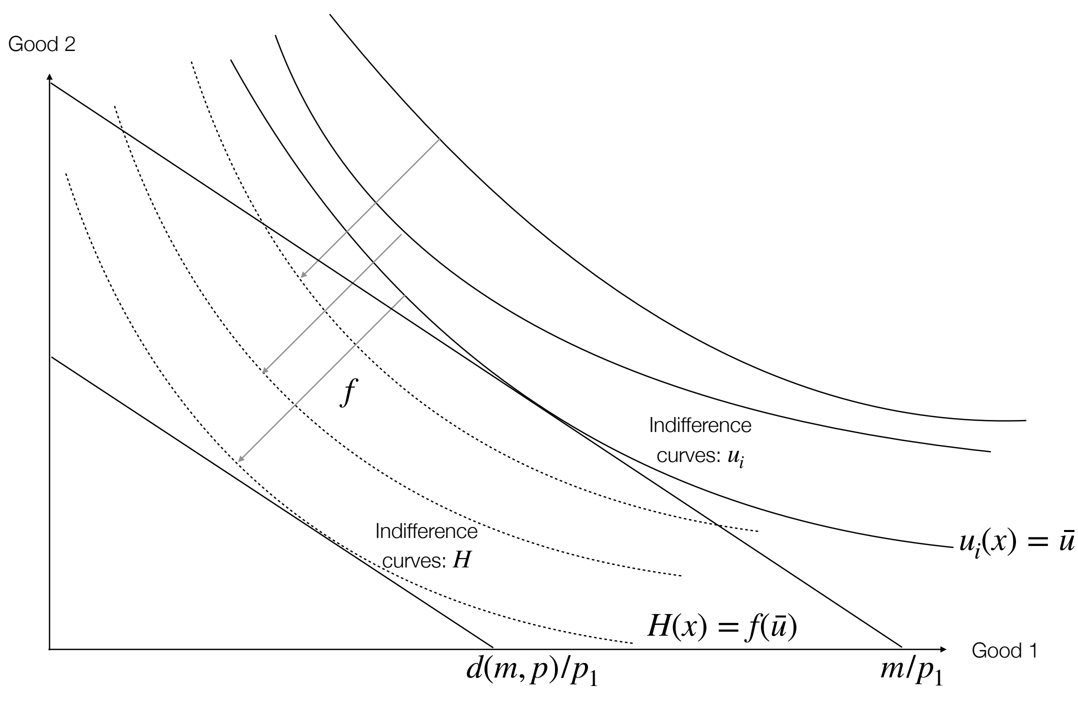
$$V(m,p|H,d) = \max_x H(x) \text{ s.t. } \sum_{j=1}^{J} p^j x^j \leq d(m,p) \tag{6}$$

This optimization problem can be interpreted as a consumer optimization problem in which the consumer's preferences are represented by the utility function H, and his budget constraint allows him to allocate $d(m,p)$ dollars towards the objective of maximizing his utility. The function d can be interpreted as a "disposable income function", as it specifies how much the consumer can spend given gross income m when faced with prices p.

Definition. *The pair* (H,d) *is an* interpretation *of the preferences* $\preceq_i$ *if* $V(\cdot,\cdot|H,d)$ *is a representation of the indirect preference relation* $\preceq_i$*.*

The pair (H,d) is an interpretation of individual i's preferences if, based on his indirect preference ranking, we cannot reject the hypothesis that the optimization problems that he faces are the ones in expression 6.

Figure 2: Constructing an interpretation



Every preference relation has many interpretations. To see this, consider the construction in Figure 2. In this figure, the solid curves are indifference curves of the preference relation of individual i, which is represented by some utility function, u_i. The dashed curves are indifference curves of another preference relation represented by H. The gray arrows represent a monotone bijection, $f : u_i(\mathbb{R}^J_{++}) \mapsto H(\mathbb{R}^J_{++})$.

Given H and the mapping f, the disposable income, $d(m,p)$, is implicitly defined based on the condition

$$f(u_i(x(m,p))) = \max_x H(x) \text{ s.t. } \sum_{j=1}^{J} p^j x^j \leq d(m,p)$$

where

$$x(m,p) = \arg\max_x u_i(x) \text{ s.t. } \sum_{j=1}^{J} p^j x^j \leq m$$

Note that $u_i(x(\cdot,\cdot))$ represents the indirect preference relation, $\preceq_i$. Because f is strictly monotone, the indirect preferences represented by $f(u_i(x(\cdot,\cdot)))$ are the same as the preferences represented by $\preceq_i$. Notice that there are infinitely many possibilities for specifying f, and that each of them implies a different d. It follows that, for each H, there are infinitely many options for specifying d such that $V(\cdot,\cdot|H,d)$ is a representation of the indirect preferences $\preceq_i$.

The theorem below uses this construction to lay out a necessary and sufficient condition for the consistency of Pareto and Income Anonymity.

Theorem 2. *There exists a social preference relation that is consistent with Pareto and Income Anonymity if and only if there exists a disposable income function, d, and homogeneous utility functions, $\{H_i\}_{i=1}^I$, such that, for every i, (H_i, d) is an interpretation of $\preceq_i$.*

This theorem provides a necessary and sufficient condition for the existence of a social preference relation that satisfies Pareto and Income Anonymity. Combined with the uniqueness result, it also implies the following corollary.

Corollary 1. *Assume that preferences are heterogeneous, and let d and $\{H_i\}_{i=1}^I$ be functions that are homogeneous of degree 1 such that $V(\cdot,\cdot|H_i,d)$ represents the preferences $\preceq_i$. Then, there exists a unique social preference relation that satisfies Pareto and Income Anonymity, and it can be represented by*

$$W(\mathbf{m},\mathbf{p}) = \sum_{i=1}^{I} \ln(V(m_i,p_i|H_i,d))$$

This corollary implies the following special cases.

Homothetic preferences. Assume that $\{H_i\}_{i=1}^I$ are representations of individual preferences that are homogeneous of degree 1, and let $d(m,p) = m$. Trivially, (H_i, d) is an interpretation of $\preceq_i$. It follows that the unique social preference relation that satisfies Pareto and Income Anonymity is represented by

$$W(\mathbf{m}, \mathbf{p}) = \sum_{i=1}^{I} \ln(H_i(\hat{x}_i(m_i, p_i))) = \sum_{i=1}^{I} \ln(m_i) + \sum_{i=1}^{I} \ln(H_i(\hat{x}_i(1, p_i)))$$

where

$$\hat{x}_i(m, p) = \arg\max_x H_i(x) \text{ s.t. } \sum_{j=1}^{J} p^j x^j \leq m \tag{7}$$

This social welfare criterion has two interesting properties, which have been pointed out elsewhere in the literature. First, as noted by Roberts [1980], this social welfare criterion generates price-independent welfare prescriptions: the ranking of income distributions does not depend on the prices, even when different people face different prices.

Second, as noted by Milleron [1970], this welfare criterion implies that, in an exchange economy, it is welfare-maximizing to divide the aggregate endowment equally between individuals. According to this welfare criterion, resource equality is always welfare-maximizing, provided that people can freely trade.

Stone-Geary preferences. The most common specification of non-homothetic preferences are Stone-Geary preferences. Assume that individual i's preferences are represented by

$$H_i(x - \bar{x})$$

where, for each j, $\bar{x}^j \geq 0$ and H_i is homogeneous of degree 1 (and, for any x such that $x^j < 0$ for some j, $H(x) \equiv 0$). One technical problem with considering these preferences is that they are not strictly monotone. However, it is possible to approximate these preferences arbitrarily-well with a strictly monotone function.[7]

The bundle $\bar{x}$ is typically interpreted as a "subsistence requirement". This interpretation implies a model in which, unless a person consumes at least $\bar{x}^j$ of good j, he will die. According to this interpretation, any non-homotheticities are due to the presence of expenditures that are necessary in order to avoid unacceptable harm.

Define $d(m, p)$ as income net of the cost of the subsistence bundle,

$$d(m, p) = m - \sum_{j=1}^{J} p^j \bar{x}^j$$

[7] For example, let $H_n(x) = H(g_n^1(x^1), ..., g_n^J(x^J))$, where $g_n^j(x^j) = \min\{\left(x^j/\bar{x}^j\right)^n, 1\}x^j$. Note that $H_n(x) \to_{n\to\infty} H(x - \bar{x})$, and that H_n is strictly monotone.

To see that (H_i, d) is an interpretation of $\preceq_i$, note that, given these preferences, consuming at least $\bar{x}^j$ of each good j always takes precedent over consuming good k in excess of $\bar{x}^k$. It follows that the objective of maximizing $H_i(x-\bar{x})$ is equivalent to the objective of maximizing $H_i(\tilde{x})$ (where $\tilde{x} = x - \bar{x}$), subject to the constraint that

$$\sum_{j=1}^{J} p^j \tilde{x}^j \leq m - \sum_{j=1}^{J} p^j \bar{x}^j = d(m, p)$$

Hence, in this setting, (H_i, d) is an interpretation of individual i's indirect preferences.

By Corollary 1, if $\{H_i\}_{i=1}^I$ are heterogeneous, then there exists a unique social preference relation that satisfies Pareto and Income Anonymity, which can be represented by

$$W(\mathbf{m}, \mathbf{p}) = \sum_{i=1}^{I} \ln(H_i(\hat{x}_i(m_i - \sum_{i=1}^{I} p_i^j \bar{x}^j, p_i))) = \sum_{i=1}^{I} \ln\left(m_i - \sum_{i=1}^{I} p_i^j \bar{x}^j\right) + \sum_{i=1}^{I} \ln(H_i(\hat{x}_i(1, p_i)))$$

(where $\hat{x}_i$ is defined in expression 7).

Gorman form preferences. Another common assumption is that indirect preferences are represented by the Gorman form,

$$v_i(p, m) = a_i(p) + b(p)m$$

where the terms $\{a_i(p)\}_{i=1}^I$ may vary across individuals, but the term $b(p)$ is common across individuals. This assumption is usually evoked for the purpose of generating aggregate demand curves that depend only on aggregate income (and not on how income is distributed across individuals).

One example of such preferences are quasilinear preferences over goods, in which individual preferences are represented by

$$u_i(x) = x^1 + g_i(x^2, ..., x^J)$$

where g_i is a strictly increasing, strictly concave function that satisfies the Inada conditions, and x^1 is a "numeraire" good (note that these preferences are defined for every $x^1 \in \mathbb{R}$ – that is, the individual may consume negative quantities of x^1). The Gorman form representation for these quasilinear preferences is

$$a_i(p) = -\frac{\sum_{j=2}^{J} p^j x_i^j(p)}{p^1} + g_i(x_i^2(p), ..., x_i^J(p)) \text{ and } b(p) = \frac{1}{p^1}$$

Consider the social preference relation represented by

$$W(\mathbf{m},\mathbf{p}) = \sum_{i=1}^{I} (a_i(p_i) + b(p_i)m_i)$$

This social preference relation satisfies Pareto because it is represented by the sum of individual indirect utility functions. It also satisfies Income Anonymity because, when $p_i = p_k$ for every i, k, then $b(p_i) = b(p_k)$ for every i, k, and hence it is symmetric in $m_1, ..., m_I$.

By Theorem 1, if $\{a_i\}_{i=1}^{I}$ are heterogeneous, then this social preference relation is the unique one that satisfies both Pareto and Income Anonymity. A surprising implication is that, under these assumptions, these two axioms are inconsistent with aversion to income inequality. According to this welfare criterion, when everyone faces the same prices, then income distributions should be ranked according to the sum of individual income levels. Only the mean of the income distribution matters, and income inequality does not.

4 A model of non-homotheticities

How reasonable is the condition in Theorem 2? This section presents a behavioral model of non-homotheticies, in which all non-homotheticities reflect the need to spend some money in order to prevent unacceptable harm. In this model, Pareto and Income Anonymity are consistent whenever people have the same necessities.

Consider a model in which each indirect preference relation, $\preceq_i$, has a correct interpretation, (H_i^*, d_i^*). In this model, individual i's true objective is to maximize the utility function $H_i^*(x)$. However, the income that he is able to allocate towards this goal is only $d_i^*(m, p)$. The optimization problem that individual i actually faces is

$$\max_x H_i^*(x) \text{ s.t. } \sum_{j=1}^{J} p^j x^j \leq d_i^*(m, p)$$

Here, the disposable income, $d_i^*(m, p)$, is interpreted broadly as income net of expenditures that are necessary for preventing unacceptable harm. This extends the standard definition of disposable income, which is income net of taxes. The rationale for excluding taxes is that people do not really choose how much taxes to pay. If a person does not pay his required taxes, he will go to prison. This unacceptable harm means that he has no real choice: how much taxes he pays tells us nothing about his "preference" for taxes or even about his "preferences" for staying out of prison.

A similar logic suggests the deduction of other expenditures that are necessary in order to prevent unacceptable harm. For example, if people do not eat, they die. If a disabled person does not have a wheelchair, she is immobile. Here, we interpret $d^*(m, p)$ as the income available for unnecessary expenditures.

Given this interpretation, the utility function H_i^* represents the individual's preferences over unnecessary expenditures. After the individual pays all of his "taxes" – both to the government and to himself – then he is free to choose how to spend the remainder of his income. The key assumption is that these preferences are homothetic: in other words, all non-homotheticicities are due to genuine needs, in the sense that they are expenditures that are meant to prevent unacceptable harm.

Assumption 1. *For every i, H_i^* is homogeneous of degree 1.*

The plausibility of this assumption depends on which harms are considered "unacceptable". If the only unacceptable harm is death, then this assumption is implausible, because many goods are necessity goods in the consumer-theory sense but are not strictly necessary for survival. For example, the expenditure share on cell phones is decreasing in income, so they are classified as necessity goods.[8] However, humanity has survived for many years without them, so they are clearly not necessary for survival.

The assumption that preferences over unnecessary expenditures are homothetic is more plausible if we consider a broader range of harms unacceptable. If social isolation is unacceptable, then expenditure on telecommunications may be a genuine necessity. Evaluating the plausibility of this model requires both a normative assessment of which harms are unacceptable, and an empirical evaluation of whether, after deducting expenditures that are necessary in order to prevent such harms, people's preferences over the remaining expenditures are homothetic.

Corollary 2. *Assume that $d_i^* = d_k^*$ for all i, k. Under Assumption 1, there exists a social preference relation that satisfies Pareto and Income Anonymity, which can be represented by*

$$W(\mathbf{m}, \mathbf{p}) = \sum_{i=1}^{I} \ln(H_i^*(\hat{x}_i(d_i^*(m_i, p_i), p_i))) = \sum_{i=1}^{I} \ln(d_i^*(m_i, p_i)) + \sum_{i=1}^{I} \ln(H_i^*(\hat{x}_i(1, p_i))) \quad (8)$$

This Corollary is immediate given Corollary 1. Under the assumption that preferences over unnecessary expenditures are homothetic, Pareto and Income Anonymity are consistent with one another whenever individuals have the same genuine necessities.

Sen [1980] points out that, when this isn't the case, Income Anonymity is ethically problematic. According to Sen, two people who have the same income but different expenditures on necessities should not be considered equal, because they have different consumption possibilities.

[8] See, for example, Breunig and McCarthy [2020].

Sen's critique suggests that we should reject Income Anonymity in situations in which $\{d_i^*\}$ are heterogeneous across individuals. Notably, in this model, Income Anonymity is consistent with Pareto whenever it is immune to Sen's critique.

5 Extension: heterogeneous necessities

Sen's critique suggests a natural modification of Income Anonymity. In this framework, heterogeneity in needs corresponds to heterogeneity in $\{d_i^*\}_{i=1}^I$. In the presence of such heterogeneity, Income Anonymity is ethically unappealing, and it makes sense to modify Income Anonymity as follows.

Axiom (Income Anonymity with Respect to Disposable Income (Income Anonymity-D))**.** *For every common price vector $p \in \mathbb{R}_{++}^J$ and income distributions $\mathbf{m}, \mathbf{m}' \in \mathbb{R}_{++}^I$ such that*

$$d_i^*(m_i, p) = d_{\sigma(i)}^*(m'_{\sigma(i)}, p)$$

for some permutation $\sigma : \{1, ..., I\} \mapsto \{1, ..., I\}$, it holds that

$$(\mathbf{m}, p) \sim (\mathbf{m}', p)$$

Income Anonymity-D is equivalent to Income Anonymity whenever individuals have the same disposable income functions. More generally, Income Anonymity-D states that, when everyone faces the same prices, income distributions can be evaluated based on the distribution of *disposable* income, from which expenditures on genuine necessities are deducted. Under this axiom, two people who have the same disposable income should be considered equal, even if they have different preferences, different cardinal utilities, or different needs. As long as they both have the same amount of money to devote to unnecessary expenditures, the social preference relation should treat them equally.

This criterion is immune to Sen's critique, because expenditures that compensate for disabilities can be considered necessary. At the same time, it maintains Dworkin's emphasis on resource equality, and does not rely on interpersonal comparisons of utilities. For example, starvation is considered equally unacceptable for everyone - both for people who enjoy life (and derive a lot of "utility" from it) and for people whose experiences are more mixed.

While satisfying Income Anonymity-D does not require cardinal information on utility, it does require information on the disposable income functions, $\{d_i^*\}_{i=1}^I$. As commented earlier, the true disposable income function cannot be inferred from the individual's ordinal preference ranking, because each ranking has infinitely many interpretations. To satisfy Income Anonymity-D, the social ranking requires independent information about individuals' circumstances. Specifically, it is necessary to know which expenditures are necessary for them to avoid unacceptable harm (and which harms are unacceptable).

The analysis above implies that the combination of Pareto and Income Anonymity-D uniquely characterizes the social preference relation in environments in which people have heterogeneous necessities and heterogeneous homothetic preferences over unnecessary expenditures.

Corollary 3. *Assume that individuals have heterogeneous preferences over unnecessary expenditure that are represented by* $\{H_i^*\}_{i=1}^I$*. Assume further that, for each* i*,* H_i^* *is homogeneous of degree 1. Let* $\{d_i^*\}_{i=1}^I$ *denote individuals' true disposable income functions. Then there exists a unique social preference relation that satisfies Pareto and Income Anonymity-D, which can be represented by equation 8.*

This welfare criterion suggests that, when people have heterogeneous necessities, the social objective should be to maximize the logarithmic sum of disposable incomes – regardless of prices, and even if people face different prices.

6 Conclusion

While there is some disagreement about the appropriate amount of inequality aversion, there is broad agreement on two principles. The first is the Pareto principle, which says that if a certain policy improves everyone's well-being, then it should be adopted. The second is an equal-treatment condition, which states that, unless people have different necessities, two people with the same incomes should be treated equally.

On their own, neither of these principles provide complete guidelines about how to rank income distributions. However, their combination has surprisingly strong implications. For example, when preferences are homothetic, it implies that the social preference relation must rank income distributions according to the sum of log incomes. More generally, there is at most one social preference relation that is consistent with these two principles, and, when it exists, its level of inequality aversion depends on the preference domain.

The equal-treatment condition is more controversial when people have heterogeneous needs. If necessities are more important than luxuries, then it may make sense to allocate more resources towards people who have more necessary expenditures. This suggests an alternative fairness criterion that is based on the amount of income that is available towards unnecessary purchases, after necessities are covered. This modified fairness criterion is consistent with the Pareto condition, provided that preferences over unnecessary expenditures are homothetic. In this case, the two conditions characterize a unique social preference relation, which ranks income distributions according to the logarithmic sum of incomes net of expenditures on necessities.

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A Proofs

The proofs rely on the following theorem.

Theorem 3. *There exist social preferences that satisfy Pareto and Income Anonymity if and only if there exists a function* $v : \mathbb{R}_{++} \times \mathbb{R}^J_{++} \mapsto \mathbb{R}$ *and functions* $\{\gamma_i : \mathbb{R}^J_{++} \mapsto \mathbb{R}\}_{i=1}^I$ *such that, for every* i, $v(m,p) + \gamma_i(p)$ *is a representation of the indirect preferences* $\preceq_i$, *and* $\gamma_1 = 0$.

A.1 Proof of Theorem 3

Assume that such a representation exists. Then, it is straightforward to verify that the social welfare function

$$W(\mathbf{m}, \mathbf{p}) = \sum_{i=1}^{I} v(m_i, p_i) + \gamma_i(p_i) \tag{9}$$

satisfies both Pareto and Income Anonymity.

Conversely, assume that there exists a social preference relation that satisfies Pareto and Income Anonymity. If all individuals have the same ordinal preferences, then any common representation of their indirect preferences satisfies the theorem's condition (with the choice $\gamma_i = 0$ for all i).

Otherwise, assume that individual preferences are heterogeneous. Define the equivalent incomes $\{e_i : \mathbb{R}_{++} \times \mathbb{R}^J_{++} \times \mathbb{R}^J_{++} \mapsto \mathbb{R}_{++}\}_{i=1}^I$ based on the indifference conditions

$$(m, p) \sim_i (e_i(m, p, p'), p') \tag{10}$$

To see that e_i exists and is uniquely defined, note that, for m' sufficiently small, $p'^j x^j(m,p) > m'$ for all j, and hence $(m,p) \succ_i (m',p')$. At the same time, for m' sufficiently large, the bundle $x(m,p)$ is affordable at the prices p', and hence $(m,p) \preceq_i (m',p')$. As individual preferences are strictly increasing and continuous, an intermediate value argument implies that there exists a unique $m' = e_i(m,p,p')$ for which the above indifference condition is satisfied.

Note that the continuity of individual preferences implies that e_i is continuous. In addition, it is useful to note that $e_i(\cdot,p,p') = e_i^{-1}(\cdot,p',p)$, because, if i is indifferent between (m,p) and (m',p'), then $e_i(m,p,p') = m'$ and $e_i(m',p',p) = e_i(e_i(m,p,p'),p',p) = m$.

Given two individuals, $i,i' \in \{1,...,I\}$, define the partial social ranking $\preceq_{i,i'}$ on allocations of the form $((m_i,m_{i'}),p)$ as follows. The partial ranking $\preceq_{i,i'}$ is defined based on the condition

$$((m_i,m_{i'}),p) \preceq_{i,i'} ((m'_i,m'_{i'}),p') \Leftrightarrow ((m_1,...,m_I),p) \preceq (m'_1,...,m'_I,p'),$$

where $m'_k = e_k(m_k,p,p')$ for all $k \neq i,i'$.

To interpret this condition, consider a price change from p to p', which is accompanied by a change in the income distribution that leaves all individuals indifferent, with the exception of individuals i and i' (who see their incomes change to m'_i and $m'_{i'}$, respectively). If this change is socially desirable regardless of the income levels of individuals $k \neq i,i'$, then it holds that $((m_i,m_{i'}),p) \preceq_{i,i'} ((m'_i,m'_{i'}),p')$.

For each pair $i,i' \in \{1,...,I\}$, define a function $z_{i,i'} : \mathbb{R}_{++} \times \mathbb{R}^J_{++} \times \mathbb{R}^J_{++} \mapsto \mathbb{R}_{++}$ as

$$z_{i,i'}(m,p,p') = e_{i'}(e_i(m,p,p'),p',p) \tag{11}$$

Figure 3 illustrates this definition, in a setting with two individuals and two goods where the price of good 1 is normalized to one.[9] This figure also illustrates the proofs of the following claims.

Claim 1. *For every $i,i' \in \{1,...,I\}$, $m \in \mathbb{R}_{++}$ and $p,p' \in \mathbb{R}^J_{++}$, it holds that if $y = z_{i,i'}(m,p,p')$ then $m = z_{i',i}(y,p,p')$.*

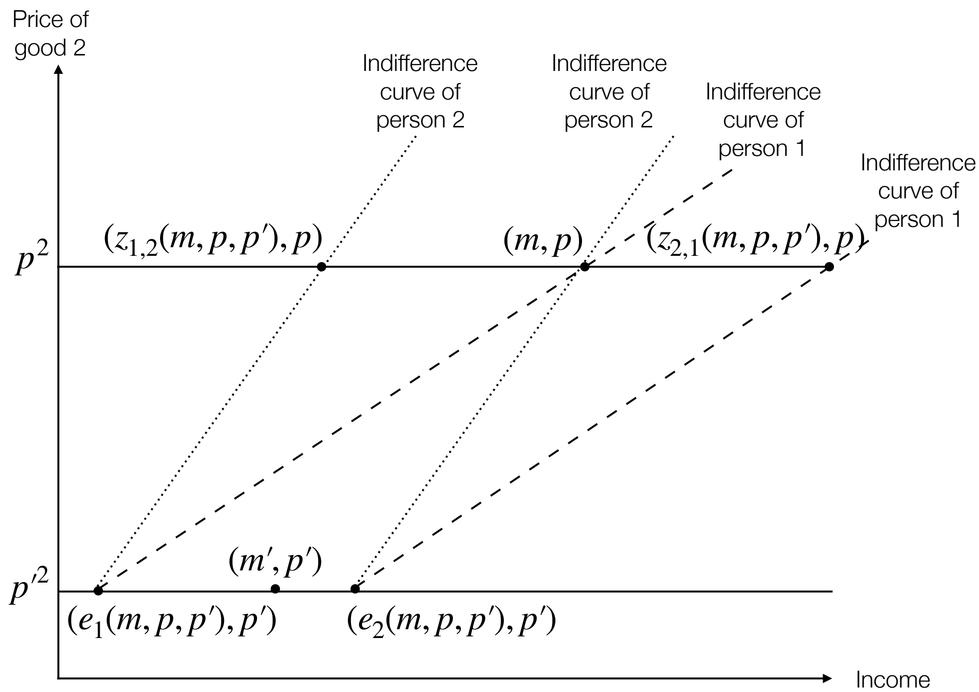
Proof. Let $y = z_{i,i'}(m,p,p')$. Given the definition of $z_{i,i'}$, individual i' is indifferent between (y,p) and $(e_i(m,p,p'),p')$. Hence, $e_{i'}(y,p,p') = e_i(m,p,p')$. Note that

$$z_{i',i}(y,p,p') = e_i(e_{i'}(y,p,p'),p',p) = e_i(e_i(m,p,p'),p',p) \tag{12}$$

Note that, as $e_i(\cdot,p,p') = e_i^{-1}(\cdot,p',p)$, it follows that $e_i(e_i(m,p,p'),p',p) = m$, thus concluding the proof. □

Claim 2. *Let there be $i,i' \in \{1,...,I\}$ such that $(m',p') \preceq_i (m,p)$ and $(m,p) \prec_{i'} (m',p')$. Then, $z_{i',i}(m,p,p') < m < z_{i,i'}(m,p,p')$.*

Figure 3: The definitions of e and z, and the proof of Claims 1 and 2



Proof. Note that $(m',p') \preceq_i (m,p) \sim_i (e_i(m,p,p'),p')$. Hence, $m' \leq e_i(m,p,p')$. It follows that

$$(m',p') \preceq_{i'} (e_i(m,p,p'),p') \sim_{i'} (e_{i'}(e_i(m,p,p'),p',p),p) = (z_{i,i'}(m,p,p'),p)$$

Given the assumption that $(m,p) \prec_{i'} (m',p')$, it follows that $m < z_{i,i'}(m,p,p')$.

Similar steps can be used to show that $z_{i',i}(m,p,p') < m$.

□

Claim 3. *Assume that Pareto and Income Anonymity hold. Then, for any m, m', p, p' and $i, i' \in \{1, ..., I\}$, it holds that*

$$((m,m'),p) \sim_{i,i'} ((z_{i,i'}(m,p,p'), z_{i',i}(m',p,p')),p)$$

Proof. Consider the permutation $\sigma(i) = i'$, $\sigma(i') = i$ and $\sigma(k) = k$ for $k \neq i, i'$. By Income Anonymity,

$$((m,m'),p) \sim_{i,i'} ((m',m),p) \tag{13}$$

By Pareto indifference,

$$((m',m),p) \sim_{i,i'} ((e_i(m',p,p'), e_{i'}(m,p,p')),p') \tag{14}$$

[9] All the other figures in this proof use this same setting.

By Income Anonymity, using the same permutation as above,

$$((e_i(m', p, p'), e_{i'}(m, p, p')), p') \sim_{i,i'} ((e_{i'}(m, p, p'), e_i(m', p, p')), p') \tag{15}$$

By Pareto indifference,

$$((e_{i'}(m, p, p'), e_i(m', p, p')), p') \sim_{i,i'} ((e_i(e_{i'}(m, p, p'), p', p), e_{i'}(e_i(m', p, p'), p', p)), p)$$

$$= ((z_{i,i'}(m, p, p'), z_{i',i}(m', p, p')), p)$$

The proof follows from the transitivity of the indifference relation. □

Claim 4. *Assume that Pareto and Income Anonymity hold. Then, for any m, m', p, p' and $i, i', k, k' \in \{1, ..., I\}$, it holds that*

$$((m, m'), p) \sim_{k,k'} (z_{i,i'}(m, p, p'), z_{i',i}(m', p, p'), p)$$

Proof. By Claim 3, it holds that $((m, m'), p) \sim_{i,i'} (z_{i,i'}(m, p, p'), z_{i',i}(m', p, p'), p)$. Let $\mathbf{m}$ be such that $m_i = m$ and $m_{i'} = m'$. Note that, for every $k \in \{1, ..., I\}$, it holds that $e_k(m_k, p, p) = m_k$. Consequently, by the definition of the partial ranking $\preceq_{i,i'}$, it holds that

$$(\mathbf{m}, p) \sim ((m_1, ..., m_{i-1}, z_{i,i'}(m, p, p'), m_{i+1}, ..., m_{i'-1}, z_{i,i'}(m', p, p'), m_{i'+1}, ..., m_I), p)$$

Define a permutation σ such that $\sigma(i) = k$ and $\sigma(i') = k'$. By Income Anonymity, the social preference relation is indifferent with respect to applying this permutation to the income distribution on both sides of the above expression. Because this indifference holds for every $\mathbf{m}$, the claim follows. □

As preferences are heterogeneous, there exists two people who have different preferences. It is straightforward to establish that this implies different indirect preferences over prices and incomes. Without loss of generality, assume that these people are 1 and 2, and that m_0, m', p_0, p_1 are such that $(m_0, p_0) \preceq_1 (m', p_1)$ but $(m_0, p_0) \succ_2 (m', p_1)$.

Starting from m_0, define a sequence $\{m_k\}_{k=-\infty}^{\infty}$ as follows: for $k > 0$, let $m_k = z_{2,1}(m_{k-1}, p_0, p_1)$, and, for $k < 0$, let $m_k = z_{1,2}(m_{k+1}, p_0, p_1)$. Figure 4 illustrates this construction.

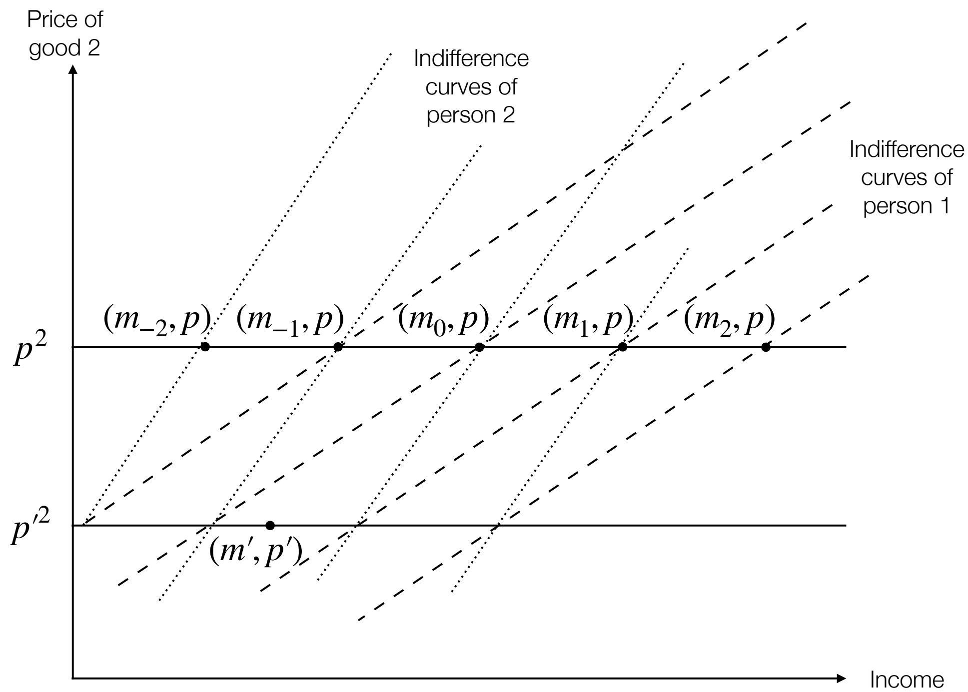
Note that, by Claim 1, for every k (both positive and negative), it holds that

$$m_k = z_{1,2}(m_{k+1}, p_0, p_1) = z_{2,1}(m_{k-1}, p_0, p_1) \tag{16}$$

Claim 5. *For every k, $m_k < m_{k+1}$.*

Proof. For $k = 0$, this follows from Claim 2. Now, suppose that the claim holds for k, that

Figure 4: Construction of $\{m_k\}_{k=-\infty}^{\infty}$



is, $m_k < m_{k+1}$. Thus, $e_2(m_k, p_0, p_1) < e_2(m_{k+1}, p_0, p_1)$. Let m be such that

$$e_2(m_k, p_0, p_1) < m < e_2(m_{k+1}, p_0, p_1) \tag{17}$$

By construction,

$$(m_{k+1}, p_0) = (z_{2,1}(m_k, p_0, p_1), p_0) = (e_1(e_2(m_k, p_0, p_1), p_1, p_0), p_0) \sim_1 (e_2(m_k, p_0, p_1), p_1) \prec_1 (m, p_1)$$

and

$$(m, p_1) \prec_2 (e_2(m_{k+1}, p_0, p_1), p_1) \sim_2 (m_{k+1}, p_0)$$

Hence, by Claim 2, it follows that

$$m_{k+1} < z_{2,1}(m_{k+1}, p_0, p_1) = m_{k+2} \tag{18}$$

Thus completing the induction.

□

Claim 6. *Assume that Pareto and Income Anonymity hold. Then, for every $y > 0$, there exist $k \in \mathbb{Z}$ such that $m_k \leq y < m_{k+1}$.*

Proof. Assume by way of contradiction that the claim is false. It follows that there exists

$y > 0$ for which either $m_k < y$ for every k, or $m_k > y$ for every k, otherwise the set $\{k \in \mathbb{Z}; m_k \leq y\}$ is non-empty and bounded from above, and so $\max\{k \in \mathbb{Z}; m_k \leq y\}$ satisfies the claim. Consider the case where $m_k < y$ for all k (the proof for the case in which $m_k \geq y$ for every m is similar and hence omitted). This implies that $\{m_k\}_{k=-\infty}^{\infty}$ is bounded from above. As $\{m_k\}_{k=0}^{\infty}$ is an increasing and bounded sequence, it converges - let m^* be its limit. By continuity of z, it follows that

$$m^* = \lim_{k\to\infty} m_k = \lim_{k\to\infty} z_{2,1}(m_{k-1}, p_0, p_1) = z_{2,1}(\lim_{k\to\infty} m_{k-1}, p_0, p_1) = z_{2,1}(m^*, p_0, p_1)$$

By Claim 4,

$$((m_0, m^*), p_0) \sim_{1,2} ((z_{1,2}(m_0, p_0, p_1), z_{2,1}(m^*, p_0, p_1)), p_0) = ((m_{-1}, m^*), p_0)$$

Given Claim 5, it holds that $m_{-1} < m_0$, and hence $(m_{-1}, p_0) \prec_1 (m_0, p_0)$. Thus, the Pareto principle requires that $((m_0, m^*), p_0) \succ_{1,2} ((m_{-1}, m^*), p_0)$, which is a contradiction to the above indifference condition.

□

In what follows, we define a sequence of functions $\{\phi_n : \mathbb{R}_{++} \mapsto \mathbb{R}\}_{n=1}^{\infty}$ and a sequence of sequences $\{\{m_{n,k}\}_{k=-\infty}^{\infty}\}_{n=1}^{\infty}$ for which the following properties hold.

Property 1. *There exists $p_n \in \mathbb{R}^J_{++}$ such that, for every k, $m_{n,k} = z_{2,1}(m_{n,k-1}, p_0, p_n)$.*

Property 2. *For every $N < n$, $m_{N,k} = m_{n,2^{n-N}k}$.*

Property 3. *ϕ_n is strictly increasing and continuous.*

Property 4. *For every $N \leq n$, $\phi_n(m_{N,k}) = \phi_N(m_{N,k}) = k/2^N$.*

Property 5. *For every k, $m_{n,k} < m_{n,k+1}$.*

For $n = 1$, define $\{m_{1,k}\}_{k=-\infty}^{\infty} = \{m_k\}_{k=-\infty}^{\infty}$. For some $r > 0$, by Claim 6, it holds that there exists k for which $m_k \leq r < m_{k+1}$. Let $\phi_1(r) = \frac{1}{2}(k + (r - m_k)/(m_{k+1} - m_k))$.

Claim 7. *The sequence $\{m_{1,k}\}_{k=-\infty}^{\infty}$ and function ϕ_1 satisfy Properties 1 to 5.*

Proof. The first property holds for p_1 by definition of $\{m_k\}_{k=-\infty}^{\infty}$. The second property trivially holds, as there is no $N < 1$. The third and fourth properties hold by definition of ϕ_1. The fifth property holds by Claim 5. □

An induction in n is now used to show that these properties hold for $n + 1$.

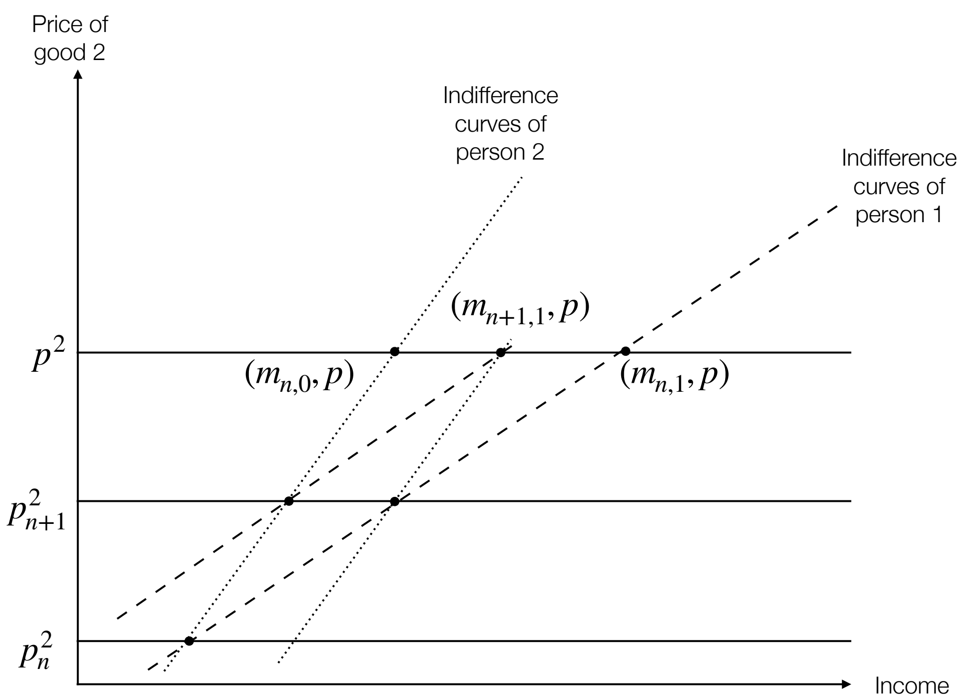
Claim 8. *Assume that Properties 1 to 5 hold for n. Then, there exists* $p_{n+1} \in \mathbb{R}^J_{++}$ *such that*

$$m_{n,1} = z_{2,1}(z_{2,1}(m_{n,0}, p_0, p_{n+1}), p_0, p_{n+1}) \tag{19}$$

Proof. This proof follows from an intermediate value argument. For $p_{n+1} = p_0$, the right hand side takes the value $m_{n,0}$, which is not a solution given that $m_{n,0} < m_{n,1}$. Alternatively, for $p_{n+1} = p_n$, it holds that $z_{2,1}(m_{n,0}, p_0, p_{n+1}) = m_{n,1}$ and hence the right hand side takes the value $m_{n,2}$: for this choice of p_{n+1}, the right hand side is strictly greater than the left hand side (by Property 5). By the continuity of individual preferences, there exists $\eta \in (0, 1)$ such that $p_{n+1} = (1 - \eta)p + \eta p_n$ is a solution to the above equation. □

Let p_{n+1} satisfy equation (19). Define the sequence $\{m_{n+1,k}\}_{k=-\infty}^{\infty}$ as follows. Let $m_{n+1,0} = m_0$, and let $\{m_{n+1,k}\}_{k=-\infty}^{\infty}$ to be consistent with Property 1: for $k > 0$, define $m_{n+1,k} = z_{2,1}(m_{n+1,k-1}, p_0, p_{n+1})$, and for $k < 0$, define $m_{n+1,k} = z_{1,2}(m_{n+1,k+1}, p_0, p_{n+1})$. Figure 5 illustrates this construction.

Figure 5: Construction of $m_{n+1,1}$ and $m_{n+1,2}$



The following claim shows that Property 2 holds.

Claim 9. *Assume that Properties 1 to 5 hold for n, and that Pareto and Income Anonymity hold. Then,*

$$m_{n+1,2k} = m_{n,k} \tag{20}$$

Proof. Note that this holds trivially for $k = 0$. To see that this holds for $k = 1$, note that $m_{n+1,2}$ is given by

$$\begin{aligned} m_{n+1,2} &= z_{2,1}(z_{2,1}(m_{n+1,0}, p_0, p_{n+1}), p_0, p_{n+1}) \\ &= z_{2,1}(z_{2,1}(m_{n,0}, p_0, p_{n+1}), p_0, p_{n+1}) \\ &= m_{n,1} \end{aligned} \tag{21}$$

where the last equality follows from expression 19.

I use induction to show that this holds for every $k > 1$ (the proof that this holds for every $k < 0$ is similar and hence omitted). Assume that equation 20 holds for $k-1$ and k. To establish that this holds for $k+1$, note that, by Claim 4,

$$((m_{n,k-1}, m_{n,k+1}), p_0) \sim_{1,2} ((m_{n,k}, m_{n,k}), p_0)$$

and, similarly,

$$((m_{n+1,2(k-1)}, m_{n+1,2(k+1)}), p_0) \sim_{1,2} ((m_{n+1,2k-1}, m_{n+1,2k+1}), p_0) \sim_{1,2} ((m_{n+1,2k}, m_{n+1,2k}), p_0)$$

as, by the induction hypothesis, equation 20 holds for k, it holds that $(m_{n+1,2k}, m_{n+1,2k}) = (m_{n,k}, m_{n,k})$, it follows that

$$((m_{n,k-1}, m_{n,k+1}), p_0) \sim_{1,2} ((m_{n+1,2(k-1)}, m_{n+1,2(k+1)}), p_0)$$

As, by the induction hypothesis, equation 20 holds for $k-1$, it follows that $m_{n,k-1} = m_{n+1,2(k-1)}$. The above indifference condition can therefore be rewritten as

$$((m_{n,k-1}, m_{n,k+1}), p_0) \sim_{1,2} ((m_{n,k-1}, m_{n+1,2(k+1)}), p_0)$$

The Pareto principle implies that $m_{n,k+1} = m_{n+1,2(k+1)}$. □

Define $\phi_{n+1}(m_{n+1,k}) = k/2^{n+1}$, and extrapolate ϕ_{n+1} to be an increasing and continuous function (to maintain consistency with Property 3). Property 4 then follows from Claim 9 by backward induction, as $\phi_{n+1}(m_{n+1,2k}) = 2k/2^{n+1} = k/2^n = \phi_n(m_{n,k})$.

Claim 10. *Assume that Properties 1 to 5 hold for n, and that Pareto and Income Anonymity hold. Then, Property 5 holds for $n+1$.*

Proof. Using Claim 9 and the induction hypothesis that Property 5 holds for n, it follows that

$$m_{n+1,2k} = m_{n,k} < m_{n,k+1} = m_{n+1,2(k+1)} \tag{22}$$

By Claim 4,

$$((m_{n+1,2k+1}, m_{n+1,2k+1}), p_0) \sim_{1,2} ((m_{n+1,2k}, m_{n+1,2(k+1)}), p_0) \tag{23}$$

If $m_{n+1,2k+1} \leq m_{n+1,2k}$, by expression 22 it follows that $m_{n+1,2k+1} < m_{n+1,2(k+1)}$. Thus, if $m_{n+1,2k+1} \leq m_{n+1,2k}$, the above indifference condition would constitute a violation of the Pareto condition. It follows that $m_{n+1,2k} < m_{n+1,2k+1}$, concluding the proof that the first property holds for $n + 1$. □

Note that Claims 5 and 6 apply to $\{m_{n,k}\}_{k=-\infty}^{\infty}$ for all $n \geq 1$, following the same proofs.

Claim 11. *Assume that, for all $n \geq 1$, the sequence $\{m_{n,k}\}_{k=-\infty}^{\infty}$ satisfies Claim 6. Then, any sequence of weakly increasing and continuous functions $\{\phi_n\}_{n=1}^{\infty}$ that is consistent with Property 4 converges to a weakly increasing and continuous limiting function, $\phi = \lim_{n\to\infty} \phi_n$.*

Proof. I establish that the sequence of function $\{\phi_n\}_{n=1}^{\infty}$ is uniformly Cauchy, and therefore (as ϕ_n is continuous for each n) uniformly converges to a weakly-increasing continuous function ϕ (note that the limit of weakly-increasing functions is weakly-increasing, and that if a sequence of continuous function uniformly converges then the limit is continuous). Let there be $\varepsilon > 0$. We show that, for each $r > 0$, there exists N such that for each $n, n' > N$, it holds that $|\phi_n(r) - \phi_{n'}(r)| < \varepsilon$. To see this, choose N such that $1/2^N < \varepsilon$. Given $r > 0$, let k be such that $m_{N,k} < r \leq m_{N,k+1}$ (such a k exists for every N by Claim 6 and Property 1). Given the definition of ϕ_N as a weakly increasing function, it holds that $\phi_N(r) \in [\phi_N(m_{N,k}), \phi_N(m_{N,k+1})] = [k/2^N, (k+1)/2^N]$. Given Property 4, it holds that, for every $n > N$, $[\phi_N(m_{N,k}), \phi_N(m_{N,k+1})] = [\phi_n(m_{N,k}), \phi_n(m_{N,k+1})]$ and hence $\phi_n(r) \in [k/2^N, (k+1)/2^N]$.

It follows that, if $n, n' > N$, then $\phi_n(r), \phi_{n'}(r) \in [m/2^N, (m+1)/2^N]$ and hence $|\phi_n(r) - \phi_{n'}(r)| \leq 1/2^N < \varepsilon$. This concludes the proof that $\{\phi_n\}_{n=1}^{\infty}$ is uniformly Cauchy, and hence the sequence converges to a weakly-increasing continuous function. □

Using the above claim, define $\phi = \lim_{n\to\infty} \phi_n$, and observe that ϕ is continuous and weakly increasing.

To establish that the limit function ϕ is strictly monotone, consider the following claim:

Claim 12. *Assume that , for all $n \geq 1$, the sequence $\{m_{n,k}\}_{k=-\infty}^{\infty}$ satisfies Claim 6. Let there be $r, r' > 0$ such that $r < r'$. There exist n, k such that $r < m_{n,k} < r'$.*

Proof. Assume by way of contradiction that there are no such n, k.

By Claim 6, there exist k and k' such that $m_{1,k} < r$ and $r' < m_{1,k'}$. Thus, the sets $\{m_{n,k} | m_{n,k} < r, n \geq 1, k \in \mathbb{Z}\}$ and $\{m_{n,k} | r' < m_{n,k}, n \geq 1, k \in \mathbb{Z}\}$ are non-empty, and there exist $\bar{m}, \bar{m} > 0$ such that

$$\bar{m} = \sup\{m_{n,k} | m_{n,k} < r, n \geq 1, k \in \mathbb{Z}\} \text{ and } \bar{m} = \inf\{m_{n,k} | r' < m_{n,k}, n \geq 1, k \in \mathbb{Z}\}$$

Note that $\bar{m} \leq r < r' \leq \bar{m}$.

Define a sequence of functions $\{\tilde{\phi}_n : \mathbb{R}_{++} \mapsto \mathbb{R}\}_{n=1}^{\infty}$ as follows:

$$\tilde{\phi}_n(y) = \begin{cases} \phi(y) & \text{if } y \leq \bar{m} \text{ or } \bar{m} \leq y \\ \psi_1(y) & \text{if } \bar{m} < y < \bar{m} \text{ and } n \text{ is odd} \\ \psi_2(y) & \text{if } \bar{m} < y < \bar{m} \text{ and } n \text{ is even.} \end{cases} \tag{24}$$

where ψ_1 and ψ_2 are two different strictly-increasing functions for which $\psi_1(\bar{m}) = \psi_2(\bar{m}) = \phi(\bar{m})$ and $\psi_1(\bar{m}) = \psi_2(\bar{m}) = \phi(\bar{m})$.

The sequence $\{\tilde{\phi}_n : \mathbb{R}_{++} \mapsto \mathbb{R}\}_{n=1}^{\infty}$ satisfies the assumptions of Claim 11: each $\tilde{\phi}_n$ is continuous, because ϕ is continuous, ψ_1 and ψ_2 are continuous, and the conditions $\psi_1(\bar{m}) = \psi_2(\bar{m}) = \phi(\bar{m})$ and $\psi_1(\bar{m}) = \psi_2(\bar{m}) = \phi(\bar{m})$ are satisfied. Similarly, each $\tilde{\phi}_n$ is weakly-increasing because ϕ, ψ_1 and ψ_2 are weakly increasing (and $\tilde{\phi}_n$ is continuous for each n).

Finally, Property 4 is satisfied because for each n, k, it holds that $\phi(m_{n,k}) = \lim_{n' \to \infty} \phi_{n'}(m_{n,k}) = \lim_{n' \to \infty} \phi_n(m_{n,k}) = \lim_{n' \to \infty} k/2^n = k/2^n$; as, for each n, k, it holds that $m_{n,k} \leq \bar{m}$ or $\bar{m} \leq m_{n,k}$, it holds that, for every n', $\tilde{\phi}_{n'}(m_{n,k}) = \phi(m_{n,k}) = k/2^n$. Hence, Property 4 holds.

By Claim 11, the sequence $\{\tilde{\phi}_n\}_{n=1}^{\infty}$ converges to some function, $\tilde{\phi}$. However, this is a contradiction because, for each n, $\tilde{\phi}_{2n} = \tilde{\phi}_0$ and $\tilde{\phi}_{2n+1} = \tilde{\phi}_1 \neq \tilde{\phi}_0$; thus, the subsequences $\{\tilde{\phi}_{2n}\}_{n=1}^{\infty}$ and $\{\tilde{\phi}_{2n+1}\}_{n=1}^{\infty}$ have different limits, which is a contradiction to the result that the sequence $\{\tilde{\phi}_n\}_{n=1}^{\infty}$ converges.

It follows that there must exist some n, k such that $r < m_{n,k} < r'$. □

Claim 13. *Assume that , for all $n \geq 1$, the sequence $\{m_{n,k}\}_{k=-\infty}^{\infty}$ satisfies Claim 6. Then, ϕ is strictly increasing, and the inverse $\phi^{-1} : \mathbb{R} \mapsto \mathbb{R}_{++}$ exists.*

Proof. Let there be $r < r'$. By Claim 12, there exists N and K such that $r < m_{N,K} < r'$. Further, there exists n, k such that $r < m_{n,k} < m_{N,K}$. If $n \geq N$, then, by Property 2, it holds that $m_{N,K} = m_{n,k'}$ for some k'; as $m_{n,k} < m_{N,K} = m_{n,k'}$, it cannot be that $k = k'$. By Property 5, it must hold that $k < k'$, and hence $k/2^n < k'/2^n = K/2^N$ (where the equality follows from Property 4). Similarly, if $n < N$, there exists some K' such that $m_{n,k} = m_{N,K'}$; it follows that $K' < K$ and hence $k/2^n = K'/2^N < K/2^N$.

Hence, as ϕ is weakly monotone,

$$\phi(r) \leq \phi(m_{n,k}) = \frac{k}{2^n} < \frac{K}{2^N} = \phi(m_{N,K}) \leq \phi(r')$$

Thereby establishing that if $r < r'$, then $\phi(r) < \phi(r')$. Thus, ϕ is strictly increasing. Since it also holds that $\phi(\mathbb{R}_{++}) = \mathbb{R}$, it follows that ϕ is invertible. □

Claim 14. *Assume that Pareto and Income Anonymity hold. Then, for each $r, r', r'' > 0$ such that $((m_0, r), p_0) \sim_{1,2} ((r', r''), p_0)$, it holds that $\phi(r) = \phi(r') + \phi(r'')$.*

Proof. Let $r, r', r'' > 0$ be such that $((m_0, r), p_0) \sim_{1,2} ((r', r''), p_0)$.

Let $\{k(n)\}_{n=1}^{\infty} \subset \mathbb{Z}$ be a sequence of integers such that $\lim_{n\to\infty} k(n)/2^n = \phi(r)$, and let $\{\ell(n)\}_{n=1}^{\infty} \subset \mathbb{Z}$ be a sequence of integers such that $\lim_{n\to\infty} \ell(n)/2^n = \phi(r')$.

I establish that, for every n,

$$\left(\left(m_0, \phi^{-1}\left(\frac{k(n)}{2^n}\right)\right), p_0\right) \sim_{1,2} \left(\left(\phi^{-1}\left(\frac{\ell(n)}{2^n}\right), \phi^{-1}\left(\frac{k(n)-\ell(n)}{2^n}\right)\right), p_0\right) \tag{25}$$

I show this by induction on $L = \ell(n)$. Without loss of generality, assume that $\ell(n) \geq 0$ (the proof for the case $\ell(n) \leq 0$ is analogous). For $L = 0$, this follows trivially from the reflexivity of the indifference relation (note that, as $m_{n,0} = m_0$ for all n, it holds that $\phi_n(m_0) = 0$ for all n, and hence $\phi(m_0) = 0$ and $\phi^{-1}(0) = m_0$). Assume that, for some $L \geq 0$,

$$\left(\left(m_0, \phi^{-1}\left(\frac{k(n)}{2^n}\right)\right), p_0\right) \sim_{1,2} \left(\left(\phi^{-1}\left(\frac{L}{2^n}\right), \phi^{-1}\left(\frac{k(n)-L}{2^n}\right)\right), p_0\right) \tag{26}$$

Note that, by Property 4,

$$\left(\left(\phi^{-1}\left(\frac{L}{2^n}\right), \phi^{-1}\left(\frac{k(n)-L}{2^n}\right)\right), p_0\right) = ((m_{n,L}, m_{n,k(n)-L}), p_0) \tag{27}$$

Using Claim 4, it follows that

$$((m_{n,L}, m_{n,k(n)-L}), p_0) \sim_{1,2} ((z_{2,1}(m_{n,L}, p_0, p_n), z_{1,2}(m_{n,k(n)-L}, p_0, p_n)), p_0) \tag{28}$$

Using Property 1 and Claim 1,

$$((z_{2,1}(m_{n,L}, p_0, p_n), z_{1,2}(m_{n,k(n)-L}, p_0, p_n)), p_0) = ((m_{n,L+1}, m_{n,k(n)-(L+1)}), p_0) = \tag{29}$$

$$\left(\left(\phi^{-1}\left(\frac{L+1}{2^n}\right), \phi^{-1}\left(\frac{k(n)-(L+1)}{2^n}\right)\right), p_0\right)$$

Using the transitivity of the indifference relation, expressions 25-29 establish that

$$\left(\left(m_0, \phi^{-1}\left(\frac{k(n)}{2^n}\right)\right), p_0\right) \sim_{1,2} \left(\left(\phi^{-1}\left(\frac{L+1}{2^n}\right), \phi^{-1}\left(\frac{k(n)-(L+1)}{2^n}\right)\right), p_0\right) \tag{30}$$

Concluding the proof by induction, and thus establishing that, for each n, expression 25 holds.

As $k(n)/2^n \to_{n\to\infty} \phi(r)$ and $\ell(n)/2^n \to_{n\to\infty} \phi(r')$, it follows that $(k(n)-\ell(n))/2^n \to_{n\to\infty} \phi(r)-\phi(r')$. Denote $s = \phi^{-1}(\phi(r)-\phi(r'))$.

Assume by way of contradiction that $s \neq r''$; in particular, without loss of generality, assume that $\phi(s) = \phi(r)-\phi(r') < \phi(r'')$ (the proof for the opposite inequality is similar).

Recall that the sequences $\{k(n)\}_{n=1}^{\infty}$ and $\{\ell(n)\}_{n=1}^{\infty}$ were specified as any sequences of integers for which $k(n)/2^n \to_{n\to\infty} \phi(r)$ and $\ell(n)/2^n \to_{n\to\infty} \phi(r')$. It is therefore possible to choose $\{k(n), \ell(n)\}_{n=1}^{\infty}$ such that $k(n)/2^n > \phi(r)$ and $\ell(n)/2^n < \phi(r')$ for every n.

As ϕ is strictly monotone, the choice $k(n)/2^n > \phi(r)$ implies that $r < \phi^{-1}(\frac{k(n)}{2^n})$. Hence, by the Pareto principle,

$$((m_0, r), p_0) \prec_{1,2} \left(\left(m_0, \phi^{-1}\left(\frac{k(n)}{2^n}\right)\right), p_0\right) \sim_{1,2} \left(\left(\phi^{-1}\left(\frac{\ell(n)}{2^n}\right), \phi^{-1}\left(\frac{k(n)-\ell(n)}{2^n}\right)\right), p_0\right)$$

where the last indifference follows from expression 25. Similarly, as ϕ is strictly monotone, the choice $\ell(n)/2^n < \phi(r')$ implies that $\phi^{-1}\left(\frac{\ell(n)}{2^n}\right) < r'$ and, hence, by the Pareto principle,

$$\left(\left(\phi^{-1}\left(\frac{\ell(n)}{2^n}\right), \phi^{-1}\left(\frac{k(n)-\ell(n)}{2^n}\right)\right), p_0\right) \prec_{1,2} \left(\left(r', \phi^{-1}\left(\frac{k(n)-\ell(n)}{2^n}\right)\right), p_0\right)$$

As $(k(n)-\ell(n))/2^n \to_{n\to\infty} \phi(s) < \phi(r'')$, for n sufficiently large, it holds that $(k(n)-\ell(n))/2^n < \phi(r'')$. As ϕ is strictly increasing and continuous, so is its inverse, and hence $\phi^{-1}\left(\frac{k(n)-\ell(n)}{2^n}\right) < r''$; thus, by the Pareto principle, for n sufficiently large it holds that

$$\left(\left(r', \phi^{-1}\left(\frac{k(n)-\ell(n)}{2^n}\right)\right), p_0\right) \prec_{1,2} ((r', r''), p_0)$$

Combining, these indifference relations imply that

$$((m_0, r), p_0) \prec_{1,2} ((r', r''), p_0)$$

in contradiction to the assumption that $((m_0, r), p_0) \sim_{1,2} ((r', r''), p_0)$. This contradiction establishes that $s = r''$, and hence $\phi(r)-\phi(r') = \phi(s) = \phi(r'')$; rearranging, this implies that $\phi(r) = \phi(r') + \phi(r'')$, concluding the proof of the claim.

□

Define the functions $\upsilon : \mathbb{R}_{++} \times \mathbb{R}^J_{++} \mapsto \mathbb{R}$ and $\{\gamma_i : \mathbb{R}^J_{++} \mapsto \mathbb{R}\}_{i=1}^{I}$ as follows.

$$\upsilon(m, p) = \phi(e_1(m, p, p_0)) \tag{31}$$

$$\gamma_i(p) = \phi(z_{1,i}(m_0, p_0, p)) \tag{32}$$

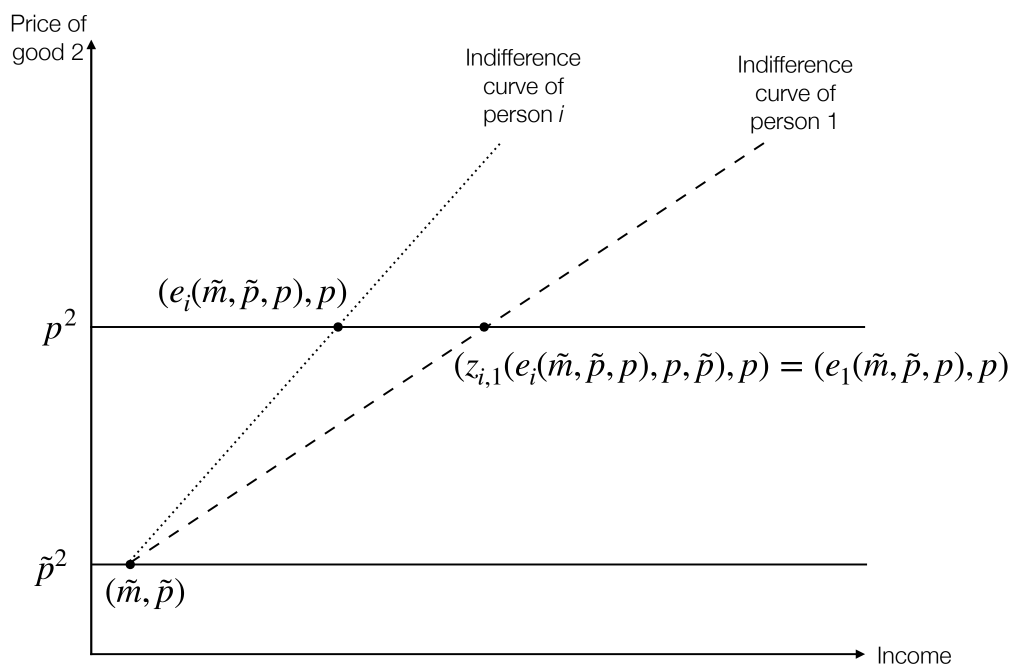
Note that υ is continuous because ϕ and $e_1(\cdot, \cdot, p_0)$ are continuous.

Claim 15. *Assume that Pareto and Income Anonymity hold. Then, for each* (m, p) *and* i*, it holds that* $\upsilon(m, p) + \gamma_i(p) = \phi(e_i(m, p, p_0))$.

Proof. The proof relies on the following identity, which is illustrated in figure 6:

$$z_{i,1}(e_i(m, p, p_0), p_0, p) = e_1(m, p, p_0) \tag{33}$$

Figure 6: The identity in equation 33



Thus, by Claim 4, it holds that

$$((m_0, e_i(m, p, p_0)), p_0) \sim_{1,i} ((z_{1,i}(m_0, p_0, p), z_{i,1}(e_i(m, p, p_0), p_0, p)), p_0) = \tag{34}$$

$$((z_{1,i}(m_0, p_0, p), e_1(m, p, p_0)), p_0)$$

By Income Anonymity and the definition of the partial ranking $\preceq_{1,2}$, it follows that

$$((m_0, e_i(m, p, p_0)), p_0) \sim_{1,2} ((z_{1,i}(m_0, p_0, p), e_1(m, p, p_0)), p_0) \tag{35}$$

By Claim 14, this implies that

$$\phi(e_i(m,p,p_0)) = \phi(z_{1,i}(m_0,p_0,p)) + \phi(e_1(m,p,p_0)) \tag{36}$$

By the definitions of $\upsilon(m,p)$ and $\gamma_i(p)$ (equations 31 and 32), the claim follows. □

Note that, since ϕ is monotonic and $e_i(\cdot,\cdot,p_0)$ is a representation of the indirect preferences of individual i over different combinations of incomes and prices, it holds that $\phi(e_i(\cdot,\cdot,p_0)) : \mathbb{R}_{++} \times \mathbb{R}^J_{++}$ also represents individual i's indirect preferences over income and prices. Using the above claim, it follows that $\upsilon(m,p) + \gamma_i(p)$ is a representation of the indirect preferences of individual i, concluding the proof of the theorem.

A.2 Proof of Theorem 1

By Theorem 3, if there exists a social preference relation that satisfies Pareto and Income Anonymity, then there exist functions $\upsilon : \mathbb{R}_{++} \times \mathbb{R}^J_{++} \mapsto \mathbb{R}$ and $\{\gamma_i\}_{i=1}^I : \mathbb{R}^J_{++} \mapsto \mathbb{R}$ such that individual preferences are represented by $\{\upsilon(m,p) + \gamma_i(p)\}_{i=1}^I$.

The proof of uniqueness is methodologically related to Mueller [1973], DeGroot [1974], Millner [2020] and Eden [2020]: the idea is to use repeated, alternating applications of the Pareto indifference condition and the Income Anonymity condition to pin down the equally-distributed equivalent of the allocation. As equally-distributed equivalents are unanimously ranked, this mapping characterizes the entire social preference relation.

Fix some p and define a function $\upsilon_p : \mathbb{R}_{++} \mapsto \mathbb{R}$ as $\upsilon_p(m) = \upsilon(m,p)$. Note that υ_p is strictly increasing because at any price level, indirect preferences are increasing in income. Hence, υ_p is invertible; let υ_p^{-1} denote its inverse. In addition, because υ is continuous, so is υ_p.

Note that the continuity of individual preferences implies that e_i is continuous. In addition, it is useful to note that $e_i(\cdot,p,p') = e_i^{-1}(\cdot,p',p)$, because, if i is indifferent between (m,p) and (m',p'), then $e_i(m,p,p') = m'$ and $e_i(m',p',p) = e_i(e_i(m,p,p'),p',p) = m$.

For each pair $i,i' \in \{1,...,I\}$, define a function $z_{i,i'} : \mathbb{R}_{++} \times \mathbb{R}^J_{++} \times \mathbb{R}^J_{++} \mapsto X$ as

$$z_{i,i'}(m,p,p') = e_{i'}(e_i(m,p,p'),p',p) \tag{37}$$

Figure 3 illustrates this definition.

Claim 16. *1. For any m, p', i and i', it holds that*

$$\upsilon_p(z_{i',i}(m,p,p')) - \upsilon_p(m) = -\left(\upsilon_p(z_{i,i'}(m,p,p')) - \upsilon_p(m)\right)$$

2. For any m'' and p', it holds that $\upsilon_p(z_{i,i'}(m'',p,p'))-\upsilon_p(m'') = \upsilon_p(z_{i,i'}(m,p,p'))-\upsilon_p(m)$.

Proof. Note that $(e_i(m,p,p'),p') \sim_i (m,p)$. As $\upsilon(\cdot,\cdot)+\gamma_i(\cdot)$ is a representation of individual i's indirect preferences, it follows that

$$\upsilon(m,p)+\gamma_i(p) = \upsilon(e_i(m,p,p'),p')+\gamma_i(p')$$

$$\Rightarrow \upsilon(e_i(m,p,p'),p') = \upsilon(m,p)+\gamma_i(p)-\gamma_i(p') \tag{38}$$

Similarly, as (by definition of z) $(e_i(m,p,p'),p') \sim_{i'} (z_{i,i'}(m,p,p'),p)$, it holds that

$$\upsilon(z_{i,i'}(m,p,p'),p)+\gamma_{i'}(p) = \upsilon(e_i(m,p,p'),p')+\gamma_{i'}(p') \tag{39}$$

Substituting in the previous equation yields

$$\upsilon(z_{i,i'}(m,p,p'),p)+\gamma_{i'}(p) = (\upsilon(m,p)+\gamma_i(p)-\gamma_i(p'))+\gamma_{i'}(p')$$

Rearranging yields

$$\upsilon(z_{i,i'}(m,p,p'),p)-\upsilon(m,p) = (\gamma_i(p)-\gamma_i(p'))-(\gamma_{i'}(p)-\gamma_{i'}(p')) \tag{40}$$

Similarly (switching i and i'),

$$\upsilon(z_{i',i}(m,p,p'),p)-\upsilon(m,p) = (\gamma_{i'}(p)-\gamma_{i'}(p'))-(\gamma_i(p)-\gamma_i(p')) \tag{41}$$

The first part of the claim follows by equation 40 and the definition of υ_p.

To prove the second part of the claim, note that, by equation 40 (replacing m with m''),

$$\upsilon(z_{i,i'}(m'',p,p'),p)-\upsilon(m'',p) = (\gamma_i(p)-\gamma_i(p'))-(\gamma_{i'}(p)-\gamma_{i'}(p')) \tag{42}$$

The second clause of the claim then follows from equation 40 and the definition of υ_p. □

Given two individuals, $i,i' \in \{1,...,I\}$, define the partial social ranking $\preceq_{i,i'}$ on allocations of the form $((m_i,m_{i'}),p)$ based on the condition

$$((m_i,m_{i'}),p) \preceq_{i,i'} ((m'_i,m'_{i'}),p') \Leftrightarrow ((m_1,...,m_I),p) \preceq (m'_1,...,m'_I,p'),$$

where $m'_k = e_k(m_k,p,p')$ for all $k \neq i,i'$.

To interpret this condition, consider a price change from p to p', which is accompanied by a change in the income distribution that leaves all individuals indifferent, with the exception of individuals i and i' (who see their incomes change to m'_i and $m'_{i'}$, respectively). If this change is socially desirable regardless of the income levels of individuals $k \neq i,i'$, then it holds that $((m_i,m_{i'}),p) \preceq_{i,i'} ((m'_i,m'_{i'}),p')$.

Claim 17. *For each* $\mathbf{m}$*, it holds that*

$$(\mathbf{m}, p) \sim \left(\left(\upsilon_p^{-1} \left(\frac{1}{I} \sum_{i=1}^{I} \upsilon_p(m_i) \right), ..., \upsilon_p^{-1} \left(\frac{1}{I} \sum_{i=1}^{I} \upsilon_p(m_i) \right) \right), p \right)$$

Proof. Define $N : \mathbb{R}_{++}^I \mapsto \{0, ..., I\}$ so that $N(\mathbf{m})$ is the number of elements in $\upsilon_p(m_1), ..., \upsilon_p(m_I)$ which are different from $\frac{1}{I}\sum_{i=1}^{I} \upsilon_p(m_i)$. The claim trivially holds for all $\mathbf{m} \in \mathbb{R}_{++}^I$ for which $N(\mathbf{m}) = 0$. The value $N(\mathbf{m}) = 1$ is not possible because there cannot be only one element that is different from the average – thus, the claim trivially holds for all $\mathbf{m}$ for which $N(\mathbf{m}) = 1$ (which is an empty set).

Assume that the claim holds for any $\mathbf{m} \in \mathbb{R}_{++}^I$ for which $N(\mathbf{m}) \leq k$, where $1 \leq k < I$. Let there be $\mathbf{m} \in \mathbb{R}_{++}^I$ for which $N(\mathbf{m}) = k + 1$. Using Income Anonymity, we can assume without loss of generality that $\upsilon_p(m_1) < \frac{1}{I}\sum_{i=1}^{I} \upsilon_p(m_i) < \upsilon_p(m_2)$.

In addition, given that preferences are heterogeneous, there exist individuals i and i' who do not have the same preferences; let m and (m', p') be such that $(m', p') \preceq_i (m, p)$ and $(m, p) \prec_{i'} (m', p')$. By Claim 2,

$$m < z_{i,i'}(m, p, p') \tag{43}$$

As υ_p is strictly monotone, it follows that $\upsilon_p(m) < \upsilon_p(z_{i,i'}(m, p, p'))$, and hence

$$\upsilon_p(z_{i,i'}(m, p, p')) - \upsilon_p(m) > 0 \tag{44}$$

Note that $z_{i,i'}(m, p, p) = m$, and that $z_{i,i'}(m, p, (1 - \eta)p + \eta p')$ is a continuous function of $\eta \in [0, 1]$ (this follows from the assumption that preferences are continuous). As υ_p is continuous, it follows that, for a sufficiently large integer K, there exists $\eta \in [0, 1]$ such that, for $p'' = (1 - \eta)p + \eta p'$,

$$\upsilon_p(z_{i,i'}(m, p, p'')) - \upsilon_p(m) = \frac{\frac{1}{I}\sum_{i=1}^{I} \upsilon_p(m_i) - \upsilon_p(m_1)}{K} \tag{45}$$

Define sequences $\{r_n\}_{n=0}^K, \{s_n\}_{n=0}^K \subset \mathbb{R}_{++}$ as follows: $r_0 = m_1$, $s_0 = m_2$, and, for $n \geq 0$, $r_{n+1} = z_{i,i'}(r_n, p, p'')$ and $s_{n+1} = z_{i',i}(s_n, p, p'')$.

The construction of r_{n+1} and s_{n+1} is illustrated in Figure 7.

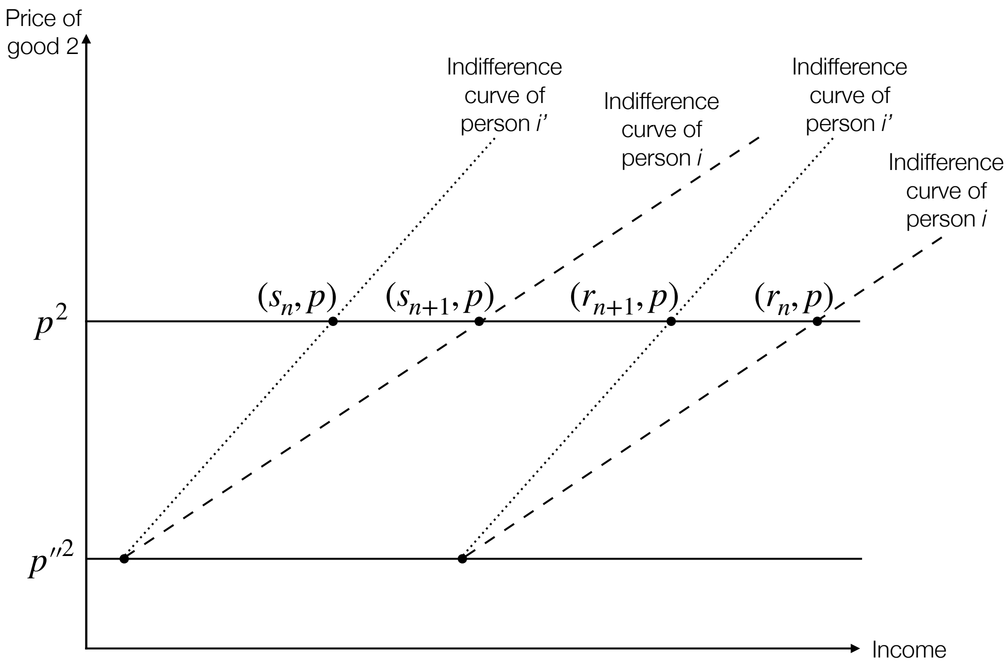
By Claim 4, it holds that

$$((r_{n+1}, s_{n+1}, m_3, ..., m_I), p) \sim ((r_n, s_n, m_3, ..., m_I), p) \tag{46}$$

Hence, by induction, it holds that

$$((r_K, s_K, m_3, ..., m_I), p) \sim (\mathbf{m}, p) \tag{47}$$

Figure 7: The construction of r_{n+1} and s_{n+1}



Further, note that, by the second clause of Claim 16,

$$\upsilon_p(r_{n+1}) - \upsilon_p(r_n) = \upsilon_p(z_{i,i'}(r_n, p, p'')) - \upsilon_p(r_n) = \tag{48}$$

$$\upsilon_p(z_{i,i'}(m, p, p'')) - \upsilon_p(m) = \frac{\frac{1}{I}\sum_{i=1}^{I} \upsilon_p(m_i) - \upsilon_p(m_1)}{K}$$

where the last equality follows from equation 45.

Consequently,

$$\upsilon_p(r_K) - \upsilon_p(r_0) = (\upsilon_p(r_K) - \upsilon_p(r_{K-1})) + ... + (\upsilon_p(r_1) - \upsilon_p(r_0)) =$$

$$K\frac{\frac{1}{I}\sum_{i=1}^{I} \upsilon_p(m_i) - \upsilon_p(m_1)}{K} = \frac{1}{I}\sum_{i=1}^{I} \upsilon_p(m_i) - \upsilon_p(m_1)$$

As $r_0 = m_1$, it follows that

$$\upsilon_p(r_K) = \frac{1}{I}\sum_{i=1}^{I} \upsilon_p(m_i) \tag{49}$$

Note that, by equation 48,

$$\upsilon_p(r_{n+1}) = \upsilon_p(r_n) + \frac{\frac{1}{I}\sum_{i=1}^{I} \upsilon_p(m_i) - \upsilon_p(m_1)}{K} \tag{50}$$

and, similarly (using the first part of Claim 16)

$$\upsilon_p(s_{n+1}) = \upsilon_p(s_n) - \frac{\frac{1}{I}\sum_{i=1}^{I}\upsilon_p(m_i) - \upsilon_p(m_1)}{K} \tag{51}$$

It thus follows inductively that $\upsilon_p(r_K) + \upsilon_p(s_K) = \upsilon_p(r_0) + \upsilon_p(s_0) = \upsilon_p(m_1) + \upsilon_p(m_2)$, and hence

$$\frac{\upsilon_p(r_K) + \upsilon_p(s_K) + \sum_{i=3}^{I}\upsilon_p(m_i)}{I} = \frac{\sum_{i=1}^{I}\upsilon_p(m_i)}{I} \tag{52}$$

Thus, by equations 49 and 52, $N(r_K, s_K, m_3, ..., m_I) \leq N(\mathbf{m}) - 1 \leq k$. By the induction hypothesis,

$$((r_K, s_K, m_3, ..., m_I), p) \sim \tag{53}$$

$$\left(\upsilon_p^{-1}\left(\frac{\upsilon(r_K) + \upsilon_p(s_K) + \sum_{i=3}^{I}\upsilon_p(m_i)}{I}\right), ..., \upsilon_p^{-1}\left(\frac{\upsilon(r_K) + \upsilon_p(s_K) + \sum_{i=3}^{I}\upsilon_p(m_i)}{I}\right), p\right)$$

By expressions 47, 52 and 53, it follows that

$$(\mathbf{m}, p) \sim ((r_K, s_K, m_3, ..., m_I), p) \sim \left(\left(\upsilon_p^{-1}\left(\frac{\sum_{i=1}^{I}\upsilon_p(m_i)}{I}\right), ..., \upsilon_p^{-1}\left(\frac{\sum_{i=1}^{I}\upsilon_p(m_i)}{I}\right)\right), p\right)$$

This concludes the proof that the claim holds for all $\mathbf{m}$ such that $N(\mathbf{m}) \leq k+1$.

By induction, it follows that the claim holds for all $\mathbf{m}$ for which $N(\mathbf{m}) \leq I$ – which is the entire set $\mathbb{R}_{++}^{I}$. □

To conclude the proof of uniqueness, note that, by the Pareto principle, it must hold that, for every $(\mathbf{m}', \mathbf{p}')$, $(\mathbf{m}', \mathbf{p}') \sim ((e_1(m_1', p_1', p), ..., e_I(m_I', p_I', p)), p)$. Thus, for every $\mathbf{m}', \mathbf{m}'', \mathbf{p}', \mathbf{p}''$,

$$(\mathbf{m}', \mathbf{p}') \preceq (\mathbf{m}'', \mathbf{p}'') \Leftrightarrow ((e_1(m_1', p_1', p), ..., e_I(m_I', p_I', p)), p) \preceq ((e_1(m_1'', p_1'', p), ..., e_I(m_I'', p_I'', p)), p)$$

By the above claim, this holds if and only if

$$\left(\left(\upsilon_p^{-1}\left(\frac{1}{I}\sum_{i=1}^{I}\upsilon_p\left(e_i\left(m_i', p_i', p\right)\right)\right), ..., \upsilon_p^{-1}\left(\frac{1}{I}\sum_{i=1}^{I}\upsilon_p\left(e_i\left(m_i', p_i', p\right)\right)\right)\right), p\right) \preceq$$

$$\left(\left(\upsilon_p^{-1}\left(\frac{1}{I}\sum_{i=1}^{I}\upsilon_p\left(e_i\left(m_i'', p_i'', p\right)\right)\right), ..., \upsilon_p^{-1}\left(\frac{1}{I}\sum_{i=1}^{I}\upsilon_p\left(e_i\left(m_i'', p_i'', p\right)\right)\right)\right), p\right)$$

By the Pareto principle, this holds if and only if

$$\upsilon_p^{-1}\left(\frac{1}{I}\sum_{i=1}^{I}\upsilon_p\left(e_i\left(m_i', p_i', p\right)\right)\right) \leq \upsilon_p^{-1}\left(\frac{1}{I}\sum_{i=1}^{I}\upsilon_p\left(e_i\left(m_i'', p_i'', p\right)\right)\right)$$

This condition uniquely characterizes the social preference relation. It follows that the social preference relation that satisfies Pareto and Income Anonymity is unique.

A.3 Proof of Theorem 2

By Theorem 3, there exists a social preference relation that satisfies Pareto and Income Anonymity if and only if there exists υ and $\{\gamma_i\}_{i=1}^{I}$ such that $\upsilon + \gamma_i$ is a representation of individual i's preferences, and $\gamma_1 = 0$.

Assume that there exists a social preference relation that satisfies Pareto and Income Anonymity. Then, there exists υ and $\{\gamma_i\}_{i=1}^{I}$ such that $\upsilon+\gamma_i$ is a representation of individual i's preferences.

Let $H^* : \mathbb{R}_{++}^J \mapsto \mathbb{R}_{++}$ denote a function that is homogeneous of degree 1, strictly concave and increasing (for example, $H^*(x) = \left(\prod_{j=1}^{J} x^j\right)^{1/J}$). Define $\tilde{x}(m,p)$ as

$$\tilde{x}(m,p) = \arg\max H^*(x) \text{ s.t. } \sum_{j=1}^{J} p^j x^j \leq m$$

Note that, as individual i's preferences are represented by $\upsilon+\gamma_i$, they are also represented by

$$(\upsilon(m,p) - \ln(H^*(\tilde{x}(1,p)))) + (\gamma_i(p) + \ln(H^*(\tilde{x}(1,p))))$$

Define

$$d(m,p) = \exp\left(\upsilon(m,p) - \ln(H^*(\tilde{x}(1,p)))\right)$$

and

$$V_i(m,p) = m \exp\left(\gamma_i(p) + \ln(H^*(\tilde{x}(1,p)))\right)$$

Using this notation, individual i's preferences are represented by

$$\frac{d(m,p)}{m} V_i(m,p)$$

Note that individual 1's indirect preferences are represented by $\upsilon(m,p)$. It follows that υ, and hence d, are strictly increasing in m. In addition, as a representation of indirect preferences, they are homogeneous of degree 0. Hence, for every $\lambda > 0$,

$$\upsilon(\lambda m, \lambda p) = \upsilon(m,p)$$

Because individual i's indirect preferences are homogeneous of degree 0, it holds that

$$\upsilon(m,p) + \gamma_i(p) = \upsilon(\lambda m, \lambda p) + \gamma_i(\lambda p) = \upsilon(m,p) + \gamma_i(\lambda p) \Rightarrow \gamma_i(p) = \gamma_i(\lambda p)$$

Hence, for every $\lambda > 0$,

$$V_i(\lambda m, \lambda p) = \lambda m \exp(\gamma_i(\lambda p))H^*(\tilde{x}(1, \lambda p)) = \lambda m \exp(\gamma_i(p))H^*(\tilde{x}(1, \lambda p)) = \exp(\gamma_i(p))H^*(\tilde{x}(\lambda m, \lambda p))$$

$$= \exp(\gamma_i(p))H^*(\tilde{x}(m, p)) = m \exp(\gamma_i(p))H^*(\tilde{x}(1, p)) = V_i(m, p)$$

Hence, V_i is homogeneous of degree 0, and strictly increasing in m.

Furthermore, note that, for every $\lambda > 0$,

$$V_i(\lambda m, p) = \lambda m \exp(\gamma_i(p))H^*(\tilde{x}(1, p)) = \lambda V_i(m, p)$$

It follows that V_i represents a homothetic preference relation. Let H_i be a function that is homogeneous of degree 1 that represents this homothetic preference relation.

Denote

$$\hat{x}(m, p) = \arg\max_x H_i(x) \text{ s.t. } \sum_{j=1}^{J} p^j x^j \leq m$$

Note that $V_i(m, p) = H_i(\hat{x}(m, p))$. Furthermore, note that, as preferences are homothetic, $\hat{x}(m, p) = m\hat{x}(1, p)$. As H_i is homogeneous of degree 1, it follows that

$$V_i(m, p) = H_i(\hat{x}(m, p)) = mH_i(1, p) = mV_i(1, p)$$

Hence,

$$\frac{d_i(m, p)}{m} V_i(m, p) = V_i(d_i(m, p), p)$$

thus establishing that individual i's preferences are represented by $V(\cdot, \cdot | H_i, d)$.

Conversely, assume that there exist functions $\{H_i\}_{i=1}^{I}$ that are homogeneous of degree 1 and a function d such that, for every i, individual i's indirect preferences are represented by $V(\cdot, \cdot | H_i, d)$.

It follows that they are also represented by

$$\ln(V(m, p|H_i, d)) = \ln(H_i(\hat{x}_i(d(m, p), p))) = \ln(d(m, p)H_i(\hat{x}_i(1, p))) = \ln(d(m, p)) + \ln(H_i(\hat{x}_i(1, p)))$$

$$= (\ln(d(m, p)) + \ln(H_1(\hat{x}_i(1, p)))) + (\ln(H_i(\hat{x}_i(1, p))) - \ln(H_1(\hat{x}_i(1, p))))$$

where $\hat{x}_i$ is defined by expression 7. Let

$$\upsilon(m, p) = (\ln(d(m, p)) + \ln(H_1(\hat{x}_i(1, p))))$$

$$\gamma_i(p) = (\ln(H_i(\hat{x}_i(1, p))) - \ln(H_1(\hat{x}_i(1, p))))$$

It follows that individual i's preferences are represented by $\upsilon + \gamma_i$, and that $\gamma_1 = 0$.

A.4 Proof of Corollary 1

The social preference relation satisfies Pareto because it is a sum of individuals' indirect utility functions (note that if $V(\cdot,\cdot|H_i,d)$ represents individual i's indirect preferences, then so does $\ln(V(\cdot,\cdot|H_i,d))$). To see that it satisfies Income Anonymity, note that

$$\ln(V(m,p|H_i,d)) = \ln(H_i(\hat{x}_i(d(m,p),p))) = \ln(d(m,p)H_i(\hat{x}_i((1,p)) = \ln(d(m,p)) + \ln(H_i(\hat{x}_i(1,p)))$$

where $\hat{x}_i$ is defined by expression 7. The second equality follows from the fact that H_i represents a homothetic preference relation, and that H_i is homogeneous of degree 1.

Using the above identity, the social welfare function can be rewritten as

$$W(\mathbf{m},\mathbf{p}) = \sum_{i=1}^{I} \ln(d(m_i,p_i)) + \sum_{i=1}^{I} \ln(H_i(\hat{x}_i(1,p_i)))$$

when $p_i = p$ for all i, this expression is symmetric in $m_1,...,m_I$. It follows that the social preference relation satisfies Income Anonymity.

Because the social preference relation satisfies Pareto and Income Anonymity, and given that preferences are heterogeneous, it follows by Theorem 1 that it is unique.

A.5 Proof of Corollary 3

Define the social preference relation $\tilde{\preceq}$ as

$$(\mathbf{m},\mathbf{p}) \preceq (\mathbf{m}',\mathbf{p}') \Leftrightarrow (d^*(\mathbf{m},\mathbf{p}),\mathbf{p}) \tilde{\preceq} (d^*(\mathbf{m}',\mathbf{p}'),\mathbf{p}')$$

where

$$d^*(\mathbf{m},\mathbf{p}) = (d_1^*(m_1,p_1),...,d_I^*(m_I,p_I))$$

Consider a fictitious social preference profile in which individual i's preferences are represented by H_i^*.

Note that $\preceq$ satisfies Pareto given the real preference profile if and only if $\tilde{\preceq}$ satisfies Pareto with respect to the fictitious preference profile. To see this, note that, as $V(\cdot,\cdot|H_i^*,d_i^*)$ represents individual i's (real) preferences,

$$(m_i,p_i) \preceq_i (m_i',p_i') \Leftrightarrow H_i^*(d_i^*(m_i,p_i),p_i) \leq H_i^*(d_i^*(m_i',p_i'),p_i')$$

Furthermore, note that $\preceq$ satisfies Income Anonymity-D if and only if $\tilde{\preceq}$ satisfies Income Anonymity.

Given that $\{H_i^*\}_{i=1}^I$ are heterogeneous and homogeneous of degree 1, the social preference relation $\tilde{\preceq}$ satisfies Pareto and Income Anonymity if and only if it can be represented by

$$\tilde{W}(d^*(\mathbf{m}, \mathbf{p}), \mathbf{p}) = \sum_{i=1}^{I} \ln(H_i^*(\hat{x}_i(d_i^*(m_i, p_i), p_i)))$$

It follows that $\preceq$ must be represented by equation 8.