Final Exam

Name:____

PSU ID: ______________@psu.edu

Econ 522: Macro Theory II, Part II

Penn State: Spring, 2022

- You have 75 minutes.
- Neither books nor notes are permitted.
- If you make any assumptions beyond what's in the text of the question, please state those assumptions clearly.
- If you need more space, please ask for additional sheets of paper; if you use more sheets, please number them, write your name, and label clearly which question you are answering.
- Unannotated scratch work will receive no credit.
- Good luck!

These problems will have you work with a model with a representative household that consumes and supplies labor, a representative firm that produces final goods, and a continuum of monopolistically competitive firms that produce intermediate goods. Time is discrete with an infinite horizon. There is no money in this economy, so all variables are real, not nominal.

1. (10 Points) We'll start by setting up the problem facing the representative household. The household has discount factor $\beta \in [0, 1)$, and the household's period-t utility is:

$$u(c_t - \eta c_{t-1}) - v(\ell_t), \tag{1}$$

where c_t is date-t consumption and ℓ_t is date-t labor supply. The function $u(\cdot)$ captures the utility of consumption, and is increasing and concave. The parameter $\eta \in [0, 1)$ captures habit formation: People care about how much they're consuming today, relative to how much they've consumed in the recent past. The function $v(\cdot)$ captures the disutility of labor, and it is assumed to be increasing and convex. The household's labor income is $w_t \ell_t$, where w_t is the real wage. The household is also assumed to own the firms in this economy, so the firms rebate a real dividend d_t to the household. For simplicity, assume that there is no borrowing nor saving. The date-t budget constraint is therefore:

$$c_t = w_t \ell_t + d_t. \tag{2}$$

From the household's perspective, w_t and d_t jointly follow an exogenous Markov process that the household takes as given. What are the household's state variables? Write the household's Bellman equation, and call the function $f(\cdot)$.

2. (15 Points) What are the first-order and envelope conditions for the household? (You don't have to prove that the Bellman equation is differentiable. Also, for the entirety of the exam, don't worry about any non-negativity constraints, and assume that the solution is interior.)

3. (10 Points) Combine the first-order and envelope conditions to eliminate any terms containing the derivative of $f(\cdot)$. Also eliminate any terms containing Lagrange multipliers, if you used them. Your answer should give you a single expectational difference equation that contains ℓ_t, w_t, and c_t (and possibly leads and lags of these variables).

4. (10 Points) The final consumption good in this economy is produced in a perfectly competitive market, so we can look at a single representative firm. The final goods firm produces output y_t using a continuum of intermediate goods $\{y_{i,t} \mid i \in [0,1]\}$ with the technology:

$$y_t = \left(\int_0^1 y_{i,t}^{1-v_t} \, di \right)^{\frac{1}{1-v_t}}. \tag{3}$$

Note that the elasticity of substitution can change over time. Assume that v_t follows an exogenous process that firm i takes as given, and $v_t \in (0,1)$ for each date t. (An explicit stochastic process for ν_t will be provided later.) The final consumption good is the numeraire, so its price is normalized to one. The price of the i^{th} intermediate good $y_{i,t}$ is denoted $p_{i,t}$, which the final-goods firm takes as given. The final-goods firm is assumed to maximize profits period-by-period. Write down the profit-maximization problem. Take the first-order condition with respect to $y_{i,t}$, and use it to provide an expression for $p_{i,t}$ in terms of $y_{i,t}$, y_t, and ν_t.

5. Each intermediate firm i produces its good with technology $y_{i,t} = \ell_{i,t}$. The real dividend that the firm rebates to the household is

$$d_{i,t} = p_{i,t} y_{i,t} - w_t \ell_{i,t}. \tag{4}$$

The intermediate firms take w_t as given. However, firm i understands that it faces a downward-sloping demand curve for its good, and the demand for good i is characterized by your solution to question 4. Assume that firm maximizes $d_{i,t}$ period-by-period, subject to the constraints implied by the demand curve and the production technology.

(a) (5 Points) Consolidate production technology and the demand curve into equation (4) to eliminate $p_{i.t}$ and $y_{i.t}$. In other words, write $d_{i,t}$ as a function only of $\ell_{i,t}$ and things that firm i takes as given.

(b) (5 Points) In light of your answer to part (a), you can think about firm i performing a static, univariate optimization where it chooses $\ell_{i,t}$ to maximize $d_{i,t}$. What is the first-order condition?

6. Given an initial condition and an exogenous stochastic process for ν_t, an equilibrium consists of (random) sequences of aggregate variables $\{c_t, \ell_t, w_t, d_t, y_t\}_{t=0}^{\infty}$ and variables for the intermediate-goods firms $\{p_{i,t}, \ell_{i,t}, d_{i,t} \mid i \in [0,1]\}_{t=0}^{\infty}$ such that:
   - Given w_t and d_t, c_t and ℓ_t solve the household's problem.
   - Given $\{p_{i,t} \mid i \in [0,1]\}$, $\{y_{i,t} \mid i \in [0,1]\}$ solves the final-goods firm's problem.
   - Given w_t, the intermediate-firm variables solve each intermediate firm's problem.
   - Markets clear: $\ell_t = \int_0^1 \ell_{i,t} di$ and $d_t = \int_0^1 d_{i,t} di$.

   (a) (5 Points) Show that the equilibrium is symmetric: $(p_{i,t}, \ell_{i,t}, d_{i,t}) = (p_{j,t}, \ell_{j,t}, d_{j,t})$ for all i and j.

   (b) (10 Points) Show that, in an equilibrium, $y_t = \ell_t = c_t$, and $w_t = 1 - \nu_t$.

7. (10 Points) It will be convenient to define $x_t \equiv \frac{v_t}{1-\nu_t}$ and declare a stochastic process for x_t, rather than ν_t. (Notice that this formulation implies $v_t = \frac{x_t}{1+x_t}$, so ν_t is guaranteed to be between zero and one. We'll be more explicit about the stochastic process for x_t later.) Combine your answer to question 3 with the results from question 6b and the definition of x_t. This should give you a single expectational difference equation that contains only y_t, y_{t-1}, y_{t+1}, and x_t.

8. (10 Points) For the remainder of the exam, assume that the utility function satisfies the functional forms:

$$u\left(c_t - \eta c_{t-1}\right) = -\frac{1}{2}\left(c_t - \eta c_{t-1} - \kappa\right)^2 \quad (5)$$

$$v\left(\ell_t\right) = \frac{1}{2}\ell_t^2, \quad (6)$$

where $\kappa > 0$. (If we assume that κ is sufficiently large, then $u\left(c_t - \eta c_{t-1}\right)$ will be increasing in c_t for "typical" values. The purpose of using quadratic functions is just to make it easy to compute the marginal utilities that appear in the equilibrium conditions.) Rewrite your answer to question 7 using these functional forms.

9. Assume that the log of x_t follows an ARMA$(1,1)$ process:

$$\log\left(x_t\right) = (1-\rho)\log\left(\bar{x}\right) + \rho\log\left(x_{t-1}\right) + \psi\epsilon_{t-1} + \epsilon_t, \quad \epsilon_t \overset{\text{i.i.d.}}{\sim} \mathrm{N}\left(0, \sigma_\epsilon^2\right). \tag{7}$$

Assume that the parameters of the ARMA$(1,1)$ process are such that $\log\left(x_t\right)$ is stationary and invertible. The equilibrium dynamics can now be summarized by two variables (x_t and y_t) and two equations (your answer to question 8 and the stochastic process for x_t).

(a) (5 Points) From equation (7), it should be clear that the steady state value of x_t is $\bar{x}$, which we're taking as given as a parameter. Let $\bar{y}$ denote the steady-state value of $\bar{y}$. Solve for $\bar{y}$ in terms of the model's parameters.

(b) (15 Points) Let $\hat{x}_t \equiv \log\left(\frac{x_t}{\bar{x}}\right)$ and $\hat{y}_t \equiv \log\left(\frac{y_t}{\bar{y}}\right)$. Log-linearize equation (7) and the difference equation you derived in question 8.

10. For the remainder of the exam, assume that $\beta = 0$, so households are myopic in their decision making. You don't have to re-solve the household's problem. You'll just be setting β to zero in the expressions you've already derived.

    (a) (10 Points) Show that, when $\beta = 0$, $\hat{y}_t$ follows an ARMA(p, q) process, where p and q are finite integers. Provide expressions for the ARMA coefficients or, equivalently, the AR and MA lag polynomials. Hint: It's easiest to use lag polynomials.

(b) (5 Points) What condition(s) must be satisfied for $\hat{y}_t$ to be stationary? Given what we've assumed about the model's parameters and the stochastic process for x_t, is $\hat{y}_t$ stationary?

(c) (5 Points) What condition(s) must be satisfied for $\hat{y}_t$ to be invertible? Given what we've assumed about the model's parameters and the stochastic process for x_t, is $\hat{y}_t$ invertible?

Final Exam: Solutions

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where c_t is date-t consumption and ℓ_t is date-t labor supply. The function $u(\cdot)$ captures the utility of consumption, and is increasing and concave. The parameter $\eta \in [0, 1)$ captures habit formation: People care about how much they're consuming today, relative to how much they've consumed in the recent past. The function $v(\cdot)$ captures the disutility of labor, and it is assumed to be increasing and convex. The household's labor income is $w_t \ell_t$, where w_t is the real wage. The household is also assumed to own the firms in this economy, so the firms rebate a real dividend d_t to the household. For simplicity, assume that there is no borrowing nor saving. The date-t budget constraint is therefore:

$$c_t = w_t \ell_t + d_t. \tag{2}$$

From the household's perspective, w_t and d_t jointly follow an exogenous Markov process that the household takes as given. What are the household's state variables? Write the household's Bellman equation, and call the function $f(\cdot)$.

Solution: The household's endogenous state variable is c_{t-1}, and the exogenous state variables are w_t and d_t. The household's Bellman equation is:

$$\begin{aligned} f\left(c_{t-1}, w_t, d_t\right) &= \max_{c_t, \ell_t} \left\{u\left(c_t - \eta c_{t-1}\right) - v\left(\ell_t\right) + \beta \mathbb{E}\left[f\left(c_t, w_{t+1}, d_{t+1}\right) \mid c_{t-1}, w_t, d_t\right]\right\} \\ & \quad \text{s.t.: } c_t = w_t \ell_t + d_t \end{aligned}$$

It's fine to write the conditional expectation as either $\mathbb{E}\left[f\left(c_t, w_{t+1}, d_{t+1}\right) \mid c_{t-1}, w_t, d_t\right]$ or $\mathbb{E}_t\left[f\left(c_t, w_{t+1}, d_{t+1}\right)\right]$.

2. (15 Points) What are the first-order and envelope conditions for the household? (You don't have to prove that the Bellman equation is differentiable. Also, for the entirety of the exam, don't worry about any non-negativity constraints, and assume that the solution is interior.)

Solution: The Lagrangian associated with the optimization problem is:

$$\mathcal{L} = u\left(c_t - \eta c_{t-1}\right) - v\left(\ell_t\right) + \beta \mathbb{E}\left[f\left(c_t, w_{t+1}, d_{t+1}\right) \mid c_{t-1}, w_t, d_t\right] + \lambda_t\left(w_t \ell_t + d_t - c_t\right).$$

The first-order conditions are:

$$\begin{aligned} \lambda_t &= u'\left(c_t - \eta c_{t-1}\right) + \beta \mathbb{E}\left[f_1\left(c_t, w_{t+1}, d_{t+1}\right) \mid c_{t-1}, w_t, d_t\right] \\ \lambda_t w_t &= v'\left(\ell_t\right). \end{aligned}$$

The envelope condition is:

$$f_1\left(c_{t-1}, w_t, d_t\right) = -\eta u'\left(c_t - \eta c_{t-1}\right).$$

3. (10 Points) Combine the first-order and envelope conditions to eliminate any terms containing the derivative of $f(\cdot)$. Also eliminate any terms containing Lagrange multipliers, if you used them. Your answer should give you a single expectational difference equation that contains ℓ_t, w_t, and c_t (and possibly leads and lags of these variables).

Solution: Evaluating the envelope condition at date $t+1$ instead of date t allows us to write:

$$f_1\left(c_t, w_{t+1}, d_{t+1}\right) = -\eta u\left(c_{t+1} - \eta c_t\right).$$

We can use this to get rid of the partial derivative of $f(\cdot)$ in the first-order condition with respect to consumption:

$$\lambda_t = u'\left(c_t - \eta c_{t-1}\right) - \eta\beta\mathbb{E}\left[u'\left(c_{t+1} - \eta c_t\right) \mid c_{t-1}, w_t, d_t\right].$$

We can combine the above with the first-order condition with respect to labor to eliminate the Lagrange multiplier:

$$v'\left(\ell_t\right) = w_t\left(u'\left(c_t - \eta c_{t-1}\right) - \eta\beta\mathbb{E}\left[u'\left(c_{t+1} - \eta c_t\right) \mid c_{t-1}, w_t, d_t\right]\right).$$

4. (10 Points) The final consumption good in this economy is produced in a perfectly competitive market, so we can look at a single representative firm. The final goods firm produces output y_t using a continuum of intermediate goods $\{y_{i,t} \mid i \in [0,1]\}$ with the technology:

$$y_t = \left(\int_0^1 y_{i,t}^{1-v_t} di\right)^{\frac{1}{1-v_t}}. \tag{3}$$

Note that the elasticity of substitution can change over time. Assume that v_t follows an exogenous process that firm i takes as given, and $v_t \in (0,1)$ for each date t. (An explicit stochastic process for ν_t will be provided later.) The final consumption good is the numeraire, so its price is normalized to one. The price of the i^{th} intermediate good $y_{i,t}$ is denoted $p_{i,t}$, which the final-goods firm takes as given. The final-goods firm is assumed to maximize profits period-by-period. Write down the profit-maximization problem. Take the first-order condition with respect to $y_{i,t}$, and use it to provide an expression for $p_{i,t}$ in terms of $y_{i,t}$, y_t, and ν_t.

Solution: The firm solves:

$$\max_{\{y_{i,t} \mid i\in[0,1]\}} \left(\int_0^1 y_{i,t}^{1-v_t} di\right)^{\frac{1}{1-v_t}} - \int_0^1 p_{i,t} y_{i,t} di.$$

The first-order condition is:

$$\left(\int_0^1 y_{j,t}^{1-v_t} dj\right)^{\frac{1}{1-v_t}-1} y_{i,t}^{-v_t} = p_{i,t}.$$

More concisely, the inverse demand curve for good i is:

$$\left(\frac{y_{i,t}}{y_t}\right)^{-v_t} = p_{i,t},$$

which uses the fact that $y_t = \left(\int_0^1 y_{i,t}^{1-v_t} di\right)^{\frac{1}{1-v_t}}$.

5. Each intermediate firm i produces its good with technology $y_{i,t} = \ell_{i,t}$. The real dividend that the firm rebates to the household is

$$d_{i,t} = p_{i,t} y_{i,t} - w_t \ell_{i,t}. \tag{4}$$

The intermediate firms take w_t as given. However, firm i understands that it faces a downward-sloping demand curve for its good, and the demand for good i is characterized by your solution to question 4. Assume that firm maximizes $d_{i,t}$ period-by-period, subject to the constraints implied by the demand curve and the production technology.

(a) (5 Points) Consolidate production technology and the demand curve into equation (4) to eliminate $p_{i.t}$ and $y_{i.t}$. In other words, write $d_{i,t}$ as a function only of $\ell_{i,t}$ and things that firm i takes as given.

Solution: Observe that:

$$\begin{aligned} d_{i,t} &= p_{i,t}y_{i,t} - w_t \ell_{i,t} \\ &= y_t^{\nu_t} y_{i,t}^{1-v_t} - w_t \ell_{i,t} \\ &= y_t^{\nu_t} \ell_{i,t}^{1-v_t} - w_t \ell_{i,t}, \end{aligned}$$

where the second line uses the inverse demand curve from question 4 to replace $p_{i,t}$, and the third line uses the production technology to replace $y_{i,t}$.

(b) (5 Points) In light of your answer to part (a), you can think about firm i performing a static, univariate optimization where it chooses $\ell_{i,t}$ to maximize $d_{i,t}$. What is the first-order condition?

Solution: The first-order condition is:

$$(1 - v_t)\, y_t^{\nu_t} \ell_{i,t}^{-v_t} = w_t.$$

6. Given an initial condition and an exogenous stochastic process for ν_t, an equilibrium consists of (random) sequences of aggregate variables $\{c_t, \ell_t, w_t, d_t, y_t\}_{t=0}^{\infty}$ and intermediate-firm variables $\{p_{i,t}, \ell_{i,t}, d_{i,t} \mid i \in [0,1]\}_{t=0}^{\infty}$ such that:
   - Given w_t and d_t, c_t and ℓ_t solve the household's problem.
   - Given $\{p_{i,t} \mid i \in [0,1]\}$, $\{y_{i,t} \mid i \in [0,1]\}$ solves the final-goods firm's problem.
   - Given w_t, the intermediate-firm variables solve each intermediate firm's problem.
   - Markets clear: $\ell_t = \int_0^1 \ell_{i,t} di$ and $d_t = \int_0^1 d_{i,t} di$.

(a) (5 Points) Show that the equilibrium is symmetric: $(p_{i,t}, \ell_{i,t}, d_{i,t}) = (p_{j,t}, \ell_{j,t}, d_{j,t})$ for all i and j.

Solution: Recall from question 5b that $(1 - v_t)\, y_t^{\nu_t} \ell_{i,t}^{-v_t} = w_t$, which implies:

$$\ell_{i,t} = \left(\frac{1 - v_t}{w_t}\right)^{\frac{1}{\nu_t}} y_t,$$

the right-hand side of which does not depend on i. Hence, $\ell_{i,t} = \ell_{j,t}$, so the production technology implies $y_{i,t} = y_{j,t}$. By extension, the solution to question 4 implies $p_{i,t} = p_{j,t}$.

(b) (10 Points) Show that, in an equilibrium, $y_t = \ell_t = c_t$, and $w_t = 1 - \nu_t$.

Solution: Because the equilibrium is symmetric, the market-clearing condition for labor implies $\ell_t = \ell_{i,t}$, and $d_t = d_{i,t}$. The production technology implies:

$$y_t = \left(\int_0^1 y_{i,t}^{1-\nu_t} di\right)^{\frac{1}{1-\nu_t}} = \left(\int_0^1 \ell_{i,t}^{1-\nu_t} di\right)^{\frac{1}{1-\nu_t}} = \ell_t,$$

where the first equality uses technology for producing final goods, the second equality uses the technology for producing intermediate goods, and the final equality uses $\ell_{i,t} = \ell_t$. Because $y_t = \ell_{i,t}$ and $d_t = d_{i,t}$, the answer to question 5a reduces to $d_t = (1 - w_t)\, y_t$, and the answer to question 4 reduces to $(1 - v_t) = w_t$. Hence, $d_t = \nu_t y_t$. The household's budget constraint becomes:

$$c_t = w_t \ell_t + d_t = (1 - v_t)\, \ell_t + v_t y_t = y_t,$$

where the second equality uses $(1 - v_t) = w_t$ and $(1 - w_t)\, y_t$, and the final equality uses $\ell_t = y_t$.

7. (10 Points) It will be convenient to define $x_t \equiv \frac{v_t}{1-\nu_t}$ and declare a stochastic process for x_t, rather than ν_t. (Notice that this formulation implies $v_t = \frac{x_t}{1+x_t}$, so ν_t is guaranteed to be between zero and one. We'll be more explicit about the stochastic process for x_t later.) Combine your answer to question 3 with the results from question 6b and the definition of x_t. This should give you a single expectational difference equation that contains only y_t, y_{t-1}, y_{t+1}, and x_t.

Solution: Replacing c_t and ℓ_t with y_t, and replacing w_t with $1-\nu_t$, we get:

$$v'(y_t) = (1-\nu_t)\left(u'(y_t - \eta y_{t-1}) - \eta\beta\mathbb{E}_t\left[u'(y_{t+1} - \eta y_t)\right]\right).$$

The definition of x_t implies $v_t = \frac{x_t}{1+x_t}$, so $1 - \nu_t = \frac{1}{1+x_t}$. Hence:

$$(1+x_t)\,v'(y_t) = u'(y_t - \eta y_{t-1}) - \eta\beta\mathbb{E}_t\left[u'(y_{t+1} - \eta y_t)\right].$$

8. (10 Points) For the remainder of the exam, assume that the utility function satisfies the functional forms:

$$u(c_t - \eta c_{t-1}) = -\frac{1}{2}(c_t - \eta c_{t-1} - \kappa)^2 \tag{5}$$

$$v(\ell_t) = \frac{1}{2}\ell_t^2, \tag{6}$$

where $\kappa > 0$. (If we assume that κ is sufficiently large, then $u(c_t - \eta c_{t-1})$ will be increasing in c_t for "typical" values. The purpose of using quadratic functions is just to make it easy to compute the marginal utilities that appear in the equilibrium conditions.) Rewrite your answer to question 7 using these functional forms.

Solution: The functional forms imply $u'(c_t - \eta c_{t-1}) = -(c_t - \eta c_{t-1} - \kappa)$ and $v'(\ell_t) = \ell_t$. The difference equation becomes:

$$(1+x_t)\,y_t = -(y_t - \eta y_{t-1} - \kappa) + \eta\beta\mathbb{E}_t\left[y_{t+1} - \eta y_t - \kappa\right].$$

9. Assume that the log of x_t follows an ARMA$(1,1)$ process:

$$\log(x_t) = (1-\rho)\log(\bar{x}) + \rho\log(x_{t-1}) + \psi\epsilon_{t-1} + \epsilon_t, \quad \epsilon_t \overset{\text{i.i.d.}}{\sim} \mathrm{N}\left(0, \sigma_\epsilon^2\right). \tag{7}$$

Assume that the parameters of the ARMA$(1,1)$ process are such that $\log(x_t)$ is stationary and invertible. The equilibrium dynamics can now be summarized by two variables (x_t and y_t) and two equations (your answer to question 8 and the stochastic process for x_t).

(a) (5 Points) From equation (7), it should be clear that the steady state value of x_t is $\bar{x}$, which we're taking as given as a parameter. Let $\bar{y}$ denote the steady-state value of $\bar{y}$. Solve for $\bar{y}$ in terms of the model's parameters.

Solution: Evaluating the answer to question 7 in a steady state, we get:

$$(1+\bar{x})\,v'(\bar{y}) = (1-\eta\beta)\,u'((1-\eta)\,\bar{y}).$$

Using the functional forms, the above becomes:

$$(1+\bar{x})\,\bar{y} = -(1-\eta\beta)\left[(1-\eta)\,\bar{y} - \kappa\right].$$

Rearranging:

$$\bar{y} = \frac{(1-\eta\beta)\,\kappa}{1+\bar{x}+(1-\eta\beta)(1-\eta)}.$$

(b) (15 Points) Let $\hat{x}_t \equiv \log\left(\frac{x_t}{\bar{x}}\right)$ and $\hat{y}_t \equiv \log\left(\frac{y_t}{\bar{y}}\right)$. Log-linearize equation (7) and the difference equation you derived in question 8.

Solution: The stochastic process for x_t implies that $\hat{x}_t$ is a mean-zero ARMA$(1,1)$ process:

$$\hat{x}_t = \rho \hat{x}_{t-1} + \epsilon_t + \psi \epsilon_{t-1}.$$

The difference equation for y_t can be log-linearized as:

$$\bar{y}\hat{y}_t + \bar{x}\bar{y}\left(\hat{x}_t + \hat{y}_t\right) = -\bar{y}\left(\hat{y}_t - \eta \hat{y}_{t-1}\right) + \bar{y}\eta\beta\mathbb{E}_t\left[\hat{y}_{t+1} - \eta\hat{y}_t\right].$$

Notice that the $\bar{y}$ falls out, so the above simplifies somewhat to:

$$\left(\bar{x} + 2 + \eta^2\beta\right)\hat{y}_t + \bar{x}\hat{x}_t = \eta\hat{y}_{t-1} + \eta\beta\mathbb{E}_t\left[\hat{y}_{t+1}\right].$$

10. For the remainder of the exam, assume that $\beta = 0$, so households are myopic in their decision making. You don't have to re-solve the household's problem. You'll just be setting β to zero in the expressions you've already derived.

(a) (10 Points) Show that, when $\beta = 0$, $\hat{y}_t$ follows an ARMA(p,q) process, where p and q are finite integers. Provide expressions for the ARMA coefficients or, equivalently, the AR and MA lag polynomials. Hint: It's easiest to use lag polynomials.

Solution: When $\beta = 0$, the difference equation from question 9b reduces to:

$$\left(\bar{x} + 2\right)\hat{y}_t + \bar{x}\hat{x}_t = \eta\hat{y}_{t-1}.$$

We can write this using the lag operator as:

$$\left(1 - \frac{\eta}{\bar{x}+2}L\right)\hat{y}_t = -\frac{\bar{x}}{\bar{x}+2}\hat{x}_t.$$

We can write the process for $\hat{x}_t$ using the lag operator as:

$$\left(1 - \rho L\right)\hat{x}_t = \left(1 + \psi L\right)\epsilon_t.$$

Because the ARMA process for $\hat{x}_t$ is assumed to be stationary, we can write the above as:

$$\hat{x}_t = \frac{1 + \psi L}{1 - \rho L}\epsilon_t.$$

We can therefore eliminate $\hat{x}_t$ in the lag polynomial for $\hat{y}_t$ to obtain:

$$\left(1 - \frac{\eta}{\bar{x}+2}L\right)\hat{y}_t = -\frac{\bar{x}}{\bar{x}+2}\frac{1 + \psi L}{1 - \rho L}\epsilon_t.$$

Equivalently:

$$\left(1 - \frac{\eta}{\bar{x}+2}L\right)\left(1 - \rho L\right)\hat{y}_t = -\frac{\bar{x}}{\bar{x}+2}\left(1 + \psi L\right)\epsilon_t.$$

We see that $\hat{y}_t$ is being multiplied by a second-degree lag polynomial, and ϵ_t is being multiplied by a first-order lag polynomial. Hence, $\hat{y}_t$ is an ARMA$(2,1)$ process.

(b) (5 Points) What condition(s) must be satisfied for $\hat{y}_t$ to be stationary? Given what we've assumed about the model's parameters and the stochastic process for x_t, is $\hat{y}_t$ stationary?

Solution: For $\hat{y}_t$ to be stationary, the polynomial $\left(1 - \frac{\eta}{\bar{x}+2}L\right)(1 - \rho L)$ must have all of its roots be greater than one in absolute value. The roots are $\frac{\bar{x}+2}{\eta}$ and $\frac{1}{\rho}$. We know that $\left|\frac{1}{\rho}\right| > 1$, because we've assumed that $\hat{x}_t$ is stationary, which requires $|\rho| < 1$. We also know that $\frac{\bar{x}+2}{\eta} > 1$, because η is assumed to be between zero and one, and because $\bar{x}$ must be positive, which ensures that $\bar{x} + 2$ is greater than one. Hence $\hat{y}_t$ is invertible.

(c) (5 Points) What condition(s) must be satisfied for $\hat{y}_t$ to be invertible? Given what we've assumed about the model's parameters and the stochastic process for x_t, is $\hat{y}_t$ invertible?

Solution: For $\hat{y}_t$ to be invertible, the polynomial $1 + \psi L$ must have all of its roots be greater than one in absolute value. This is identical to the condition that ensures that $\hat{x}_t$ is invertible. Because we assumed that $\hat{x}_t$ is invertible, it must also be the case that $\hat{y}_t$ is invertible.