Finals: Econ 521 Spring 2022

August 18, 2022

There are two parts in this exam. Each part has three problem and **you can choose any two of the three problems to answer**. Each problem is worth 50 points. You have three hours. Good luck.

Part A

Problem 1

Let $u : [0, \infty) \to \mathbb{R}$ be a strictly increasing, twice continuously differentiable, concave function. Assume that $\lim_{x \to 0} u'(x) = +\infty$. Consider a decision maker with preferences over $\mathbb{R}^2_+$ given by the following utility function:

$$U_{\alpha,\beta}(x_1, x_2) = \min \{\alpha u(x_1) + (1 - \alpha)u(x_2), \beta u(x_1) + (1 - \beta)u(x_2)\}$$

where $0 \leq \alpha \leq 1/2 \leq \beta \leq 1$.

Part a: Is the utility function $U_{\alpha,\beta}$ quasi-concave? If so, prove. Otherwise, provide a concrete counterexample.

Part b: Let $c_{(\alpha,\beta)}(x_1, x_2)$ be the unique value that satisfies the following equation:

$$U_{\alpha,\beta}\left(c_{(\alpha,\beta)}(x_1, x_2), c_{(\alpha,\beta)}(x_1, x_2)\right) = U_{\alpha,\beta}(x_1, x_2).$$

In other words, $c_{\alpha,\beta}(x_1, x_2)$ is the amount consumed in equal quantity of commodities 1 and 2 that yields the same utility as the original bundle (x_1, x_2). Suppose that $\alpha' \leq \alpha \leq 1/2 \leq \beta \leq \beta'$. How does $c_{\alpha,\beta}(x_1, x_2)$ compare to $c_{\alpha',\beta'}(x_1, x_2)$?

Part c: Solve for the demand function of the decision maker with utility function $U_{\alpha,\beta}$ when $u(x) = x$.

Problem 2

Consider the following pure exchange economy with two goods denoted x and y. There are two types of consumers. Type 1 consumers have the utility function $u_1 = \min\{x, y\}$

while type 2 consumers have utility function $u_2(x,y) = \frac{2}{3}x + \frac{1}{3}y$ where $\alpha \in (0,1)$. The endowments of type 1 consumers are given by $\omega_1 = (4,1)$ while the endowments of type 2 consumers are given by $\omega_2 = (1,4)$.[1] Suppose there are n consumers of each type.

Part a: Suppose that $n = 1$. What are all of the core allocations of this economy? Justify your answer.

Part b: Compute the set of all symmetric Walrasian equilibrium allocations when $n = 1$.[2]

Part c: Consider the allocation in which all consumers of type 1 consume $(1,1)$ and all consumers of type 2 consume $(4,4)$. Is this allocation in the core when $n = 2$? Justify your answer rigorously.

Part d: Compute the set of all symmetric core allocations when $n = 2$. Justify your answer rigorously.

Problem 3

Consider the following two-person pure exchange economy with uncertainty. There are two equally likely states of nature, $\theta \in \{g, b\}$. There is only one physical good, denoted by x. Suppose that the two agents are both expected utility maximizers:

$$U_1(x_g, x_b) = \frac{1}{2}u_1(x_g) + \frac{1}{2}u_1(x_b)$$
$$U_2(x_g, x_b) = \frac{1}{2}u_2(x_g) + \frac{1}{2}u_2(x_b),$$

where both u_1, u_2 are twice continuously differentiable, strictly increasing functions with $\lim_{x\to 0} u_1'(x) = \lim_{x\to 0} u_2'(x) = 0$ and $u_1'', u_2'' < 0$. Suppose that the initial endowments of the two consumers are

$$(\omega_g^1, \omega_b^1) = (3,3), (\omega_g^2, \omega_b^2) = (6,1).$$

Assume throughout that there is a complete set of contingent markets.

Part a: Given any Arrow-Debreu equilibrium of this economy, let p_θ be the price of the commodity in state θ. In an Arrow-Debreu equilibrium, is $p_g > p_b$ or $p_g < p_b$? Justify your answer.

Part b: Suppose that individual 2's endowment changes from $(6,1)$ to $(7,1)$. Everything else in the economy remains unchanged. Is it clear what happens to the price ratio $\frac{p_g}{p_b}$ in Arrow-Debreu equilibrium? Explain.

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[1] Here the first coordinate of ω_i represents the quantity of x initially owned by consumer i, and the second coordinate represents the corresponding quantity of y.

[2] Recall that an allocation is symmetric if all consumers of the same type all consume the same consumption bundle.

Part B

Problem 1

I) A crime is observed by a group of $n \geq 2$ people. Everybody would like the police to be informed about the crime but everybody prefers that someone else makes a call. They choose simultaneously whether to call the police or not. When nobody calls, everybody's payoff is 0. If anybody calls the police, those who call the police receive $v - c$ and those who do not receive v, where $v > c > 0$.

(a) Find all pure strategy Nash equilibria.

(b) Using the following steps find a symmetric mixed strategy Nash equilibrium.

(i) Suppose that each player other than player i calls the police with probability p. What is the expected payoff player i gets if he calls the police and what is his expected payoff if he does not?

(ii) Find a symmetric mixed strategy Nash equilibrium in which each player calls with probability p^*. Is it unique?

(iii) In the Nash equilibrium you found in (ii), is the probability that a particular player reports the crime increasing or decreasing in n? Is the probability that the police is informed about the crime (i.e. at least one player calls) increasing or decreasing in n? What happens as n approaches infinity? Interpret the result.

II) Now, consider a model where player i's payoff when he reports is now $v - c_i$, instead of $v - c$ and c_i is only known to himself/herslef. Everybody believes that each c_i is identically and independently distributed over $[\underline{c}, \overline{c}]$. Let F be the distribution function of c_i and f is the associated density function with $f > 0$ for all $c_i \in [\underline{c}, \overline{c}]$.

(a) Give a realistic story(s) that would justify an uncertain c_i.

(b) Show that for any strategy profile of all others, a player's best response is given by the following cutoff strategy $s_i(c)$ with some $c \in [\underline{c}, \overline{c}]$ such that he calls the police if $c_i < c$ and does not if $c_i > c$.

(c) Find a symmetric pure strategy Bayesian Nash equilibrium in which every player plays $s_i(c^*(n))$. (i.e. find a value of such $c^*(n)$) What is the probability that a player *does* not call the police? What is the probability that *nobody* calls the police?

(d) Compare your answer in part (c) above to part (b) in the previous model, when $\underline{c}$ and $\overline{c}$ are very close to one another. Comment (think Harsayni!)

Problem 2

A) Consider a Cournot model with two firms where the demand function is given by $p = a - Q$, for $Q = q_1 + q_2$. The marginal costs of production of the two firms are given by c_1 and c_2 where $c_1, c_2 < a$. The fixed costs are zero.

1. Find the unique Nash equilibrium of this game.
2. Explain the restrictions on the parameters which ensure that the market operates as a monopoly, duopoly or shuts down.
3. Explain the comparative statics of a firm's profits with respect to its own cost and to its competitor's costs.

B) Suppose $c_1 = c_2 = c$, q_m is the profit maximizing quantity for a monopoly and q_c is the Nash equilibrium quantity. The two firms play this game repeatedly. They follow the grim trigger strategy where $q_1 = q_2 = q^* \in [\frac{1}{2}q_m, q_c]$ is played in the first period, and then is played in every subsequent period if no one deviated from it. In all other scenarios $q_1 = q_2 = q_c$ is played. Find the (lowest) value of q^* as a function of a, c and δ such that the this grim trigger is a subgame perfect equilibrium.

Problem 3

Consider a variant of the Prisoner's Dilemma game in which player 2 observes 1's play before moving. The payoffs are $(1, 1)$ if both cooperate, $(0, 0)$ if both defect, and α for the defector and β for the sucker if only one defects. Assume $\alpha > 1$, $\beta < 0$, $\alpha + \beta < 2$.

1. What is the unique subgame perfect equilibrium for the N-time repeated version of the game? Does it have a unique Nash equilibrium?
2. Suppose that with probability $\varepsilon > 0$, player 2 (the second to move) is a type that must reciprocate, that is, he must cooperate whenever player 1 does and defect whenever 1 does. Find the (unique) sequential equilibrium of the *twice* repeated version of this game, in which player 1 does not know the type of player 2 she is facing.
3. Find the sequential equilibrium of the N-times repeated version of this game. *Remark:* In a knife-edge case where ε and β are related in a particular way equilibrium is not strictly unique. You may ignore such non-generic cases.