A Second Welfare Theorem for Economies with Search, Matching, and Investments*

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Abstract

We study a search-and-matching model where heterogeneous agents invest to acquire skills before entering the market. Agents produce in pairs and must engage in costly search for partners: in every period, agents incur an additive search cost, pairs meet at random, and can either accept and bargain over their joint output or reject and continue searching for a better match. Potential sources for inefficiencies are the hold-up problem and mismatches between skills. Despite these, we prove a second welfare theorem: the *constrained efficient* allocation is an equilibrium. We also establish a general assortative matching result, equilibrium existence, provide conditions for uniqueness, and derive novel economic implications: the efficient outcome can be discriminatory in the marriage market.

1 Introduction

This paper re-examines classical questions regarding the efficiency and structure of equilibrium in markets with search frictions. We consider a model where heterogeneous agents make costly investments to acquire skills before entering a matching market. Agents produce in pairs, and their match output depends on the skills that they have acquired. To form productive matches, agents engage in costly search:

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each agent in the market incurs a per-period search cost, pairs meet at random and can either accept and bargain over their joint output or reject and continue searching for a better match. Our main result establishes a new second welfare theorem for economies with explicit search costs: every constrained efficient allocation, regarding both the skills acquired and who matches with whom, is obtainable as an equilibrium outcome. We also prove a general sorting result, establish conditions for a unique equilibrium, and derive novel economic implications.

To motivate our model, let us mention a few settings where investment and search play an important role. Individuals in the marriage market make premarital investments in their education and career before looking for a partner. In the labor market, workers acquire human capital before searching for jobs, while firms adopt technologies before hiring workers. A similar situation occurs in other settings: in the real estate market, developers often build before finding prospective buyers; in a financial market, entrepreneurs invest time and money developing start-ups prior to seeking venture capital funding; and in product markets, sellers make investments in quality before seeking potential buyers.

In such settings, agents are usually heterogeneous, and the payoff from investing depends on who matches with whom and the search duration. For example, in the labor market, a worker's return to schooling depends on which types of firms may potentially hire her and how long it will take her to find a job. Likewise, a firm's benefit from adopting a new technology depends both on the skills of workers it may potentially hire and how long it will take to fill vacancies. Therefore, search frictions impact each agent's matching outcome and the incentive to invest.

Building on the foundational Diamond-Mortensen-Pissarides model, the prevailing view in the literature is that efficiency fails in decentralized markets with search frictions. First, when investments are sunk by the time agents meet, a hold-up problem may reduce the incentive to invest (because agents bear the entire cost of investment and receive only a fraction of the additional output). Second, heterogeneous agents can mismatch because they fail to internalize how their decisions to accept/reject partners affects other agents. For instance, a social planner may want an agent to match with a partner who is not highly productive, but the agent prefers to continue searching for a more productive match.

In what follows, we will challenge the view that search frictions necessarily create inefficiencies. We develop a search-and-matching model with transfers between two populations of agents, called buyers and sellers, but one can equally consider workers and firms, men and women, or any other two groups that invest and then match. What is important is that output is produced by pairs of agents, one from each side of the market. The model has two key ingredients. First, agents invest in skills before entering the market. The agents are heterogeneous in their investment costs and buyer-seller pairs produce output according to their skills. Second, we consider the standard random search and bargaining process with additive search costs rather than discounting, as in Atakan [2006].

Specifically, in every period, each agent in the market incurs the same search cost and randomly meets an agent from the other side. When two agents meet, they can either agree to match or continue searching. If both agree, they exit the market and divide their output according to Nash bargaining. A new cohort of agents is born in every period, acquires skills, and then enters the market. We analyze a *steady-state equilibrium* where, for every skill, the inflow of agents to the market equals the outflow, and so the distribution of skills in the market is in a steady state.

The term skill refers to investments that enhance productivity. For instance, in the labor market, a worker may acquire some education level which is their skill, and a firm may adopt a particular technology which is the firm's skill. In the product market, a seller's investment may reduce their production cost, a buyer's investment may increase their value, and the output function is the difference between the buyer's value and the seller's cost. In the marriage market, the output function typically depends on the premarital investments of both agents, and we assume ex-ante symmetry -- men and women can acquire the same skills and have the same cost distribution.

The market is competitive in that every skill has a value and agents optimize given these values. Thus, each agent invests by comparing the marginal value of each skill to the agent's marginal cost of acquiring that skill. In addition, when two agents meet they will accept (reject) the match if their joint output is greater (smaller) than the sum of their values. As in standard search and matching models, these values are endogenously determined in an equilibrium and must be consistent with the steady-state conditions, the search strategies, and Nash bargaining (see e.g., Burdett and Coles 1999; Shimer and Smith 2000). An important and novel feature of our model is that the values serve *double duty*: creating incentives to invest and to accept matches.

Despite the potential inefficiencies, we prove that every constrained efficient allocation is an equilibrium outcome.[1] The proof constructs market values that satisfy the standard equilibrium conditions while perfectly aligning the agents' incentives with the planner. Strikingly, these values *simultaneously* solve the investment and matching problems. This theorem also establishes the existence of equilibrium.

Notice that when agents invest in skills and decide whether to accept/reject a potential partner, they impose externalities on other agents. Regarding investment, the social planner must weigh each agent's marginal investment cost against the marginal effect on productivity and search costs of all agents. In particular, increasing the inflow of one skill affects the productivity and search costs of the agents that now acquire it and their partners, but there is also an *indirect effect* on other agents via the change in the steady-state skill composition. Regarding matching, when the planner decides that two skills should reject rather than accept, the planner forgoes their match output and incurs a higher search cost to form more productive partnerships, but must also consider the change in the steady-state skill composition. In contrast, in equilibrium, each agent invests and accepts or rejects partners simply by their private incentives, as determined by the value of each skill in the market. Remarkably, the equilibrium values make the agents internalize the direct and indirect steady-state search externalities.

Our second main point is that the equilibria have a clear and simple structure. We prove that there is assortative matching if the production function is super/submodular. Furthermore, if the production function is additively separable, then the equilibrium is unique and it achieves the first-best allocation. Economies with non-separable production functions can have multiple equilibria and the agents may fail to coordinate on the efficient one and so there is scope for policy interventions.[2]

The tradeoff between investment, search, and matching has important implications. For example, in the marriage market, a gender gap in skill acquisition not only arises, but can be efficient, even when the two populations are ex-ante identical. The key tradeoff is that asymmetric investments induce a higher total investment cost (due to a misallocation of talent), but the resulting lopsided skill distributions can facilitate search-and-matching (since it is more likely that agents with opposite skills

[1]The constrained efficient allocation solves the problem faced by a social planner who controls the agents' decisions while respecting the steady-state condition. Since utility is transferable, the Pareto-optimal outcomes are the constrained efficient ones.

[2]For example, a no-investment equilibrium may occur if not investing is self-reinforcing: agents do not invest because all others do not.

meet). As a result, in the efficient equilibrium, men and women may receive different payoffs from their investments in the market.

These results have practical implications. For example, in the labor market, we establish when sorting occurs (high-tech firms match with high-skill workers) and the model captures how it impacts investment. In product markets, the joint output function is often assumed additively separable, and we show that there exists a unique equilibrium which achieves the first-best allocation. In the marriage market, we establish that a gender gap can occur. Finally, we generalize our model by incorporating a discount factor and show that the main results are robust to small modifications in the time costs.

Related Literature

Our paper is the first to provide a general and tractable model incorporating three components: (i) random search and bargaining, (ii) matching between heterogeneous agents, and (iii) pre-entry investments. These three components have not been studied together and have novel implications when studied jointly (e.g., the discrimination outcome in Section 3). Table 1 summarizes the models and results of the central papers in the strands of the literature most closely related to our work. We elaborate on these and other papers below.

Papers	Search	Matching	Investment	Results
Cole et al. [2001] Noldeke and Samuelson [2015]	No	Yes	Yes	Efficiency
Shimer and Smith [2000] Atakan [2006]	Yes	Yes	No	Sorting (single population)
Shimer and Smith [2001]	Yes	Yes	No	Inefficiency
Acemoglu [1996] Masters [1998] Acemoglu and Shimer [1999a]	Yes	No	Yes	Inefficiency
Hosios [1990]	Yes	No	No	Efficiency
Gale [1987] Mortensen and Wright [2002] Lauermann [2013]	Yes	No	No	Convergence to First Best
This paper	Yes	Yes	Yes	Constrained Efficiency + Sorting (two populations) + Convergence to Second Best

Table 1: Literature Comparison

Previous work on two-sided matching with transfers has extended the classical assignment model of Shapley and Shubik [1971] to settings with ex-ante investments and perfect *frictionless matching* (components (ii) and (iii)). These models typically find that the first-best allocation is a competitive equilibrium outcome, but there may exist additional inefficient equilibria (see, e.g., Cole et al. 2001, Mailath et al. 2013, Noldeke and Samuelson 2015, Dizdar 2018, Chade and Lindenlaub 2022). We contribute to this literature by adding search frictions and establishing that the *constrained efficient* allocation is an equilibrium outcome. One novel implication of our model is that in the marriage market, a gender gap in skill acquisition can be efficient, even when the two populations are ex-ante identical. In contrast, in frictionless models, the efficient outcome is always symmetric.

An important strand of the literature studies the random search and bargaining model with heterogeneous agents (components (i) and (ii) above, see Burdett and Coles 1999, Shimer and Smith 2000, and Atakan 2006). As in Atakan [2006], we consider an additive search cost, whereas most of this literature assumes time discounting. We extend these models by adding pre-entry investment, which endogenizes the skills in the search market. We contribute to this literature by proving a second welfare theorem and a general sorting result. In particular, our sorting result applies to *two-sided* matching markets, whereas the previous results of Shimer and Smith [2000] and Atakan [2006] establish sorting in a one-population search model (the one-population model is a special case of our two-population model).[3]

Building on the foundational Diamond-Mortensen-Pissarides model, the prevailing view in the search literature is that decentralized markets with search frictions fail to achieve efficient outcomes regarding matching and investments. First, regarding matching, Shimer and Smith [2001] consider the random search and bargaining model with heterogeneous agents but without investments (components (i) and (ii) above) and show that agents mismatch in the following way: low-types reject too frequently while high-types accept too often. Second, regarding investments, several papers

[3]The sorting result is non-trivial and important for applications, as both the labor market and product markets are two-sided. The Welfare Theorem also establishes the existence of a steady-state equilibrium. Existence proofs can be tricky in other search models (see, e.g., Manea 2017 and Lauermann et al. 2020). Both results are computationally useful since the planner's problem is often more amenable to numerical analysis than the equilibrium conditions, and the sorting result drastically limits the policy space.

have studied search models with pre-entry investments but sidestep the matching problem by assuming *homogeneous* agents (components (i) and (iii) above) and show that the equilibrium investments are always inefficient due to the hold-up problem (Acemoglu 1996, Masters 1998, Acemoglu and Shimer 1999a). In contrast, we prove a general second welfare theorem in a model with heterogeneous agents and pre-entry investments. Remarkably, the equilibrium values simultaneously solve both the investment and the matching problems.

Efficiency fails in those models because when agents discount time, they incur implicit search costs (due to delayed payoffs), which are proportional to the continuation values and heterogeneous across skills. Thus, acquiring a higher skill also entails a higher search cost which diminishes the incentive to invest. Furthermore, the implicit search costs also distort the bargaining splits away from the efficient ones, which leads to inefficient matching. In our model, all agents incur the same additive search cost (see Section 2 for details).

Hosios' [1990] classic paper considers a standard search and bargaining model with *homogeneous* agents who choose their search intensity. He proves that the equilibrium can achieve the constrained efficient outcome, provided that the meeting function exhibits constant returns to scale and the bargaining weight equals the elasticity of the meeting function. The underlying point is that with the "right" bargaining weight, the search externalities that agents impose on each other perfectly offset in equilibrium. We derive a similar result for an economy with heterogeneous agents and pre-entry investment (see Proposition 3).

Our paper contributes to the literature on equilibrium convergence in economies with decentralized exchange initiated by the classic work of Rubinstein and Wolinsky [1985, 1990] (see also Gale 1987, Mortensen and Wright 2002, and Lauermann 2013). The previous work in this literature does not consider models with pre-entry investments or with matching. In the context of the product market, our model accommodates heterogeneous goods, whereas those models consider homogeneous goods. In Section 6.1, we extend our model by adding a discount factor δ and show that the equilibrium converges, as $\delta \to 1$, to our baseline equilibrium and thus to the constrained efficient allocation.

Finally, we mention several papers which are important but less related to ours. Burdett and Coles [2001] consider a marriage market with premarital investments, but they assume homogeneous investment costs and a very specific form of *non-transferable utility*. They show that an equilibrium exists and that it is inefficient.[4] In the literature on directed search, sellers post prices to attract buyers, and the equilibrium can achieve an efficient allocation, overcoming the hold-up and matching problems (e.g., Acemoglu and Shimer 2000, 1999b, Shi 2001, Jerez 2017). However, in these models, the matching process and the price-determination mechanism are substantially different than in random search and bargaining models.

2 The Model

There is a continuum population of buyers $\beta \sim F^b$ and sellers $\sigma \sim F^s$. Each buyer chooses one skill from a finite set $I \subset \mathbb{N}$ and each seller chooses one skill from a finite set $J \subset \mathbb{N}$. The cost of skill i to buyer β is $C^b(i, \beta)$ and the cost of skill j to seller σ is $C^s(j, \sigma)$. Output is produced by buyer-seller pairs according to their skills and is summarized by the matrix $G = [g_{ij}]$, where the entry $g_{ij} \geq 0$ denotes the output of a pair with skills i, j. Agents have transferable utility and incur a fixed per-period search cost $c > 0$.

The type distributions F^b and F^s are continuous and strictly increasing over their connected supports: $\mathcal{B} = supp(F^b) \subseteq \mathbb{R}$ and $\mathcal{S} = supp(F^s) \subseteq \mathbb{R}$. The match output g_{ij} is strictly increasing in skills. The cost functions are non-negative, strictly increasing, bounded and continuous. Furthermore, they satisfy increasing differences: the difference $C^b(i', \beta) - C^b(i, \beta)$ is strictly increasing in β whenever $i' > i$ and the difference $C^s(j', \sigma) - C^s(j, \sigma)$ is strictly increasing in σ whenever $j' > j$. That is, a higher skill enhances match output, but is more costly to acquire, and higher types have higher costs and higher marginal costs.

Definition. An *economy* is a tuple $\langle F^b, F^s, I, J, C^b, C^s, G, c\rangle$ consisting of prior distributions, skill sets, investment cost functions, the output function, and a search cost. The economy is *symmetric* if $F^b = F^s, I = J$, $C^b = C^s$, and $g_{ij} = g_{ji}, \forall i, j$.

Timing. Search and matching takes place in discrete time periods over an infinite horizon. In every period, a unit measure of buyers and a unit measure of sellers are born. Each newborn agent chooses a skill and then enters the matching market.

[4] In the non-trivial case of high investment costs, agents overinvest to appeal to better partners, and they search too much in that agents are too selective.

Each agent in the market incurs the search cost c and randomly meets a partner. When two agents meet, they can either accept the match or continue searching in the hope of finding a better partner. If both agents accept the match, then they exit the market and divide their output according to Nash bargaining. If at least one rejects, then they both remain in the market. In the next period, a new cohort enters the market and the process repeats itself. We refer to the agents in the market as the stock population, the agents entering the market as the inflow population, and the agents exiting the market as the outflow population.

Steady State. The economy is in a *steady state* if in the stock population the measure of agents with each skill is constant over time. Therefore, for each skill, the inflow of agents equals the outflow. In a steady state, we denote the measures of skill i buyers and skill j sellers in the stock population by b_i and s_j. The total measures of buyers and sellers in the market are $B = \sum_{i\in I} b_i$ and $S = \sum_{j\in J} s_j$, and the *proportions* of skill i buyers and skill j sellers are $x_i = b_i/B$ and $y_j = s_j/S$ (notice that $B \geq 1$ and $S \geq 1$). The notation (x_i) and (y_j) denotes the profile of buyer and seller proportions. We let $z = \langle (x_i), (y_j), B, S\rangle$ be the *state variable* where the set of all state variables is $\mathcal{Z} = \Delta(I) \times \Delta(J) \times [1, \infty)^2$.

Meetings. An agent can meet at most one partner in each period and pairs meet at random. The total number of meetings per period is $\mu(B, S) = \min(B, S)$. Therefore, if the market is balanced, i.e. $B = S$, then every agent randomly draws a partner in each period. For now, we will assume that the market is balanced, and denote the market size by $N = B = S$ and the state by $z = \langle (x_i), (y_j), N\rangle$. If the market is unbalanced, agents on the long side of the market would need to be rationed, but this cannot occur in equilibrium (see Lemma 1). In Section 6.3, we extend the analysis to consider more general meeting functions.

Strategies. An agent's strategy specifies their choice of skill and which agents they accept. We assume Markov strategies. The *investment strategy* of buyer β is $\mathcal{I}^\beta : \mathcal{Z} \to I$ and that of seller σ is $\mathcal{I}^\sigma : \mathcal{Z} \to J$. The *acceptance strategy* of a buyer with skill i is $A^b_i : \mathcal{Z} \times J \to [0, 1]$, which specifies the probability she accepts a seller with skill j upon meeting. For a seller with skill j, it is $A^s_j : \mathcal{Z} \times I \to [0, 1]$. Note that the acceptance strategies do not depend on the agents' identities because the match output depends only on skills. To simplify, we will suppress the state variable in the strategies. It will be convenient to summarize the acceptance strategies by a matching matrix $M = [m_{ij}]$, where the element $m_{ij} = A^b_i(j) \cdot A^s_j(i)$ is the probability that buyer i and seller j both agree to match, conditional on meeting.

Remark 1. The search cost c captures in reduced form the wide range of costs people explicitly incur as they search. These include the opportunity cost of time (think of the man-hours firms spend screening and interviewing candidates; while candidates forgo some income, say from driving an Uber, as they go through ads, apply, and prepare to interview); the flow payments to search intermediaries and online platforms (such as monthly advertising fees or hiring talent recruiters in-house); cognitive effort costs (browsing and comparing products online for hours, or the negative mental health impact of unemployment); or even direct payments (e.g., singles paying per date). In contrast, in a discount factor model, agents incur an implicit search cost as their payoffs are delayed. Which costs are more salient depends upon the economic situation being modeled, but there are certainly situations where additive costs are predominant.[5] In Section 6, we generalize the model by adding a discount factor.

2.1 Equilibrium

Every skill has a value in the market and agents optimize given the values and the steady state. We denote the values of a skill i buyer by v_i, and of a skill j seller by w_j. The profiles of buyer and seller values are (v_i) and (w_j), respectively. As is standard in the search and matching literature, we define an equilibrium using the matching matrix and values, rather than the strategies.

Definition. A *steady state equilibrium* $\langle z, M, (v_i), (w_j)\rangle$ consists of a state variable, a matching matrix, and market values satisfying conditions (1), (3), and (4) below.

The first condition is that acceptance decisions are individually optimal. When two agents with skills i and j meet, the *surplus* is $s_{ij} = g_{ij} - v_i - w_j$, and the acceptance decisions satisfies the *Efficient Matching* condition:

$$m_{ij} = \begin{cases} 1 & \text{if } s_{ij} > 0 \\ 0 & \text{if } s_{ij} < 0 \end{cases} \tag{1}$$

[5] For example, when search transpires over a short period of time and does not affect the consumption date (think of the time spent searching online for a product that will be delivered tomorrow or college students applying for jobs which they will take after graduation).

The condition is intuitive because an agent will accept a match precisely when her payoff from doing so is greater than her continuation value. When the surplus is negative, i.e. $v_i + w_j > g_{ij}$, the match is always rejected because both agents cannot receive at least their value, while when the surplus is positive, the agents will reach a mutually beneficial agreement. If the surplus is exactly zero, then m_{ij} is unrestricted, i.e. $0 \leq m_{ij} \leq 1$.

When two agents accept each other, each receives their own value and half of the match surplus. This division rule is the Nash bargaining solution and also is a subgame perfect equilibrium of a strategic bargaining game (see, e.g., Atakan 2006). The second condition is that the values are self-consistent, and therefore satisfy the following recursive equation:

$$\begin{aligned} v_i &= \sum_{j \in J} y_j \left[m_{ij} \left(v_i + \tfrac{s_{ij}}{2} \right) + (1 - m_{ij}) v_i \right] - c, \forall i \\ w_j &= \sum_{i \in I} x_i \left[m_{ij} \left(w_j + \tfrac{s_{ij}}{2} \right) + (1 - m_{ij}) w_j \right] - c, \forall j \end{aligned} \tag{2}$$

That is, in every period, buyer i pays the search cost c and meets seller j with probability y_j. If a match is accepted, the buyer receives her continuation value and half of the surplus, whereas if the match is rejected, she attains her continuation value v_i. Simplifying, we obtain the *Constant Surplus* equations:

$$\begin{aligned} \sum_{j \in J} y_j m_{ij} s_{ij} &= 2c, \forall i \\ \sum_{i \in I} x_i m_{ij} s_{ij} &= 2c, \forall j \end{aligned} \tag{3}$$

The investment decisions are individually optimal: $\mathcal{I}^\beta \in \arg\max_{i \in I} v_i - C(i, \beta), \forall \beta$ and $\mathcal{I}^\sigma \in \arg\max_{j \in J} w_j - C(j, \sigma), \forall \sigma$. Since the cost function satisfies strictly increasing differences, the set of cost types who choose each skill is an interval (and hence measurable). Furthermore, at most one type can be indifferent between any two skills,[6] and thus the values (v_i) and (w_j) uniquely determine the inflows (up to measure zero). Formally, we denote by $F^b(A) = \int_A dF^b$ the measure of set A according to F^b. The measure of buyers who choose skill i is $F^b\left(\left\{\beta : \mathcal{I}^\beta = i\right\}\right) = F^b\left(\left\{\beta : i \in \arg\max_{i' \in I} v_{i'} - C^b(i', \beta)\right\}\right)$, and analogously for sellers.

[6] If buyer $\hat{\beta}$ is indifferent between acquiring skills i and i', where $i' > i$, then all buyers $\beta < \hat{\beta}$ strictly prefer skill i' to skill i and all buyers $\beta > \hat{\beta}$ strictly prefer skill i to skill i'.

The final set of conditions is that the economy is in a steady state. We refer to Equations (4) as the *Inflow=Outflow* equations:

$$\begin{aligned} \overbrace{F^b\left(\left\{\beta : i \in \arg\max_{i' \in I} v_{i'} - C^b(i', \beta)\right\}\right)}^{inflow} &= \overbrace{Nx_i \sum_{j \in J} y_j m_{ij}}^{outflow}, \forall i \in I \\ F^s\left(\left\{\sigma : j \in \arg\max_{j' \in j} w_{j'} - C^s(j', \sigma)\right\}\right) &= \ Ny_j \sum_{i \in I} x_i m_{ij}, \forall j \in J \end{aligned} \tag{4}$$

The inflow is the measure of buyers who choose skill i. The outflow is the measure of skill i buyers in the market, Nx_i, times the probability of exiting (each buyer meets a skill j with probability, y_j, and they accept each other with probability, m_{ij}). The seller Inflow=Outflow equations are analogous.

2.2 Equilibrium Properties

The next two lemmas will be useful. The first states that unbalanced states do not occur in equilibria.

Lemma 1. *(No Rationing) In any equilibrium, $B = S$.*

Proof. WLOG, suppose that $B \geq S$. Then, a buyer meets a seller with probability $\rho = S/B$, and a seller meets a buyer with probability 1. Therefore, the values satisfy:

$$\begin{aligned} \forall i : v_i =& \rho \sum_{j \in J} y_j \left[m_{ij}\left(v_i + \frac{s_{ij}}{2}\right) + (1 - m_{ij})v_i\right] + (1-\rho)v_i - c \Rightarrow \sum_{j \in J} y_j m_{ij} s_{ij} = \frac{2c}{\rho} \\ \forall j : w_j =& \sum_{i \in I} x_i \left[m_{ij}\left(w_j + \frac{s_{ij}}{2}\right) + (1 - m_{ij})w_j\right] - c \Rightarrow \sum_{i \in I} x_i m_{ij} s_{ij} = 2c \end{aligned}$$

Therefore, since $\sum_{i \in I} x_i = \sum_{j \in J} y_j = 1$:

$$\frac{2c}{\rho} = \sum_{i \in I} x_i \sum_{j \in J} y_j m_{ij} s_{ij} = \sum_{j \in J} y_j \left(\sum_{i \in I} x_i m_{ij} s_{ij}\right) = 2c \Rightarrow B = S$$

□

The next lemma states that, in equilibrium, the agents' values are increasing and the marginal values are bounded by the expected marginal productivity.

Lemma 2. *In any equilibrium,*

$$\frac{\sum_{j\in J} y_j m_{i'j}(g_{i'j}-g_{ij})}{\sum_{j\in J} y_j m_{i'j}} \geq v_{i'} - v_i \geq \frac{\sum_{j\in J} y_j m_{ij}(g_{i'j}-g_{ij})}{\sum_{j\in J} y_j m_{ij}} > 0,\ \forall i' > i$$

$$\frac{\sum_{i\in I} x_i m_{ij'}(g_{ij'}-g_{ij})}{\sum_{i\in I} x_i m_{ij'}} \geq w_{j'} - w_j \geq \frac{\sum_{i\in I} x_i m_{ij}(g_{ij'}-g_{ij})}{\sum_{i\in I} x_i m_{ij}} > 0,\ \forall j' > j$$

In particular, if $m_{ij} = 1, \forall i, j$*, then the marginal value equals the expected marginal productivity:* $v_{i'} - v_i = \sum_{j\in J} y_j(g_{i'j} - g_{ij})$ *and* $w_{j'} - w_j = \sum_{i\in I} x_i(g_{ij'} - g_{ij})$*.*

Proof. The Constant Surplus and Efficient Matching conditions imply that:

$$\sum_{j\in J} y_j m_{ij} s_{ij} = 2c = \sum_{j\in J} y_j m_{i'j} s_{i'j} \geq \sum_{j\in J} y_j m_{ij} s_{i'j}$$

Subtracting the RHS from the LHS, and normalizing:

$$v_{i'} - v_i \geq \frac{\sum_j y_j m_{ij}(g_{i'j}-g_{ij})}{\sum_j y_j m_{ij}} > 0$$

The upper bound is derived analogously by switching i and i'. □

Lemma 2 also implies that there is a uniform bound on marginal values: $\max_j g_{i'j} - g_{ij} \geq v_{i'} - v_i \geq \min_j g_{i'j} - g_{ij}$.

Remark 2. The Constant Surplus equations have two further implications: First, they determine the values for unchosen (measure 0) skills, and therefore we are not free to set those values arbitrarily (for instance, to minus infinity). Second, every agent has at least one partner with whom the surplus is positive. Furthermore, that partner is not of measure 0, which implies that there are no pathological equilibria where an agent searches forever.

Remark 3. If $\langle z, M, (v_i), (w_j)\rangle$ is an equilibrium, then so is $\langle z, M, (v_i + t), (w_j - t)\rangle$ for any transfer $t \in \mathbb{R}$. Therefore, there is at least one degree of freedom in the equilibrium values. We now show that there is in fact exactly one degree of freedom. This is because the marginal values, i.e. Δv_i, are uniquely pinned down by the investment decisions and a Constant Surplus equation imposes an additional condition on the value functions.

3 Illustrative Examples

We now illustrate the model with two examples. We consider a symmetric economy with two skills, $I = J = \{0, 1\}$. Each agent can either invest and become skilled, $i = j = 1$, or not invest and remain unskilled, $i = j = 0$. The cost of becoming skilled is the agent's type, and types are uniformly distributed, $\beta, \sigma \sim F = U[a, d]$. To simplify notation, let $x \equiv x_1$ and $y \equiv y_1$, denote the proportion of skilled buyers and skilled sellers. We consider the following supermodular and submodular production matrices:

$$G^{sup} = \begin{bmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad G^{sub} = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$$

In both matrices, skilled-skilled pairs produce $g_{11} = 4$ and unskilled-unskilled pairs produce $g_{00} = 1$. The first production matrix is *supermodular* because skilled-unskilled pairs produce $g_{10} = g_{01} = 2$, so an agent's marginal productivity is greater when matched with a skilled agent than when matched with an unskilled agent, $g_{11} - g_{01} = 2 > 1 = g_{10} - g_{00}$. Conversely, the second production matrix is *submodular* because $g_{10} = g_{01} = 3$, and so $g_{11} - g_{01} = 1 < 2 = g_{10} - g_{00}$.

In each case, we will demonstrate the constrained efficient allocation and the equilibria. The constrained efficient allocation solves the planner's problem which is to choose the investment thresholds, matching rule and skill distribution to maximize per-period welfare[7]

$$W(x, y, N, [m_{ij}], \beta_1, \sigma_1) = \overbrace{\sum_{i=0}^{1}\sum_{j=0}^{1} N x_i y_j m_{ij} g_{ij}}^{\text{Productivity}} - \overbrace{2Nc}^{\text{Search Costs}} - \overbrace{\int_a^{\beta_1} \beta dF(\beta) - \int_a^{\sigma_1} \sigma dF(\sigma)}^{\text{Investment Costs}}$$

subject to the steady state constraints:

$$Nx \sum_{j=0}^{1} y_j m_{1j} = F(\beta_1) \text{ and } N(1-x) \sum_{j=0}^{1} y_j m_{0j} = 1 - F(\beta_1)$$

$$Ny \sum_{i=0}^{1} x_i m_{i1} = F(\sigma_1) \text{ and } N(1-y) \sum_{i=0}^{1} x_i m_{j0} = 1 - F(\sigma_1)$$

[7] The first term in the welfare function is the per-period output ($N x_i y_j m_{ij}$ is the measure of accepted matches between skills i, j and g_{ij} is their output), the second term is the search cost, and the final terms are the investment costs.

Supermodular Production

In this case, $G = G^{sup}$ and to simplify, we fix the distribution parameters $a = 0.8$ and $d = 2.8$. The planner's optimal policy depends upon c and the matching rule is either:

1) **All Skills Match:** Agents accept any partner ($m_{ij} = 1, \forall i, j$). Total per-period welfare is:

$$\begin{aligned} W^{All}(N, x, y, \beta_1, \sigma_1) =& N\left[4xy + 2y(1-x) + 2x(1-y) + (1-x)(1-y)\right] \\ & - 2Nc - \int_a^{\beta_1} \beta dF(\beta) - \int_a^{\sigma_1} \sigma dF(\sigma) \end{aligned}$$

and the steady state equations are:

$$\begin{aligned} Nx = F(\beta_1) \text{ and } N(1-x) = 1 - F(\beta_1) \\ Ny = F(\sigma_1) \text{ and } N(1-y) = 1 - F(\sigma_1) \end{aligned}$$

which imply $N = 1$ and $x = F(\beta_1)$ and $y = F(\sigma_1)$, and so the planner's optimization problem is two-dimensional (β_1, σ_1). The optimal investment thresholds are $\beta_1 = \sigma_1 \equiv \beta^{*ALL}$, the optimal state $x = y \equiv x^{*ALL}$, and they do not depend on c.

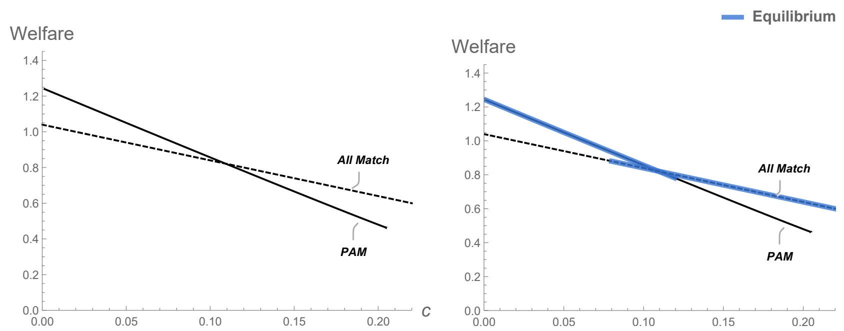
2) **Positive Assortative Matching (PAM):** Same skills accept and opposite skills reject, that is, $M = \left[\begin{smallmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Total per-period welfare is:

$$W^{PAM}(N, x, y, \beta_1, \sigma_1) = N\left[4xy + (1-x)(1-y)\right] - 2Nc - \int_a^{\beta_1} \beta dF(\beta) - \int_a^{\sigma_1} \sigma dF(\sigma)$$

The steady state equations are:

$$\begin{aligned} Nxy = F(\beta_1) \text{ and } N(1-x)(1-y) = 1 - F(\beta_1) \\ Nyx = F(\sigma_1) \text{ and } N(1-y)(1-x) = 1 - F(\sigma_1) \end{aligned}$$

which imply $\beta_1 = \sigma_1$ and $N = \frac{1}{xy+(1-x)(1-y)}$, and so the planner's optimization problem is three dimensional (β_1, x, y). The solution is always symmetric, $x = y \equiv x^{*PAM}$ and $\beta_1 = \sigma_1 \equiv \beta^{*PAM}$, but its flows and stocks depend on c.



(a) Welfare (b) Equilibrium Regions

Figure 1: Welfare Comparison and Equilibrium Regions for $F = U[0.8, 2.8]$

Figure 1, Panel (a) depicts the welfare of these two allocations as a function of the search cost c, assuming that the planner chooses the investment thresholds optimally. In the All Match allocation (1), the planner's optimal policy does not depend on c, and so this curve is linear. In contrast, in the PAM allocation (2), the planner's optimal stocks and flows change with c, and thus the PAM curve is convex (though it is hard to see in this graph). The constrained efficient allocation is the upper envelope of the PAM and All Match curves. The basic trade-off is between higher productivity (PAM) versus lower search cost (All Match). Panel (b) depicts the regions where the PAM and All Match allocations are an equilibrium.[8] The figure visually demonstrates a Second Welfare Theorem: the upper envelope of the two curves, which is the constrained efficient allocation, is always an equilibrium.

The following bullets explain how the equilibrium regions are derived:

- **The All Match allocation is an equilibrium if and only if $c \geq 0.08$.** To see why: In an equilibrium where all skills match, Lemma 2 implies that the marginal values must equal the marginal productivities:

$$\Delta v = y(g_{11} - g_{01}) + (1 - y)(g_{10} - g_{00}) = 1 + y$$
$$\Delta w = x(g_{11} - g_{10}) + (1 - x)(g_{01} - g_{00}) = 1 + x$$

and the steady state equations are $F(\Delta v) = x$ and $F(\Delta w) = y$. These equations have a unique solution $\Delta v = \Delta w \equiv \Delta \bar{v}$ and $x = y \equiv \bar{x}$, and the candidate state $\bar{x}$ induces values $(\bar{v}), (\bar{w})$ that must solve: i) the corresponding Constant Surplus

[8] We depict a certain economy parameterized by F and G^{sup}, but a similar picture would arise for most two-skill supermodular symmetric economies.

equations ($\sum_{i=0}^{1} \bar{x}_i s_{ij} = 2c, \forall j$); and ii) the Efficient Matching conditions so that all pairs indeed want to match ($\bar{v}_i + \bar{w}_j \leq g_{ij}$, $\forall i, j$). "$\Leftarrow$" On the one hand, if $c < 0.08$, the values that solve the Constant Surplus equations are too high and would violate the Efficient Matching condition because agents with opposite skills would reject each other (that is, $\bar{v}_i + \bar{w}_j > g_{ij}$ whenever $i \neq j$), and so there does not exist an equilibrium where all pairs match. "$\Rightarrow$" On the other hand, if $c \geq 0.08$, the induced values are sufficiently low and so the state $x = y = \bar{x}$ and marginal values $\Delta\bar{v} = \Delta\bar{w}$ constitute the unique equilibrium where all pairs match. Remarkably, the equilibrium stocks and flows coincide with the All Match allocation, $\bar{x} = x^{*ALL}$ and $\Delta\bar{v} = \beta^{*ALL}$, and the equilibrium exists whenever this allocation is efficient.

- **The PAM allocation is an equilibrium if and only if $c \leq c_1 \approx 0.12$.** To see why: in an equilibrium with PAM, it must be that $\Delta v = \Delta w$ (because the outflow of skilled buyers = the outflow of skilled sellers); and $x = y$ (because of the Constant Surplus equations $x s_{11} = 2c = y s_{11}$); together implying[9]

$$F(\Delta v) = \frac{x^2}{x^2 + (1-x)^2}; \; \Delta v = \frac{g_{11}}{2} - \frac{c}{x} - \left(\frac{g_{00}}{2} - \frac{c}{1-x} \right)$$

The states x that solves these two equations are the only candidates for an equilibrium with PAM and every candidate x induces values that must solve: i) the corresponding Constant Surplus equations ($x s_{11} = 2c$ and $(1-x)s_{00} = 2c$); and ii) the Efficient Matching conditions so that same skills accept ($v_i + w_j \leq g_{ij}$ for $i = j$) and opposite skills reject ($v_i + w_j \geq g_{ij}$ for $i \neq j$). "$\Leftarrow$" If $c > c_1$, then the values that solve the Constant Surplus equation are too low and would violate the Efficient Matching condition because agents with opposite skills would accept each other (that is, $v_i + w_j < g_{ij}$ when $i \neq j$), and so the PAM allocation cannot be supported by an equilibrium. "$\Rightarrow$" If $c \leq c_1$, then there is a *unique* candidate state $\hat{x}$ whose induced values $(\hat{v}), (\hat{w})$ satisfy these two conditions. The state $y = x = \hat{x}$, size $\hat{N} = \frac{1}{\hat{x}^2 + (1-\hat{x})^2}$, and marginal values $\Delta\hat{v} = \Delta\hat{w}$ constitute the unique equilibrium with PAM. Remarkably, its stocks and flows coincide with the PAM allocation, $\hat{x} = x^{*PAM}$ and $\Delta\hat{v} = \beta^{*PAM}$, and the equilibrium exists whenever this allocation is efficient.

[9] To get the first equation, divide the two steady state equations $Nx^2 = F(\Delta v)$ and $N(1-x)^2 = 1 - F(\Delta v)$; and to get the second, subtract the two Constant Surplus equations $x s_{11} = 2c$ and $(1-x)s_{00} = 2c$ and use $\Delta v = \Delta w$.

Notice that in Figure 1, there can be multiple equilibria as the PAM and All Match regions overlap, but the overlap region is relatively small. In addition, the planner could implement a wide range of other policies, varying either the investment thresholds or the matching rule, but those other policies would generate lower welfare, and they cannot be supported by an equilibrium (generically). In this example, for any c, there are at most three equilibria: the two depicted above and possibly a mixed one where skilled-unskilled pairs match with a strictly positive probability (which has lower welfare).

Submodular Production

In this case, $G = G^{sub}$ and we fix the average cost type to be $(a + d)/2 = 1.5$. The constrained efficient allocation is spanned by the following three simple allocations:

1) **Negative Assortative Matching (NAM):** Agents with below-average costs invest, those with above-average costs do not, and only opposite skills match, $M = \left[\begin{smallmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. The steady state is $x = y = 1/2$ and $N = 2$. Total per-period welfare is:

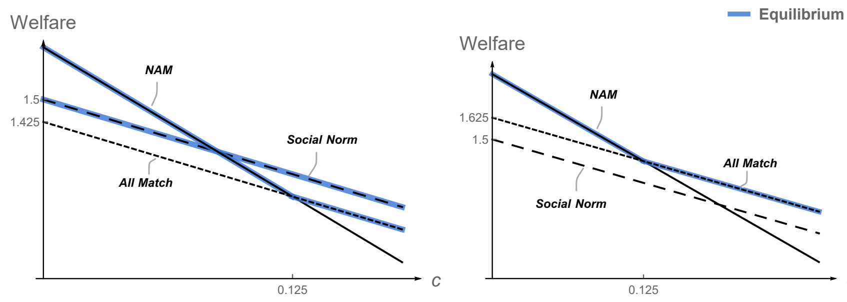
$$\mathcal{W}^{NAM} = g_{10} - 2Nc - \int_a^\mu \beta f(\beta) d\beta - \int_a^\mu \sigma f(\sigma) d\sigma = 3 - 4c - (3a + d)/4$$

2) **All Skills Match:** The investment thresholds are the same as in the NAM allocation, but now all pairs match, $M = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. Thus the stock population is $N = 1$, and $x = y = 1/2$. Total per-period welfare is:

$$\begin{aligned} \mathcal{W}^{All} &= \frac{1}{4}(g_{11} + g_{10} + g_{01} + g_{00}) - 2Nc - \int_a^\mu \beta f(\beta) d\beta - \int_a^\mu \sigma f(\sigma) d\sigma \\ &= 2.75 - 2c - (3a + d)/4 \end{aligned}$$

3) **Social Norm (one-sided investment):** Every buyer invests and becomes skilled and every seller does not invest and remains unskilled. Agents accept any partner. Since the market clears in every period, the stock population is $N = 1$. The total per-period welfare is:

$$\mathcal{W}^{SN} = g_{10} - 2Nc - \int_a^d \beta f(\beta) d\beta = 3 - 2c - (a + d)/2$$



(a) $l < 1$ (b) $l > 1$

Figure 2: Equilibrium Regions

Figure 2 illustrates the welfare of these allocations as a function of c:

$$\begin{aligned} \mathcal{W}^{NAM} &= 3 \;\; -4c - \left(\tfrac{3}{4}a + \tfrac{1}{4}d\right) \\ \mathcal{W}^{All} &= 2.75 - 2c - \left(\tfrac{3}{4}a + \tfrac{1}{4}d\right) \\ \mathcal{W}^{SN} &= 3 \;\; -2c - \left(\tfrac{1}{2}a + \tfrac{1}{2}d\right) \end{aligned}$$

The equilibrium regions are shaded blue. Panel (a) depicts the case where the distribution F has small support, $l = d - a < 1$ and Panel (b) depicts a large support[10] $l > 1$. These three allocations demonstrate the trade-off between productivity, investment cost, and search cost. Each allocation optimizes two components at the expense of the third (see Table 2). Notice each allocation is supported by an equilibrium whenever it is efficient.

	Productivity	Search Cost	Investment Cost
NAM	✓	×	✓
All Skills Match	×	✓	✓
Social Norm	✓	✓	×

Table 2: Welfare Comparisons

- **The NAM allocation is an equilibrium if and only if the search cost** $c \leq 1/8$. The argument is similar to the PAM equilibrium in the supermodular case (see above). This allocation maximizes productivity, but every agent must search twice (on average) to find the most productive partner.

[10]The specific parameters illustrated are $l = 0.5$ and $l = 1.5$.

- ∘ **The All Match allocation is an equilibrium if and only if the search cost $c \geq 1/8$.** The argument is similar to the All Match equilibrium in the supermodular case (see above). This allocation benefits from lower search costs but has lower productivity because agents mismatch (both unskilled-unskilled and skilled-skilled matches occur).
- ∘ **The Social Norm allocation is an equilibrium if and only if the support**[11] **$l \leq 1$.** The Social Norm allocation maximizes productivity and minimizes the search cost but has a higher total investment cost because high-cost buyers invest while low-cost sellers do not. This talent misallocation problem becomes more severe as we stretch the support of the cost distribution.

Takeaways

The key takeaways from these two examples are:

1) The Second Welfare Theorem. In each example, the constrained efficient allocation, depicted by the upper envelope of the lines, is an equilibrium allocation. In the next section, we establish a general second welfare theorem.

2) Assortative Matching. In the first example, the production function was supermodular, and agents with the same skills matched. In the second example, the production function was submodular, and agents with opposite skills matched. In Section 5.1, we establish general sorting patterns: if the production function is super/submodular, there is positive/negative assortative matching in equilibrim.

3) Discrimination. When the production function is submodular, the efficient outcome can be discriminatory. Discrimination induces the two groups to invest differently and thereby minimizes search costs and enhances productivity, but at the expense of higher investment costs.

4) Multiple Equilibria: The First Welfare Theorem does not hold as there can be multiple equilibria. However, the equilibria set is small and tractable.

[11]"$\Leftarrow$" Suppose $l = d - a > 1$. By Lemma 2, the marginal values are bounded by the marginal productivities: $1 \leq \Delta v \leq 2$. Since the average cost-type is 1.5, if $l > 1$, then $a < 1$ and the buyers/sellers with investment costs less than 1 will invest in every equilibrium. "$\Rightarrow$" Suppose $l \leq 1$. The supporting values are: $v_1 = 2.5 - c$, $v_0 = 0.5 - c$, $w_1 = 1.5 - c$, and $w_0 = 0.5 - c$. Since the average cost-type is 1.5, it follows that $1 < a < d < 2$. Therefore, all buyers want to invest because $\beta \leq d \leq 2 = \Delta v$ and no seller wants to invest because $\Delta w = 1 \leq a \leq \sigma$, and the values satisfy the CS equations.

4 The Second Welfare Theorem

To simplify notation, we label the skills as $I = \{0, 1, \dots, |I| - 1\}$ and $J = \{0, 1, \dots, |J| - 1\}$. The constrained efficient allocation is the solution to the problem of a social planner who chooses the investment and acceptance strategies and sets the stock in the matching market, in order to maximize per-period total welfare, subject to the condition that the economy is in a steady state. Without loss of generality: i) the planner chooses a balanced state,[12] $B = S = N$; ii) the matching strategies are represented by a matching matrix; and iii) since the investment cost functions satisfy strictly increasing differences, the planner's optimal investment strategies can be defined by thresholds $\beta_0 \geq \beta_1 \geq \dots \geq \beta_I$ and $\sigma_0 \geq \sigma_1 \geq \dots \geq \sigma_J$, so that all buyers of type $\beta \in (\beta_{i+1}, \beta_i)$ choose skill i and all sellers of type $\sigma \in (\sigma_{j+1}, \sigma_j)$ choose skill j. Notice that the thresholds are descending because costs increase with type, so higher types choose lower skills. The planner chooses a tuple $\langle z, M, (\beta_i), (\sigma_j) \rangle$ of steady state, matching matrix, and investment thresholds in order to maximize:

$$\begin{aligned} \mathcal{W}\left(\langle z, M, (\beta_i), (\sigma_j) \rangle\right) = & \sum_{i \in I} \sum_{j \in J} N x_i y_j m_{ij} g_{ij} - 2Nc - \sum_{i \in I} \int_{\beta_{i+1}}^{\beta_i} C^b(i, \beta) f^b(\beta) d\beta \\ & - \sum_{j \in J} \int_{\sigma_{j+1}}^{\sigma_j} C^s(j, \sigma) f^s(\sigma) d\sigma \end{aligned} \tag{5}$$

subject to

$$\begin{aligned} flow_i^b &:= \left(F^b(\beta_i) - F^b(\beta_{i+1})\right) - N x_i \sum_{j \in J} y_j m_{ij} = 0, \forall i \\ flow_j^s &:= \left(F^s(\sigma_j) - F^s(\sigma_{j+1})\right) - N y_j \sum_{i \in I} x_i m_{ij} = 0, \forall j \\ x_i &\geq 0, \forall i \\ y_j &\geq 0, \forall j \\ X &:= 1 - \sum_{i \in I} x_i = 0 \\ Y &:= 1 - \sum_{j \in J} y_j = 0 \\ 1 &\geq m_{ij} \geq 0, \forall i, j \\ F^b(\beta_{|I|}) &= F^s(\sigma_{|J|}) = 0 \\ F^b(\beta_0) &= F^s(\sigma_0) = 1 \end{aligned}$$

[12]If $B > S$, then there exists another state with lower total search cost and identical output and investment cost.

The first term in the objective function is per-period total output (the measure of formed matches between buyer i and seller j is $Nx_iy_jm_{ij}$ and the match output is g_{ij}), the second term is the per-period total search cost, and the last two terms are the per-period total investment costs. The first constraint is that inflow equals outflow. The other conditions stipulate that x_i, y_j are proportions, m_{ij} are probabilities, and that the planner must assign a skill to every agent.

Remark 4. Notice that the maximization problem does not explicitly require that $\beta_0 \geq \beta_1 \geq \ldots \geq \beta_I$ and $\sigma_0 \geq \sigma_1 \geq \ldots \geq \sigma_J$, nor that $N > 0$, because these conditions are implied by the other constraints (see proof).

Theorem 1. *(Second Welfare Theorem) For every economy $\langle F^b, F^s, I, J, C^b, C^s, G, c\rangle$:*

i) There exists an optimal policy $\langle z, M, (\beta_i), (\sigma_j)\rangle$.

ii) Every optimal policy $\langle z, M, (\beta_i), (\sigma_j)\rangle$ can be decentralized. That is, there are values (v_i^*), (w_j^*), and a matching matrix M^* such that $\langle z, M^*, (v_i^*), (w_j^*)\rangle$ is an equilibrium, where $m_{ij}^* = m_{ij}$ for all i, j such that x_i, $y_j > 0$.

The theorem demonstrates that any optimum policy can be decentralized as an equilibrium. The proof shows that the equilibrium values that decentralize the optimal allocation are the shadow values of the flow constraints in the dual problem. We show that these values are internally self-consistent with the bargaining procedure, that is, they satisfy the Constant Surplus equations; and also motivate the agents to invest and match efficiently. For instance, if the planner wants buyer β and seller σ to choose skill i^* and j^*, then $i^* \in \arg\max_{i\in I} v_i - C^b(i, \beta)$ and $j^* \in \arg\max_{j\in J} w_j - C^s(j, \sigma)$; and if the planner wants them to accept (reject) each other, then $v_{i^*} + w_{j^*} \geq g_{i^*j^*}$ ($v_{i^*} + w_{j^*} \leq g_{i^*j^*}$).

Proof. First, we show that the constraints of the problem imply that $N > 0$, and $\beta_i \geq \beta_{i+1}$ for all i, and $\sigma_j \geq \sigma_{j+1}$ for all j. To see this, observe that $F^b(\beta_{|I|}) = 0$ and $F^b(\beta_0) = 1$, and so there exists a skill i such that $F(\beta_i) > F(\beta_{i+1})$. By constraint $flow_i^b$, it must be that $Nx_i \sum_{j\in J} y_jm_{ij} > 0$. Since x_i, y_j, m_{ij} are all non-negative, it follows that $N > 0$. Thus, the outflow of every skill is non-negative, and from the flow conditions, it must be that $\beta_i \geq \beta_{i+1}$ for all i, and likewise $\sigma_j \geq \sigma_{j+1}$ for all j.

(i) Existence: To demonstrate existence, since the objective is continuous, all we need to show is that the policy space is compact. First, there is a uniform upper bound

$\overline{N}$ so that in any optimum, $N \leq \overline{N}$ (recall that $N \geq 0$). For the upper bound, notice that the Inflow=Outflow constraints imply $\sum_{i\in I}\sum_{j\in J} N x_i y_j m_{ij} = 1$, and therefore the first term in the welfare expression is a convex combination of g_{ij} and therefore is uniformly bounded by $\max g_{ij}$. Thus, $\lim_{N\to\infty} \mathcal{W} = -\infty$ and so the optimal policy cannot involve arbitrarily large N. The planner can choose quantiles $F(\beta_i)$ instead of thresholds β_i, and since the objective is also continuous in the quantiles and the quantile space is bounded, a maximum indeed exists.

(ii) Decentralizing optimal allocations: The dual problem is

$$\begin{aligned}\mathcal{L}\left(\langle z, M, (\beta_i), (\sigma_j)\rangle\right) = &\sum_{i\in I}\sum_{j\in J} N x_i y_j m_{ij} g_{ij} - 2Nc \\ &- \sum_{i\in I}\int_{B_i} C^b(i,\beta) f^b(\beta) d\beta - \sum_{j\in J}\int_{S_j} C^s(j,\sigma) f^s(\sigma) d\sigma \\ &+ \sum_{i\in I} v_i \cdot flow_i^b + \sum_{j\in J} w_j \cdot flow_j^s + \sum_i \phi_i x_i + \sum_j \psi_j y_j + \gamma X + \lambda Y \\ &+ \sum_{i\in I}\sum_{j\in J}\left(\eta_{ij} m_{ij} + \hat{\eta}_{ij}(1 - m_{ij})\right)\end{aligned}$$

We will first show that a constraint qualification holds and then construct an equilibrium using the shadow values from the KKT conditions.

1) The Constraint Qualifications: Since the problem is not convex, we use the constant rank regularity condition, which requires that for each subset of the gradients of the active inequality constraints and the equality constraints, the rank in the vicinity of the optimal point is constant (Janin [1984]). The formal proof is given in Lemma 5 in the Appendix.

2) Deriving values from the KKT conditions: Due to the constraint qualification above, the first order conditions (FOC) of the dual problem $\mathcal{L}$ are necessary at any optimum:

$$\text{FOC(N): } \sum_{i\in I}\sum_{j\in J} x_i y_j m_{ij} g_{ij} - 2c - \sum_{i\in I} v_i \left(x_i \sum_{j\in J} y_j m_{ij}\right) - \sum_{j\in J} w_j \left(y_j \sum_{i\in I} x_i m_{ij}\right) = 0$$

$$\iff \sum_i \sum_j x_i y_j m_{ij}(g_{ij} - v_i - w_j) = 2c$$

$$\text{FOC}(x_i)\text{: } N\sum_j y_j m_{ij} g_{ij} - v_i N \sum_j y_j m_{ij} - N\sum_j w_j m_{ij} y_j - \gamma + \phi_i = 0$$
$$\iff N\sum_j y_j m_{ij}\left(g_{ij} - v_i - w_j\right) = \gamma - \phi_i$$
$$\text{FOC}(y_j)\text{: } N\sum_i x_i m_{ij} g_{ij} - N\sum_i v_i x_i m_{ij} - w_j N \sum_i m_{ij} x_i - \lambda + \psi_j = 0$$
$$\iff N\sum_i x_i m_{ij}\left(g_{ij} - v_i - w_j\right) = \lambda - \psi_j$$

Complementary slackness: $\phi_i x_i = 0$ and $y_j \psi_j = 0$ and $\phi_i, \psi_j \geq 0$.

$$\text{FOC}(m_{ij})\text{: } N x_i y_j g_{ij} - v_i N x_i y_j - w_j N x_i y_j + \eta_{ij} - \hat{\eta}_{ij} = 0$$
$$\iff N x_i y_j (g_{ij} - v_i - w_j) = -\eta_{ij} + \hat{\eta}_{ij}$$

Complementary slackness: $\eta_{ij} m_{ij} = 0$ and $\hat{\eta}_{ij}(1 - m_{ij}) = 0$ and $\eta_{ij}, \hat{\eta}_{ij} \geq 0$.

$$\text{FOC}(\beta_i)\text{: } f^b(\beta_i)(v_i - v_{i-1}) = f^b(\beta_i)\left(C(i,\beta_i) - C(i-1,\beta_i)\right), \text{ for } i \in \{1,\ldots,I-1\}$$

$$\text{FOC}(\sigma_j)\text{: } f^s(\sigma_j)(w_j - w_{j-1}) = f^s(\sigma_j)\left(C(j,\sigma_j) - C(j-1,\sigma_j)\right), \text{ for } j \in \{1,\ldots,J-1\}$$

We now show that the shadow values v_i, w_j, together with the matching matrix M and state z, constitute an equilibrium.

Decentralizing the constrained optimal allocation when z is interior (ii): To verify the Constant Surplus equations, notice that:

$$N \cdot 2c = N\sum_I \sum_J x_i y_j m_{ij}(g_{ij} - v_i - w_j) = \sum_I x_i N \sum_J y_j m_{ij}(g_{ij} - v_i - w_j)$$
$$= \sum_I x_i(\gamma + \phi_i) = \sum_I \gamma x_i + \phi_i x_i = \sum_I \gamma x_i = \gamma$$

The first line uses FOC(N), while the second line uses FOC(x_i), complementary slackness ($\phi_i x_i = 0$), and the condition $\sum_I x_i = 1$. Therefore $\gamma = 2cN$. Since z is interior, $\phi_i = 0$, and the FOC(x_i) is $\sum_J y_j m_{ij}\left(g_{ij} - v_i - w_j\right) = 2c$, which is the Constant Surplus equation for skill i. An analogous argument holds for the sellers' Constant Surplus equations.

To verify the Efficient Matching conditions, notice that if $g_{ij} - v_i - w_j > 0$, the FOC for m_{ij} requires that $\hat{\eta}_{ij} > 0$ and therefore $m_{ij} = 1$. Similarly, if $g_{ij} - v_i - w_j < 0$, the FOC for m_{ij} requires that $\eta_{ij} > 0$ and therefore $m_{ij} = 0$.

To verify that the investments are incentive compatible, we show that for any type $\beta \in [\beta_{i+1}, \beta_i]$, their most preferred skill is i. To see this, for any lower skill, $i' \leq i$, the FOC for the threshold $\beta_{i'}$ is $f(\beta_{i'})(v_{i'} - v_{i'-1}) = f(\beta_{i'})(C(i', \beta_{i'}) - C(i' - 1, \beta_{i'}))$ and recall that $\beta_{i'} \geq \beta$. Since $f > 0$ everywhere, this can be simplified to $v_{i'} - C(i', \beta_{i'}) = v_{i'-1} - C(i' - 1, \beta_{i'})$. Since type $\beta_{i'}$ is indifferent between the skills i' and $i' - 1$, by single-crossing, type β weakly prefers skill i' to skill $i' - 1$. Thus, type β weakly prefers i to any lower skill i'. An analogous argument applies for higher skills.

The case of a non-interior z can be found in the Appendix. □

It immediately follows from Theorem 1 that an equilibrium exists.

Corollary 1. *An equilibrium exists.*

The following proposition demonstrates some comparative statics for welfare.

Proposition 1. *The welfare function $\mathcal{W}$ is continuous, strictly decreasing, and convex in c. Moreover, the population size N is weakly decreasing in c.*

The proof is in the Appendix. It relies on the observation that $\partial \mathcal{W} / \partial c = -2N$, which follows immediately from the envelope theorem, implying that a shock to c has greater impact on welfare when c is small than when c is large.

Remark 5. (Matching and Values of Unrealized Skills) Theorem 1 proves that any optimum can be decentralized (modulo matching between unrealized skills). The planner can match unrealized types in any fashion because they have no impact on welfare, and thus the optimization problem places no restriction on their matching. However, the equilibrium conditions (the Constant Surplus equations and Efficient Matching conditions) apply for all skills, including unrealized ones. In the Appendix, we construct the matching and values for these unrealized skills.

4.1 Outside Options and Endogenous Entry

We now extend the efficiency result to the case where agents have outside options. Suppose that every new-born agent can either invest and enter the market or opt out and receive the outside payoff equal to u^b for buyers and u^s for sellers. In equilibrium, buyer β enters the market if and only if $\max_i v_i - C(i,\beta) \geq u^b$, and seller σ enters if and only if $\max w_j - C(j,\sigma) \geq u^s$. We focus on the interesting case where there are gains to trade, and so for at least two types, β and σ, $\max_{i\in I, j\in J} g_{ij} - 2c - C(i,\beta) - C(j,\sigma) > u^b + u^s$. The only difference from the baseline model is that the planner now also chooses the entry thresholds β_0 and σ_0 in order to maximize:

$$\mathcal{W} = N\sum_{i\in I}\sum_{j\in J} x_i y_j m_{ij} g_{ij} - 2Nc - \sum_{i\in I}\int_{\beta_{i+1}}^{\beta_i} C(i,\beta) f^b(\beta)d\beta - \sum_{j\in J}\int_{\sigma_{j+1}}^{\sigma_j} C(j,\sigma) f^s(\sigma)d\sigma$$
$$+ \int_{\beta_0}^{\infty} u^b f^b(\beta)d\beta + \int_{\sigma_0}^{\infty} u^s f^s(\sigma)d\sigma$$

and the boundary conditions $F^b(\beta_0) = 1$ and $F^s(\sigma_0) = 1$ are removed.

Corollary 2. *In a model with outside options, the constrained efficient outcome is an equilibrium.*

The proof shows that the shadow values still constitute an equilibrium (see Appendix). As before, v_0 is the shadow value of the skill 0 flow constraint. However, there is an additional first-order condition since β_0 is now endogenous: $v_0 - C(0, \beta_0) = u^b$ which is precisely the equilibrium entry condition for buyers. An analogous argument holds for sellers.

Remark 6. In the baseline model, there is exactly one degree of freedom in the equilibrium values (see Remark 3). In the model with outside options, there is an additional entry condition and thus the values are unique.

5 Equilibrium Sorting and Uniqueness

In this section, we show that the equilibria have a clear and simple structure: Section 5.1 shows that every equilibrium exhibits assortative matching if the production function is super/submodular. Section 5.2 considers an additively separable production function (product market) and shows that the equilibrium is unique. Furthermore, these results show that for our second welfare theorem, the efficient allocation is not caught in a widely cast net.

5.1 Assortative Matching

Denote the matching set of skill-i buyers by $M_i = \{j : m_{ij} > 0\} \subseteq J$, this is the set of seller skills with whom buyer i matches. Similarly, for sellers, $M_j = \{i : m_{ij} > 0\} \subseteq I$. The maxima and minima of these sets are denoted $\overline{m}_i = \max M_i$, $\underline{m}_i = \min M_i$, $\overline{m}_j = \max M_j$ and $\underline{m}_j = \min M_j$. We say that a buyer's matching set M_i is convex if $\underline{m}_i < j < \overline{m}_i$ implies that $m_{ij} = 1$ (this is stronger than stating that the matching sets are intervals because it requires that only boundary types can match probabilistically). Convexity is defined analogously for sellers. A matching matrix M exhibits *positive assortative matching* (PAM) if the matching sets are convex and the maxima/minima are weakly increasing. Likewise, M exhibits *negative assortative matching* (NAM) if the matching sets are convex and the maxima/minima are weakly decreasing. Finally, we say that *All Skills Match* if $m_{ij} = 1$ for all i, j.

m_{ij}	j_1	j_2	j_3	j_4	j_5
i_1					
i_2					
i_3					
i_4					
i_5					

Table 3: A PAM matrix: $m_{ij} = 1$ (blue), $0 < m_{ij} < 1$ (green), and $m_{ij} = 0$ (blank)

In Table 3, we depict a matching matrix that satisfies PAM. To maintain PAM, this matrix cannot be modified so that buyer 1 matches with seller 3 (pure or mixed) because that would violate the convexity condition for buyer 1. Likewise, it cannot be that buyer 2 matches with seller 5 because that would violate monotonicity.

The production function G is *supermodular (submodular)* if the marginal productivity of every skill i, $g_{(i+1)j} - g_{ij}$, is strictly increasing (decreasing) in j, and the marginal productivity of every skill j, $g_{i(j+1)} - g_{ij}$, is strictly increasing (decreasing) in i; G is *separable* if the marginal productivity of every skill i is constant in j, and the marginal productivity of every skill j is constant in i.

Previous work established sufficient conditions for positive/negative assortative matching for a single population of agents (Shimer and Smith [2000], Atakan [2006]). However, the single population model is restrictive and does not cover many important settings where there are two different populations, such as labor and product markets.

An open question in the literature is whether assortative matching holds when the two populations are not identical.[13] The next result shows that the answers is a firm yes. To our knowledge, this is the first paper which establishes assortativity beyond the single-population framework.

Theorem 2. *In equilibrium, there is PAM whenever G is supermodular, NAM whenever G is submodular, and All Skills Match whenever G is separable.*

To outline the argument, we first show that the surplus function s_{ij} inherits super/submodularity from G. We use this observation and Lemma 2 to establish that the bounds of the matching sets are monotonic. We prove convexity from algebraic manipulations of the Constant Surplus equations. In contrast, existing proofs rely heavily on symmetry (Shimer and Smith 2000; Atakan 2006). In the discounting case, to show that the matching sets are convex, Shimer and Smith [2000] place further restriction on the production function which imply that the surplus function s_{ij} is convex[14] whereas our proof works without further restrictions.

Proof. Demonstrating PAM requires demonstrating two components, that the bounds of the matching set are weakly increasing and that the matching set is convex. Throughout, we will use the following key fact: if G is supermodular, then so are the surpluses $[s_{ij}]$.

Increasing Upper Bounds: Fix two buyer skills $i_2 > i_1$. Suppose that $\overline{m}_{i_2} < \overline{m}_{i_1}$. Denote these as $j_2 = \overline{m}_{i_2}$ and $j_1 = \overline{m}_{i_1}$. By Efficient Matching, it must be that $s_{i_1 j_1} \geq 0 \geq s_{i_2 j_1}$. By supermodularity, then it must be that for every $j < j_1$ it is the case that $s_{i_1 j} > s_{i_2 j}$. This violates the Constant Surplus equations because

$$2c = \sum_{j \in J} y_j m_{i_2 j} s_{i_2 j} = \sum_{j \in M_{i_2}} y_j s_{i_2 j} < \sum_{j \in M_{i_2}} y_j s_{i_1 j} \leq \sum_{j \in M_{i_1}} y_j s_{i_1 j} = \sum_{j \in J} y_j m_{i_1 j} s_{i_1 j} = 2c$$

The case for lower bounds and for submodular G are analogous.

Convexity: Suppose not. That is, there is a buyer i and sellers $j_1 < j < j_2$ such that $m_{ij} < 1$, and $m_{ij_1}, m_{ij_2} > 0$. Then, it must be the case that seller j has a strictly

[13] Furthermore, even when the populations are ex-ante symmetric, their investments may be asymmetric and hence the equilibrium will not be symmetric (see Example 2).

[14] In fact, there are examples where G is supermodular and s_{ij} is not convex, and yet there is PAM.

positive surplus with a lower buyer and that buyer is present with non-zero measure. Otherwise

$$2c = \sum_{i'>i} x_i s^+_{i'j} < \sum_{i'>i} x_i s^+_{i'j_2} \leq 2c$$

with the inequality being due to the fact that $s_{i'j_2} \geq s_{ij} + s_{i'j_2} > s_{ij_2} + s_{i'j} \geq s_{ij_2}$ for every $i' > i$ due to the supermodularity of s. Therefore, there is some $i' < i$ such that $x_{i'} > 0$ and $s_{i'j} > 0$.

An analogous argument demonstrates that there is:

1. A higher buyer $i' > i$ such that $x_{i'} > 0$ and $s_{i'j} > 0$.
2. A lower seller $j' < j$ such that $y_{j'} > 0$ and $s_{ij'} > 0$.
3. A higher seller $j' > j$ such that $y_{j'} > 0$ and $s_{ij'} > 0$.

Let $\underline{j} = arg\max_{j' \leq j} s_{ij'}$ and likewise $\overline{j} = arg\max_{j' \geq j} s_{ij'}$. Similarly, let $\underline{i} = arg\max_{i' \leq i} s_{i'j}$ and likewise $\overline{i} = arg\max_{i' \geq i} s_{i'j}$. See below for an illustration of the matching matrix.

	$\dots$	$\underline{j}$	$\dots$	j	$\dots$	$\overline{j}$	$\dots$
$\dots$				0			
$\underline{i}$				1			
$\dots$							
i	0	1		$m_{ij} < 1$		1	0
$\dots$							
$\overline{i}$				1			
$\dots$				0			

Define $y = y_j$, $\underline{y} = \sum_{j'<j} y_{j'}$ and $\overline{y} = \sum_{j'>j} y_{j'}$. Similarly, $x = x_i$, $\underline{x} = \sum_{i'<i} x_{i'}$, and $\overline{x} = \sum_{i'>i} x_{i'}$. Notice that $\overline{x}, \underline{x}, \overline{y}, \underline{y} > 0$ as shown above.

By the supermodularity of s, for any $i' > i$, it is the case that $s_{i'\overline{j}} + s_{ij} > s_{i'j} + s_{i\overline{j}}$ and since $s_{ij} \leq 0$, it follows that $s_{i'\overline{j}} > s_{i'j} + s_{i\overline{j}}$. Thus,

$$2c \geq \sum_{i' \geq i} x_{i'} s_{i'\overline{j}} > \sum_{i' \geq i} x_{i'}(s_{i'j} + s_{i\overline{j}}) = \left(\sum_{i' \geq i} x_{i'} s_{i'j} \right) + (x + \overline{x}) s_{i\overline{j}} \tag{6}$$

The strict inequality use the fact that $x_{i'} > 0$ for some $i' > i$.

Next, notice that $s_{\underline{i}j} \geq s_{i'j}$ for all $i' < i$. Therefore,

$$\underline{x} s_{\underline{i}j} = \sum_{i'<i} x_{i'} s_{\underline{i}j} \geq \sum_{i'<i} x_{i'} s_{i'j} \tag{7}$$

Adding equations (6) and (7) gives

$$2c + \underline{x}s_{\underline{i}j} > \sum_{i'} x_{i'}s_{i'j} + (x + \overline{x})s_{i\overline{j}}$$

And therefore,

$$\underline{x}s_{\underline{i}j} > (x + \overline{x})s_{i\overline{j}} \tag{8}$$

Similarly, it can be observed that:

$$s_{\bar{i}j'} > s_{ij'} + s_{\bar{i}j} \text{ for all } j > j'$$
$$s_{\underline{i}j'} > s_{ij'} + s_{\underline{i}j} \text{ for all } j' > j$$
$$s_{i'\underline{j}} > s_{i'j} + s_{i\underline{j}} \text{ for all } j' < j$$

Repeating the same arguments:

$$\overline{y}s_{i\underline{j}} > (\underline{y} + y)s_{\bar{i}j} \tag{9}$$
$$\overline{y}s_{i\overline{j}} > (y + \underline{y})s_{\underline{i}j} \tag{10}$$
$$\overline{x}s_{\bar{i}j} > (\underline{x} + x)s_{i\underline{j}} \tag{11}$$

As shown earlier, all of the surpluses, $s_{\underline{i}j}, s_{\bar{i}j}, s_{i\underline{j}}, s_{i\overline{j}}$ are positive. Taking the product of Inequalities (8)–(11) and dividing by the surpluses yields:

$$\underline{x}\overline{x}\underline{y}\overline{y} > (\underline{x} + x)(\overline{x} + x)(\underline{y} + y)(\overline{y} + y)$$

which is a contradiction due to the strict inequality.

Separability Implies All Skills Match: By Lemma 2, it is the case that for any two sellers, $w_{j'} - w_j = g_{j'} - g_j$. Therefore, the surplus function is constant $s_{ij'} = g_i + g_{j'} - v_i - w_{j'} = g_i + g_j - v_i - w_j$ and by the Constant Surplus equations, it must be that $s_{ij} = 2c$ for all i, j. So, every pair of agents accept their match. □

Remark 7. The assortative matching result is useful for numerical analysis. For example, in the 5×5 case depicted in Table 3, there are $2^{25} \approx 33.6$ million pure matching matrices, but only 2,762 of them satisfy PAM. In the 5×7 case, there are $2^{35} \approx 34$ trillion pure matching matrices, of which only 21,659 satisfy PAM.[15]

[15] At 1000 calculations per second, this is the difference between a program taking a millennium and 21 seconds.

5.2 Uniqueness: Separable Production

We now demonstrate that when the production function is separable, i.e. $g_{ij} = g_i + g_j$, there is a unique equilibrium. To relate to previous work, e.g. Rubinstein and Wolinsky [1985], Gale [1987], we phrase this subsection in the language of a product market. Each seller can produce one unit of a homogeneous good and each buyer desires a single unit. A buyer that invests in skill i receives the payoff α_i from consuming the good and a seller that invests in skill j can produce the good at a cost κ_j. The consumption value α_i is increasing in i and the cost κ_j is decreasing in j. When a buyer and seller meet, their output is $g_{ij} = \alpha_i - \kappa_j$. This production function is separable because the marginal productivity $g_{i'j} - g_{ij}$ is independent of j. As in Gale [1987], we allow for endogenous entry, with outside payoffs equal to u^b for buyers and u^s for sellers. To focus on the interesting case, we ignore the trivial equilibrium where no agent enters, and we assume that there are gains to trade, and so for at least two types, β and σ, $\max_{i\in I, j\in J} g_{ij} - 2c - C(i,\beta) - C(j,\sigma) > u^b + u^s$ and that not all agents enter, so there are at least two types for which the opposite inequality holds.

Proposition 2. *Any economy with a separable production function (with or without outside options) has a* ***unique*** *equilibrium and its allocation achieves the first best.*

Theorem 2 demonstrates that with a separable production function, in any equilibrium, All Skills Match. The rest of the proof immediately follows from Lemma 2: since All Skills Match, the marginal values equal marginal productivities, and by separability, $\Delta v_i = g_i$ and $\Delta w_j = g_j$. Thus, the flows and stocks are uniquely pinned down. Moreover, the surpluses s_{ij} are constant, and so **a law of one price** prevails (all trades occur at one price) and endogenous entry uniquely pins down the price that equates supply and demand. Finally, the agents' private incentives to invest are exactly aligned with the planner, so the equilibrium achieves the first-best. For a formal proof, see Appendix.

6 Robustness

In this section, we will show that the efficiency and sorting results are robust to small modifications in the time costs.

6.1 Discounting

We extend our baseline model to an economy $\mathcal{E}_{\delta,c} = \langle F^b, F^s, I, J, C^b, C^S, G, c, \delta \rangle$ where agents both incur an additive search cost c and discount time at the rate $\delta \in [0, 1]$. Since the agents' continuation values are now discounted, the match surplus becomes

$$s_{ij} = g_{ij} - \delta v_i - \delta w_j$$

and the surplus equations are now:

$$v_i = \sum_{j \in J} y_j \left[m_{ij} \left(\delta v_i + \frac{s_{ij}}{2} \right) + (1 - m_{ij})\, \delta v_i \right] - c$$

$$\Rightarrow \sum_{j \in J} y_j m_{ij} s_{ij} = 2 \left[c + (1 - \delta) v_i \right]$$

Notice that agents incur both an explicit search cost of c and an implicit search cost of $(1 - \delta)v_i$ because their payoffs are delayed. The implicit search costs are increasing in skills which affects the agents' decisions. First, acquiring a higher skill entails a higher implicit search cost, which reduces the incentive to invest. Second, high-skill agent (who have high search costs) may accept too frequently while low-skill agents (with low search costs) may reject too often [Shimer and Smith, 2001]. Nevertheless, we will demonstrate that our main results are robust: as the discount factor heads to 1, we prove an equilibrium convergence result, a general sorting result, and under further conditions, an approximate efficiency result.

The equilibrium conditions for the extended economy are the same except that the s_{ij} are modified:

$$\sum_{j \in J} y_j m_{ij} s_{ij} = 2 \left[c + (1 - \delta) v_i \right], \forall i$$

$$\sum_{i \in I} x_i m_{ij} s_{ij} = 2 \left[c + (1 - \delta) w_j \right], \forall j$$

$$m_{ij} = \begin{cases} 1 & \text{if } s_{ij} > 0 \\ 0 & \text{if } s_{ij} < 0 \end{cases}, \forall i, j$$

Fixing $c > 0$, let E_δ be the set of equilibria of this economy and likewise E_1 denote the equilibria of our baseline model (since c is fixed, we suppress it as an index to reduce notation).

Lemma 3. *(Upper Hemicontinuity) The equilibrium correspondence E_δ is bounded and upper hemicontinuous as $\delta \to 1$. Thus, every sequence of equilibria $e_\delta \in E_\delta$ where $\delta \to 1$ has a subsequence whose limit is $e^* \in E_1$.*

The proof is given in the Appendix. It shows that the equilibrium variables N, v, w are bounded. Upper hemicontinuity then follows by continuity of the equilibrium conditions.

Upper hemicontinuity states that for δ large enough, any equilibrium from E_δ must be close to an equilibrium of the baseline model. We now establish lower hemicontinuity: every equilibrium of the baseline model has a nearby equilibrium in E_δ. Lower hemicontinuity is more difficult to prove because it requires establishing equilibrium existence. To simplify the problem, we assume that:

A1. There is a single population.

A2. The production function is strictly supermodular.

A3. All equilibria of the baseline model are interior (that is, $x_i > 0$ for all i).[16]

Lemma 4. *(Lower hemicontinuity of the equilibrium set). Under A1-A3, there exists $c^* > 0$ such that for every $c < c^*$, every equilibrium of the baseline economy is a limit of equilibria in E_δ as $\delta \to 1$.*

Proof. The proof applies the implicit function theorem around our baseline equilibrium. In order to do so, we will show that there exists a c^* sufficiently small, so that for all $c < c^*$, the Jacobian of the equilibrium conditions are invertible at any equilibrium evaluated at $\delta = 1$. The implicit function theorem then stipulates that for δ sufficiently close to 1, there is a nearby E_δ-equilibrium.

We first establish that, for sufficiently low c, the equilibrium matching rule is $M = M^{PAM}$.

Claim. There exists $c_2 > 0$ such that for every $c < c_2$, every equilibrium has $M = M^{PAM}$, that is, $m_{ii} = 1$ and $m_{ij} = 0$ for every pair $i \neq j$.

Proof of Claim. By Theorem 4, $m_{ii} = 1$ for every i. Therefore, $2c \geq x_i s_{ii} \geq a s_{ii}$ and so $v_i \geq \frac{g_{ii}}{2} - \frac{c}{a}$. That is, $v_i \to \frac{g_{ii}}{2}$ as $c \to 0$, and thus, when $i \neq j$, it holds that $s_{ij} = g_{ij} - v_i - w_j \to g_{ij} - \frac{g_{ii}}{2} - \frac{g_{jj}}{2} < 0$ where the last inequality is due to strict

[16] A natural assumption that the investment cost functions are sufficiently rich guarantees that all equilibria are interior.

supermodularity. Thus, for every pair $i \neq j$, there is a c sufficiently small such that i and j do not match, and so $M = M^{PAM}$. $\square$

We now write the equilibrium conditions (using the fact that $M = M^{PAM}$). The equilibrium conditions are:

$$\sum_i x_i = 1 \tag{12}$$

$$N x_i^2 = F(\beta_i) - F(\beta_{i+1}) \text{ for } i = 0, \dots, n-1 \tag{13}$$

$$x_i(g_{ij} - 2\delta v_i) = 2\left[c + v_i(1-\delta)\right] \text{ for } i = 0, \dots, n-1 \tag{14}$$

$$C(i, \beta_i) - C(i-1, \beta_i) = v_i - v_{i-1} \text{ for } i = 1, \dots, n-1 \tag{15}$$

Notice that the Jacobian is a square matrix: there is 1 boundary conditions, n inflow=outflow conditions, n surplus conditions, and $n-1$ optimal investment conditions, for a total of $3n$ conditions; furthermore, there are $n-1$ thresholds (β_i) for $i = 1, \dots, n-2$, and $2n+1$ state variables (x_i, v_i, and N).

To simplify notation, we will write the Jacobian in a block form. Define $C_{ii} = \frac{\partial}{\partial \beta_i}\left[C(i, \beta_i) - C(i-1, \beta_i)\right]$ and notice that $C_{ii} > 0$ because of strictly increasing differences. Also, some entries won't be material for calculating the determinant, so we simply summarize those blocks with single letters.

	N	x_0	x_1	$\dots$	x_{n-1}	β_1	β_2	$\dots$	β_{n-1}	v_0	v_1	$\dots$	v_{n-1}
$\sum x_i = 1$	0	1	1	$\dots$	1								
0 inflow=outflow	x_0^2	$2Nx_0$	0	$\dots$	0								
1 inflow=outflow	x_1^2	0	$2Nx_1$	$\dots$	0		$\hat{B}$				0		
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$								
$n-1$ inflow=outflow	x_{n-1}^2	0	0	$\dots$	$2Nx_{n-1}$								
$0 \to 1$ Invest						C_{11}	0	$\dots$	0				
$1 \to 2$ Invest						0	C_{22}	$\dots$	0				
$\dots$			0			$\dots$	$\dots$	$\dots$	$\dots$		D		
$n-2 \to n-1$ Invest						0	0	$\dots$	$C_{n-1,n-1}$				
0 CS	0	s_{00}	0	$\dots$	0					$-2x_0$	0	$\dots$	0
1 CS	0	0	s_{11}	$\dots$	0					0	$-2x_1$	$\dots$	0
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$		0			$\dots$	$\dots$	$\dots$	$\dots$
$n-1$ CS	0	0	0	$\dots$	$s_{n-1,n-1}$					0	0	$\dots$	$-2x_{n-1}$

Table 4: The Jacobian Matrix

We do algebraic manipulations (which do not affect the rank of the matrix). First, we change the top section by dividing the inflows by $-2Nx_i$ and then add to the first row and then multiply the first column by -$2N$. Second, we change the bottom section by multiplying the CS equations by x_i and using the Constant surplus equations $x_i s_{ii} = 2c$ (recall that we are evaluating at $\delta = 1$). The Jacobian becomes (in block matrix form):

$$J = \left[\begin{array}{c|c|c} \begin{matrix} 1 & 0 & 0 \dots 0 \\ x_0 & -1 & 0 \dots 0 \\ x_1 & 0 & -1 \dots 0 \\ \dots & \dots & \dots \\ x_{n-1} & 0 & 0 \dots -1 \end{matrix} & B & 0 \\ \hline 0 & \begin{matrix} C_{11} & 0 \dots & 0 \\ 0 & C_{22} \dots & 0 \\ \dots & \dots & \dots \\ 0 & 0 \dots & C_{n-1,n-1} \end{matrix} & D \\ \hline \begin{matrix} 0 & 2c & 0 \dots 0 \\ 0 & 0 & 2c \dots 0 \\ \dots & \dots & \dots \\ 0 & 0 & 0 \dots 2c \end{matrix} & 0 & \begin{matrix} -2x_0^2 & 0 & \dots & 0 \\ 0 & -2x_1^2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -2x_{n-1}^2 \end{matrix} \end{array}\right] =: \begin{bmatrix} A & B & 0 \\ 0 & C & D \\ E & 0 & F \end{bmatrix}$$

Notice that if $E = 0$ then, $det(J) = det(A)det(C)det(F)$. As c appears linearly in E, it follows that $det(J) = det(A)det(C)det(F) + c*$(additional terms). As A is lower triangular and C, F are both diagonal, their determinants are the product of their diagonals. As all parameters are uniformly bounded from 0 for all c, there are uniform bounds z_1 and z_2 which are independent of c and satisfy: $z_1 < |det(A)det(C)det(F)|$ and $z_2 >$ |additional terms|. That is, the bounds z_1, z_2 are independent of c. Therefore, if $c < z_1/z_2$, then the above matrix has positive determinant.

3) Since the Jacobian is invertible (as it has non-zero determinant), by the implicit function theorem for every $\epsilon > 0$ there is a sufficiently large δ so that there is an equilibrium candidate $e_{\delta,c}$ which satisfy Equations 12-15. For it to be an equilibrium, it also must satisfy efficient matching condition $s_{ii} \geq 0$ and $s_{ij} \leq 0$ for every $i, j \neq i$. These conditions are satisfied because the limit equilibrium has strict surpluses: $s^*_{ii} > 0$ and $s^*_{ij} < 0$. $\square$

Remark 8. Previous work analysed the conditions under which search equilibrium converge to the first-best outcome. However, there are no convergence results for economies with neither investment nor matching. We established convergence to the *constrained efficient* outcome because the limit economy still has the friction c>0.

We now prove a robust version of our efficiency result. Recall that c^* is the threshold cost from the previous Lemma.

Theorem 3. *(Approximate efficiency) Under A1-A3, for every* $\epsilon > 0$ *and* $c < c^*$*, there exists a* $\delta_c < 1$ *such that, if* $\delta > \delta_c$ *then there is an equilibrium* $e_\delta \in E_\delta$ *which is* ϵ*-efficient.*

Proof. Take $c < c^*$. Then, by Lemma 4, for sufficiently high δ, there is an equilibrium in E_δ which is arbitrarily close to the efficient equilibrium from E_1. Since the welfare criterion is also continuous, for δ large enough, the equilibrium in E_δ is also arbitrarily close to the welfare-optimal allocation for $\mathcal{E}_{\delta,c}$. □

Finally, we establish a more general sorting result.

Theorem 4. *(Exact Assortativity) If G is strictly supermodular (submodular), then for sufficiently large* $\delta < 1$*, every equilibrium in* E_δ *has positive (negative) assortative matching.*

The formal proof is in the Appendix. The challenge in the proof is that convergence of equilibria is not enough because the limit of non-assortative matrices could be assortative. To address this, the key insight is that we work directly with the surpluses s_{ij} and use the strong convexity of the limit equilibria: the proof of Theorem 2 established that the interior surpluses are *strictly positive*, and if they are strictly positive in the limit, then they must be strictly positive for sufficiently large δ.

Remark 9. In single-population models, [Shimer and Smith, 2000] established assortative matching provided that the production function and its partials: g, $\log g_x$ and $\log g_{xy}$ are supermodular. Theorem 4 shows that in a two-population model (which includes the one-population setting as a special case) with discounting and additive search costs, for δ sufficiently large, the latter two conditions are unnecessary.

6.2 Asymmetric Search Costs

Our results are robust to changes in other parameters as well. We now return to our baseline model, but with a modification that buyers and sellers may differ in their search costs, c^b and c^s, and bargaining weights, α and $1-\alpha$. When a buyer with skill i and a seller with skill j accept each other, the buyer receives $v_i + \alpha s_{ij}$ and the seller receives $w_j + (1-\alpha)s_{ij}$. In the baseline model, Lemma 1 establishes that the equilibrium state is balanced, $B = S$, and the proof turned on the assumptions that $c^b = c^s = c$ and $\alpha = 1/2$. If either assumption does not hold, then the equilibrium state can be unbalanced. In an unbalanced economy where $B > S$, every buyer meets a seller with probability S/B and every seller always meets a buyer (and vice-versa if $S > B$).

Proposition 3. *Given any bargaining weight α, and search costs c^b and c^s:*

1. *An equilibrium exists and the steady state satisfies the balance condition*

$$\frac{B}{S} = \frac{\alpha}{1-\alpha}\frac{c^s}{c^b}$$

2. *All equilibria exhibit PAM (NAM) whenever G is strictly supermodular (submodular).*
3. *The constrained efficient allocation is an equilibrium if and only if $\alpha = \frac{c^b}{c^s+c^b}$*

Proof. Define $\mu = min(B, S)$. In equilibrium, the values satisfy:

$$v_i = (\mu/B)\left(\sum_{j\in J} y_j\left[m_{ij}\left(v_i + \alpha s_{ij}\right) + (1-m_{ij})v_i\right]\right) + (1-\mu/B)\,v_i - c^b, \forall i$$
$$w_j = (\mu/S)\left(\sum_{i\in I} x_i\left[m_{ij}\left(w_j + (1-\alpha)s_{ij}\right) + (1-m_{ij})w_j\right]\right) + (1-\mu/S)w_j - c^s, \forall j$$

Rewriting, we obtain the modified Constant Surplus equations:

$$\sum_{j\in J} y_j m_{ij} s_{ij} = \frac{c^b}{\alpha\left(\mu/B\right)}, \forall i \qquad (16)$$
$$\sum_{i\in I} x_i m_{ij} s_{ij} = \frac{c^s}{(1-\alpha)\left(\mu/S\right)}, \forall j$$

$$\Rightarrow \frac{c^b}{\alpha\left(\mu/B\right)} = \sum_{i\in I} x_i \sum_{j\in J} y_j m_{ij} s_{ij} = \sum_{j\in J} y_j \sum_{i\in I} x_i m_{ij} s_{ij} = \frac{c^s}{\left(1-\alpha\right)\left(\mu/S\right)}$$

$$\Rightarrow \frac{B}{S} = \frac{\alpha}{1-\alpha} \cdot \frac{c^s}{c^b} \tag{17}$$

We first note that the tuple $\langle \hat{B}, \hat{S}, (x_i), (y_j), [m_{ij}], (v_i), (w_j) \rangle$ is an equilibrium of an economy with asymmetric search costs and bargaining parameters $(\hat{c}^b, \hat{s}^s, \hat{\alpha})$ if and only if the tuple $\langle \hat{N}, (x_i), (y_j), [m_{ij}], (v_i), (w_j) \rangle$, where $\hat{N} = \min\{\hat{B}, \hat{S}\}$, is an equilibrium of a symmetric economy where $c^b = c^s = c = \max\{\frac{\hat{c}^b}{2\hat{\alpha}}, \frac{\hat{c}^s}{2(1-\hat{\alpha})}\}$ and $\alpha = 1/2$. This is because the equilibrium conditions are identical.

1. By the above equivalence and Corollary 1, an equilibrium exists.
2. By the above equivalence and Theorem 2, sorting still holds.
3. In the constrained efficient allocation, the state must always be balanced. Otherwise, the planner can increase welfare by equalizing the state $B = S$ without affecting productivity or the investment cost. By equation (17) , every equilibrium state is balanced if and only if $\alpha = \frac{c^b}{c^s+c^b}$, and so this condition is necessary for efficiency. For sufficiency, notice that when $\alpha = \frac{c^b}{c^s+c^b}$, it holds that $B = S$ and the constant surplus equations are identical to those where buyers and sellers have the same search cost $c' = (c^s + c^b)/2$, and same bargaining weight $\alpha' = 1/2$. Moreover, the two models also have identical constrained efficient allocations and equilibria. Therefore, applying our second welfare theorem to that model yields equilibrium values (v_i) and (w_j) that support the constrained efficient outcome as an equilibrium of this model. □

The above proposition shows that the equilibrium existence and sorting results still hold. Furthermore, our welfare theorem generalizes for the "right" bargaining weight, which equals the relative share of total search costs $\alpha^* = \frac{c^b}{c^s+c^b}$.

Remark 10. It is surprising that adjusting only the bargaining weights is sufficient to achieve efficiency, and the "right" bargaining weight α^* is the most natural one: each side's bargaining power should be their fraction of the search cost (e.g. if $c^s = 4c^b$, then buyers should have 20% weight and sellers have 80%). For any other bargaining weight, the steady state is unbalanced and the equilibrium is inefficient: there are too many buyers when they have too much bargaining weight ($B > S$ whenever $\alpha > \alpha^*$), and too few when they have too little weight ($B < S$ whenever $\alpha < \alpha^*$). In the online appendix we quantify the maximal welfare loss in equilibria and show that it is proportional to the search cost difference $|c^b - c^s|$.

Remark 11. Whether the welfare result of Proposition 3 is a positive or negative result is in the eye of the beholder: a positive view would be that the bargaining weights adjust to the correct ones through some social process, and a negative view is that they do not. In either case, the result is helpful for designing policies. For instance, subsidizing the search cost improves welfare only if it targets the disadvanatged side (which is the short side of the market), that is, if buyer's bargaining power is too low, then subsidizing their search costs improves welfare while subsidizing the sellers does not.

6.3 The Hosios Condition

Finally, we consider a general meeting function where $\mu(B, S)$ is the total number of meetings in a period. In every period, each agent can meet at most one other agent, and so $\mu(B, S) \leq \min\{B, S\}$. Meetings are still random and the probability that a buyer meets a seller is $\mu(B, S)/B$, while the probability that a seller meets a buyer is $\mu(B, S)/S$. As is standard, we take μ to be homogeneous of degree 1 and differentiable.

Corollary 3. *The constrained efficient allocation is an equilibrium if and only if*

$$\alpha = \frac{B^* c^b}{B^* c^b + S^* c^s} = \frac{\partial \mu(B^*, S^*)/\partial B}{\mu(B^*, S^*)/B^*}$$

where B^*, S^* are the constrained efficient stock.

In words, the constrained efficient allocation can be decentralized as an equilibrium if and only if the bargaining weight of each side equals their share of the overall search costs, which also equals the elasticity of the meeting function at the optimum (the Hosios condition). The proof closely follows that of the welfare theorem (see Appendix). Hosios (1990) shows that when agents are *homogeneous* the search externalities that they impose on each other are perfectly offset under the "right" sharing rule. In contrast, in our model, agents are heterogeneous and they make ex-ante investments. Remarkably, the same sharing rule still works. This result also provides a new interpretation to the Hosios' Condition: the "right" bargaining weight should give each side their share of the total search cost.

7 Discussion

This paper developed and analyzed a search-and-matching model with heterogeneous agents and pre-entry investments. Our main result establishes a second welfare theorem: every constrained efficient allocation, regarding both the investment and the matching rule, is an equilibrium. The result is surprising as the decisions to invest and to accept/reject partners impose externalities on other agents. In particular, notice that if the highest productivity agents decide to match more frequently or less agents acquire the highest skill, then the overall pool of agents' skills in the market changes (the number of agents with this skill should diminish and other skills may increase or decrease). The planner's solution takes into account these steady-state search externalities, and the equilibrium values must make agents internalize them.

In addition, we analyzed the equilibrium structure by establishing sufficient conditions for sorting and uniqueness. The sorting result is significant because it applies to two-sided search markets, whereas previous results applied only to single population models (the single population model is a special case of our two population model). We demonstrated novel economic implications (such as discrimination in the marraige market or subsidizing search costs on one side) due to the tradeoff between investment, search, and matching. Finally, we showed that our main results are robust to small modifications in the time costs (Section 6). First, we added a discount rate δ and showed that as $\delta \to 1$: the equilirum converge to our baseline model, the sorting result holds exactly for sufficiently large $\delta < 1$ (due to upper hemi-continuity) and the welfare result holds approximately (under further conditions that we used to demonstrate the lower hemi-continuity). Second, we showed that the results continue to hold under appropriate conditions for economies with outside options, asymmetric search costs and CRS meeting functions.

As previously mentioned, the search cost c captures in reduced form the wide range of costs people *explicitly* incur as they search. In contrast, when agents discount time, they incur *implicit* search costs as payoffs are delayed. These implicit search costs are proportional to their continuation values which has consequences. First, acquiring a higher skill entails a higher implicit search cost, which reduces the incentives to invest. Second, agents may mismatch in the following way: high-skill agents may accept too frequently because they have high implicit search costs, while low-skill agents may reject too often because they have low implicit search costs, as in Shimer and Smith [2001]. By severing the implicit link between values and search costs, our model delivers powerful results: a second welfare theorem, a general sorting result, existence, and the equilibria have a clear and intuitive structure.

Our analysis shows that inefficiencies are not endemic in markets with search frictions, but rather depend on the nature of the search costs. For applications, we believe both types of costs are important, but which is more salient depends upon the economic situation being modeled.[17] Our results have several implications:

Labor Market - A central question regarding sorting in the labor market is when do high-skill workers match with high-tech firms? The previous one-population sorting results of Shimer and Smith [2000] and Atakan [2006] do not apply to the labor market because the agents on opposite sides are different. Our results provide a theoretical foundation for assortative matching in two-sided markets. In addition, the mismatch between labor skills and production technologies has been extensively studied, both theoretically and empirically. However, the skill-technology mismatch also affects (and is affected by) investment in human and physical capital. For instance, a lower search cost generally leads to finer sorting, which affects the marginal productivity of some skills and thereby the incentive to invest. Alternatively, a change in the investment costs changes the composition of skills in the market, which may further impact search and matching. Our model provides a general framework to study investment and matching together and do comparitive statics.

Product and Marriage Markets - In a product market, the joint production function is typically separable, and we showed that the equilibrium is unique and efficient. On the other hand, in a marriage market, the joint household production function typically has complementarities between skills, which can generate multiple equilibria. It is not surprising that a symmetric economy has symmetric equilibria, but we show that there can also be asymmetric equilibrium, which can even be efficient. The asymmetric equilibrium is discriminatory in the sense that the return on investment depends on gender, which generates a gap in skill acquisition. This gender gap can persist even when it is inefficient (see Section 3) and in some cases can be corrected by a policy intervention such as an investment subsidy or tax.

[17] For example, when search transpires over a short period of time and does not affect the consumption date (think of the time spent today searching online for a product that will be delivered tomorrow or college students applying for jobs which they will take after graduation), the explicit costs are important.

Policy Intervention - It immediately follows from our second welfare theorem that at the efficient equilibrium, any policy intervention causes a harmful distortion. However, there is still room for policy interventions at inefficient equilibria. For example, in the marriage market with submodular production, an investment subsidy can boost welfare by eliminating inefficient discriminatory equilibria without affecting the efficient one.

Applications and Simulations: The welfare and sorting results are also useful tools for applying and simulating the model. In particular, the planner's problem is more amenable to numerical simulations than the equilibrium conditions, as there are less conditions and values need not be derived. For an n-skill economy, the endogenous variables $(v_i), (w_j), (x_i), (y_j), (\beta_i), (\sigma_j)$ are of order n, but the matching matrix $[m_{ij}]$ is of order n^2. The assortative matching result facilitates simulations by reducing the number of matching variables (from n^2 to $2n$), which brings the whole problem from $O(n^2)$ to $O(n)$. It remains to be seen whether the model can be calibrated to derive useful empirical predictions, but the theoretical results found here are promising.

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8 Appendix

Remaining Proofs for Theorem 1:

We first prove the non-interior case and then the constant rank constraint qualification.

Proof. z **is non-interior:**

Given any optimal policy $\langle z, M, (\beta_i), (\sigma_j)\rangle$, the FOCs imply that there are shadow values (v_i), (w_j) such that (see proof of Theorem 1 in text):

$$\sum_j y_j m_{ij} (g_{ij} - v_i - w_j) \geq 2c \text{ with equality when } x_i > 0$$

$$\sum_i x_i m_{ij} (g_{ij} - v_i - w_j) \geq 2c \text{ with equality when } y_j > 0$$

$$N x_i y_j (g_{ij} - v_i - w_j) = -\eta_{ij} + \hat{\eta}_{ij}$$

where $\eta_{ij} m_{ij} = 0$ and $\hat{\eta}_{ij}(1 - m_{ij}) = 0$ and $\eta_{ij}, \hat{\eta}_{ij} \geq 0$.

The above equations demonstrate the Constant Surplus equations for all i where $x_i > 0$. But, the Constant Surplus equation may not hold for skills i where $x_i = 0$. Therefore, for any skill i where $x_i = 0$, we define v_i^* to be the unique value which solves $\sum_j y_j \max\{g_{ij} - v_i^* - w_j, 0\} = 2c$. For any skill i where $x_i > 0$, we define $v_i^* = v_i$. Likewise, for sellers j where $y_j = 0$, define w_j^* to be the unique value which solves $\sum_j y_j \max\left(g_{ij} - v_i - w_j^*, 0\right) = 2c\ y_j > 0$. For sellers j where $y_j > 0$, define $w_j^* = w_j$. Define a matching matrix by $m_{ij}^* = \mathbf{1}_{g_{ij} - v_i^* - w_j^* > 0}$ whenever $x_i = 0$ or $y_j = 0$ and setting $m_{ij}^* = m_{ij}$ otherwise.

It now remains to be seen that $\langle z, M^*, (v_i^*), (w_j^*)\rangle$ satisfies the equilibrium constraints.

The Constant Surplus Equations hold: For any skill i where $x_i > 0$, from the above, we have that $\sum_j y_j m_{ij}^* \left(g_{ij} - v_i^* - w_j^*\right) = \sum_j y_j m_{ij} (g_{ij} - v_i - w_j) = 2c$ because $v_i^* = v_i$ and whenever $y_j > 0$, then $m_{ij} = m_{ij}^*$ and $w_j = w_j^*$. For any skill i

where $x_i = 0$,

$$\sum_j y_j m^*_{ij}\left(g_{ij} - v^*_i - w^*_j\right) = \sum_j y_j \max\left(g_{ij} - v^*_i - w^*_j, 0\right)$$
$$= \sum_j y_j \max\left(g_{ij} - v^*_i - w_j, 0\right) = 2c$$

because $w^*_j = w_j$ whenever $y_j > 0$. The same argument demonstrates the Constant Surplus equations for the sellers.

Efficient Matching holds: For any two skills i, j where $x_i = 0$ or $y_j = 0$, the efficient matching condition holds by definition. For any two skills i, j where $x_i > 0$ and $y_j > 0$, then $v^*_i = v_i$, $w^*_j = w_j$, and $m^*_{ij} = m_{ij}$ and the Efficient Matching condition is a direct consequence of FOC(m_{ij}).

Optimal Investments: Regarding optimal investments, just as in the proof in the main section, here the values (v_i) satisfy incentive compatibility for investments. However, it is not readily evident that the values (v^*_i) satisfy incentive compatibility because the values for unrealized skills are modified, and may be increased. We now show that for all unrealized skills $v_i \geq v^*_i$.

Since $m_{ij} x_i y_j = m^*_{ij} x_i y_j$ for any two skills i, j, the policy $\langle z, M^*, (\beta_i), (\sigma_j)\rangle$ is admissible and optimal. By the constraint qualifications, there are values $(\hat{v}_i)$, $(\hat{w}_j)$ which satisfy the FOCs for $\langle z, M^*, (\beta_i), (\sigma_j)\rangle$. From FOC($\beta_i$), we have that the marginal values are equal for all i, $\hat{v}_i - \hat{v}_{i-1} = C(i, \beta_i) - C(i-1, \beta_i) = v_i - v_{i-1}$. Likewise, for all sellers j, $\hat{w}_j - \hat{w}_{j-1} = w_j - w_{j-1}$. Thus, there is a constant t such that $\hat{v}_i + \hat{w}_j = v_i + w_j + t$ for all i, j. For any skill i such that $x_i > 0$,

$$2c = \sum_j y_j m^*_{ij}(g_{ij} - \hat{v}_i - \hat{w}_j) = \sum_j y_j m^*_{ij}(g_{ij} - v_i - w_j - t)$$
$$= \sum_j y_j m_{ij}(g_{ij} - v_i - w_j - t) = 2c - t \sum_{ij} y_j m_{ij}$$

Therefore, $t = 0$ and so $\hat{v}_i + \hat{w}_j = v_i + w_j$ for all i, j.

For any unchosen skill i,

$$\sum_j y_j m^*_{ij}\left(g_{ij} - v^*_i - w_j\right) = 2c \geq \sum_j y_j m^*_{ij}\left(g_{ij} - \hat{v}_i - \hat{w}_j\right) = \sum_j y_j m^*_{ij}\left(g_{ij} - v_i - w_j\right)$$

Therefore, we can conclude that $v_i \geq v_i^*$. This demonstrates incentive compatibility. For every skill i, $v_i \geq v_i^*$ with equality if $x_i > 0$. As (v_i) satisfied incentive compatibility and (v_i^*) differs by only lowering the value of unrealized skills, the values (v_i^*) also satisfy incentive compatibility. This establishes that for the values $(v_i^*), (w_j^*)$, no agent wishes to choose any unchosen skill and completes the proof. □

Constraint Qualification

Lemma 5. *The planner's optimization problem satisfies the Constant Rank Constraint Qualification.*

Proof. We show that for each subset of the gradients of the active inequality constraints and the equality constraints, the rank in a vicinity of the optimal point is constant (Janin [1984]).

There is an immediate linear dependency among the gradients:

$$\sum_{i \in I} \alpha \nabla flow_i^b - \sum_{j \in J} \alpha \nabla flow_j^s = 0$$

which follows from

$$\sum_{i \in I} flow_i^b - \sum_{j \in J} flow_j^s = 0$$

We will show that this is the only linear dependency, which suffices for the constant rank constraint qualification. Suppose that $\sum_n \alpha_n \nabla_n = 0$ where the summation is over all the active gradients. To simplify notation, we label the skills as $I = \{0, \ldots, k\}$ and $J = \{0, \ldots, l\}$. Notice first that (β_i) and (σ_j) appear only in the flow constraints:

∇	β_1	β_2	β_3	$\ldots$	β_k	N	$\sigma_j, x_i,$ y_j, m_{ij}
$\nabla flow_0^b$	$-f^b(\beta_1)$	0	0	0	0	$-x_0 \sum_{j \in J} y_j m_{0j}$	$\ldots$
$\nabla flow_1^b$	$f^b(\beta_1)$	$-f^b(\beta_2)$	0	0	0	$-x_1 \sum_{j \in J} y_j m_{1j}$	$\ldots$
$\nabla flow_2^b$	0	$f^b(\beta_2)$	$-f^b(\beta_3)$	0	0	$-x_2 \sum_{j \in J} y_j m_{2j}$	$\ldots$
$\ldots$	0	0	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$\nabla flow_{k-1}^b$	0	0	0	$f^b(\beta_{k-1})$	$-f^b(\beta_k)$	$-x_{k-1} \sum_{j \in J} y_j m_{k-1,j}$	$\ldots$
$\nabla flow_k^b$	0	0	0	0	$f^b(\beta_k)$	$-x_k \sum_{j \in J} y_j m_{k,j}$	$\ldots$

Since β_i only shows up in up in $flow_i^b$, $flow_{i-1}^b$ it must be that

$$0 = \sum_n \alpha_n \frac{\partial f_n}{\partial \beta_{i'}} = \sum_{i \in I} \alpha_i \frac{\partial flow_i^b}{\partial \beta_{i'}} = f(\beta_{i'})\alpha_{i'} - f(\beta_{i'})\alpha_{i'+1} \text{ for all } i'$$

Thus, there is an α such that $\alpha_i = \alpha$ for all the coefficients of the constraints $\nabla flow_i^b$. Similarly, there is a χ so that $\alpha_j = \chi$ for all the coefficients of the constraints $\nabla flow_j^s$. Furthermore, N only shows up in the flow constraints, so it must be that

$$-\alpha \sum_i x_i \sum_j y_j m_{ij} - \chi \sum_j y_j \sum_i x_i m_{ij} = 0$$

which implies $\chi = -\alpha$ (notice that $\sum_i x_i \sum_j y_j m_{ij} = 1/N$). Therefore, there is exactly one linear dependency

$$\sum \alpha_i \nabla flow_i^b + \sum_j \alpha_j \nabla flow_j^s = \alpha \left(\sum_i \nabla flow_i^b - \sum_j \nabla flow_j^s \right) = 0$$

Second, the coefficients on $\nabla(x_i \geq 0)$ and ∇X are all zeros. The reason is that x_i appears in the flow constraints and the constraints $x_i \geq 0$ and $X = 0$. By the previous step, in any linear dependence, the flow constraints cancel each other out, so only the constraints $x_i \geq 0$ and $X = 0$ are relevant. . Therefore, if $\sum_i \xi_i \nabla(x_i \geq 0) + \xi \nabla X = 0$, then $0 = \xi_i \frac{\partial x_i}{\partial x_i} + \xi \frac{\partial X}{\partial x_i} = \xi_i - \xi$, and so $\xi_i = \xi$ for all i. If $\xi \neq 0$, then it must be that every inequality on x is active, so $x_i = 0$ for every i, contradicting $0 = X = 1 - \sum_i x_i$, which holds in any admissible tuple. The same argument applies to the y_j. So $\xi_i = \xi = \xi_j = 0$ for all i, j.

Third, the coefficients on the m_{ij} constraints are zeros. The reason is that the variable m_{ij} appears only in the flow equations and the inequality constraints on m_{ij}. The flow constraints cancel each other out. For the m_{ij} constraints, $\nabla(1 \geq m_{ij} \geq 0) = (0, \ldots 0, \pm 1, 0 \ldots)$ and at most one of the m_{ij} constraints can be active where the only non-zero element is in the m_{ij} coordinate and therefore these gradients coefficients must be 0. □

Proof of Proposition 1:

Proof. Consider the economies $\Gamma_c = \langle F^b, F^s, I, J, C^b, C^s, G, c\rangle$ indexed by their search cost c and denote its constrained efficient welfare as $\mathcal{W}_c$. Denote an optimal allocation as x_c with associated population N_c (there may be multiple optimal allocations). Notice that by an imitation argument, $\mathcal{W}_c \geq \mathcal{W}_{c'} + 2N(c')(c'-c)$ because the planner could implement $x_{c'}$ when faced with the economy x_c. This implies that welfare is decreasing in c, as expected. Reversing c and c' gives $2N(c)(c'-c) + \mathcal{W}_{c'} \geq \mathcal{W}_c$. Taking $c' > c$. this implies that $|\mathcal{W}_c - \mathcal{W}_{c'}| \leq 2N(c)(c'-c)$. That is, when $N(c)$ is unique, it is the case that $\frac{\partial \mathcal{W}_c}{\partial c} = -2N(c)$ and otherwise the left-derivative is $\sup -2N(c)$ and the right-derivative is $\inf -2N(c)$. To see convexity of $\mathcal{W}_c$, it suffices to demonstrate that N is increasing in c. Take $c' > c$. Since $\mathcal{W}_c \geq \mathcal{W}_{c'} + 2N(c')(c'-c)$, and similarly $\mathcal{W}_{c'} \geq \mathcal{W}_c + 2N(c)(c-c')$. Adding these two equations together gives $0 > 2(N(c') - N(c))(c'-c)$ and therefore $N(c) \geq N(c')$. □

Proof of Proposition 2:

Proof. The first-best allocation is unique and satisfies:

First-Best Matching: All pairs match. Since the marginal productivity of an agent is not affected by the skills of her partner, all pairs match to minimize the search cost.

First-Best Investment: Buyer β and seller σ acquire the skills: $i^*(\beta) = \arg\max_i \alpha_i - C^b(i,\beta)$ and $j^*(\sigma) = \arg\max_j -\kappa_j - C^s(j,\sigma)$. Denote by $C^{b*}(\beta) = C^b(i^*(\beta),\beta)$ the investment cost buyer β pays to acquire the efficient skill, and likewise $C^{s*}(\sigma) = C^s(j^*(\sigma),\sigma)$.

The social welfare of a match between buyer β and seller σ is $\omega(\beta,\sigma) = \alpha_{i^*(\beta)} - C^{b*}(\beta) - \kappa_{j^*(\sigma)} - C^{s*}(\sigma) - 2c$. The assumption before the proof implies that there are types, $\beta', \sigma', \hat{\beta}, \hat{\sigma}$ such that $\omega(\beta',\sigma') > u^b + u^s > \omega(\hat{\beta},\hat{\sigma})$. So, in the first-best, some agents enter and others don't.[18]

First-Best Entry: Buyer β and seller σ enter iff $\beta \leq \beta_0$ and $\sigma \leq \sigma_0$. The entry thresholds are pinned down by[19] $F^b(\beta_0) = F^s(\sigma_0)$ and $\omega(\beta_0,\sigma_0) = u^b + u^s$.

Since g is separable, Lemma 2 implies that in equilibrium, the marginal value equal the marginal productivity: $\Delta v_i = \alpha_{i+1} - \alpha_i$, for every i, and $\Delta w_j = -(\kappa_{j+1} -$

[18] The case where everyone enters is trivial.

[19] Since buyers and sellers exit in equal numbers, in a steady state they must also enter in equal numbers.

κ_j), for every j. Therefore, the match surplus $s_{ij} = \alpha_i - \kappa_j - v_i - w_j$ is constant. As a result:

Equilibrium Matching: Theorem 2 demonstrates that in every equilibrium, all skills match.

Equilibrium Investment: The individually optimal investments satisfy

$$\arg\max_i \left\{v_i - C^b(i,\beta)\right\} = \arg\max_i \left\{\alpha_i - C^b(i,\beta)\right\}, \text{ for evey } \beta$$
$$\arg\max_j \left\{w_j - C^s(j,\sigma)\right\} = \arg\max_j \left\{-\kappa_j - C^s(j,\sigma)\right\}, \text{ for every } \sigma$$

The maximizers are equal because $\alpha_i - v_i$ and $-\kappa_j - w_j$ are constant

Equilibrium Entry: First, we show that there is entry. If not, then $v_{i^*(\beta)} - C^{b*}(\beta) \leq u^b$ and $w_{j^*(\sigma)} - C^{s*}(\sigma) \leq u^s$, for all β, σ, and so $v_{i^*(\beta)} - C^{b*}(\beta) + w_{j^*(\sigma)} - C^{s*}(\sigma) \leq u^b + u^s$. Substituting in the Constant Surplus equations, it follows that, $\alpha_{i^*(\beta)} - C^{b*}(\beta) - \kappa_{j^*(\sigma)} - C^{s*}(\sigma) - 2c \leq u^b + u^s$, which violates the assumption that there are types, β', σ' such that $\omega(\beta', \sigma') > u^b + u^s$. By a similar argument, it cannot be that all agents enter. Second, since some agents enter and others do not, denote by $\underline{\beta}, \underline{\sigma}$ the threshold types for whom the entry constraints hold with equality, notice that

$$\begin{aligned} u^b + u^s &= v_{i^*(\underline{\beta})} - C^{b*}\left(\underline{\beta}\right) + w_{j^*(\underline{\sigma})} - C^{s*}\left(\underline{\sigma}\right) \\ &= \alpha_{i^*(\underline{\beta})} - C^{b*}\left(\underline{\beta}\right) - \kappa_{j^*(\underline{\sigma})} - C^{s*}\left(\underline{\sigma}\right) - 2c = \omega(\underline{\beta}, \underline{\sigma}) \end{aligned}$$

The second equality follows from the Constant Surplus equation, $v_i + w_j = \alpha_i - \kappa_j - 2c$. In a steady state, the same measure of buyers and sellers enter, $F^b(\underline{\beta}) = F^s(\underline{\sigma})$. These two equations are the same as the equations that characterized the first-best entry decisions, and therefore it must be that $\underline{\beta} = \beta_0$ and $\underline{\sigma} = \sigma_0$. □

Proof of Proposition 3:

Proof. **Boundedness:** Take a sequence of equilibria $e_\delta \in E_\delta$ where $e_\delta = \langle z, M, (v_i), (w_j)\rangle$.

The only variables that can be unbounded are the values, and the population size N. We first establish that the population size is bounded. Suppose the equilibria are such that $\lim_{\delta \to 1} N_\delta \to \infty$. All equilibrium variables depend upon δ and c, but we suppress the subscripts in order to simplify notation.The first step will show that for some agent i, $v_i \to -\infty$: The surplus equations imply:

$$\begin{aligned}\textstyle\sum_j y_j m_{ij} s_{ij} - 2(1-\delta) v_i = 2c &\Longrightarrow \textstyle\sum_i \sum_j x_i y_j m_{ij} s_{ij} - 2(1-\delta) \sum_i x_i v_i = 2c \\ \textstyle\sum_i x_i m_{ij} s_{ij} - 2(1-\delta) w_j = 2c &\Longrightarrow \textstyle\sum_i \sum_j x_i y_j m_{ij} s_{ij} - 2(1-\delta) \sum_j y_j w_j = 2c\end{aligned}$$

and thus, $\sum_i x_i v_i = \sum_j y_j w_j$. If the s_{ij} are bounded above, then

$$\begin{aligned}2c &= \sum_i \sum_j x_i y_j m_{ij} s_{ij} - 2(1-\delta) \sum_i x_i v_i \\ &\leq \sum_i \frac{p_i}{N} \bar{s} - 2(1-\delta) \sum_i x_i v_i = \frac{\bar{s}}{N} - 2(1-\delta) \sum_i x_i v_i\end{aligned}$$

(the inequality uses the inflow-outflow condition: $N x_i \sum_j y_j m_{ij} = p_i$), and so $v_i \to -\infty$ for some i. If the s_{ij} are not bounded above, then there exists some agent (buyer or seller, here we suppose buyer without loss of generality) ι for whom $v_\iota \to -\infty$.

The second step is to show that if $v_i \to -\infty$, then for every i', $v_{i'} \to -\infty$. Notice that

$$\left.\begin{aligned}\textstyle\sum_j y_j m_{ij} s_{ij} - 2(1-\delta) v_i &= 2c \\ \textstyle\sum_j y_j m_{i'j} s_{i'j} - 2(1-\delta) v_{i'} &= 2c\end{aligned}\right\} \Longrightarrow \sum_j y_j (m_{ij} s_{ij} - m_{i'j} s_{i'j}) = 2(1-\delta)(v_i - v_{i'})$$

For any i', if $v_{i'} \not\to -\infty$, then $s_{ij} - s_{i'j} \to \infty$ for every j. So any agent who matches with i' matches with i

$$\sum_{j \in M(i')} y_j (s_{ij} - s_{i'j}) \leq 2(1-\delta)(v_i - v_{i'}) \leq 0$$

But, $\sum_{j \in M(i')} y_j (s_{ij} - s_{i'j}) > 0$ since $s_{ij} - s_{i'j} \to \infty$, a contradiction. Thus, $v_{i'} \to -\infty$ for all i'.

The third step shows that for all sellers, it must also be that $w_j \to -\infty$: As shown earlier, since $\sum_i x_i v_i = \sum_j y_j w_j$ and $v_i \to -\infty$ for every i, it follows that $w_j \to -\infty$ for some j, and then by an analogous argument, $w_{j'} \to -\infty$ for all j'. But, then $s_{ij} \to \infty$ for all i, j, and so $m_{ij} = 1$ for all i, j. But, this violates the surplus equation: $2c = \sum_j y_j m_{ij} s_{ij} - 2(1-\delta) v_i \geq \sum_j y_j s_{ij}$ since $s_{ij} \to \infty$.

The argument that values are bounded is similar. If an agent's value goes to $-\infty$, then the argument above can be repeated from the second step. If any agent's value goes to ∞, then a similar argument can be used to show that all values go to ∞, then $s_{ij} \to -\infty$, and so every $m_{ij} = 0$, a contradiction.

Finally, since the equilibrium values and population size are all bounded, there exists a converging subsequence.

Upper hemicontinuity: By continuity of the equilibrium conditions. $\square$

Proof of Theorem 4:

Proof. Take G strictly supermodular.

Convexity: By contradiction, if there is no sufficiently large δ, then there is a sequence $\delta_n \to 1$ such that $e_n \in E_{\delta_n}$ and the matching in e_n is not convex. Then, for each n, it must be that there is an i and $j_1 < j_2 < j_3$ such that $m_{ij_1}, m_{ij_3} > 0$, and $m_{ij_2} < 1$. But, then $s_{ij_1}, s_{ij_3} \geq 0$ and $s_{ij_2} \leq 0$. Pass to a subsequence so that the same indices are used for every element in the subsequence. But, by Proposition 3, there is a limiting equilibrium e^* which has $s^*_{ij_1}, s^*_{ij_3} \geq 0$ and $s^*_{ij_2} \leq 0$, which contradicts the proof of Theorem 2 (in that proof, in order to demonstrate assortativity, we showed that such a surplus configuration cannot occur).

Upper monotonicity: Take any equilibrium $e \in E_\delta$ and by contradiction, suppose that there are two skills $i < i'$ such that $\bar{m}_i = \max M_i > \max M_{i'}$. The constant surplus equations stipulate that:

$$2c + 2(1-\delta)v_i = \sum_{j \leq \bar{m}_i} y_j s^+_{ij} \geq \sum_{j < \bar{m}_i} y_j s^+_{ij}$$

$$2c + 2(1-\delta)v_{i'} = \sum_{j \leq \bar{m}_{i'}} y_j s^+_{i'j} = \sum_{j < \bar{m}_i} y_j s^+_{i'j}$$

By the above matching, if follows that $s_{i\bar{m}_i} \geq 0 \geq s_{i'\bar{m}_i}$. Since g is supermodular, s is also supermodular, and so, for lower skills $j < \bar{m}_i$, it holds that $s_{ij} + s_{i'\bar{m}_i} > s_{i'j} + s_{i\bar{m}_i}$, and so $s_{ij} > s_{i'j}$. So $2c + 2(1-\delta)v_i > 2c + 2(1-\delta)v_{i'}$ and therefore $v_i > v_{i'}$, a contradiction.

Lower monotonicity: Upper and lower monotonicity are not the same. Upper monotonicity holds for any δ, whereas we will only be able to demonstrate lower monotonicity for sufficiently high δ.

By contradiction, if there is no sufficiently large δ, then there is a sequence $\delta_n \to 1$ such that $e_n \in E_{\delta_n}$ and the matching in e_n is not lower monotonic. That is, there are two skills $i < i'$ such that $\min M_i > \min M_{i'} = \underline{m_{i'}}$. The constant surplus equations stipulate that:

$$2c + 2(1-\delta)v_i = \sum_{j \geq \underline{m_{i'}}} y_j s_{ij}^{+} = \sum_{j > \underline{m_{i'}}} y_j s_{ij}^{+}$$

$$2c + 2(1-\delta)v_{i'} = \sum_{j \geq \underline{m_{i'}}} y_j s_{i'j}^{+} \geq \sum_{j > \underline{m_{i'}}} y_j s_{ij}^{+}$$

But, since g is strictly supermodular, so is s. Therefore, for any $j > z$, $\Delta + s_{ij} < s_{i'j}$ for some $\Delta > 0$. But, then $2(1-\delta)(v_{i'} - v_i) \geq \sum_{j > \underline{m_{i'}}} y_j m_{ij} \Delta$, a contradiction as the LHS vanishes as $\delta \to 1$ and the RHS does not (recall that in the limit $\sum_{j > \underline{m_{i'}}} y_j m_{ij} s_{ij} = 2c$, so it holds that $\sum_{j > \underline{m_{i'}}} y_j m_{ij} > 0$). An analogous argument applies for the case where G is strictly submodular. $\square$

9 Online Appendix

We now return to the asymmetric cost setting studied in Section 6.2.

To quantify the inefficiencies, we fix the the production matrix G, type distributions F^b and F^s, and the bargaining weight $\alpha = 1/2$. Let $W^*(c^b, c^s)$ and $W^E(c^b, c^s)$ denote, respectively, the constrained efficient welfare and the maximal equilibrium welfare for the search costs c^b and c^s. Let $N^*(c^b, c^s)$ denote the measure of agents on each side in the constrained efficient steady state.

Proposition 4. *Assume $\alpha = 1/2$ and $c^b > c^s$. The efficiency gap:*

$$N^*(c^b, c^b)(c^b - c^s) \leq W^*(c^b, c^s) - W^E(c^b, c^s) \leq N^*\left(\tfrac{c^b+c^s}{2}, \tfrac{c^b+c^s}{2}\right)(c^b - c^s)$$

Thus, as $c^b \to c^s$,

$$W^*(c^b, c^s) - W^E(c^b, c^s) \to 0$$

See Proof in Appendix. Since there is a uniform bound on the population N^*, the above proposition effectively says that the welfare gap is on the order of the search cost gap $c^b - c^s$.

Proof of Proposition 4:

Proof. As $c^b > c^s$, it must be that in equilibrium, $S > B$. Notice that any equilibrium $\langle B, S, (x_i), (y_j), [m_{ij}], (v_i), (w_j)\rangle$ in the (c^b, c^s) economy gives rise to a corresponding equilibrium $\langle B, B, x, y, M, v, w\rangle$ in the (c^b, c^b) economy with the same investments, matches, and welfare. Thus, $W^E(c^b, c^s) = W^E(c^b, c^b) = W^*(c^b, c^b)$ where the last equality is the welfare theorem. Likewise, any constrained efficient allocation in the (c^b, c^s) economy is also constrained efficient in the (c, c) economy where $c = \frac{c^b+c^s}{2}$. Thus, $W^*(c^b, c^s) = W^*(c, c)$. Together, we have

$$W^*(c^b, c^s) - W^E(c^b, c^s) = W^*(c, c) - W^*(c^b, c^b)$$

The convexity of the welfare function (Proposition 1) implies the bounds above. □

Proof of Corollaries 2 and 3:

We will now prove a more general version of these two corollaries together:

Generalized Corollary: *In a model with outside options, and a CRS meeting function, the constrained efficient outcome is an equilibrium if and only if*

$$\alpha = \frac{B^* c^b}{B^* c^b + S^* c^s} = \frac{\partial \mu(B^*, S^*)/\partial B}{\mu(B^*, S^*)/B^*}$$

where B^*, S^* *are the constrained efficient stock.*

Proof. To simplify, we focus on the case where the state is interior and the proof repeats that argument with the appropriate modifications. The same could be done for the boundary case as well. Recall that $\mu(B, S)$ is the number of meetings in every period. The original planner's problem 5 is modified because the agents have an outside option and there is a general meeting function, and so the measure of buyers B need not equal the measure of sellers S. The planner now chooses the state $z = (B, S, (x_i), (y_j))$ instead of $z = (N, (x_i), (y_j))$, the investment thresholds, and the matching rule to maximize

$$\mathcal{W} = \mu(B, S) \sum_{i \in I} \sum_{j \in J} x_i y_j m_{ij} g_{ij} - Bc^b - Sc^s - \sum_{i \in I} \int_{\beta_{i+1}}^{\beta_i} C^b(i, \beta) f^b(\beta) d\beta$$
$$- \sum_{j \in J} \int_{\sigma_{j+1}}^{\sigma_j} C^s(j, \sigma) f^s(\sigma) d\sigma + \int_{\beta_0}^{\infty} u^b f^b(\beta) d\beta + \int_{\sigma_0}^{\infty} u^s f^s(\sigma) d\sigma$$

subject to the steady state conditions,

$$
\begin{aligned}
flow_i = \int_{\beta_{i+1}}^{\beta_i} f^b(\beta)d\beta - x_i\mu(B,S)\sum_{j\in J} y_j m_{ij} &= 0, \forall i \\
flow_j = \int_{\sigma_{j+1}}^{\sigma_j} f^s(\sigma)d\sigma - y_j\mu(B,S)\sum_{i\in I} x_i m_{ij} &= 0, \forall j \\
B, S &\geq 0 \\
x_i &\geq 0, \forall i \\
y_j &\geq 0, \forall j \\
X = 1 - \sum_{i\in I} x_i &= 0 \\
Y = 1 - \sum_{j\in J} y_j &= 0 \\
1 \geq m_{ij} &\geq 0, \forall i, j \\
F^b(\beta_{|I|}) = F^s(\sigma_{|J|}) &= 0
\end{aligned}
$$

Notice that taking weighted sums of the flow conditions implies that $F^b(\beta_0) = F^s(\sigma_0)$. The planner's problem is modified in three ways: i) agents can take an outside option which is included in the objective function and the conditions $F(\beta_0) = 1$ and $F(\sigma_0) = 1$ are removed; ii) the measure of buyers B and sellers S may differ and since we assumed that the are gains to trade, the conditions $B, S \geq 0$ will not bind at the efficient solution; iii) the Inflow=Outflow equations are modified because the outflow of buyers and sellers is

$$
\begin{aligned}
(Bx_i)\left(\frac{\mu(B,S)}{B}\right)\sum_{j\in J} y_j m_{ij} &= x_i\mu(B,S)\sum_{j\in J} y_j m_{ij}, \ \forall i \\
(Sy_j)\left(\frac{\mu(B,S)}{S}\right)\sum_{i\in I} x_i m_{ij} &= y_j\mu(B,S)\sum_{i\in I} x_i m_{ij}, \ \forall j
\end{aligned}
$$

The KKT conditions regularity conditions continue to hold, by the same arguments as in Theorem 1 (because the linear dependencies of the gradients do not change).

The Lagrangian is

$$\begin{aligned}
\mathcal{L} =& \mu(B,S)\sum_I\sum_j x_i y_j m_{ij} g_{ij} - Bc^b - Sc^s - \sum_I \int_{\beta_{i+1}}^{\beta_i} c(i,\beta) f^b(\beta) d\beta \\
& - \sum_J \int_{\sigma_{j+1}}^{\sigma_j} c(j,\sigma) f^s(\sigma) d\sigma + \int_{\beta_0}^{\infty} u^b f^b(\beta) d\beta + \int_{\sigma_0}^{\infty} u^s f^s(\sigma) d\sigma \\
& + \sum_{i\in I} v_i flow_i + \sum_{j\in J} w_j flow_j + \sum_i \phi_i x_i + \sum_j \psi_j y_j + \gamma X + \lambda Y \\
& + \sum_{i\in I}\sum_{j\in J} (\eta_{ij} m_{ij} + \hat{\eta}_{ij}(1-m_{ij}))
\end{aligned}$$

FOC(B): $$(\partial\mu/\partial B)\left(\sum_{i\in I}\sum_{j\in J} x_i y_j m_{ij}\left(g_{ij} - v_i - w_j\right)\right) - c^b = 0$$

$$\Rightarrow \sum_{i\in I}\sum_{j\in J} x_i y_j m_{ij} s_{ij} = \frac{c^b}{\partial\mu/\partial B}$$

FOC($\mathbf{x_i}$): $$\mu \sum_{j\in J} y_j m_{ij} g_{ij} - v_i \mu \sum_{j\in J} y_j m_{ij} - \mu \sum_{j\in J} w_j y_j m_{ij} - \gamma - \phi_i = 0$$

$$\sum_{j\in J} y_j m_{ij} s_{ij} = \frac{\gamma + \phi_i}{\mu}$$

and $x_i\phi_i = 0$.

Thus, substituting FOC(x_i) into FOC(B), the second into the first, we get $\frac{\gamma}{\mu} = \frac{c^b}{\partial\mu/\partial B}$ (because $\sum_{i\in I} x_i = 1$ and $x_i\phi_i = 0$). Thus

$$\sum_J y_j m_{ij} s_{ij} = \frac{c^b}{\partial\mu/\partial B} + \frac{\phi_i}{\mu}$$

and if $\phi_i = 0$, then

$$\sum_J y_j m_{ij} s_{ij} = \frac{c^b}{\partial\mu/\partial B} \tag{18}$$

We now do the same for the sellers.

FOC(S): $$\sum_{j\in J}\sum_{i\in I} x_i y_j m_{ij} s_{ij} = \frac{c^s}{\partial\mu/\partial S}$$

FOC($\mathbf{y_j}$):

$$\mu\sum_{I}x_im_{ij}g_{ij}-w_j\mu\sum_{I}x_im_{ij}-\sum_{I}v_i\mu x_im_{ij}-\eta-\psi_j=0$$

$$\sum_{I}x_im_{ij}s_{ij}=\frac{\lambda+\psi_j}{\mu}$$

and $\psi_jy_j=0$. Thus,

$$\sum_{I}x_im_{ij}s_{ij}=\frac{c^s}{\partial\mu/\partial S} \tag{19}$$

Decentralizing the optimal allocation: we show that the shadow values v_i,w_j together with the matching matrix M and state z constitute an equilibrium, provided that the bargaining weight is $\alpha=\frac{\partial\mu/\partial B}{\mu/B}$. To see why, substitute $\partial\mu/\partial B=\alpha\left(\mu/B\right)$ into condition (18)

$$\sum_{j\in J}y_jm_{ij}s_{ij}=\frac{c^b}{\alpha\left(\mu/B\right)},\forall i$$

which is the Constant Surplus equation for skill i.

For sellers, since μ is homogeneous of degree 1,[20] $\frac{\partial\mu/\partial S}{\mu/S}=1-\frac{\partial\mu/\partial B}{\mu/B}$, and thus, $1-\alpha=\frac{\partial\mu/\partial S}{\mu/S}$. Substituting into equation (19) gives the sellers' Constant Surplus equations:

$$\sum_{i\in I}x_im_{ij}s_{ij}=\frac{c^s}{\left(1-\alpha\right)\left(\mu/S\right)},\forall j$$

The FOC(β_0) condition is precisely the equilibrium entry condition, $v_0-C(0,\beta_0)=u^b$, and so the shadow value v_0 and threshold β_0 satisfy the equilibrium entry condition. Likewise, the seller's entry condition holds as well. The proofs that the Efficient Matching conditions and individual optimal investments hold are the same as in Theorem 1.

Furthermore, by FOC(B) and FOC(S), we have that $c^s\left(\partial\mu/\partial B\right)=c^b\left(\partial\mu/\partial S\right)$. By homogeneity of degree 1,

$$B\left(\partial\mu/\partial B\right)+S(\partial\mu/\partial S)=\mu\Rightarrow c^b\left[B\left(\partial\mu/\partial B\right)+S(\partial\mu/\partial S)\right]=c^b\mu$$

Substituting in gives:

$$Bc^b\left(\partial\mu/\partial B\right)+Sc^s\left(\partial\mu/\partial B\right)=c^b\mu\Rightarrow\frac{\partial\mu/\partial B}{\mu}=\frac{c^b}{Bc^b+Sc^s}$$

[20] Homogeneity of degree 1 implies $B\left(\partial\mu/\partial B\right)+S(\partial\mu/\partial S)=\mu\iff\left(\partial\mu/\partial B\right)/\left(\mu/B\right)+(\partial\mu/\partial S)/\left(\mu/S\right)=1$.

Therefore, the buyers' bargaining weight $\alpha = \frac{\partial\mu/\partial B}{\mu/B} = \frac{Bc^b}{Bc^b+Sc^s}$ and so the seller's bargaining weight is $1-\alpha = \frac{Sc^s}{Bc^b+Sc^s}$.

$\Leftarrow$ Recall that, in an equilibrium, the Constant Surplus equations imply Equation (17) $\frac{B}{S} = \frac{\alpha}{1-\alpha} \cdot \frac{c^s}{c^b}$. Therefore, if the constrained efficient solution is an equilibrium, it must be that $\frac{c^s}{c^b} = \frac{(1-\alpha)B}{\alpha S}$ and therefore, it must be that $\alpha = \frac{Bc^b}{Bc^b+Sc^s}$. $\square$