2022 Comprehensive Exam in Econometrics

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Four hours for eight questions

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Good Luck!

1. (10%) The legislatures of Utopia are trying to determine whether to impeach their President. The Congress meets first and can decide: (a) to impeach, (b) to pass the buck to the Senate, or (c) to not impeach. If decision (b) is made, then in the next round the Senate gets to decide on the same three options [(b) in this case is "pass the buck back to the Congress"]. The probabilities of each decision in each round are $p_C, q_C, 1 - p_C - q_C$ for the Congress and $p_S, q_S, 1 - p_S - q_S$ for the Senate, respectively.

   (i) What is the probability p_I that the President is impeached?

   (ii) If each decision (i.e. a decision in Congress and also a decision in the Senate) takes 1 month and $q := q_C = q_S$ what is the expected length of time before the process is terminated?

2. (14%) a) Let $(\Omega_i, \mathcal{F}_i)$, $i = 1, 2$, be measurable spaces. Assume $\mathcal{E}_2 \subset \mathcal{F}_2$ is such that $\sigma(\mathcal{E}_2) = \mathcal{F}_2$. Show $T : \Omega_1 \to \Omega_2$ is measurable (with respect to the sigma algebras $\mathcal{F}_1$ and $\mathcal{F}_2$) if $T^{-1}(E_2) \in \mathcal{F}_1$ for all $E_2 \in \mathcal{E}_2$.

   Let $(\Omega_i, \mathcal{F}_i)$, $i = 1, 2$ be measurable spaces again, $\Omega := \Omega_1 \times \Omega_2$, and let $\rho_i : \Omega \to \Omega_i$ be the projection mapping onto the ith component, i.e. $\rho_i((w_1, w_2)) = w_i$, for $i = 1, 2$. Assume $\mathcal{E}_i \subset \mathcal{F}_i$ is such that i) $\sigma(\mathcal{E}_i) = \mathcal{F}_i$ for $i = 1, 2$ and ii) for each $i = 1, 2$ there are non-decreasing sequences $E_{mi} \in \mathcal{E}_i$ ($m \in \mathbb{N}$) such that $E_{mi} \uparrow \Omega_i$.

   b) Let $\mathcal{A}$ be an arbitrary sigma-algebra in Ω. Show that each ρ_i is measurable (with respect to $\mathcal{A}$ and $\mathcal{F}_i$) if and only if

$$E_1 \times E_2 \in \mathcal{A}$$

   for any $E_i \in \mathcal{E}_i$, $i = 1, 2$. [Hint: use part a).]

   c) Denote by $\otimes_{i=1}^2 \mathcal{F}_i$ the smallest sigma algebra on Ω such that both the projection mappings ρ_i are measurable (with respect to $\mathcal{F}_i$) for $i = 1, 2$. Show that $\otimes_{i=1}^2 \mathcal{F}_i = \sigma(E_1 \times E_2; E_i \in \mathcal{E}_i, i = 1, 2)$.

   d) Give an example that shows that the statement in c) no longer holds true if condition ii) is not assumed..

3. (14%) Suppose that

$$Y_i = \beta X_i + \varepsilon_i,$$

where (X_i, ε_i) are i.i.d., X_i and ε_i are independent, $EX_i = \mu \neq 0$, and $E\varepsilon_i = 0$. Define

$$\widehat{\beta} = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n X_i^2}, \ \overline{\beta} = \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n X_i}, \text{ and } T_n = \exp(\widehat{\beta} + \overline{\beta}).$$

The objective is to work out the asymptotic distribution of T_n as $n \to \infty$ (after proper centering and normalization) following the steps (a)-(d) below. State any additional technical assumption (such as finiteness of moments) you require in your derivations, precisely define any notation you introduce, precisely state any result from the lecture you use, and show all your work.

(a) Derive the probability limits of $\widehat{\beta}$, $\overline{\beta}$, and T_n.

(b) Define the vector

$$M_n = \begin{pmatrix} n^{-1} \sum_{i=1}^n X_i \varepsilon_i \\ n^{-1} \sum_{i=1}^n \varepsilon_i \end{pmatrix}.$$

Derive the asymptotic distribution of M_n (after proper centering and normalization).

(c) Consider

$$\widetilde{T}_n = \exp(2\beta) \exp\left(\frac{n^{-1} \sum_{i=1}^n X_i \varepsilon_i}{EX_i^2} + \frac{n^{-1} \sum_{i=1}^n \varepsilon_i}{\mu}\right).$$

Derive the asymptotic distribution of $\widetilde{T}_n$ (after proper centering and normalization).

(d) Show that $\sqrt{n}(\widetilde{T}_n - T_n) = o_p(1)$. What is the asymptotic distribution of T_n?

4. (12%) True or false? Explain! (The explanation is what counts!)

(a) Let $X_n \in R$ be a sequence of integrable random variables. Then, if $X_n \to_p b \in R$ then $EX_n \to b$.

(b) Assume we test

$$H_0 : \mu_1 = \mu_2 = 0 \text{ vs } H_1 : \text{not } H_0$$

based on a sample of iid bivariate random vectors $X_i \sim N(\mu, I_2)$, $i = 1, ..., n$, where $\mu = (\mu_1, \mu_2)'$. Assume for some $0 < \alpha < 1/2$ the null is rejected if $|t_{nj}| > z_{1-\alpha/2}$ for $j = 1$ or $j = 2$, where

$$t_{nj} = n^{1/2}(\overline{X}_{nj} - \mu_j),$$

$\overline{X}_{nj} = n^{-1} \sum_{i=1}^{n} X_{ij}$ for $j = 1, 2$, and $X_i = (X_{i1}, X_{i2})'$, $i = 1, ..., n$. Then the asymptotic null rejection probability of the test is α.

(c) By the Hölder inequality, $E(1/X) \leq 1/E(X)$ for any positive random variable $X > 0$.

(d) A sufficient statistic is one whose marginal distribution does not depend on any unknown parameters.

(e) If $X \sim \chi^2(m)$, $Y \sim \chi^2(n)$, and $m > n$, then $X - Y \sim \chi^2(m - n)$.

(f) Let X and Y be two mean zero random variables. Then, "$E(X|Y) = 0$ almost surely Y and $E(Y|X) = 0$ almost surely X" if and only if "X and Y are uncorrelated."

For the remaining questions, bold face means random variable.

5. Consider

$$\theta_0 = \operatorname*{argmin}_{\theta \in \mathbb{R}} \mathbb{E}\big[3 \log\{1 + \exp(\theta - \boldsymbol{x}_1)\} - (\theta - \boldsymbol{x}_1)\big], \tag{1}$$

where $\boldsymbol{x}_1$ is integrable and drawn from a nondegenerate distribution. Suppose that you have an i.i.d. sample $\{\boldsymbol{x}_i\}$ at your disposal. Propose an estimator $\hat{\boldsymbol{\theta}}$ of θ_0, prove its consistency, prove its asymptotic normality, and derive its asymptotic distribution. [**15%**]

6. For each pair (x, y) of the following concepts, indicate if (necessarily) $x \Rightarrow y$, $x \Rightarrow \neg y$, or $x \Leftrightarrow y$ (omit if there is no implication): *(a)* identified *(b)* weakly identified *(c)* exactly identified *(d)* overidentified *(e)* underidentified. No points without a correct explanation. *Correctly identified relationships earn points, incorrectly identified ones lose points.* [**5%**]

7. Suppose $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ are iid draws from the distribution with probability mass function $p(x) = \gamma_0 I(x = 0) + \lambda_0^x \exp(-\lambda_0)(1 - \gamma_0)/x!$, for $x = 0, 1, \ldots,$ where $-1/(e^{\lambda_0} - 1) \leq \gamma_0 \leq 1$ and $\lambda_0 > 0$. **[10%]**

   (a) Derive the score test statistic for $\gamma_0 = 0$.

   (b) Suppose you conducted both the score test and the likelihood ratio test for $\gamma_0 = 0$ and found that the likelihood ratio test produced a p-value equal to 0.03 and the score test produced a p-value equal to 0.45. How would you explain that contrast in this case?

8. Suppose that $\boldsymbol{y}_i$ denotes the product purchased by individual $i = 1, \ldots, n$, $\boldsymbol{x}_{ij}$ some observed product characteristics, and $\boldsymbol{\xi}_j$ an unobserved product characteristic, where $j = 1, \ldots, J$. There is an 'outside good' (no purchase), product 0. Suppose further that we know that

$$\Pr(\boldsymbol{y}_i = j \mid \boldsymbol{x}_{i\cdot}, \boldsymbol{\xi}) = \frac{\exp(\boldsymbol{x}_{ij}^T \theta_0 + \boldsymbol{\xi}_j)}{1 + \sum_{t=1}^J \exp(\boldsymbol{x}_{it}^T \theta_0 + \boldsymbol{\xi}_t)}. \tag{2}$$

Finally, suppose that the object of estimation is θ_0, that J is fixed, and that $n \to \infty$. Use GMM for the questions below, even though it is not the most natural estimation method here. **[20%]**

(a) What conditional moments conditions would you use?

(b) Derive the optimal instruments.

(c) What would have been a more natural estimation method here?