5 Internet Appendix

5.1 Internet Appendix Tables

Table A1: Estimated Bounds on Average Effects on Health Care Utilization with a Valid IV

	Prescription Drugs				Outpatient Visits			
	Any		Number		Any		Number	
Bounds on ATE								
Proposition 1	-.348	.362	-3.854	13.20	-.288	.423	-4.278	17.04
Bounded Outcome (A5)	(-.362, .374)		(-4.087, 13.97)		(-.300, .433)		(-4.474, 17.30)	
Proposition 2	.028	.362	.113	13.20	.062	.423	.316	17.04
Monotonicity (A6)	(.014, .374)		(.027, 13.97)		(.050, .433)		(.226, 17.30)	
Proposition 3c'	-.348	.110	-3.854	.619	-.288	.187	-4.278	1.049
Mean Dominance (A7c')	(-.362, .133)		(-4.088, .780)		(-.300, .208)		(-4.474, 1.227)	
Proposition 6'	.028	.110	.113	.619	.062	.187	.316	1.049
A5 & A6 &A7c'	(.012, .136)		(.017, .799)		(.049, .210)		(.217, 1.245)	
Bounds on ATT								
Proposition 1	-.058	.424	-9.826	1.749	.012	.495	-12.27	2.200
Bounded Outcome (A5)	(-.085, .452)		(-10.59, 1.911)		(-.011, .520)		(-12.82, 2.364)	
Proposition 2	.050	.424	.202	1.749	.112	.495	.564	2.200
Monotonicity (A6)	(.025, .452)		(.044, 1.915)		(.089, .520)		(.398, 2.370)	
Proposition 3c'	-.058	.133	-9.826	.651	.012	.258	-12.27	1.482
Mean Dominance (A7c')	(-.087, .182)		(-10.59, .910)		(-.013, .300)		(-12.82, 1.741)	
Proposition 6'	.050	.133	.202	.651	.112	.258	.564	1.482
A5 & A6 &A7c'	(.021, .184)		(.027, .945)		(.086, .301)		(.385, 1.770)	
Bounds on $LATE_{nt}$								
Proposition 2	0	.397	0	21.97	0	.440	0	28.26
Monotonicity (A6)	(0, .410)		(0, 23.25)		(0, .452)		(0, 28.33)	
Proposition 3'	-.602	.101	-2.027	.653	-.560	.146	-1.744	.830
Mean Dominance (A7c')	(-.615, .126)		(-2.099, .837)		(-.572, .170)		(-1.824, 1.052)	
Proposition 6'	0	.101	0	.653	0	.146	0	.830
A5& A6 &A7c'	(0, .129)		(0, .855)		(0, .171)		(0, 1.070)	
Bounds on $LATE_{at}$								
Proposition 2	0	.775	0	3.208	0	.794	0	3.392
Monotonicity (A6)	(0, .797)		(0, 3.376)		(0, .815)		(0, 3.608)	
Proposition 3'	-.225	.171	-20.79	.931	-.205	.303	-.26.61	1.904
Mean Dominance (A7c')	(-.249, .222)		(-22.06, 1.225)		(-.227, .346)		(-26.82, 2.225)	
Proposition 6'	0	.171	0	.931	0	.303	0	1.904
A5 & A6 &A7c'	(0, .225)		(0, 1.259)		(0, .348)		(0, 2.250)	

Table A2: Estimated Bounds on Average Effects on Preventive Care with a Valid IV

	Blood Cholesterol Checked		Blood Tested for High Blood Sugar/Diabetes		Mammogram (Women ≥40)		Pap Test (Women)	
Bounds on ATE								
Proposition 1	-.380	.331	-.360	.350	-.176	.528	-.234	.485
Bounded Outcome (A5)	(-.392, .342)		(-.372, .361)		(-.194, .546)		(-.249, .499)	
Proposition 2	.032	.331	.025	.350	.053	.528	.051	.485
Monotonicity (A6)	(.020, .342)		(.013, .361)		(.033, .546)		(.035, .499)	
Proposition 3c'	-.380	.085	-.360	.078	-.176	.151	-.234	.154
Mean Dominance (A7c')	(-.392, .107)		(-.372, .099)		(-.195, .188)		(-.249, .184)	
Proposition 6'	.032	.085	.025	.078	.053	.151	.051	.154
A5 & A6 &A7c'	(.018, .110)		(.011, .102)		(.030, .191)		(.033, .186)	
Bounds on ATT								
Proposition 1	-.118	.365	-.102	.380	-.111	.331	-.129	.404
Bounded Outcome (A5)	(-.142, .391)		(-.126, .405)		(-.154, .371)		(-.159, .431)	
Proposition 2	.058	.365	.045	.380	.100	.331	.085	.404
Monotonicity (A6)	(.035, .391)		(.022, .405)		(.059, .372)		(.058, .431)	
Proposition 3c'	-.118	.073	-.102	.102	-.111	.230	-.129	.236
Mean Dominance (A7c')	(-.144, .118)		(-.128, .144)		(-.156, .295)		(-.160, .286)	
Proposition 6'	.058	.073	.045	.102	.100	.230	.085	.236
A5 & A6 &A7c'	(.031, .121)		(.019, .147)		(.056, .299)		(.055, .289)	
Bounds on $LATE_{nt}$								
Proposition 2	0	.369	0	.402	0	.706	0	.605
Monotonicity (A6)	(0, .381)		(0, .414)		(0, .724)		(0, .620)	
Proposition 3c'	-.631	.085	-.598	.064	-.294	.109	-.395	.103
Mean Dominance (A7c')	(-.642, .109)		(-.609, .088)		(-.313, .153)		(-.410, .137)	
Proposition 6	0	.085	0	.064	0	.109	0	.103
A5 & A6 &A7c'	(0, .111)		(0, .091)		(0, .156)		(0, .141)	
Bounds on $LATE_{at}$								
Proposition 2	0	.636	0	.695	0	.523	0	.599
Monotonicity (A5)	(0, .661)		(0, .718)		(0, .569)		(0, .628)	
Proposition 3	-.364	.032	-.305	.119	-.477	.296	-.401	.283
Mean Dominance (A6)	(-.390, .081)		(-.329, .164)		(-.523, .370)		(-.431, .337)	
Proposition 4	0	.032	0	.119	0	.296	0	.283
A4 & A5 &A6	(0, .086)		(0, .168)		(0, .375)		(0, .341)	

Table A3: Estimated Bounds on Average Effects on Self-Reported Health (Binary) with a Valid IV

	Not Fair or Poor		Not Poor		Same or Gotten Better		Not Screen Positive for Depression	
Bounds on ATE								
Proposition 1	-.370	.340	-.511	.200	-.443	.267	-.444	.266
Bounded Outcome (A5)	(-.383, .351)		(-.521, .209)		(-.455, .278)		(-.456, .277)	
Proposition 2	.038	.340	.029	.200	.032	.267	.024	.266
Monotonicity (A6)	(.025, .351)		(.020, .209)		(.021, .278)		(.012, .277)	
Proposition 3c	.050	.340	.036	.200	.037	.267	-.001	.266
Mean Dominance (A7c)	(.027, .352)		(.020, .209)		(.016, .278)		(-.023, .278)	
Proposition 6	.044	.340	.032	.200	.038	.267	.025	.266
A5 & A6 &A7c	(.024, .352)		(.019, .209)		(.024, .278)		(.009, .277)	
Bounds on ATT								
Proposition 1	-.162	.321	-.032	.450	-.083	.399	-.137	.345
Bounded Outcome (A5)	(-.187, .345)		(-.047, .471)		(-.104, .424)		(-.161, .371)	
Proposition 2	.068	.321	.051	.450	.058	.399	.043	.345
Monotonicity (A6)	(.044, .346)		(.036, .471)		(.037, .424)		(.020, .371)	
Proposition 3	.092	.321	.062	.450	.092	.399	.059	.345
Mean Dominance (A7c)	(.048, .347)		(.036, .472)		(.055, .425)		(.018, .372)	
Proposition 6	.087	.321	.059	.450	.088	.399	.055	.345
A5 & A6 &A7c	(.045, .347)		(.035, .472)		(.052, .425)		(.017, .372)	
Bounds on $LATE_{nt}$								
Proposition 2	0	.402	0	.104	0	.242	0	.274
Monotonicity (A6)	(0, .413)		(0, .111)		(0, .252)		(0, .285)	
Proposition 3c	.008	.402	.007	.104	-.009	.242	-.051	.274
Mean Dominance (A7c)	(-.017, .413)		(-.012, .112)		(-.032, .252)		(-.076, .285)	
Proposition 6	.008	.402	.007	.104	-.000	.242	-.000	.274
A5 & A6 &A7c	(-.000, .413)		(-.000, .111)		(-.000, .252)		(-.000, .285)	
Bounds on $LATE_{at}$								
Proposition 2	0	.523	0	.827	0	.708	0	.626
Monotonicity (A6)	(0, .549)		(0, .847)		(0, .729)		(0, .649)	
Proposition 3c	.049	.523	.023	.827	.071	.708	.034	.626
Mean Dominance (A7c)	(.001, .550)		(-.009, .847)		(.031, .730)		(-.011, .649)	
Proposition 6	.048	.523	.023	.827	.071	.708	.033	.626
A5 & A6 &A7c	(.003, .549)		(-.000, .847)		(.032, .730)		(-.000, .649)	

Table A4: Estimated Bounds on Average Effects on Self-Reported Health (# of days) with a Valid IV

	Physical Health Good		Mental Health Good		Poor Physical or Mental Health Did not Impair Usual Activity	
Bounds on ATE						
Proposition 1	-13.48	7.832	-12.61	8.706	-14.49	6.820
Bounded Outcome (A5)	(-13.80, 8.119)		(-12.93, 9.012)		(-14.79, 7.093)	
Proposition 2	.417	7.832	.589	8.706	.333	6.820
Monotonicity (A6)	(.123, 8.119)		(.269, 9.012)		(.061, 7.094)	
Proposition 3c	-.156	7.832	.099	8.706	-.703	6.820
Mean Dominance (A7c)	(-.691, 8.122)		(-.471, 9.015)		(-1.198, 7.096)	
Proposition 6	.347	7.832	.594	8.706	.268	6.820
A5 & A6 &A7c	(.070, 8.119)		(.212, 9.012)		(.012, 7.094)	
Bounds on ATT						
Proposition 1	-4.689	9.780	-4.848	9.621	-4.441	10.03
Bounded Outcome (A5)	(-5.257, 10.42)		(-5.463, 10.30)		(-4.964, 10.67)	
Proposition 2	.745	9.780	1.053	9.621	.595	10.03
Monotonicity (A6)	(.214, 10.43)		(.472, 10.31)		(.104, 10.68)	
Proposition 3c	.532	9.780	1.425	9.621	.086	10.03
Mean Dominance (A7c)	(-.418, 10.45)		(.380, 10.33)		(-.775, 10.69)	
Proposition 6	.685	9.780	1.317	9.621	.536	10.03
A5 & A6 &A7c	(.169, 10.43)		(.408, 10.33)		(.060, 10.68)	
Bounds on $LATE_{nt}$						
Proposition 2	0	8.494	0	9.941	0	6.690
Monotonicity (A6)	(0, 8.757)		(0, 10.23)		(0, 6.937)	
Proposition 3c	-.890	8.494	-1.030	9.941	-1.549	6.690
Mean Dominance (A7c)	(-1.507, 8.760)		(-1.657, 10.23)		(-2.119, 6.940)	
Proposition 6	-.000	8.494	-.000	9.941	-.000	6.690
A5 & A6 &A7c	(-.000, 8.757)		(-.000, 10.23)		(-.000, 6.937)	
Bounds on $LATE_{at}$						
Proposition 2	0	18.73	0	17.77	0	19.56
Monotonicity (A6)	(0, 19.28)		(0, 18.34)		(0, 20.11)	
Proposition 3c	-.442	18.73	.772	17.77	-1.054	19.56
Mean Dominance (A7c)	(-1.493, 19.28)		(-.341, 18.34)		(-2.030, 20.11)	
Proposition 6	-.000	18.73	.767	17.77	-.000	19.56
A5 & A6 &A7c	(-.000, 19.28)		(-.000, 18.34)		(-.000, 20.11)	

Table A5: Estimated Bounds on Average Effects on the Alleviation of Financial Strain with a Valid IV

	No out of pocket medical expenses		Not owe for medical expenses currently		Not borrow money to pay medical bills		Not be refused treatment due to medical debt	
Bounds on ATE								
Proposition 1	-.260	.450	-.284	.426	-.359	.351	-.534	.176
Bounded Outcome (A5)	(-.273, .461)		(-.296, .437)		(-.372, .362)		(-.544, .184)	
Proposition 2	.056	.450	.052	.426	.045	.351	.010	.176
Monotonicity (A6)	(.044, .461)		(.040, .437)		(.033, .362)		(.003, .184)	
Proposition 3c'	-.260	.187	-.284	.104	-.359	.137	-.534	.019
Mean Dominance (A7c')	(-.273, .210)		(-.296, .126)		(-.372, .157)		(-.544, .030)	
Proposition 4'	.056	.187	.052	.104	.045	.137	.010	.019
A5 & A6 &A7c'	(.043, .211)		(.038, .129)		(.032, .159)		(.002, .032)	
Bounds on ATT								
Proposition 1	-.139	.343	-.196	.286	-.063	.419	-.018	.464
Bounded Outcome (A5)	(-.165, .366)		(-.222, .310)		(-.086, .443)		(-.031, .485)	
Proposition 2	.101	.343	.093	.286	.081	.419	.018	.464
Monotonicity (A6)	(.077, .367)		(.070, .311)		(.059, .444)		(.006, .485)	
Proposition 3'	-.139	.123	-.196	.122	-.063	.123	-.018	.028
Mean Dominance (A7c')	(-.167, .166)		(-.223, .163)		(-.088, .165)		(-.032, .052)	
Proposition 4'	.101	.123	.093	.122	.081	.123	.018	.028
A5 & A5 &A7c'	(.074, .170)		(.067, .168)		(.056, .167)		(.004, .053)	
Bounds on $LATE_{nt}$								
Proposition 2	0	.567	0	.556	0	.367	0	.073
Monotonicity (A6)	(0, .579)		(0, .568)		(0, .378)		(0, .079)	
Proposition 3c'	-.443	.217	-.444	.076	-.633	.138	-.927	.010
Mean Dominance (A7c')	(-.446, .243)		(-.456, .103)		(-.645, .162)		(-.934, .024)	
Proposition 6'	0	.217	0	.076	0	.138	0	.010
A5 & A6 &A7c'	(0, .245)		(0, .106)		(0, .164)		(0, .026)	
Bounds on $LATE_{at}$								
Proposition 2	0	.502	0	.401	0	.701	0	.925
Monotonicity (A6)	(0, .527)		(0, .426)		(0, .723)		(0, .938)	
Proposition 3c'	-.498	.046	-.599	.060	-.299	.086	-.075	.022
Mean Dominance (A7c')	(-.524, .093)		(-.624, 107)		(-.322, .132)		(-.088, .049)	
Proposition 6'	0	.046	0	.060	0	.086	0	.022
A5 & A6 &A7c'	(0, .100)		(0, .113)		(0, .136)		(0, .050)	

5.2 Derivation of the Bounds

We start by writing the parameter of interest $\Delta(z)$ for $z=0$ and $z=1$, separately.

$$\begin{aligned}
& \Delta(1) \\
= \; & \pi_{at}(E[Y(1)|at]-E[Y(1,0)|at])+\pi_{nt}(E[Y(1,1)|nt]-E[Y(1)|nt])+\pi_c LMATE_c^1 \quad (16)\\
= \; & \pi_{at}(E[Y(1)|at]-E[Y(1,0)|at])+\pi_{nt}(E[Y(1,1)|nt]-E[Y(1)|nt]) \\
& +E[Y|Z=1]-E[Y|Z=0]-\pi_{at}LNATE_{at}^0-\pi_{nt}LNATE_{nt}^0-\pi_c LNATE_c^0 \quad (17)\\
= \; & p_{1|1}\overline{Y}^{11}-p_{0|1}\overline{Y}^{10}-\pi_{at}E[Y(1,0)|at]+\pi_{nt}E[Y(1,1)|nt]-\pi_c E[Y(1,D_0)|c] \quad (18)
\end{aligned}$$

$$\begin{aligned}
& \Delta(0) \\
= \; & \pi_{at}(E[Y(0)|at]-E[Y(0,0)|at])+\pi_{nt}(E[Y(0,1)|nt]-E[Y(0)|nt])+\pi_c LMATE_c^0 \quad (19)\\
= \; & \pi_{at}(E[Y(0)|at]-E[Y(0,0)|at])+\pi_{nt}(E[Y(0,1)|nt]-E[Y(0)|nt]) \\
& +E[Y|Z=1]-E[Y|Z=0]-\pi_{at}LNATE_{at}^1-\pi_{nt}LNATE_{nt}^1-\pi_c LNATE_c^1 \quad (20)\\
= \; & p_{1|0}\overline{Y}^{01}-p_{0|0}\overline{Y}^{00}-\pi_{at}E[Y(0,0)|at]+\pi_{nt}E[Y(0,1)|nt]+\pi_c E[Y(0,D_1)|c] \quad (21)
\end{aligned}$$

Similarly, by the definitions of $LMATE_k^z$ and $LNATE_k^z$, for $k=at,nt,c$, under Assumptions 1 through 3 we have

$$\begin{aligned}
LMATE_c^z &= E[Y(z,D_1)-Y(z,D_0)|c]=\begin{cases} E[Y(1)|c]-E[Y(1,D_0)|c], \text{ for } z=1\\ E[Y(0,D_1)|c]-E[Y(0)|c], \text{ for } z=0\end{cases}\\
LMATE_{at}^z &= E[Y(z,D_1)-Y(z,D_0)|at]=0, \text{ for } z=0,1\\
LMATE_{nt}^z &= E[Y(z,D_1)-Y(z,D_0)|nt]=0, \text{ for } z=0,1 \quad (22)
\end{aligned}$$

and

$$\begin{aligned}
LNATE_c^z &= E[Y(1,D_z)-Y(0,D_z)|c]=\begin{cases} E[Y(1)|c]-E[Y(0,D_1)|c], \text{ for } z=1\\ E[Y(1,D_0)|c]-E[Y(0)|c], \text{ for } z=0\end{cases}\\
LNATE_{at}^z &= E[Y(1,D_z)-Y(0,D_z)|at]=E[Y(1)|at]-E[Y(0)|at], \text{ for } z=0,1\\
LNATE_{nt}^z &= E[Y(1,D_z)-Y(0,D_z)|nt]=E[Y(1)|nt]-E[Y(0)|nt], \text{ for } z=0,1 \quad (23)
\end{aligned}$$

In the main text, the relevant point identified objects in our setting are: $\pi_{nt}=p_{0|1}$, $\pi_{at}=p_{1|0}$, $\pi_c=p_{1|1}-p_{1|0}=p_{0|0}-p_{0|1}$, $E[Y(1)]=E[Y|Z=1]$, $E[Y(0)]=E[Y|Z=0]$, $E[Y(1)|nt]=\overline{Y}^{10}$, $E[Y(0)|at]=\overline{Y}^{01}$. The partially identified average outcomes of the corresponding strata obtained by the trimming procedure are given by: $L^{0,nt}\le E[Y(0)|nt]\le U^{0,nt}$, $L^{1,at}\le E[Y(1)|at]\le U^{1,at}$, $L^{0,c}\le E[Y(0)|c]\le U^{0,c}$ and $L^{1,c}\le E[Y(1)|c]\le U^{1,c}$, where

$$\begin{aligned}
L^{0,nt}=E[Y|Z=0,D=0,Y\le y^{00}_{(p_{0|1}/p_{0|0})}], U^{0,nt}=E[Y|Z=0,D=0,Y\ge y^{00}_{1-(p_{0|1}/p_{0|0})}];\\
L^{1,at}=E[Y|Z=1,D=1,Y\le y^{11}_{(p_{1|0}/p_{1|1})}], U^{1,at}=E[Y|Z=1,D=1,Y\ge y^{11}_{1-(p_{1|0}/p_{1|1})}];\\
L^{0,c}=E[Y|Z=0,D=0,Y\le y^{00}_{1-(p_{0|1}/p_{0|0})}], U^{0,c}=E[Y|Z=0,D=0,Y\ge y^{00}_{(p_{0|1}/p_{0|0})}];\\
L^{1,c}=E[Y|Z=1,D=1,Y\le y^{11}_{1-(p_{1|0}/p_{1|1})}], U^{1,c}=E[Y|Z=1,D=1,Y\ge y^{11}_{(p_{1|0}/p_{1|1})}].
\end{aligned}$$

Similar to the equations $\pi_{nt}E[Y(0)|nt] + \pi_c E[Y(0)|c] = p_{0|0}\overline{Y}^{00}$ and $\pi_{at}E[Y(1)|at] + \pi_c E[Y(1)|c] = p_{1|1}\overline{Y}^{11}$ in the respective trimming cells $\{Z=0, D=0\}$ and $\{Z=1, D=1\}$, by the nature of the trimming bounds, the following equations also hold: $\pi_{nt}L^{0,nt} + \pi_c U^{0,c} = p_{0|0}\overline{Y}^{00}$, $\pi_{nt}U^{0,nt} + \pi_c L^{0,c} = p_{0|0}\overline{Y}^{00}$, $\pi_{at}L^{1,at} + \pi_c U^{1,c} = p_{1|1}\overline{Y}^{11}$ and $\pi_{at}U^{1,at} + \pi_c L^{1,c} = p_{1|1}\overline{Y}^{11}$. These equations are used to compare bounds generated by different equations for $\Delta(z)$.

5.2.1 Proof of Proposition 2

We start by deriving bounds for the non-point identified mean potential outcomes of the strata, and for all the local net and mechanism average treatment effects.

Bounds for $E[Y(0)|nt]$: A5.2 implies $E[Y(1)|nt] = \overline{Y}^{10} \geq E[Y(0)|nt]$. A5 does not provide any additional information for a lower bound of $E[Y(0)|nt]$. Since $U^{0,nt}$ can be above or below $\overline{Y}^{10}$, we have: $L^{0,nt} \leq E[Y(0)|nt] \leq \min\{U^{0,nt}, \overline{Y}^{10}\}$.[17]

Bounds for $E[Y(1)|at]$: A5.2 implies $E[Y(1)|at] \geq E[Y(0)|at] = \overline{Y}^{01}$. A5 does not provide any additional information for an upper bound of $E[Y(1)|at]$. Thus we have: $\max\{L^{1,at}, \overline{Y}^{01}\} \leq E[Y(1)|at] \leq U^{1,at}$.

Bounds for $E[Y(0)|c]$: A5.1 and A5.2 imply $E[Y(1)|c] \geq E[Y(0)|c]$, which implies that $U^{1,c}$ is another upper bound for $E[Y(0)|c]$. A5 does not provide any additional information for a lower bound of $E[Y(0)|c]$. Hence, $L^{0,c} \leq E[Y(0)|c] \leq \min\{U^{0,c}, U^{1,c}\}$.

Bounds for $E[Y(1)|c]$: A5.1 and A5.2 imply $E[Y(1)|c] \geq E[Y(0)|c]$, which implies that $L^{0,c}$ is another lower bound for $E[Y(1)|c]$. Hence, $\max\{L^{0,c}, L^{1,c}\} \leq E[Y(1)|c] \leq U^{1,c}$.

Bounds for $E[Y(1, D_0)|c]$: A5.1 and A5.2 imply $E[Y(1)|c] \geq E[Y(1, D_0)|c] \geq E[Y(0)|c]$, which combined with the results above gives $L^{0,c} \leq E[Y(1, D_0)|c] \leq U^{1,c}$.

Bounds for $E[Y(0, D_1)|c]$: A5.1 and A5.2 imply $E[Y(1)|c] \geq E[Y(0, D_1)|c] \geq E[Y(0)|c]$, which combined with the results above gives $L^{0,c} \leq E[Y(0, D_1)|c] \leq U^{1,c}$.

Bounds for $LNATE^z_{nt}$, for $z = 0, 1$: From (23), $LNATE^z_{nt} = \overline{Y}^{10} - E[Y(0)|nt]$. Using the bounds previously derived for $E[Y(0)|nt]$, we have[18]:
$\max\{0, \overline{Y}^{10} - U^{0,nt}\} \leq LNATE^z_{nt} \leq \overline{Y}^{10} - L^{0,nt}$, for $z = 0, 1$.

Bounds for $LNATE^z_{at}$, for $z = 0, 1$: From (23), $LNATE^z_{at} = E[Y(1)|at] - \overline{Y}^{01}$. Using the bounds previously derived for $E[Y(1)|at]$, we have:
$\max\{0, L^{1,at} - \overline{Y}^{01}\} \leq LNATE^z_{at} \leq U^{1,at} - \overline{Y}^{01}$, for $z = 0, 1$.

[17] For brevity, in what follows we omit explicitly specifying when some quantities can be greater or lower than others unless we believe it is necessary. Hence, when min (or max) operators are present, it implies that none of the terms inside them is always lower (greater) than the other(s).

[18] The following equalities are helpful for the rest of the proofs. For scalars a, b, c and d we have: (i) $a - \max\{c, d\} = \min\{a - c, a - d\}$; (ii) $a - \min\{c, d\} = \max\{a - c, a - d\}$; (iii) $\max\{a, b\} - c = \max\{a - c, b - c\}$; (iv) $\min\{a, b\} - c = \min\{a - c, b - c\}$; (v) $\max\{a, b\} - \min\{c, d\} = \max\{a - c, a - d, b - c, b - d\}$; (vi) $\min\{a, b\} - \max\{c, d\} = \min\{a - c, a - d, b - c, b - d\}$.

Bounds for $LNATE_c^0$: From (23), $LNATE_c^0 = E[Y(1,D_0)|c] - E[Y(0)|c]$. A5.2 directly states $LNATE_c^0 \geq 0$. Using the bounds previously obtained for the components in $LNATE_c^0$ we obtain two additional lower bounds: $L^{0,c} - U^{0,c}$ and $L^{0,c} - U^{1,c}$. By definition, $L^{0,c} - U^{0,c} \leq 0$. Also, employing A5.1 and A5.2, we have $U^{1,c} \geq E[Y(1)|c] \geq E[Y(0)|c] \geq L^{0,c}$, so $L^{0,c} - U^{1,c} \leq 0$. Hence, the lower bound for $LNATE_c^0$ is 0. Using the bounds previously derived for the components of $LNATE_c^0$, we have the upper bound is $U^{1,c} - L^{0,c}$. Thus, $0 \leq LNATE_c^0 \leq U^{1,c} - L^{0,c}$.

Bounds for $LNATE_c^1$: From (23), $LNATE_c^1 = E[Y(1)|c] - E[Y(0,D_1)|c]$. A5.2 directly states $LNATE_c^1 \geq 0$. Using the bounds previously obtained for the components in $LNATE_c^1$ we obtain two additional lower bounds: $L^{0,c} - U^{1,c}$ and $L^{1,c} - U^{1,c}$. By definition, $L^{1,c} - U^{1,c} \leq 0$. Also, employing A5.1 and A5.2, we have $U^{1,c} \geq E[Y(1)|c] \geq E[Y(0)|c] \geq L^{0,c}$, so $L^{0,c} - U^{1,c} \leq 0$. Hence, the lower bound for $LNATE_c^0$ is 0. Using the bounds previously derived for the components of $LNATE_c^0$, we have the upper bound is $U^{1,c} - L^{0,c}$. Thus, $0 \leq LNATE_c^0 \leq U^{1,c} - L^{0,c}$.

Bounds for $LMATE_c^1$: $LMATE_c^1 = E[Y(1)|c] - E[Y(1,D_0)|c]$. A5.1 directly implies $LMATE_c^1 \geq 0$. Using the bounds previously obtained for the components of $LMATE_c^1$ we obtain two additional lower bounds: $L^{1,c} - U^{1,c}$ and $L^{0,c} - U^{1,c}$. Since $L^{1,c} - U^{1,c} \leq 0$ (by definition) and $L^{0,c} - U^{1,c} \leq 0$ (from above), the lower bound for $LMATE_c^1$ is 0. Using the bounds previously derived for the components of $LMATE_c^1$, we have the upper bound is $U^{1,c} - L^{0,c}$. Thus, $0 \leq LMATE_c^1 \leq U^{1,c} - L^{0,c}$.

Bounds for $LMATE_c^0$: $LMATE_c^0 = E[Y(0,D_1)|c] - E[Y(0)|c]$. A5.1 directly implies $LMATE_c^0 \geq 0$. Using the bounds previously obtained for the components of $LMATE_c^0$ we obtain two additional lower bounds: $L^{0,c} - U^{0,c}$ and $L^{0,c} - U^{1,c}$. Since $L^{0,c} - U^{0,c} \leq 0$ (by definition) and $L^{0,c} - U^{1,c} \leq 0$ (from above), the lower bound for $LMATE_c^0$ is 0. Using the bounds previously derived for the components of $LMATE_c^0$, we have the upper bound is $U^{1,c} - L^{0,c}$. Thus, $0 \leq LMATE_c^0 \leq U^{1,c} - L^{0,c}$.

Bounds for $E[Y(1,1)|nt]$: A4 and A5.3 imply $y^u \geq E[Y(1,1)|nt] \geq \overline{Y}^{10}$. And thus $E[Y(1,1)|nt] - E[Y(1)|nt]$ in (16) and (17) has the following bounds $0 \leq E[Y(1,1)|nt] - E[Y(1)|nt] \leq y^u - \overline{Y}^{10}$.

Bounds for $E[Y(0,1)|nt]$: A4 and A5.3 imply $y^u \geq E[Y(0,1)|nt] \geq E[Y(0)|nt]$. Thus, we have $y^u \geq E[Y(0,1)|nt] \geq L^{0,nt}$. A5.3 directly states $E[Y(0,1)|nt] - E[Y(0)|nt] \geq 0$. Using the bounds previously obtained for $E[Y(0)|nt]$ we obtain two additional lower bounds for $E[Y(0,1)|nt] - E[Y(0)|nt]$: $L^{0,nt} - U^{0,nt}$ and $L^{0,nt} - \overline{Y}^{10}$. By definition, $L^{0,nt} - U^{0,nt} \leq 0$. Also, employing A5.2, we have $\overline{Y}^{10} = E[Y(1)|nt] \geq E[Y(0)|nt] \geq L^{0,nt}$, so $L^{0,nt} - \overline{Y}^{10} \leq 0$. Hence, the lower bound for $E[Y(0,1)|nt] - E[Y(0)|nt]$ is 0. Using the bounds previously derived for $E[Y(0)|nt]$, we have the upper bound is $y^u - L^{0,nt}$. Thus, $E[Y(0,1)|nt] - E[Y(0)|nt]$ in (19) and (20) has the following bounds

$0 \le E[Y(0,1)|nt] - E[Y(0)|nt] \le y^u - L^{0,nt}$.

Bounds for $E[Y\left(1,0\right)|at]$: A4 and A5.3 imply $E[Y\left(1\right)|at] \ge E[Y\left(1,0\right)|at] \ge y^l$. Thus, we have $U^{1,at} \ge E[Y\left(1,0\right)|at] \ge y^l$. A5.3 directly states $E[Y(1)|at] - E[Y(1,0)|at] \ge 0$. Using the bounds previously obtained for $E[Y\left(1\right)|at]$ we obtain two additional lower bounds for $E[Y(1)|at] - E[Y(1,0)|at]$: $L^{1,at} - U^{1,at}$ and $\overline{Y}^{01} - U^{1,at}$. By definition, $L^{1,at} - U^{1,at} \le 0$. Also, employing A5.2, we have $\overline{Y}^{01} = E[Y\left(0\right)|at] \le E[Y(1)|at] \le U^{1,at}$, so $\overline{Y}^{01} - U^{1,at} \le 0$. Hence, the lower bound for $E[Y(1)|at] - E[Y(1,0)|at]$ is 0. Using the bounds previously derived for $E[Y\left(1\right)|at]$, we have the upper bound is $U^{1,at} - y^l$. Thus, $E[Y(1)|at] - E[Y(1,0)|at]$ in (16) and (17) has the following bounds $0 \le E[Y(1)|at] - E[Y(1,0)|at] \le U^{1,at} - y^l$.

Bounds for $E[Y\left(0,0\right)|at]$: A4 and A5.3 imply $\overline{Y}^{01} \ge E[Y\left(0,0\right)|at] \ge y^l$. Thus, $E[Y(0)|at] - E[Y(0,0)|at]$ in (19) and (20) has the following bounds $0 \le E[Y(0)|at] - E[Y(0,0)|at] \le \overline{Y}^{01} - y^l$.

Bounds on $\Delta(z)$. We now derive the bounds for $\Delta(z)$, for $z = 0, 1$, starting with the lower bound of $\Delta(1)$. We use equations (16) through (18) to derive potential lower bounds for $\Delta(1)$ by plugging in the appropriate bounds derived above into the terms that are not point identified. The corresponding three potential lower bounds are:

$$\begin{aligned}
LB^1_\alpha &= 0 \\
LB^1_\beta &= E[Y|Z=1] - E[Y|Z=0] - p_{1|0}(U^{1,at} - \overline{Y}^{01}) - p_{0|1}(\overline{Y}^{10} - L^{0,nt}) \\
&\quad -(p_{1|1} - p_{1|0})(U^{1,c} - L^{0,c}) \\
LB^1_\gamma &= p_{1|1}\overline{Y}^{11} - p_{0|1}\overline{Y}^{10} - p_{1|0}U^{1,at} + p_{0|1}\overline{Y}^{10} - (p_{1|1} - p_{1|0})U^{1,c}
\end{aligned}$$

$LB^1_\beta = p_{1|1}\overline{Y}^{11} - p_{0|0}\overline{Y}^{00} - p_{1|0}U^{1,at} - (p_{1|1} - p_{1|0})U^{1,c} + p_{0|1}L^{0,nt} + (p_{0|0} - p_{0|1})L^{0,c} = (p_{1|1} - p_{1|0})(L^{1,c} - U^{1,c}) + (p_{0|0} - p_{0|1})(L^{0,c} - U^{0,c}) \le 0$, where the second equality is obtained by $\pi_{nt}L^{0,nt} + \pi_c U^{0,c} = p_{0|0}\overline{Y}^{00}$, and $\pi_{at}U^{1,at} + \pi_c L^{1,c} = p_{1|1}\overline{Y}^{11}$.

$LB^1_\gamma = (p_{1|1} - p_{1|0})(L^{1,c} - U^{1,c}) \le 0$, where the equality follows $\pi_{at}U^{1,at} + \pi_c L^{1,c} = p_{1|1}\overline{Y}^{11}$.

Hence, the lower bound for $\Delta(1) = LB^1_\alpha = 0$.

Similarly, we use equations (19) through (21) to derive potential lower bounds for $\Delta(0)$:

$$\begin{aligned}
LB^0_\alpha &= 0 \\
LB^0_\beta &= E[Y|Z=1] - E[Y|Z=0] - p_{1|0}(U^{1,at} - \overline{Y}^{01}) - p_{0|1}(\overline{Y}^{10} - L^{0,nt}) \\
&\quad -(p_{1|1} - p_{1|0})(U^{1,c} - L^{0,c}) \\
LB^0_\gamma &= p_{1|0}\overline{Y}^{01} - p_{0|0}\overline{Y}^{00} - p_{1|0}\overline{Y}^{01} + p_{0|1}L^{0,nt} + (p_{1|1} - p_{1|0})L^{0,c}
\end{aligned}$$

After comparison, we find that $LB^0_\beta = LB^1_\beta \le 0$. Intuitively, this is because the parallel assumptions are imposed for $z = 0$, and $z = 1$, respectively.

$LB^0_\gamma = (p_{0|0} - p_{0|1})(L^{0,c} - U^{0,c}) \le 0$, where the equality follows $\pi_{nt}L^{0,nt} + \pi_c U^{0,c} = p_{0|0}\overline{Y}^{00}$.

Hence, the lower bound for $\Delta(0) = LB^0_\alpha = 0$.

We now use equations (16) through (18) to derive potential upper bounds for $\Delta(1)$:

$$\begin{aligned} UB^1_\alpha &= p_{1|0}(U^{1,at} - y^l) + p_{0|1}(y^u - \overline{Y}^{10}) + (p_{1|1} - p_{1|0})(U^{1,c} - L^{0,c}) \\ UB^1_\beta &= p_{1|0}(U^{1,at} - y^l) + p_{0|1}(y^u - \overline{Y}^{10}) + E[Y|Z=1] - E[Y|Z=0] \\ &\quad - p_{1|0}\max\{0, L^{1,at} - \overline{Y}^{01}\} - p_{0|1}\max\{0, \overline{Y}^{10} - U^{0,nt}\} \\ UB^1_\gamma &= p_{1|1}\overline{Y}^{11} - p_{0|1}\overline{Y}^{10} - p_{1|0}y^l + p_{0|1}y^u - (p_{1|1} - p_{1|0})L^{0,c} \end{aligned}$$

To compare the above upper bounds, we obtain their difference.
$UB^1_\beta - UB^1_\alpha = E[Y|Z=1] - E[Y|Z=0] - p_{1|0}\max\{0, L^{1,at} - \overline{Y}^{01}\} - p_{0|1}\max\{0, \overline{Y}^{10} - U^{0,nt}\} - (p_{1|1} - p_{1|0})U^{1,c} + (p_{0|0} - p_{0|1})L^{0,c} = p_{0|1}[(\overline{Y}^{10} - U^{0,nt}) - \max\{0, \overline{Y}^{10} - U^{0,nt}\}] + p_{1|0}[(L^{1,at} - \overline{Y}^{01}) - \max\{0, L^{1,at} - \overline{Y}^{01}\}] = p_{0|1}\min\{\overline{Y}^{10} - U^{0,nt}, 0\} + p_{1|0}\min\{L^{1,at} - \overline{Y}^{01}, 0\} \leq 0$. The second equality is derived by the following equations: $E[Y|Z=z] = p_{1|z}\overline{Y}^{z1} + p_{0|z}\overline{Y}^{z0}$, for $z = 0, 1$, and $\pi_{nt}U^{0,nt} + \pi_c L^{0,c} = p_{0|0}\overline{Y}^{00}$, $\pi_{at}L^{1,at} + \pi_c U^{1,c} = p_{1|1}\overline{Y}^{11}$. Thus, UB^1_β dominates UB^1_α.
$UB^1_\beta - UB^1_\gamma = p_{1|0}U^{1,at} - p_{1|0}\overline{Y}^{01} - p_{0|1}U^{0,nt} + p_{0|1}\overline{Y}^{10} - p_{1|0}\max\{0, L^{1,at} - \overline{Y}^{01}\} - p_{0|1}\max\{0, \overline{Y}^{10} - U^{0,nt}\} = p_{1|0}[(U^{1,at} - \overline{Y}^{01}) - \max\{0, L^{1,at} - \overline{Y}^{01}\}] + p_{0|1}[(\overline{Y}^{10} - U^{0,nt}) - \max\{0, \overline{Y}^{10} - U^{0,nt}\}] = p_{1|0}\min\{U^{1,at} - \overline{Y}^{01}, U^{1,at} - L^{1,at}\} + p_{0|1}\min\{\overline{Y}^{10} - U^{0,nt}, 0\}$. Because $U^{1,at} \geq E[Y(1)|at] \geq E[Y(0)|at] = \overline{Y}^{01}$, the first component involving min. operator is non-negative, while the second component is non-positive. Thus, the upper bound for $\Delta(1) = \min\{UB^1_\beta, UB^1_\gamma\}$.
We now use equations (19) through (21) to derive potential upper bounds for $\Delta(0)$:

$$\begin{aligned} UB^0_\alpha &= p_{1|0}(\overline{Y}^{01} - y^l) + p_{0|1}(y^u - L^{0,nt}) + (p_{1|1} - p_{1|0})(U^{1,c} - L^{0,c}) \\ UB^0_\beta &= p_{1|0}(\overline{Y}^{01} - y^l) + p_{0|1}(y^u - L^{0,nt}) + E[Y|Z=1] - E[Y|Z=0] \\ &\quad - p_{1|0}\max\{0, L^{1,at} - \overline{Y}^{01}\} - p_{0|1}\max\{0, \overline{Y}^{10} - U^{0,nt}\} \\ UB^0_\gamma &= p_{1|0}\overline{Y}^{01} - p_{0|0}\overline{Y}^{00} - p_{1|0}y^l + p_{0|1}y^u + (p_{1|1} - p_{1|0})U^{1,c} \end{aligned}$$

$UB^0_\beta - UB^0_\alpha = E[Y|Z=1] - E[Y|Z=0] - p_{1|0}\max\{0, L^{1,at} - \overline{Y}^{01}\} - p_{0|1}\max\{0, \overline{Y}^{10} - U^{0,nt}\} - (p_{1|1} - p_{1|0})(U^{1,c} - L^{0,c}) = p_{0|1}\overline{Y}^{10} + p_{1|0}L^{1,at} - p_{1|0}\overline{Y}^{01} - p_{0|1}U^{0,nt} - p_{1|0}\max\{0, L^{1,at} - \overline{Y}^{01}\} - p_{0|1}\max\{0, \overline{Y}^{10} - U^{0,nt}\} = p_{0|1}[(\overline{Y}^{10} - U^{0,nt}) - \max\{0, \overline{Y}^{10} - U^{0,nt}\}] + p_{1|0}[(L^{1,at} - \overline{Y}^{01}) - \max\{0, L^{1,at} - \overline{Y}^{01}\}] = p_{0|1}\min\{\overline{Y}^{10} - U^{0,nt}, 0\} + p_{1|0}\min\{L^{1,at} - \overline{Y}^{01}, 0\} \leq 0$. The second equality is derived by the following equations: $E[Y|Z=z] = p_{1|z}\overline{Y}^{z1} + p_{0|z}\overline{Y}^{z0}$, for $z = 0, 1$, and $\pi_{nt}U^{0,nt} + \pi_c L^{0,c} = p_{0|0}\overline{Y}^{00}$, $\pi_{at}L^{1,at} + \pi_c U^{1,c} = p_{1|1}\overline{Y}^{11}$. Thus, UB^0_β dominates UB^0_α.
$UB^0_\beta - UB^0_\gamma = -p_{0|1}L^{0,nt} + p_{0|1}\overline{Y}^{10} + p_{1|0}L^{1,at} - p_{1|0}\overline{Y}^{01} - p_{1|0}\max\{0, L^{1,at} - \overline{Y}^{01}\} - p_{0|1}\max\{0, \overline{Y}^{10} - U^{0,nt}\} = p_{1|0}[(L^{1,at} - \overline{Y}^{01}) - \max\{0, L^{1,at} - \overline{Y}^{01}\}] + p_{0|1}[(\overline{Y}^{10} - L^{0,nt}) - \max\{0, \overline{Y}^{10} - U^{0,nt}\}] =$

$p_{1|0}\min\{L^{1,at}-\overline{Y}^{01},0\}+p_{0|1}\min\{\overline{Y}^{10}-L^{0,nt},U^{0,nt}-L^{0,nt}\}$. Because $\overline{Y}^{10}=E[Y(1)|nt]\geq E[Y(0)|nt]\geq L^{0,nt}$, the first component involving min. operator is non-positive, while the second component is non-negative. Thus, the upper bound for $\Delta(0)=\min\{UB^0_\beta,UB^0_\gamma\}$.
Finally, the bounds for $E[\Delta(z)]$ are obtained by directly plugging the corresponding terms into the equation $\Pr(Z=1)\Delta(1)+\Pr(Z=0)\Delta(0)$.

Bounds on ATT**.** We now derive the bounds on the ATT under the same set of the assumptions. Because $ATT=\frac{w_1}{r_1}\Gamma(1)+\frac{w_0}{r_1}\Gamma(0)$, with $\Gamma(1)=p_{1|1}\overline{Y}^{11}-\pi_{at}E[Y(1,0)|at]-\pi_cE[Y(1,D_0)|c]$ and $\Gamma(0)=p_{1|0}(\overline{Y}^{01}-E[Y(0,0)|at])$. We start by deriving bounds on $\Gamma(0)$, and then derive bounds on $\Gamma(1)$ according to equations (13), (14) and (15).
Under A4 and A5.3, $0\leq\Gamma(0)\leq p_{1|0}(\overline{Y}^{01}-y^l)$. The lower bounds on $\Gamma(1)$ are:

$$
\begin{aligned}
lb^1_\alpha &= 0\\
lb^1_\beta &= p_{1|0}(\overline{Y}^{01}-U^{1,at})+E[Y|Z=1]-E[Y|Z=0]\\
&\quad -p_{0|1}(\overline{Y}^{10}-L^{0,nt})-(p_{1|1}-p_{1|0})(U^{1,c}-L^{0,c})\\
lb^1_\gamma &= p_{1|1}\overline{Y}^{11}-p_{1|0}U^{1,at}-(p_{1|1}-p_{1|0})U^{1,c}
\end{aligned}
$$

After arrangement, $lb^1_\beta=(p_{1|1}\overline{Y}^{11}-p_{1|0}U^{1,at}-\pi_cU^{1,c})+(p_{0|1}L^{0,nt}+\pi_cL^{0,c}-p_{0|0}\overline{Y}^{00})=\pi_c(L^{1,c}-U^{1,c})+\pi_c(L^{0,c}-U^{0,c})\leq 0$. $lb^1_\gamma=\pi_c(L^{1,c}-U^{1,c})\leq 0$. Thus, $lb^1=lb^1_\alpha=0$. The upper bounds on $\Gamma(1)$ are:

$$
\begin{aligned}
ub^1_\alpha &= p_{1|0}(U^{1,at}-y^l)+(p_{1|1}-p_{1|0})(U^{1,c}-L^{0,c})\\
ub^1_\beta &= p_{1|0}(\overline{Y}^{01}-y^l)+E[Y|Z=1]-E[Y|Z=0]-p_{0|1}\max\{0,\overline{Y}^{10}-U^{0,nt}\}\\
ub^1_\gamma &= p_{1|1}\overline{Y}^{11}-p_{1|0}y^l-(p_{1|1}-p_{1|0})L^{0,c}
\end{aligned}
$$

$ub^1_\alpha-ub^1_\gamma=p_{1|0}U^{1,at}+(p_{1|1}-p_{1|0})U^{1,c}-p_{1|1}\overline{Y}^{11}=p_{1|0}(U^{1,at}-L^{1,at})\geq 0$.
$ub^1_\beta-ub^1_\gamma=-p_{0|0}\overline{Y}^{00}+p_{0|1}\overline{Y}^{10}+(p_{1|1}-p_{1|0})L^{0,c}-p_{0|1}\max\{0,\overline{Y}^{10}-U^{0,nt}\}=p_{0|1}[(\overline{Y}^{10}-U^{0,nt})-\max\{0,\overline{Y}^{10}-U^{0,nt}\}]$. If $\overline{Y}^{10}\geq U^{0,nt}$, $ub^1_\beta=ub^1_\gamma$; if $\overline{Y}^{10}\leq U^{0,nt}$, $ub^1_\beta-ub^1_\gamma=p_{0|1}(\overline{Y}^{10}-U^{0,nt})\leq 0$. Thus, $ub^1=ub^1_\beta$. Specifically, if $\overline{Y}^{10}\geq U^{0,nt}$, $ub^1=ub^1_\beta=ub^1_\gamma$; if $\overline{Y}^{10}\leq U^{0,nt}$, $ub^1=ub^1_\beta=p_{1|0}(\overline{Y}^{01}-y^l)+E[Y|Z=1]-E[Y|Z=0]$.
According to $ATT=\frac{w_1}{r_1}\Gamma(1)+\frac{w_0}{r_1}\Gamma(0)$, $0\leq ATT\leq\frac{w_1}{r_1}ub^1_\beta+\frac{w_0p_{1|0}}{r_1}(\overline{Y}^{01}-y^l)$. After rearrangement, we have $ub_a=\frac{w_1}{r_1}ub^1_\beta+\frac{w_0p_{1|0}}{r_1}(\overline{Y}^{01}-y^l)$ if $\overline{Y}^{10}\leq U^{0,nt}$, and $ub_b=\frac{w_1}{r_1}ub^1_\gamma+\frac{w_0p_{1|0}}{r_1}(\overline{Y}^{01}-y^l)$ if $\overline{Y}^{10}\geq U^{0,nt}$. Q.D.E

5.2.2 Proof of Proposition 3

As before, we first derive bounds for the non-point identified mean potential outcomes of the strata, and for all the local net and mechanism average treatment effects.

Bounds for $E[Y(0)|nt]$: A6.2 implies $\overline{Y}^{01} = E[Y(0)|at] \geq E[Y(0)|nt]$. A6.2 and the equation $\pi_{nt}E[Y(0)|nt] + \pi_c E[Y(0)|c] = p_{0|0}\overline{Y}^{00}$ imply $E[Y(0)|nt] \leq \overline{Y}^{00}$. Since $\overline{Y}^{00} \leq U^{0,nt}$ by definition and $\overline{Y}^{00} \leq \overline{Y}^{01}$ by A6.2 , the upper bound is $\overline{Y}^{00}$. A6 does not provide any additional information for a lower bound of $E[Y(0)|nt]$. Thus, $L^{0,nt} \leq E[Y(0)|nt] \leq \overline{Y}^{00}$.
Bounds for $E[Y(1)|at]$: A6.2 implies $E[Y(1)|at] \geq E[Y(1)|nt] = \overline{Y}^{10}$. A6.2 and the equation $\pi_{at}E[Y(1)|at] + \pi_c E[Y(1)|c] = p_{1|1}\overline{Y}^{11}$ yield $E[Y(1)|at] \geq \overline{Y}^{11}$. Since $\overline{Y}^{11} \geq L^{1,at}$ by definition and $\overline{Y}^{11} \geq \overline{Y}^{10}$ by A 6.2 , the lower bound is $\overline{Y}^{11}$. A6 does not provide any additional information for an upper bound of $E[Y(1)|at]$. Thus, $\overline{Y}^{11} \leq E[Y(1)|at] \leq U^{1,at}$.
Bounds for $E[Y(0)|c]$: A6.2 and the equation $\pi_{nt}E[Y(0)|nt] + \pi_c E[Y(0)|c] = p_{0|0}\overline{Y}^{00}$ yield $E[Y(0)|c] \geq \overline{Y}^{00}$, where by definition $\overline{Y}^{00} \geq L^{0,c}$. As for the upper bound, A6.2 implies $E[Y(0)|c] \leq E[Y(0)|at] = \overline{Y}^{01}$, which can be greater or less than $U^{0,c}$. Thus, $\overline{Y}^{00} \leq E[Y(0)|c] \leq \min\{U^{0,c}, \overline{Y}^{01}\}$.
Bounds for $E[Y(1)|c]$: A6.2 and the equation $\pi_{at}E[Y(1)|at] + \pi_c E[Y(1)|c] = p_{1|1}\overline{Y}^{11}$ yield $E[Y(1)|c] \leq \overline{Y}^{11}$, where by definition $U^{1,c} \geq \overline{Y}^{11}$. As for the lower bound, A6.2 implies $E[Y(1)|c] \geq E[Y(1)|nt] = \overline{Y}^{10}$, which can be greater or less than $L^{1,c}$. Thus, $\max\{L^{1,c}, \overline{Y}^{10}\} \leq E[Y(1)|c] \leq \overline{Y}^{11}$.
Bounds for $E[Y(1, D_0)|c]$: A6.1 implies $E[Y(1, D_0)|c] \geq E[Y(1)|nt] = \overline{Y}^{10}$. Combining with the bounds previously derived for $E[Y(1)|at]$ yields $E[Y(1, D_0)|c] \leq E[Y(1)|at] \leq U^{1,at}$. Hence, $\overline{Y}^{10} \leq E[Y(1, D_0)|c] \leq U^{1,at}$.
Bounds for $E[Y(0, D_1)|c]$: A6.1 implies $E[Y(0, D_1)|c] \leq E[Y(0)|at] = \overline{Y}^{01}$. Combining with the bounds previously derived for $E[Y(0)|nt]$ yields $E[Y(0, D_1)|c] \geq E[Y(0)|nt] \geq L^{0,nt}$. Hence, $L^{0,nt} \leq E[Y(0, D_1)|c] \leq \overline{Y}^{01}$.
Bounds for $LNATE^z_{nt}$, for $z = 0, 1$: From (23), $LNATE^z_{nt} = \overline{Y}^{10} - E[Y(0)|nt]$. Using the bounds previously derived for $E[Y(0)|nt]$: $\overline{Y}^{10} - \overline{Y}^{00} \leq LNATE^z_{nt} \leq \overline{Y}^{10} - L^{0,nt}$, for $z = 0, 1$.
Bounds for $LNATE^z_{at}$, for $z = 0, 1$: From (23), $LNATE^z_{at} = E[Y(1)|at] - \overline{Y}^{01}$. Using the bounds previously derived for $E[Y(1)|at]$: $\overline{Y}^{11} - \overline{Y}^{01} \leq LNATE^z_{at} \leq U^{1,at} - \overline{Y}^{01}$, for $z = 0, 1$.
Bounds for $LNATE^0_c$: From (23), $LNATE^0_c = E[Y(1, D_0)|c] - E[Y(0)|c]$. Using the bounds previously obtained for the components in $LNATE^0_c$, we obtain $\overline{Y}^{10} - \min\{U^{0,c}, \overline{Y}^{01}\} \leq LNATE^0_c \leq U^{1,at} - \overline{Y}^{00}$.
Bounds for $LNATE^1_c$: From (23), $LNATE^1_c = E[Y(1)|c] - E[Y(0, D_1)|c]$. Using the bounds previously derived for the components of $LNATE^1_c$, we have $\max\{L^{1,c}, \overline{Y}^{10}\} - \overline{Y}^{01} \leq LNATE^1_c \leq \overline{Y}^{11} - L^{0,nt}$.
Bounds for $LMATE^1_c$: $LMATE^1_c = E[Y(1)|c] - E[Y(1, D_0)|c]$. Using the bounds previously derived for the components of $LMATE^1_c$, we have $\max\{L^{1,c}, \overline{Y}^{10}\} - U^{1,at} \leq LMATE^1_c \leq \overline{Y}^{11} - \overline{Y}^{10}$.
Bounds for $LMATE^0_c$: $LMATE^0_c = E[Y(0, D_1)|c] - E[Y(0)|c]$. Using the bounds previously derived for the components of $LMATE^0_c$, we have

$L^{0,nt} - \min\{U^{0,c}, \overline{Y}^{01}\} \leq LMATE_c^0 \leq \overline{Y}^{01} - \overline{Y}^{00}$.

Bounds for $E[Y(1,1)|nt]$: A4 and A6.3 imply $E[Y(1)|c] \geq E[Y(1,1)|nt] \geq y^l$. Combining with the bounds previously derived for $E[Y(1)|c]$ yields $\overline{Y}^{11} \geq E[Y(1,1)|nt] \geq y^l$. And thus $E[Y(1,1)|nt] - E[Y(1)|nt]$ in (16) and (17) has the following bounds

$y^l - \overline{Y}^{10} \leq E[Y(1,1)|nt] - E[Y(1)|nt] \leq \overline{Y}^{11} - \overline{Y}^{10}$.

Bounds for $E[Y(0,1)|nt]$: A4 and A6.3 imply $E[Y(0,D_1)|c] \geq E[Y(0,1)|nt] \geq y^l$. Combining with the bounds previously derived for $E[Y(0,D_1)|c]$ yields $\overline{Y}^{01} \geq E[Y(0,1)|nt] \geq y^l$. Using the bounds previously derived for $E[Y(0)|nt]$, we have $E[Y(0,1)|nt] - E[Y(0)|nt]$ in (19) and (20) has the following bounds $y^l - \overline{Y}^{00} \leq E[Y(0,1)|nt] - E[Y(0)|nt] \leq \overline{Y}^{01} - L^{0,nt}$.

Bounds for $E[Y(1,0)|at]$: A4 and A6.3 imply $y^u \geq E[Y(1,0)|at] \geq E[Y(1,D_0)|c]$. Combining with the bounds previously derived for $E[Y(1,D_0)|c]$ yields $y^u \geq E[Y(1,0)|at] \geq \overline{Y}^{10}$. Using the bounds previously derived for $E[Y(1)|at]$, we have $E[Y(1)|at] - E[Y(1,0)|at]$ in (16) and (17) has the following bounds $\overline{Y}^{11} - y^u \leq E[Y(1)|at] - E[Y(1,0)|at] \leq U^{1,at} - \overline{Y}^{10}$.

Bounds for $E[Y(0,0)|at]$: A4 and A6.3 imply $y^u \geq E[Y(0,0)|at] \geq E[Y(0)|c]$. Combining with the bounds previously derived for $E[Y(0)|c]$ yields $y^u \geq E[Y(0,0)|at] \geq \overline{Y}^{00}$. And thus $E[Y(0)|at] - E[Y(0,0)|at]$ in (19) and (20) has the following bounds

$\overline{Y}^{01} - y^u \leq E[Y(0)|at] - E[Y(0,0)|at] \leq \overline{Y}^{01} - \overline{Y}^{00}$.

Bounds on $\Delta(z)$. We now derive the lower bound of $\Delta(1)$ by the use of the equations (16) through (18). The corresponding three potential lower bounds are:

$$\begin{aligned}
LB_\alpha^1 &= p_{1|0}(\overline{Y}^{11} - y^u) + p_{0|1}(y^l - \overline{Y}^{10}) + (p_{1|1} - p_{1|0})(\max\{L^{1,c}, \overline{Y}^{10}\} - U^{1,at}) \\
LB_\beta^1 &= p_{1|0}(\overline{Y}^{11} - y^u) + p_{0|1}(y^l - \overline{Y}^{10}) + E[Y|Z=1] - E[Y|Z=0] \\
&\quad - p_{1|0}(U^{1,at} - \overline{Y}^{01}) - p_{0|1}(\overline{Y}^{10} - L^{0,nt}) - (p_{1|1} - p_{1|0})(U^{1,at} - \overline{Y}^{00}) \\
LB_\gamma^1 &= p_{1|1}\overline{Y}^{11} - p_{0|1}\overline{Y}^{10} - p_{1|0}y^u + p_{0|1}y^l - (p_{1|1} - p_{1|0})U^{1,at}
\end{aligned}$$

After some algebra, we have

$LB_\beta^1 - LB_\alpha^1 = E[Y|Z=1] - E[Y|Z=0] - p_{1|0}(U^{1,at} - \overline{Y}^{01}) - p_{0|1}(\overline{Y}^{10} - L^{0,nt}) + (p_{1|1} - p_{1|0})(\overline{Y}^{00} - \max\{L^{1,c}, \overline{Y}^{10}\}) = (p_{1|1} - p_{1|0})(L^{1,c} - U^{0,c} + \overline{Y}^{00} - \max\{L^{1,c}, \overline{Y}^{10}\}) = (p_{1|1} - p_{1|0})[(\overline{Y}^{00} - U^{0,c}) + (L^{1,c} - \max\{L^{1,c}, \overline{Y}^{10}\})] \leq 0$. The second equality is derived by the following equations: $E[Y|Z=z] = p_{1|z}\overline{Y}^{z1} + p_{0|z}\overline{Y}^{z0}$, for $z = 0, 1$, and $\pi_{nt}L^{0,nt} + \pi_c U^{0,c} = p_{0|0}\overline{Y}^{00}$, and $\pi_{at}U^{1,at} + \pi_c L^{1,c} = p_{1|1}\overline{Y}^{11}$. The inequality is because $\overline{Y}^{00} \leq U^{0,c}$ and $L^{1,c} \leq \max\{L^{1,c}, \overline{Y}^{10}\}$ by definition. Hence, LB_α^1 dominates LB_β^1.

$LB_\gamma^1 - LB_\alpha^1 = p_{1|1}\overline{Y}^{11} - p_{1|0}\overline{Y}^{11} - (p_{1|1} - p_{1|0})\max\{L^{1,c}, \overline{Y}^{10}\} = (p_{1|1} - p_{1|0})(\overline{Y}^{11} - \max\{L^{1,c}, \overline{Y}^{10}\}) \geq 0$. The inequality is because $\overline{Y}^{11} \geq L^{1,c}$ by definition and $\overline{Y}^{11} \geq \overline{Y}^{10}$ by A6.2. Hence, LB_γ^1 dominates LB_α^1. Therefore, the lower bound for $\Delta(1) = LB_\gamma^1 = p_{1|1}(\overline{Y}^{11} - U^{1,at}) + p_{0|1}(y^l - \overline{Y}^{10}) + p_{1|0}(U^{1,at} - y^u)$.

Similarly, we use equations (19) through (21) to derive potential lower bounds for $\Delta(0)$:

$$\begin{aligned}
LB^0_\alpha &= p_{1|0}(\overline{Y}^{01} - y^u) + p_{0|1}(y^l - \overline{Y}^{00}) + (p_{1|1} - p_{1|0})(L^{0,nt} - \min\{U^{0,c}, \overline{Y}^{01}\}) \\
LB^0_\beta &= p_{1|0}(\overline{Y}^{01} - y^u) + p_{0|1}(y^l - \overline{Y}^{00}) + E[Y|Z=1] - E[Y|Z=0] \\
&\quad -p_{1|0}(U^{1,at} - \overline{Y}^{01}) - p_{0|1}(\overline{Y}^{10} - L^{0,nt}) - (p_{1|1} - p_{1|0})(\overline{Y}^{11} - L^{0,nt}) \\
LB^0_\gamma &= p_{1|0}\overline{Y}^{01} - p_{0|0}\overline{Y}^{00} - p_{1|0}y^u + p_{0|1}y^l + (p_{1|1} - p_{1|0})L^{0,nt}
\end{aligned}$$

After some algebra, we have
$LB^0_\beta - LB^0_\alpha = E[Y|Z=1] - E[Y|Z=0] - p_{1|0}(U^{1,at} - \overline{Y}^{01}) - p_{0|1}(\overline{Y}^{10} - L^{0,nt}) + (p_{1|1} - p_{1|0})(\min\{U^{0,c}, \overline{Y}^{01}\} - \overline{Y}^{11}) = (p_{1|1} - p_{1|0})(L^{1,c} - U^{0,c} + \min\{U^{0,c}, \overline{Y}^{01}\} - \overline{Y}^{11}) = (p_{1|1} - p_{1|0})[(\min\{U^{0,c}, \overline{Y}^{01}\} - U^{0,c}) + (L^{1,c} - \overline{Y}^{11})] \leq 0$. The second equality is derived by the following equations: $E[Y|Z=z] = p_{1|z}\overline{Y}^{z1} + p_{0|z}\overline{Y}^{z0}$, for $z = 0, 1$, and $\pi_{nt}L^{0,nt} + \pi_c U^{0,c} = p_{0|0}\overline{Y}^{00}$, and $\pi_{at}U^{1,at} + \pi_c L^{1,c} = p_{1|1}\overline{Y}^{11}$. The inequality is because $L^{1,c} \leq \overline{Y}^{11}$ and $\min\{U^{0,c}, \overline{Y}^{01}\} \leq U^{0,c}$ by definition. Hence, LB^0_α dominates LB^0_β.
$LB^0_\gamma - LB^0_\alpha = p_{0|1}\overline{Y}^{00} - p_{0|0}\overline{Y}^{00} + (p_{1|1} - p_{1|0})\min\{U^{0,c}, \overline{Y}^{01}\} = (p_{1|1} - p_{1|0})(\min\{U^{0,c}, \overline{Y}^{01}\} - \overline{Y}^{00}) \geq 0$. The inequality is because $U^{0,c} \geq \overline{Y}^{00}$ by definition and $\overline{Y}^{01} \geq \overline{Y}^{00}$ by A6.2. Hence, LB^0_γ dominates LB^0_α. Therefore, the lower bound for $\Delta(0) = LB^0_\gamma = p_{1|0}(\overline{Y}^{01} - y^u) + p_{0|0}(L^{0,nt} - \overline{Y}^{00}) + p_{0|1}(y^l - L^{0,nt})$.
We now use equations (16) through (18) to derive potential upper bounds for $\Delta(1)$:

$$\begin{aligned}
UB^1_\alpha &= p_{1|0}(U^{1,at} - \overline{Y}^{10}) + p_{0|1}(\overline{Y}^{11} - \overline{Y}^{10}) + (p_{1|1} - p_{1|0})(\overline{Y}^{11} - \overline{Y}^{10}) \\
UB^1_\beta &= p_{1|0}(U^{1,at} - \overline{Y}^{10}) + p_{0|1}(\overline{Y}^{11} - \overline{Y}^{10}) + E[Y|Z=1] - E[Y|Z=0] \\
&\quad -p_{1|0}(\overline{Y}^{11} - \overline{Y}^{01}) - p_{0|1}(\overline{Y}^{10} - \overline{Y}^{00}) - (p_{1|1} - p_{1|0})(\overline{Y}^{10} - \min\{U^{0,c}, \overline{Y}^{01}\}) \\
UB^1_\gamma &= p_{1|1}\overline{Y}^{11} - p_{0|1}\overline{Y}^{10} - p_{1|0}\overline{Y}^{10} + p_{0|1}\overline{Y}^{11} - (p_{1|1} - p_{1|0})\overline{Y}^{10}
\end{aligned}$$

$UB^1_\beta - UB^1_\alpha = E[Y|Z=1] - E[Y|Z=0] - p_{1|0}(\overline{Y}^{11} - \overline{Y}^{01}) - p_{0|1}(\overline{Y}^{10} - \overline{Y}^{00}) + (p_{1|1} - p_{1|0})(\min\{U^{0,c}, \overline{Y}^{01}\} - \overline{Y}^{11}) = p_{1|1}\overline{Y}^{11} - p_{1|0}\overline{Y}^{11} - p_{0|0}\overline{Y}^{00} + p_{0|1}\overline{Y}^{00} + (p_{1|1} - p_{1|0})(\min\{U^{0,c}, \overline{Y}^{01}\} - \overline{Y}^{11}) = (p_{1|1} - p_{1|0})(\min\{U^{0,c}, \overline{Y}^{01}\} - \overline{Y}^{00}) \geq 0$.
The second equality is derived by the equations: $E[Y|Z=z] = p_{1|z}\overline{Y}^{z1} + p_{0|z}\overline{Y}^{z0}$, for $z = 0, 1$. The inequality is because $U^{0,c} \geq \overline{Y}^{00}$ by definition and $\overline{Y}^{01} \geq \overline{Y}^{00}$ by A6.2. Thus, UB^1_α dominates UB^1_β. $UB^1_\gamma - UB^1_a = p_{1|1}\overline{Y}^{11} - p_{1|0}U^{1,at} - (p_{1|1} - p_{1|0})\overline{Y}^{11} = p_{1|0}(\overline{Y}^{11} - U^{1,at}) \leq 0$, where the inequality follows the definition $\overline{Y}^{11} \leq U^{1,at}$. Thus, UB^1_γ dominates UB^1_a. Therefore, the upper bound for $\Delta(1) = UB^1_\gamma = \overline{Y}^{11} - p_{0|1}\overline{Y}^{10} - p_{1|1}\overline{Y}^{10} = \overline{Y}^{11} - \overline{Y}^{10}$.
We now use equations (19) through (21) to derive potential upper bounds for $\Delta(0)$:

$$\begin{aligned}
UB^0_\alpha &= p_{1|0}(\overline{Y}^{01} - \overline{Y}^{00}) + p_{0|1}(\overline{Y}^{01} - L^{0,nt}) + (p_{1|1} - p_{1|0})(\overline{Y}^{01} - \overline{Y}^{00}) \\
UB^0_\beta &= p_{1|0}(\overline{Y}^{01} - \overline{Y}^{00}) + p_{0|1}(\overline{Y}^{01} - L^{0,nt}) + E[Y|Z=1] - E[Y|Z=0] \\
&\quad -p_{1|0}(\overline{Y}^{11} - \overline{Y}^{01}) - p_{0|1}(\overline{Y}^{10} - \overline{Y}^{00}) - (p_{1|1} - p_{1|0})(\max\{L^{1,c}, \overline{Y}^{10}\} - \overline{Y}^{01}) \\
UB^0_\gamma &= p_{1|0}\overline{Y}^{01} - p_{0|0}\overline{Y}^{00} - p_{1|0}\overline{Y}^{00} + p_{0|1}\overline{Y}^{01} + (p_{1|1} - p_{1|0})\overline{Y}^{01}
\end{aligned}$$

$UB^0_\beta - UB^0_\alpha = E[Y|Z=1] - E[Y|Z=0] - p_{1|0}(\overline{Y}^{11} - \overline{Y}^{01}) - p_{0|1}(\overline{Y}^{10} - \overline{Y}^{00}) + (p_{1|1} - p_{1|0})(\overline{Y}^{00} - \max\{L^{1,c}, \overline{Y}^{10}\}) = p_{1|1}\overline{Y}^{11} - p_{0|0}\overline{Y}^{00} - p_{1|0}\overline{Y}^{11} + p_{0|1}\overline{Y}^{00} + (p_{1|1} - p_{1|0})(\overline{Y}^{00} - \max\{L^{1,c}, \overline{Y}^{10}\}) = (p_{1|1} - p_{1|0})(\overline{Y}^{11} - \max\{L^{1,c}, \overline{Y}^{10}\}) \geq 0$. The second equality is derived by the equations: $E[Y|Z=z] = p_{1|z}\overline{Y}^{z1} + p_{0|z}\overline{Y}^{z0}$, for $z = 0, 1$. The inequality is because $\overline{Y}^{11} \geq L^{1,c}$ by definition and $\overline{Y}^{11} \geq \overline{Y}^{10}$ by A6.2. Thus, UB^0_α dominates UB^0_β.
$UB^0_\gamma - UB^0_a = p_{0|1}L^{0,nt} - p_{0|0}\overline{Y}^{00} + (p_{0|0} - p_{0|1})\overline{Y}^{00} = p_{0|1}(L^{0,nt} - \overline{Y}^{00}) \leq 0$, where the inequality follows the definition $L^{0,nt} \leq \overline{Y}^{00}$. Thus, UB^0_γ dominates UB^0_a. Therefore, the upper bound for $\Delta(0) = UB^0_\gamma = p_{0|1}\overline{Y}^{01} + p_{1|1}\overline{Y}^{01} - \overline{Y}^{00} = \overline{Y}^{01} - \overline{Y}^{00}$.
Finally, the bounds for $E[\Delta(z)]$ are obtained by directly plugging the corresponding terms into the equation $\Pr(Z=1)\Delta(1) + \Pr(Z=0)\Delta(0)$.

Bounds on ATT. Under A4 and A6.3, $p_{1|0}(\overline{Y}^{01} - y^u) \leq \Gamma(0) \leq p_{1|0}(\overline{Y}^{01} - \overline{Y}^{00})$. The lower bounds on $\Gamma(1)$ are:

$$\begin{aligned}
lb^1_\alpha &= p_{1|0}(\overline{Y}^{11} - y^u) + (p_{1|1} - p_{1|0})(\max\{L^{1,c}, \overline{Y}^{10}\} - U^{1,at}) \\
lb^1_\beta &= p_{1|0}(\overline{Y}^{01} - y^u) + E[Y|Z=1] - E[Y|Z=0] \\
&\quad - p_{0|1}(\overline{Y}^{10} - L^{0,nt}) - (p_{1|1} - p_{1|0})(U^{1,at} - \overline{Y}^{00}) \\
lb^1_\gamma &= p_{1|1}\overline{Y}^{11} - p_{1|0}y^u - (p_{1|1} - p_{1|0})U^{1,at}
\end{aligned}$$

$lb^1_\alpha - lb^1_\gamma = \pi_{at}\overline{Y}^{11} + \pi_c \max\{L^{1,c}, \overline{Y}^{10}\} - p_{1|1}\overline{Y}^{11} = \pi_c(\max\{L^{1,c}, \overline{Y}^{10}\} - \overline{Y}^{11}) \leq 0$, by $E[Y(1)|nt] \leq \overline{Y}^{11}$, and $L^{1,c} \leq \frac{\pi_{at}}{p_{1|1}}U^{1,at} + \frac{\pi_c}{p_{1|1}}L^{1,c} = \overline{Y}^{11}$.
$lb^1_\beta - lb^1_\gamma = -p_{0|0}\overline{Y}^{00} + \pi_{nt}L^{0,nt} + \pi_c\overline{Y}^{00} = \pi_{nt}(L^{0,nt} - \overline{Y}^{00}) \leq 0$, by $L^{0,nt} \leq \frac{\pi_c}{p_{0|0}}U^{0,c} + \frac{\pi_{nt}}{p_{0|0}}L^{0,nt} = \overline{Y}^{00}$. Thus, $lb^1 = lb^1_\gamma$. The upper bounds on $\Gamma(1)$ are:

$$\begin{aligned}
ub^1_\alpha &= p_{1|0}(U^{1,at} - \overline{Y}^{10}) + (p_{1|1} - p_{1|0})(\overline{Y}^{11} - \overline{Y}^{10}) \\
ub^1_\beta &= p_{1|0}(\overline{Y}^{01} - \overline{Y}^{10}) + E[Y|Z=1] - E[Y|Z=0] \\
&\quad - p_{0|1}(\overline{Y}^{10} - \overline{Y}^{00}) - (p_{1|1} - p_{1|0})(\overline{Y}^{10} - \min\{U^{0,c}, \overline{Y}^{01}\}) \\
ub^1_\gamma &= p_{1|1}\overline{Y}^{11} - p_{1|0}\overline{Y}^{10} - (p_{1|1} - p_{1|0})\overline{Y}^{10}
\end{aligned}$$

$ub^1_\alpha - ub^1_\gamma = \pi_{at}U^{1,at} + \pi_c\overline{Y}^{11} - p_{1|1}\overline{Y}^{11} = \pi_{at}(U^{1,at} - \overline{Y}^{11}) \geq 0$, by $U^{1,at} \geq E[Y(1)|at] \geq E[Y(1)|c]$.
$ub^1_\beta - ub^1_\gamma = -p_{0|0}\overline{Y}^{00} + \pi_{nt}\overline{Y}^{00} + \pi_c \min\{U^{0,c}, \overline{Y}^{01}\} = \pi_c(\min\{U^{0,c}, \overline{Y}^{01}\} - \overline{Y}^{00}) \geq 0$, by $\min\{U^{0.c}, E[Y(0)|at]\} \geq E[Y(0)|c] \geq E[Y(0)|nt]$. Thus, $ub^1 = ub^1_\gamma$. According to $ATT = \frac{w_1}{r_1}\Gamma(1) + \frac{w_0}{r_1}\Gamma(0)$, we have $lb = \frac{w_1}{r_1}lb^1_\gamma + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - y^u)$ and $ub = \frac{w_1}{r_1}ub^1_\gamma + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - \overline{Y}^{00})$. Q.D.E

5.2.3 Proof of Proposition 4

Bounds for $E[Y(0)|nt]$: A5.2 implies $E[Y(0)|nt] \leq E[Y(1)|nt] = \overline{Y}^{10}$; and A6.2 implies $E[Y(0)|nt] \leq \overline{Y}^{00}$ (see proof of Prop. 3), where by definition $U^{0,nt} \geq \overline{Y}^{00}$. Combining A5 and A6 does not yield any additional upper bound for $E[Y(0)|nt]$ that could be lower than $\overline{Y}^{00}$ or $\overline{Y}^{10}$.[19] By the equation $\pi_{nt}E[Y(0)|nt] + \pi_c E[Y(0)|c] = p_{0|0}\overline{Y}^{00}$ and A6.2 $E[Y(0)|nt] \leq E[Y(0)|c]$, we have that $E[Y(1)|nt]$ can be greater or less than $E[Y(0)|c]$. Hence, the upper bound for $E[Y(0)|nt]$ is $\min\{\overline{Y}^{10}, \overline{Y}^{00}\}$. A5 and A6 do not provide any additional information for a lower bound of $E[Y(0)|nt]$. Thus, $L^{0,nt} \leq E[Y(0)|nt] \leq \min\{\overline{Y}^{10}, \overline{Y}^{00}\}$.

Bounds for $E[Y(1)|at]$: A5.2 implies $E[Y(1)|at] \geq E[Y(0)|at] = \overline{Y}^{01}$; and A6.2 implies $E[Y(1)|at] \geq \overline{Y}^{11}$ (see proof of Prop. 3), where by definition $\overline{Y}^{11} \geq L^{1,at}$. Combining A5 and A6 does not yield any additional lower bound for $E[Y(1)|at]$ that could be greater than $\overline{Y}^{11}$ or $\overline{Y}^{01}$. By the equation $\pi_{at}E[Y(1)|at] + \pi_c E[Y(1)|c] = p_{1|1}\overline{Y}^{11}$ and A6.2 $E[Y(1)|at] \geq E[Y(1)|c]$, we have that $E[Y(0)|at]$ can be greater or less than $E[Y(1)|c]$. Hence, the lower bound for $E[Y(1)|at]$ is $\max\{\overline{Y}^{11}, \overline{Y}^{01}\}$. A5 and A6 do not provide any additional information for an upper bound of $E[Y(1)|at]$. Thus, $\max\{\overline{Y}^{11}, \overline{Y}^{01}\} \leq E[Y(1)|at] \leq U^{1,at}$.

Bounds for $E[Y(0)|c]$: A6.2 and the equation $\pi_{nt}E[Y(0)|nt] + \pi_c E[Y(0)|c] = p_{0|0}\overline{Y}^{00}$ yield $E[Y(0)|c] \geq \overline{Y}^{00}$, where by definition $\overline{Y}^{00} \geq L^{0,c}$. Regarding an upper bound, the trimming procedure implies $E[Y(0)|c] \leq U^{0,c}$. A6.2 implies $E[Y(0)|c] \leq E[Y(0)|at] = \overline{Y}^{01}$. Finally, A5 implies $E[Y(1)|c] \geq E[Y(0)|c]$. Below we show that the upper bound for $E[Y(1)|c]$ under A1, A3, A5 and A6 equals $\overline{Y}^{11}$, so $E[Y(0)|c] \leq \overline{Y}^{11}$. Depending on the data, any of the previous three upper bounds for $E[Y(0)|c]$ can be less than the other two. Thus, we obtain $\overline{Y}^{00} \leq E[Y(0)|c] \leq \min\{U^{0,c}, \overline{Y}^{01}, \overline{Y}^{11}\}$.

Bounds for $E[Y(1)|c]$: A6.2 and the equation $\pi_{at}E[Y(1)|at] + \pi_c E[Y(1)|c] = p_{1|1}\overline{Y}^{11}$ yield $E[Y(1)|c] \leq \overline{Y}^{11}$, where by definition $\overline{Y}^{11} \leq U^{1,c}$. Regarding a lower bound, the trimming procedure implies $E[Y(1)|c] \geq L^{1,c}$. A6.2 implies $E[Y(1)|c] \geq E[Y(1)|nt] = \overline{Y}^{10}$. Finally, A5 implies $E[Y(1)|c] \geq E[Y(0)|c]$. Above we showed that the lower bound for $E[Y(0)|c]$ under A1, A3, A5 and A6 equals $\overline{Y}^{00}$, so $E[Y(1)|c] \geq \overline{Y}^{00}$. Depending on the data, any of the previous three lower bounds for $E[Y(1)|c]$ can be greater than the other two. Thus, we obtain $\max\{L^{1,c}, \overline{Y}^{10}, \overline{Y}^{00}\} \leq E[Y(1)|c] \leq \overline{Y}^{11}$.

Bounds for $E[Y(1, D_0)|c]$: A5.2 implies $E[Y(1, D_0)|c] \geq E[Y(0)|c]$. From above, the lower bound for $E[Y(0)|c]$ equals $\overline{Y}^{00}$. A6.1 implies $E[Y(1, D_0)|c] \geq E[Y(1)|nt] = \overline{Y}^{10}$, which can be greater or less than $\overline{Y}^{00}$ (see above). Hence, $E[Y(1, D_0)|c] \geq \max\{\overline{Y}^{00}, \overline{Y}^{10}\}$. A5.1 implies

[19] For instance, combining A6.2 and 5.2 yields $E[Y(1)|at] \geq E[Y(0)|at] \geq E[Y(0)|c] \geq E[Y(0)|nt]$, which implies $E[Y(0)|at] = \overline{Y}^{01} \geq E[Y(0)|nt]$ and $U^{1,at} \geq E[Y(1)|at] \geq E[Y(0)|nt]$. However, we have that $\overline{Y}^{01} \geq \overline{Y}^{00}$ and $U^{1,at} \geq Y^{11} \geq \overline{Y}^{00}$.

$E[Y(1)|c] \geq E[Y(1,D_0)|c]$. From above, the upper bound for $E[Y(1)|c]$ equals $\overline{Y}^{11}$. Note that A6.1 implies $E[Y(1,D_0)|c] \leq E[Y(1)|at] \leq U^{1,at}$, but by definition $\overline{Y}^{11} \leq U^{1,at}$. Therefore, $\max\{\overline{Y}^{10}, \overline{Y}^{00}\} \leq E[Y(1,D_0)|c] \leq \overline{Y}^{11}$.

Bounds for $E[Y(0,D_1)|c]$: A5.2 implies $E[Y(0,D_1)|c] \leq E[Y(1)|c]$. From above, the upper bound for $E[Y(1)|c]$ equals $\overline{Y}^{11}$. A6.1 implies $E[Y(0,D_1)|c] \leq E[Y(0)|at] = \overline{Y}^{01}$, which can be greater or less than $\overline{Y}^{11}$ (see above). Hence, $E[Y(0,D_1)|c] \leq \min\{\overline{Y}^{11}, \overline{Y}^{01}\}$. A5.1 implies $E[Y(0,D_1)|c] \geq E[Y(0)|c]$. From above, the lower bound for $E[Y(0)|c]$ equals $\overline{Y}^{00}$. Note that A6.1 implies $E[Y(0,D_1)|c] \geq E[Y(0)|nt] \geq L^{0,nt}$, but by definition $\overline{Y}^{00} \geq L^{0,nt}$. Therefore, $\overline{Y}^{00} \leq E[Y(0,D_1)|c] \leq \min\{\overline{Y}^{11}, \overline{Y}^{01}\}$.

Bounds for $LNATE^z_{nt}$, for $z = 0,1$: From (23), $LNATE^z_{nt} = \overline{Y}^{10} - E[Y(0)|nt]$. Using the bounds previously derived for $E[Y(0)|nt]$ we have:
$\max\{0, \overline{Y}^{10} - \overline{Y}^{00}\} \leq LNATE^z_{nt} \leq \overline{Y}^{10} - L^{0,nt}$, for $z = 0,1$.

Bounds for $LNATE^z_{at}$, for $z = 0,1$: From (23), $LNATE^z_{at} = E[Y(1)|at] - \overline{Y}^{01}$. Using the bounds previously derived for $E[Y(1)|at]$ we have:
$\max\{0, \overline{Y}^{11} - \overline{Y}^{01}\} \leq LNATE^z_{at} \leq U^{1,at} - \overline{Y}^{01}$, for $z = 0,1$.

Bounds for $LNATE^0_c$: From (23), $LNATE^0_c = E[Y(1,D_0)|c] - E[Y(0)|c]$. A5.2 directly implies $LNATE^0_c \geq 0$. Using the bounds previously obtained for the components of $LNATE^0_c$ we obtain six additional potential lower bounds: $\overline{Y}^{10} - U^{0,c}$, $\overline{Y}^{10} - \overline{Y}^{11}$, $\overline{Y}^{10} - \overline{Y}^{01}$, $\overline{Y}^{00} - U^{0,c}$, $\overline{Y}^{00} - \overline{Y}^{11}$ and $\overline{Y}^{00} - \overline{Y}^{01}$. Note that: $\overline{Y}^{10} - \overline{Y}^{11} \leq 0$, $\overline{Y}^{00} - U^{0,c} \leq 0$, $\overline{Y}^{00} - \overline{Y}^{11} \leq 0$, and $\overline{Y}^{00} - \overline{Y}^{01} \leq 0$. Hence, $LNATE^0_c \geq \max\{\overline{Y}^{10} - U^{0,c}, \overline{Y}^{10} - \overline{Y}^{01}, 0\}$. We have the upper bound $\overline{Y}^{11} - \overline{Y}^{00}$. Thus, $\max\{\overline{Y}^{10} - U^{0,c}, \overline{Y}^{10} - \overline{Y}^{01}, 0\} \leq LNATE^0_c \leq \overline{Y}^{11} - \overline{Y}^{00}$.

Bounds for $LNATE^1_c$: From (23), $LNATE^1_c = E[Y(1)|c] - E[Y(0,D_1)|c]$. A5.2 directly implies $LNATE^1_c \geq 0$. Using the bounds previously obtained for the components of $LNATE^1_c$ we obtain six additional potential lower bounds: $L^{1,c} - \overline{Y}^{11}$, $\overline{Y}^{10} - \overline{Y}^{11}$, $\overline{Y}^{00} - \overline{Y}^{11}$, $L^{1,c} - \overline{Y}^{01}$, $\overline{Y}^{10} - \overline{Y}^{01}$ and $\overline{Y}^{00} - \overline{Y}^{01}$. Note that: $L^{1,c} - \overline{Y}^{11} \leq 0$, $\overline{Y}^{10} - \overline{Y}^{11} \leq 0$, $\overline{Y}^{00} - \overline{Y}^{11} \leq 0$, and $\overline{Y}^{00} - \overline{Y}^{01} \leq 0$. Hence, $LNATE^1_c \geq \max\{L^{1,c} - \overline{Y}^{01}, \overline{Y}^{10} - \overline{Y}^{01}, 0\}$. We have the upper bound $\overline{Y}^{11} - \overline{Y}^{00}$. Thus, $\max\{L^{1,c} - \overline{Y}^{01}, \overline{Y}^{10} - \overline{Y}^{01}, 0\} \leq LNATE^1_c \leq \overline{Y}^{11} - \overline{Y}^{00}$.

Bounds for $LMATE^1_c$: $LMATE^1_c = E[Y(1)|c] - E[Y(1,D_0)|c]$. A5.1 directly implies $LMATE^1_c \geq 0$. Using the bounds previously obtained for the components of $LMATE^1_c$ we obtain three additional potential lower bounds: $L^{1,c} - \overline{Y}^{11}, \overline{Y}^{10} - \overline{Y}^{11}$ and $\overline{Y}^{00} - \overline{Y}^{11}$. Each of these three expressions is less than or equal to zero. We have the upper bound $\overline{Y}^{11} - \max\{\overline{Y}^{10}, \overline{Y}^{00}\}$. Thus, $0 \leq LMATE^1_c \leq \overline{Y}^{11} - \max\{\overline{Y}^{10}, \overline{Y}^{00}\}$.

Bounds for $LMATE^0_c$: $LMATE^0_c = E[Y(0,D_1)|c] - E[Y(0)|c]$. A5.1 directly implies $LMATE^0_c \geq 0$. Using the bounds previously obtained for the components of $LMATE^0_c$ we obtain three additional potential lower bounds: $\overline{Y}^{00} - U^{0,c}, \overline{Y}^{00} - \overline{Y}^{01}, \overline{Y}^{00} - \overline{Y}^{11}$. Each of these three expressions is less than or equal to zero. We have the upper bound $\min\{\overline{Y}^{11}, \overline{Y}^{01}\} - \overline{Y}^{00}$. Thus, $0 \leq LMATE^0_c \leq \min\{\overline{Y}^{11}, \overline{Y}^{01}\} - \overline{Y}^{00}$.

Bounds for $E[Y(1,1)|nt]$: A5.3 and A6.3 imply $E[Y(1)|c] \geq E[Y(1,1)|nt] \geq E[Y(1)|nt]$. Combining with the bounds previously derived for $E[Y(1)|c]$ yields $\overline{Y}^{11} \geq E[Y(1,1)|nt] \geq \overline{Y}^{10}$. And thus $E[Y(1,1)|nt] - E[Y(1)|nt]$ in (16) and (17) has the following bounds $0 \leq E[Y(1,1)|nt] - E[Y(1)|nt] \leq \overline{Y}^{11} - \overline{Y}^{10}$.
Bounds for $E[Y(0,1)|nt]$: A5.3 and A6.3 imply $E[Y(0,D_1)|c] \geq E[Y(0,1)|nt] \geq E[Y(0)|nt]$. Combining with the bounds derived for $E[Y(0,D_1)|c]$ and $E[Y(0)|nt]$ yields $\min\{\overline{Y}^{11}, \overline{Y}^{01}\} \geq E[Y(0,1)|nt] \geq L^{0,nt}$. Thus, we have $E[Y(0,1)|nt] - E[Y(0)|nt]$ in (19) and (20) has the following bounds $0 \leq E[Y(0,1)|nt] - E[Y(0)|nt] \leq \min\{\overline{Y}^{11}, \overline{Y}^{01}\} - L^{0,nt}$.
Bounds for $E[Y(1,0)|at]$: A5.3 and A6.3 imply $E[Y(1)|at] \geq E[Y(1,0)|at] \geq E[Y(1,D_0)|c]$. Combining with the bounds previously derived for $E[Y(1,D_0)|c]$ and $E[Y(1)|at]$ yields $U^{1,at} \geq E[Y(1,0)|at] \geq \max\{\overline{Y}^{10}, \overline{Y}^{00}\}$. Thus, we have $E[Y(1)|at] - E[Y(1,0)|at]$ in (16) and (17) has the following bounds $0 \leq E[Y(1)|at] - E[Y(1,0)|at] \leq U^{1,at} - \max\{\overline{Y}^{10}, \overline{Y}^{00}\}$.
Bounds for $E[Y(0,0)|at]$: A5.3 and A6.3 imply $E[Y(0)|at] \geq E[Y(0,0)|at] \geq E[Y(0)|c]$. Combining with the bounds previously derived for $E[Y(0)|c]$ yields $\overline{Y}^{01} \geq E[Y(0,0)|at] \geq \overline{Y}^{00}$. And thus $E[Y(0)|at] - E[Y(0,0)|at]$ in (19) and (20) has the following bounds $0 \leq E[Y(0)|at] - E[Y(0,0)|at] \leq \overline{Y}^{01} - \overline{Y}^{00}$.

Bounds on $\Delta(z)$. We now use equations (16) through (18) to derive the lower bounds for $\Delta(1)$. The corresponding three potential lower bounds are:

$$\begin{aligned}
LB^1_\alpha &= 0 \\
LB^1_\beta &= E[Y|Z=1] - E[Y|Z=0] - p_{1|0}(U^{1,at} - \overline{Y}^{01}) \\
&\quad - p_{0|1}(\overline{Y}^{10} - L^{0,nt}) - (p_{1|1} - p_{1|0})(\overline{Y}^{11} - \overline{Y}^{00}) \\
LB^1_\gamma &= p_{1|1}\overline{Y}^{11} - p_{0|1}\overline{Y}^{10} - p_{1|0}U^{1,at} + p_{0|1}\overline{Y}^{10} - (p_{1|1} - p_{1|0})\overline{Y}^{11}
\end{aligned}$$

After some algebra, we have $LB^1_\beta = (p_{1|1} - p_{1|0})L^{1,c} - (p_{1|1} - p_{1|0})U^{0,c} - (p_{1|1} - p_{1|0})(\overline{Y}^{11} - \overline{Y}^{00}) = (p_{1|1} - p_{1|0})[(L^{1,c} - \overline{Y}^{11}) + (\overline{Y}^{00} - U^{0,c})] \leq 0$. The second equality is derived by the following equations: $E[Y|Z=z] = p_{1|z}\overline{Y}^{z1} + p_{0|z}\overline{Y}^{z0}$, for $z = 0, 1$, and $\pi_{nt}L^{0,nt} + \pi_c U^{0,c} = p_{0|0}\overline{Y}^{00}$, and $\pi_{at}U^{1,at} + \pi_c L^{1,c} = p_{1|1}\overline{Y}^{11}$. The inequality is because $L^{1,c} \leq \overline{Y}^{11}$ and $\overline{Y}^{00} \leq U^{0,c}$. Thus, $LB^1_\beta \leq LB^1_\alpha$. $LB^1_\gamma = (p_{1|1} - p_{1|0})L^{1,c} - (p_{1|1} - p_{1|0})\overline{Y}^{11} = (p_{1|1} - p_{1|0})(L^{1,c} - \overline{Y}^{11}) \leq 0$, where the first equality is derived from $\pi_{at}U^{1,at} + \pi_c L^{1,c} = p_{1|1}\overline{Y}^{11}$ and the inequality is because $\overline{Y}^{11} \geq L^{1,c}$ by definition. Hence, $LB^1_\gamma \leq LB^1_\alpha$. Therefore, the lower bound for $\Delta(1) = LB^1_\alpha = 0$.

Similarly, we use equations (19) through (21) to derive potential lower bounds for $\Delta(0)$:

$$\begin{aligned}
LB^0_\alpha &= 0\\
LB^0_\beta &= E[Y|Z=1]-E[Y|Z=0]-p_{1|0}(U^{1,at}-\overline{Y}^{01})\\
&\quad -p_{0|1}(\overline{Y}^{10}-L^{0,nt})-(p_{1|1}-p_{1|0})(\overline{Y}^{11}-\overline{Y}^{00})\\
LB^0_\gamma &= p_{1|0}\overline{Y}^{01}-p_{0|0}\overline{Y}^{00}-p_{1|0}\overline{Y}^{01}+p_{0|1}L^{0,nt}+(p_{1|1}-p_{1|0})\overline{Y}^{00}
\end{aligned}$$

After comparison, we have $LB^0_\beta = LB^1_\beta \leq 0$. Thus, $LB^0_\beta \leq LB^0_\alpha$.
$LB^1_\gamma = -(p_{0|0}-p_{0|1})U^{0,c}+(p_{0|0}-p_{0|1})\overline{Y}^{00} = (p_{1|1}-p_{1|0})(\overline{Y}^{00}-U^{0,c}) \leq 0$, where the first equality is derived from $\pi_{nt}L^{0,nt}+\pi_c U^{0,c} = p_{0|0}\overline{Y}^{00}$ and the inequality is because $\overline{Y}^{00} \leq U^{0,c}$ by definition. Hence, $LB^0_\gamma \leq LB^0_\alpha$. Therefore, the lower bound for $\Delta(0) = LB^0_\alpha = 0$.
We now use equations (16) through (18) to derive potential upper bounds for $\Delta(1)$:

$$\begin{aligned}
UB^1_\alpha &= p_{1|0}(U^{1,at}-\max\{\overline{Y}^{10},\overline{Y}^{00}\})+p_{0|1}(\overline{Y}^{11}-\overline{Y}^{10})+(p_{1|1}-p_{1|0})(\overline{Y}^{11}-\max\{\overline{Y}^{10},\overline{Y}^{00}\})\\
UB^1_\beta &= p_{1|0}(U^{1,at}-\max\{\overline{Y}^{10},\overline{Y}^{00}\})+p_{0|1}(\overline{Y}^{11}-\overline{Y}^{10})+E[Y|Z=1]-E[Y|Z=0]\\
&\quad -p_{1|0}\max\{0,\overline{Y}^{11}-\overline{Y}^{01}\}-p_{0|1}\max\{0,\overline{Y}^{10}-\overline{Y}^{00}\}\\
&\quad -(p_{1|1}-p_{1|0})\max\{\overline{Y}^{10}-U^{0,c},\overline{Y}^{10}-\overline{Y}^{01},0\}\\
UB^1_\gamma &= p_{1|1}\overline{Y}^{11}-p_{0|1}\overline{Y}^{10}-p_{1|0}\max\{\overline{Y}^{10},\overline{Y}^{00}\}+p_{0|1}\overline{Y}^{11}-(p_{1|1}-p_{1|0})\max\{\overline{Y}^{10},\overline{Y}^{00}\}
\end{aligned}$$

To compare, $UB^1_\gamma - UB^1_\alpha = p_{1|1}\overline{Y}^{11}-p_{1|0}U^{1,at}-(p_{1|1}-p_{1|0})\overline{Y}^{11} = p_{1|0}(\overline{Y}^{11}-U^{1,at}) \leq 0$, where the inequality follows the definition $\overline{Y}^{11} \leq U^{1,at}$. Thus, UB^1_γ dominates UB^1_α.
$UB^1_\beta - UB^1_\gamma = p_{1|0}(U^{1,at}-\max\{\overline{Y}^{11},\overline{Y}^{01}\})+p_{0|1}(\min\{\overline{Y}^{10},\overline{Y}^{00}\}-\overline{Y}^{00})+(p_{1|1}-p_{1|0})(\max\{\overline{Y}^{10},\overline{Y}^{00}\}-\overline{Y}^{00}-\max\{\overline{Y}^{10}-U^{0,c},\overline{Y}^{10}-\overline{Y}^{01},0\})$. If $\overline{Y}^{10} \geq \overline{Y}^{00}$,
$UB^1_\beta - UB^1_\gamma = p_{1|0}(U^{1,at}-\max\{\overline{Y}^{11},\overline{Y}^{01}\})+(p_{1|1}-p_{1|0})(\min\{U^{0,c},\overline{Y}^{01},\overline{Y}^{10}\}-\overline{Y}^{00}) \geq 0$, because $U^{1,at} \geq \max\{\overline{Y}^{11},\overline{Y}^{01}\}$ and $\min\{U^{0,c},\overline{Y}^{01},\overline{Y}^{10}\} \geq \overline{Y}^{00}$. Thus, UB^1_γ dominates UB^1_β when $\overline{Y}^{10} \geq \overline{Y}^{00}$, $UB^1_\gamma = \overline{Y}^{11}-\overline{Y}^{10}$. If $\overline{Y}^{10} \leq \overline{Y}^{00}$,
$UB^1_\beta - UB^1_\gamma = p_{1|0}(U^{1,at}-\max\{\overline{Y}^{11},\overline{Y}^{01}\})+p_{0|1}(\overline{Y}^{10}-\overline{Y}^{00})$. We cannot determine the sign because $U^{1,at} \geq \max\{\overline{Y}^{11},\overline{Y}^{01}\}$ and $\overline{Y}^{10} \leq \overline{Y}^{00}$. $UB^1_\gamma = \overline{Y}^{11}-p_{0|1}\overline{Y}^{10}-p_{1|1}\overline{Y}^{00}$, and
$UB^1_\beta = \overline{Y}^{11}-E[Y|Z=0]+p_{1|0}(U^{1,at}-\overline{Y}^{00}-\max\{0,\overline{Y}^{11}-\overline{Y}^{01}\})$. Thus,
$UB^1_a = \overline{Y}^{11}-E[Y|Z=0]+p_{1|0}(U^{1,at}-\overline{Y}^{00}-\max\{0,\overline{Y}^{11}-\overline{Y}^{01}\})$ and $UB^1_b = UB^1_\gamma$. Finally, we have to show that when $\overline{Y}^{10} \geq \overline{Y}^{00}$, $UB^1_\gamma \leq UB^1_a$.
$UB^1_\gamma - UB^1_a = \pi_{nt}(\overline{Y}^{00}-\overline{Y}^{10})+\pi_c(\overline{Y}^{00}-\overline{Y}^{10})+\pi_{at}(\overline{Y}^{00}-\overline{Y}^{10}+\max\{\overline{Y}^{11},\overline{Y}^{01}\}-U^{1,at}) \leq 0$, because $\overline{Y}^{00} \leq \overline{Y}^{10}$ and $\max\{\overline{Y}^{11},\overline{Y}^{01}\} \leq U^{1,at}$.

We now use equations (19) through (21) to derive potential upper bounds for $\Delta(0)$:

$$
\begin{aligned}
UB_\alpha^0 &= p_{1|0}(\overline{Y}^{01} - \overline{Y}^{00}) + p_{0|1}(\min\{\overline{Y}^{11}, \overline{Y}^{01}\} - L^{0,nt}) + (p_{1|1} - p_{1|0})(\min\{\overline{Y}^{11}, \overline{Y}^{01}\} - \overline{Y}^{00}) \\
UB_\beta^0 &= p_{1|0}(\overline{Y}^{01} - \overline{Y}^{00}) + p_{0|1}(\min\{\overline{Y}^{11}, \overline{Y}^{01}\} - L^{0,nt}) + E[Y|Z=1] - E[Y|Z=0] \\
&\quad - p_{1|0}\max\{0, \overline{Y}^{11} - \overline{Y}^{01}\} - p_{0|1}\max\{0, \overline{Y}^{10} - \overline{Y}^{00}\} \\
&\quad - (p_{1|1} - p_{1|0})\max\{L^{1,c} - \overline{Y}^{01}, \overline{Y}^{10} - \overline{Y}^{01}, 0\} \\
UB_\gamma^0 &= p_{1|0}\overline{Y}^{01} - p_{0|0}\overline{Y}^{00} - p_{1|0}\overline{Y}^{00} + p_{0|1}\min\{\overline{Y}^{11}, \overline{Y}^{01}\} + (p_{1|1} - p_{1|0})\min\{\overline{Y}^{11}, \overline{Y}^{01}\}
\end{aligned}
$$

Similarly, $UB_\gamma^0 - UB_\alpha^0 = -p_{0|0}\overline{Y}^{00} + p_{0|1}L^{0,nt} + (p_{0|0} - p_{0|1})\overline{Y}^{00} = p_{0|1}(L^{0,nt} - \overline{Y}^{00}) \leq 0$, where the inequality follows the definition $L^{0,nt} \leq \overline{Y}^{00}$. Thus, UB_γ^0 dominates UB_α^0.
$UB_\beta^0 - UB_\gamma^0 = p_{0|1}(\min\{\overline{Y}^{10}, \overline{Y}^{00}\} - L^{0,nt}) + p_{1|0}(\overline{Y}^{11} - \max\{\overline{Y}^{11}, \overline{Y}^{01}\}) + (p_{1|1} - p_{1|0})(\overline{Y}^{11} - \max\{L^{1,c} - \overline{Y}^{01}, \overline{Y}^{10} - \overline{Y}^{01}, 0\} - \min\{\overline{Y}^{11}, \overline{Y}^{01}\})$. If $\overline{Y}^{11} \geq \overline{Y}^{01}$,
$UB_\beta^0 - UB_\gamma^0 = p_{0|1}(\min\{\overline{Y}^{10}, \overline{Y}^{00}\} - L^{0,nt}) + (p_{1|1} - p_{1|0})(\overline{Y}^{11} - \max\{L^{1,c}, \overline{Y}^{10}, \overline{Y}^{01}\}) \geq 0$,
because $\min\{\overline{Y}^{10}, \overline{Y}^{00}\} \geq L^{0,nt}$, and $\overline{Y}^{11} \geq \max\{L^{1,c}, \overline{Y}^{10}, \overline{Y}^{01}\}$. Thus, UB_γ^0 dominates UB_β^0 when $\overline{Y}^{11} \geq \overline{Y}^{01}$, $UB_\gamma^0 = p_{1|0}\overline{Y}^{01} - p_{0|0}\overline{Y}^{00} - p_{1|0}\overline{Y}^{00} + p_{0|0}\overline{Y}^{01} = \overline{Y}^{01} - \overline{Y}^{00}$. If $\overline{Y}^{11} \leq \overline{Y}^{01}$,
$UB_\beta^0 - UB_\gamma^0 = p_{0|1}(\min\{\overline{Y}^{10}, \overline{Y}^{00}\} - L^{0,nt}) + p_{1|0}(\overline{Y}^{11} - \overline{Y}^{01})$. We cannot determine the sign because $\min\{\overline{Y}^{10}, \overline{Y}^{00}\} \geq L^{0,nt}$ and $\overline{Y}^{11} \leq \overline{Y}^{01}$.
$UB_\beta^0 = E[Y|Z=1] - \overline{Y}^{00} + p_{0|1}(\overline{Y}^{11} - L^{0,nt} - \max\{0, \overline{Y}^{10} - \overline{Y}^{00}\})$, and
$UB_\gamma^0 = p_{1|0}\overline{Y}^{01} - \overline{Y}^{00} + p_{0|0}\overline{Y}^{11}$. Thus,
$UB_a^0 = E[Y|Z=1] - \overline{Y}^{00} + p_{0|1}(\overline{Y}^{11} - L^{0,nt} - \max\{0, \overline{Y}^{10} - \overline{Y}^{00}\})$ and $UB_b^0 = UB_\gamma^0$. Finally, we have to show that when $\overline{Y}^{11} \geq \overline{Y}^{01}$, $UB_\gamma^0 \leq UB_a^0$.
$UB_\gamma^0 - UB_a^0 = \pi_{at}(\overline{Y}^{01} - \overline{Y}^{11}) + \pi_c(\overline{Y}^{01} - \overline{Y}^{11}) + \pi_{nt}(\overline{Y}^{01} - \overline{Y}^{11} + L^{0,nt} - \min\{\overline{Y}^{10}, \overline{Y}^{00}\}) \leq 0$, because $\overline{Y}^{01} \leq \overline{Y}^{11}$, and $L^{0,nt} \leq \min\{\overline{Y}^{10}, \overline{Y}^{00}\}$.
Finally, the bounds for $E[\Delta(z)]$ are obtained by directly plugging the corresponding terms into the equation $\Pr(Z=1)\Delta(1) + \Pr(Z=0)\Delta(0)$.

Bounds on ATT. Under the same set of the assumptions, $0 \leq \Gamma(0) \leq p_{1|0}(\overline{Y}^{01} - \overline{Y}^{00})$. The lower bounds on $\Gamma(1)$ are:

$$
\begin{aligned}
lb_\alpha^1 &= 0 \\
lb_\beta^1 &= p_{1|0}(\overline{Y}^{01} - U^{1,at}) + E[Y|Z=1] - E[Y|Z=0] \\
&\quad - p_{0|1}(\overline{Y}^{10} - L^{0,nt}) - (p_{1|1} - p_{1|0})(\overline{Y}^{11} - \overline{Y}^{00}) \\
lb_\gamma^1 &= p_{1|1}\overline{Y}^{11} - p_{1|0}U^{1,at} - (p_{1|1} - p_{1|0})\overline{Y}^{11}
\end{aligned}
$$

After arrangement, $lb_\beta^1 = -\pi_{at}U^{1,at} + p_{1|1}\overline{Y}^{11} - p_{0|0}\overline{Y}^{00} + \pi_{nt}L^{0,nt} - \pi_c(\overline{Y}^{11} - \overline{Y}^{00}) = \pi_{at}(\overline{Y}^{11} - U^{1,at}) + \pi_{nt}(L^{0,nt} - \overline{Y}^{00}) \leq 0$. $lb_\gamma^1 = \pi_{at}(\overline{Y}^{11} - U^{1,at}) \leq 0$. Thus, $lb^1 = lb_\alpha^1 = 0$. The upper bounds on $\Gamma(1)$ are:

$$\begin{aligned} ub^1_\alpha &= p_{1|0}(U^{1,at} - \max\{\overline{Y}^{10}, \overline{Y}^{00}\}) + (p_{1|1} - p_{1|0})(\overline{Y}^{11} - \max\{\overline{Y}^{10}, \overline{Y}^{00}\}) \\ ub^1_\beta &= p_{1|0}(\overline{Y}^{01} - \max\{\overline{Y}^{10}, \overline{Y}^{00}\}) + E[Y|Z=1] - E[Y|Z=0] \\ &\quad -p_{0|1}\max\{0, \overline{Y}^{10} - \overline{Y}^{00}\} - (p_{1|1} - p_{1|0})\max\{\overline{Y}^{10} - U^{0,c}, \overline{Y}^{10} - \overline{Y}^{01}, 0\} \\ ub^1_\gamma &= p_{1|1}\overline{Y}^{11} - p_{1|0}\max\{\overline{Y}^{10}, \overline{Y}^{00}\} - (p_{1|1} - p_{1|0})\max\{\overline{Y}^{10}, \overline{Y}^{00}\} \end{aligned}$$

$ub^1_\alpha - ub^1_\gamma = \pi_{at}U^{1,at} + \pi_c\overline{Y}^{11} - p_{1|1}\overline{Y}^{11} = \pi_{at}(U^{1,at} - \overline{Y}^{11}) \geq 0$, by $U^{1,at} \geq E[Y(1)|at] \geq E[Y(1)|c]$. $ub^1_\beta - ub^1_\gamma = -p_{0|0}\overline{Y}^{00} + p_{0|1}\overline{Y}^{10} - p_{0|1}\max\{0, \overline{Y}^{10} - \overline{Y}^{00}\} - (p_{1|1} - p_{1|0})\max\{\overline{Y}^{10} - U^{0,c}, \overline{Y}^{10} - \overline{Y}^{01}, 0\} + (p_{1|1} - p_{1|0})\max\{\overline{Y}^{10}, \overline{Y}^{00}\}$. If $\overline{Y}^{10} \geq \overline{Y}^{00}$, $ub^1_\beta - ub^1_\gamma = -p_{0|0}\overline{Y}^{00} + \pi_{nt}\overline{Y}^{00} - \pi_c\max\{\overline{Y}^{10} - U^{0,c}, \overline{Y}^{10} - \overline{Y}^{01}, 0\} + \pi_c\overline{Y}^{10} = \pi_c(\overline{Y}^{10} - \overline{Y}^{00} - \max\{\overline{Y}^{10} - U^{0,c}, \overline{Y}^{10} - \overline{Y}^{01}, 0\}) = \pi_c(\min\{U^{0,c}, \overline{Y}^{01}, \overline{Y}^{10}\} - \overline{Y}^{00}) \geq 0$. That is, $ub^1_\beta \geq ub^1_\gamma$. If $\overline{Y}^{10} \leq \overline{Y}^{00}$, $ub^1_\beta - ub^1_\gamma = -p_{0|0}\overline{Y}^{00} + \pi_{nt}\overline{Y}^{10} + \pi_c\overline{Y}^{00} = \pi_{nt}(\overline{Y}^{10} - \overline{Y}^{00}) \leq 0$. That is, $ub^1_\beta \leq ub^1_\gamma$. Thus, if $\overline{Y}^{10} \geq \overline{Y}^{00}$, $ub^1 = ub^1_\gamma = p_{1|1}\overline{Y}^{11} - p_{1|1}\max\{\overline{Y}^{10}, \overline{Y}^{00}\} = p_{1|1}\overline{Y}^{11} - p_{1|1}\overline{Y}^{10}$; and if $\overline{Y}^{10} \leq \overline{Y}^{00}$, $ub^1 = ub^1_\beta = p_{1|0}(\overline{Y}^{01} - \overline{Y}^{00}) + E[Y|Z=1] - E[Y|Z=0] = E[Y|Z=1] - p_{0|0}\overline{Y}^{00} - p_{1|0}\overline{Y}^{00}$. Therefore, according to $ATT = \frac{w_1}{r_1}\Gamma(1) + \frac{w_0}{r_1}\Gamma(0)$, $0 \leq ATT \leq \frac{w_1}{r_1}\min\{ub^1_\beta, ub^1_\gamma\} + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - \overline{Y}^{00})$. After rearrangement, $ub_a = \frac{w_1}{r_1}ub^1_\beta + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - \overline{Y}^{00})$ if $\overline{Y}^{10} \leq \overline{Y}^{00}$, and $ub_a = \frac{w_1}{r_1}ub^1_\gamma + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - \overline{Y}^{00})$ if $\overline{Y}^{10} \geq \overline{Y}^{00}$. Q.D.E.

5.3 Sharpness of the Bounds

We prove that our bounds in Propositions 1 through 4 are sharp in the sense that they are the narrowest bounds under their corresponding assumptions and they are consistent with the data as well. The proof is completed by showing the sharpness of the bounds on $\Delta(1)$ and on $\Delta(0)$ separately. The logic of the proof consists of writing $\Delta(1)$ (and $\Delta(0)$) as a weighted average of counterfactual and observed conditional average outcomes and then defining conditional distributions that can generate such means. These defined distributions satisfy the employed assumptions and are also consistent with the observed data. In each proposition, we also show that the bounds on the ATT are sharp.

5.3.1 Proof of Proposition 1

Bounds on $\Delta(z)$. Let τ^1 denote the value of $\Delta(1)$. We will show that $\forall \tau^1 \in [LB^1, UB^1]$ in Proposition 1, there exist distributions consistent with the observed data and Assumptions 1 through 4, and $\Delta(1) = \tau^1$ evaluated under such distributions.

$$\begin{aligned} \Delta(1) &\equiv E[Y(1,1) - Y(1,0)|Z=1] \\ &= p_{1|1}E[Y(1,1)-Y(1,0)|Z=1,D=1] + p_{0|1}E[Y(1,1)-Y(1,0)|Z=1,D=0] \\ &= p_{1|1}\overline{Y}^{11} - p_{1|1}E[Y(1,0)|Z=1,D=1] + p_{0|1}E[Y(1,1)|Z=1,D=0] - p_{0|1}\overline{Y}^{10} \end{aligned} \tag{24}$$

The second equality is derived from the Law of Iterated Expectations. Denote $E[Y(1,0)|Z=1,D=1]$ as q_{10}^{11} and $E[Y(1,1)|Z=1,D=0]$ as q_{11}^{10}. Equation (24) shows that for any value of $\Delta(1)$, it can be written as $\Delta(1) = p_{1|1}\overline{Y}^{11} - p_{1|1}q_{10}^{11} + p_{0|1}q_{11}^{10} - p_{0|1}\overline{Y}^{10}$. Let $F_{Y|Z,D}(y|z,d)$ denote the observed distribution of Y conditional on $Z=z$ and $D=d$. Let y_{zd} denote the value of the potential outcome $Y(z,d)$. Then, let $F_{Y(1,0)|Z,D}(y_{10}|1,d)$ denote the distribution of the potential outcome $Y(1,0)$ conditional on $Z=1$ and $D=d$. Similarly, $F_{Y(1,1)|Z,D}(y_{11}|1,d)$ denotes the distribution of the potential outcome $Y(1,1)$ conditional on $Z=1$ and $D=d$. We define them as follows:

$$F_{Y(1,0)|Z,D}(y_{10}|1,d) = \begin{cases} F_{Y|Z,D}(y|1,0), & \text{if } d=0 \\ 1[y_{10} \geq q_{10}^{11}], & \text{if } d=1 \end{cases} \tag{25}$$

and

$$F_{Y(1,1)|Z,D}(y_{11}|1,d) = \begin{cases} F_{Y|Z,D}(y|1,1), & \text{if } d=1 \\ 1[y_{11} \geq q_{11}^{10}], & \text{if } d=0 \end{cases}, \tag{26}$$

where $q_{10}^{11} \in [y^l, y^u]$ and $q_{11}^{10} \in [y^l, y^u]$. The ranges of q_{10}^{11} and q_{11}^{10} are consistent with theirs under Assumption 4. Under such distributions, $\Delta(1) = \tau^1$ for $\forall \tau^1 \in [LB^1, UB^1]$ in Proposition 1. In particular, LB^1 is achieved when $q_{10}^{11} = y^u$ and $q_{11}^{10} = y^l$, while UB^1 is achieved when $q_{10}^{11} = y^l$ and $q_{11}^{10} = y^u$.

Similarly, let τ^0 denote the value of $\Delta(0)$. We will show that $\forall \tau^0 \in [LB^0, UB^0]$ in Proposition 1, there exist distributions consistent with the observed data and Assumptions 1 through 4, and $\Delta(0) = \tau^0$ evaluated under such distributions.

$$\begin{aligned} \Delta(0) &\equiv E[Y(0,1) - Y(0,0)|Z=0] \\ &= p_{1|0}E[Y(0,1)-Y(0,0)|Z=0,D=1] + p_{0|0}E[Y(0,1)-Y(0,0)|Z=0,D=0] \\ &= p_{1|0}\overline{Y}^{01} - p_{1|0}E[Y(0,0)|Z=0,D=1] + p_{0|0}E[Y(0,1)|Z=0,D=0] - p_{0|0}\overline{Y}^{00} \end{aligned} \tag{27}$$

Denote $E[Y(0,0)|Z=0,D=1]$ as q_{00}^{01} and $E[Y(0,1)|Z=0,D=0]$ as q_{01}^{00}. Equation (27) shows that for any value of $\Delta(0)$, it can be written as $\Delta(0) = p_{1|0}\overline{Y}^{01} - p_{1|0}q_{00}^{01} + p_{0|0}q_{01}^{00} - p_{0|0}\overline{Y}^{00}$. Let $F_{Y(0,0)|Z,D}(y_{00}|0,d)$ denote the distribution of the potential outcome $Y(0,0)$ conditional on $Z=0$ and $D=d$. Similarly, $F_{Y(0,1)|Z,D}(y_{01}|0,d)$ denotes the distribution of the potential outcome $Y(0,1)$ conditional on $Z=0$ and $D=d$. We define them as follows:

$$F_{Y(0,0)|Z,D}(y_{00}|0,d) = \begin{cases} F_{Y|Z,D}(y|0,0), & \text{if } d=0 \\ 1[y_{00} \geq q_{00}^{01}], & \text{if } d=1 \end{cases} \tag{28}$$

and

$$F_{Y(0,1)|Z,D}(y_{01}|0,d) = \begin{cases} F_{Y|Z,D}(y|0,1), & \text{if } d = 1 \\ 1[y_{01} \geq q_{01}^{00}], & \text{if } d = 0 \end{cases}, \tag{29}$$

where $q_{00}^{01} \in [y^l, y^u]$ and $q_{01}^{00} \in [y^l, y^u]$. The ranges of q_{00}^{01} and q_{01}^{00} are consistent with theirs under Assumption 4. Under such distributions, $\Delta(0) = \tau^0$ for $\forall \tau^0 \in [LB^0, UB^0]$ in Proposition 1. In particular, LB^0 is achieved when $q_{00}^{01} = y^u$ and $q_{01}^{00} = y^l$, while UB^0 is achieved when $q_{00}^{01} = y^l$ and $q_{01}^{00} = y^u$.

Bounds on ATT. Equation (12) shows that for any value of ATT, it can be written as $ATT = E[Y|D=1] - \frac{w_1 p_{1|1}}{r_1} q_{10}^{11} - \frac{w_0 p_{1|0}}{r_1} q_{00}^{01}$. Let $F_{Y(1,0)|Z,D}(y_{10}|1,1)$ denote the distribution of the potential outcome $Y(1,0)$ conditional on $Z=1$ and $D=1$, and $F_{Y(0,0)|Z,D}(y_{00}|0,1)$ denote the distribution of the potential outcome $Y(0,0)$ conditional on $Z=0$ and $D=1$. Follow the same distributions defined in equations (25) and (28), with the same ranges of $q_{10}^{11} \in [y^l, y^u]$ and $q_{00}^{01} \in [y^l, y^u]$. The ranges of q_{10}^{11} and q_{00}^{01} equal their respective ranges under Assumption 4. Under such distributions, $ATT = \rho$ for $\forall \rho \in [lb, ub]$. In particular, lb is achieved when $q_{10}^{11} = y^u$ and $q_{00}^{01} = y^u$, while ub is achieved when $q_{10}^{11} = y^l$ and $q_{00}^{01} = y^l$. Q.D.E.

5.3.2 Proof of Proposition 2

Bounds on $\Delta(z)$**.** Since the upper bounds in Proposition 2 involve min operators, the sharpness proof is completed conditional on the specific values of the upper bounds. When deriving the upper bound on $\Delta(1)$ in Proposition 2, we have derived the difference between its two values, UB_a^1 and UB_b^1. According to the notation in the Internet Appendix,
$UB_a^1 - UB_b^1 = UB_\beta^1 - UB_\gamma^1 = p_{1|0} \min\{U^{1,at} - \overline{Y}^{01}, U^{1,at} - L^{1,at}\} + p_{0|1} \min\{\overline{Y}^{10} - U^{0,nt}, 0\}$.
Let us denote this difference as θ^1. When $\theta^1 \geq 0$, $UB^1 = UB_b^1$, otherwise $UB^1 = UB_a^1$. The following proof regarding the bounds on $\Delta(1)$ is discussed based on the value of θ^1.
We start by showing that when $\theta^1 \geq 0$, $\forall \tau^1 \in [0, UB_b^1]$ in Proposition 2, there exist distributions consistent with the observed data and Assumptions 1 through 5, and $\Delta(1) = \tau^1$ evaluated under such distributions. According to equation (24), we define $F_{Y(1,0)|Z,D}(y_{10}|1,d)$ and $F_{Y(1,1)|Z,D}(y_{11}|1,d)$ in a similar way to equations (25) and (26), but modify the ranges of q_{10}^{11} and q_{11}^{10} subject to the constraints implied by Assumptions 1 through 5. By the principal stratification under Assumptions 1 through 3, $q_{10}^{11} = \frac{\pi_{at}}{p_{1|1}} E[Y(1,0)|at] + \frac{\pi_c}{p_{1|1}} E[Y(1,D_0)|c]$, and $q_{11}^{10} = E[Y(1,1)|nt]$. When deriving the bounds in Proposition 2, we have shown that Assumptions 4 and 5 imply $E[Y(1,0)|at] \in [y^l, U^{1,at}]$, $E[Y(1,D_0)|c] \in [L^{0,c}, U^{1,c}]$, and $E[Y(1,1)|nt] \in [\overline{Y}^{10}, y^u]$. Thus, the ranges of q_{10}^{11} and q_{11}^{10} are
$q_{10}^{11} \in [\frac{p_{1|0}}{p_{1|1}} y^l + \frac{p_{1|1} - p_{1|0}}{p_{1|1}} L^{0,c}, \frac{p_{1|0}}{p_{1|1}} U^{1,at} + \frac{p_{1|1} - p_{1|0}}{p_{1|1}} U^{1,c}]$ and $q_{11}^{10} \in [\overline{Y}^{10}, y^u]$. Since these are the ranges under the current assumptions, the ranges for defining $F_{Y(1,0)|Z,D}(y_{10}|1,d)$ and $F_{Y(1,1)|Z,D}(y_{11}|1,d)$ should be subsets of the above ranges. By equation (24), $LB^1 = 0$ is

achieved when $q_{10}^{11} = \overline{Y}^{11}$ and $q_{11}^{10} = \overline{Y}^{10}$. Rearrange the terms in UB_b^1 by equation (24), we have:

$$\begin{aligned} UB_b^1 &= p_{1|1}\overline{Y}^{11} - p_{1|0}y^l - (p_{1|1} - p_{1|0})L^{0,c} + p_{0|1}y^u - p_{0|1}\overline{Y}^{10} \\ &= p_{1|1}\overline{Y}^{11} - p_{1|1}(\frac{p_{1|0}}{p_{1|1}}y^l + \frac{p_{1|1} - p_{1|0}}{p_{1|1}}L^{0,c}) + p_{0|1}y^u - p_{0|1}\overline{Y}^{10}. \end{aligned}$$

Thus, UB_b^1 is achieved when $q_{10}^{11} = \frac{p_{1|0}}{p_{1|1}}y^l + \frac{p_{1|1}-p_{1|0}}{p_{1|1}}L^{0,c}$ and $q_{11}^{10} = y^u$. As a result, the defined ranges of q_{10}^{11} and q_{11}^{10} are $q_{10}^{11} \in [\frac{p_{1|0}}{p_{1|1}}y^l + \frac{p_{1|1}-p_{1|0}}{p_{1|1}}L^{0,c}, \overline{Y}^{11}]$ and $q_{11}^{10} \in [\overline{Y}^{10}, y^u]$. For q_{10}^{11}, its defined range $[\frac{p_{1|0}}{p_{1|1}}y^l + \frac{p_{1|1}-p_{1|0}}{p_{1|1}}L^{0,c}, \overline{Y}^{11}]$ is a subset of $[\frac{p_{1|0}}{p_{1|1}}y^l + \frac{p_{1|1}-p_{1|0}}{p_{1|1}}L^{0,c}, \frac{p_{1|0}}{p_{1|1}}U^{1,at} + \frac{p_{1|1}-p_{1|0}}{p_{1|1}}U^{1,c}]$ because $\overline{Y}^{11} = \frac{p_{1|0}}{p_{1|1}}U^{1,at} + \frac{p_{1|1}-p_{1|0}}{p_{1|1}}L^{1,c} \leq \frac{p_{1|0}}{p_{1|1}}U^{1,at} + \frac{p_{1|1}-p_{1|0}}{p_{1|1}}U^{1,c}$. For q_{11}^{10}, its defined range equals its range implied by the assumptions. At last, it is straightforward to show that the defined ranges of q_{10}^{11} and q_{11}^{10} are valid. That is, $\frac{p_{1|0}}{p_{1|1}}y^l + \frac{p_{1|1}-p_{1|0}}{p_{1|1}}L^{0,c} \leq \overline{Y}^{11}$ and $\overline{Y}^{10} \leq y^u$. The latter is implied by Assumption 4, while the former is derived by $y^l \leq L^{1,at}$, $L^{0,c} \leq U^{1,c}$, and $\overline{Y}^{11} = \frac{\pi_{at}}{p_{1|1}}L^{1,at} + \frac{\pi_c}{p_{1|1}}U^{1,c}$. Therefore, when $\theta^1 \geq 0$, $\forall \tau^1 \in [0, UB_b^1]$ in Proposition 2, there exist distributions $F_{Y(1,0)|Z,D}(y_{10}|1,d)$ and $F_{Y(1,1)|Z,D}(y_{11}|1,d)$ defined in equations (25) and (26), where $q_{10}^{11} \in [\frac{p_{1|0}}{p_{1|1}}y^l + \frac{p_{1|1}-p_{1|0}}{p_{1|1}}L^{0,c}, \overline{Y}^{11}]$ and $q_{11}^{10} \in [\overline{Y}^{10}, y^u]$, and $\Delta(1) = \tau^1$ evaluated under these distributions.

Now we show that when $\theta^1 \leq 0$, $\forall \tau^1 \in [0, UB_a^1]$ in Proposition 2, there exist distributions $F_{Y(1,0)|Z,D}(y_{10}|1,d)$ and $F_{Y(1,1)|Z,D}(y_{11}|1,d)$ that are consistent with the observed data and Assumptions 1 through 5, and $\Delta(1) = \tau^1$ evaluated under these distributions. Since UB_a^1 is derived from equation (17), by expending the local net effects of the different strata, we write $\Delta(1)$ as:

$$\begin{aligned} \Delta(1) &= \pi_{at}(E[Y(1)|at] - E[Y(1,0)|at]) + \pi_{nt}(E[Y(1,1)|nt] - E[Y(1)|nt]) + E[Y|Z=1] \\ &\quad -E[Y|Z=0] - \pi_{at}(E[Y(1)|at] - E[Y(0)|at]) - \pi_{nt}(E[Y(1)|nt] - E[Y(0)|nt]) \\ &\quad -\pi_c(E[Y(1,D_0)|c] - E[Y(0)|c]) \\ &= E[Y|Z=1] - 2\pi_{nt}E[Y(1)|nt] - \pi_{at}E[Y(1,0)|at] - \pi_c E[Y(1,D_0)|c] \\ &\quad +\pi_{nt}E[Y(1,1)|nt] \\ &= E[Y|Z=1] - 2p_{0|1}\overline{Y}^{10} - p_{1|1}E[Y(1,0)|Z=1,D=1] + p_{0|1}E[Y(1,1)|Z=1,D=0]. \end{aligned} \tag{30}$$

The last equality is derived from the principal stratification under Assumptions 1 through 3. Equation (30) shows that for any value of $\Delta(1)$, it can be written as $\Delta(1) = E[Y|Z=1] - 2p_{0|1}\overline{Y}^{10} - p_{1|1}q_{10}^{11} + p_{0|1}q_{11}^{10}$. According to equations (24) and (30), we redefine $F_{Y(1,0)|Z,D}(y_{10}|1,d)$ and $F_{Y(1,1)|Z,D}(y_{11}|1,d)$ by modifying the ranges of q_{10}^{11} and q_{11}^{10}. By equation (24), $LB^1 = 0$ is achieved when $q_{10}^{11} = \overline{Y}^{11}$ and $q_{11}^{10} = \overline{Y}^{10}$. Rearranging the terms in UB_a^1 according to equation (30), we have:

$$\begin{aligned} UB_a^1 &= p_{1|0}(U^{1,at}-y^l)+p_{0|1}(y^u-\overline{Y}^{10})+E[Y|Z=1]-E[Y|Z=0] \\ &\quad -p_{1|0}(\max\{L^{1,at},\overline{Y}^{01}\}-\overline{Y}^{01})-p_{0|1}(\max\{\overline{Y}^{10},U^{0,nt}\}-U^{0,nt}) \\ &\quad -(p_{1|1}-p_{1|0})(L^{0,c}-L^{0,c}) \\ &= -p_{1|0}y^l-(p_{1|1}-p_{1|0})L^{0,c}+E[Y|Z=1]-2p_{0|1}\overline{Y}^{10} \\ &\quad +p_{0|1}(y^u-\max\{\overline{Y}^{10},U^{0,nt}\}+\overline{Y}^{10})+p_{1|0}(U^{1,at}-\max\{L^{1,at},\overline{Y}^{01}\}) \\ &= E[Y|Z=1]-2p_{0|1}\overline{Y}^{10}-p_{1|1}(\frac{p_{1|0}}{p_{1|1}}y^l+\frac{p_{1|1}-p_{1|0}}{p_{1|1}}L^{0,c}) \\ &\quad +p_{0|1}[y^u-\max\{0,U^{0,nt}-\overline{Y}^{10}\}+\frac{p_{1|0}}{p_{0|1}}(U^{1,at}-\max\{L^{1,at},\overline{Y}^{01}\})]. \end{aligned}$$

UB_a^1 is achieved when $q_{10}^{11}=\frac{p_{1|0}}{p_{1|1}}y^l+\frac{p_{1|1}-p_{1|0}}{p_{1|1}}L^{0,c}$ and $q_{11}^{10}=y^u-\max\{0,U^{0,nt}-\overline{Y}^{10}\}+\frac{p_{1|0}}{p_{0|1}}(U^{1,at}-\max\{L^{1,at},\overline{Y}^{01}\})$. As a result, the defined ranges of q_{10}^{11} and q_{11}^{10} are $q_{10}^{11}\in[\frac{p_{1|0}}{p_{1|1}}y^l+\frac{p_{1|1}-p_{1|0}}{p_{1|1}}L^{0,c},\overline{Y}^{11}]$ and $q_{11}^{10}\in[\overline{Y}^{10},y^u-\max\{0,U^{0,nt}-\overline{Y}^{10}\}+\frac{p_{1|0}}{p_{0|1}}(U^{1,at}-\max\{L^{1,at},\overline{Y}^{01}\})]$. Again, these defined ranges should be subsets of the ranges of q_{10}^{11} and q_{11}^{10} under Assumptions 1 through 5. In the case of $\theta^1\geq 0$, we have shown that the defined range of q_{10}^{11}, $[\frac{p_{1|0}}{p_{1|1}}y^l+\frac{p_{1|1}-p_{1|0}}{p_{1|1}}L^{0,c},\overline{Y}^{11}]$, is a subset of $[\frac{p_{1|0}}{p_{1|1}}y^l+\frac{p_{1|1}-p_{1|0}}{p_{1|1}}L^{0,c},\frac{p_{1|0}}{p_{1|1}}U^{1,at}+\frac{p_{1|1}-p_{1|0}}{p_{1|1}}U^{1,c}]$. To show the defined range of q_{11}^{10} is a subset of $[\overline{Y}^{10},y^u]$, we have to show that $y^u-\max\{0,U^{0,nt}-\overline{Y}^{10}\}+\frac{p_{1|0}}{p_{0|1}}(U^{1,at}-\max\{L^{1,at},\overline{Y}^{01}\})\leq y^u$. That is, $\frac{p_{1|0}}{p_{0|1}}(U^{1,at}-\max\{L^{1,at},\overline{Y}^{01}\})\leq\max\{0,U^{0,nt}-\overline{Y}^{10}\}$. The condition $\theta^1\leq 0$ suffices to show this inequality, where $\theta^1=p_{1|0}\min\{U^{1,at}-\overline{Y}^{01},U^{1,at}-L^{1,at}\}+p_{0|1}\min\{\overline{Y}^{10}-U^{0,nt},0\}$. $\theta^1\leq 0$ implies $p_{1|0}(U^{1,at}-\max\{\overline{Y}^{01},L^{1,at}\})\leq p_{0|1}\max\{U^{0,nt}-\overline{Y}^{10},0\}$. Finally, we show that these defined ranges themselves are valid. The validity of the defined range of q_{10}^{11} is shown in the case of $\theta^1\geq 0$. To show the defined range of q_{11}^{10} is also valid, we use the condition $\theta^1\leq 0$. Because $\max\{U^{0,nt}-\overline{Y}^{10},0\}\geq 0$ and $U^{1,at}-\max\{\overline{Y}^{01},L^{1,at}\}\geq 0$, $\theta^1\leq 0$ implies $\frac{p_{1|0}}{p_{0|1}}\leq\frac{\max\{U^{0,nt}-\overline{Y}^{10},0\}}{U^{1,at}-\max\{\overline{Y}^{01},L^{1,at}\}}$. For positive values of π_{at} and π_{nt}, "$\leq$" holds when $U^{0,nt}\geq\overline{Y}^{10}$. Thus, $[y^u-\max\{0,U^{0,nt}-\overline{Y}^{10}\}+\frac{p_{1|0}}{p_{0|1}}(U^{1,at}-\max\{L^{1,at},\overline{Y}^{01}\})]-\overline{Y}^{10}=$ $y^u-U^{0,nt}+\overline{Y}^{10}+\frac{p_{1|0}}{p_{0|1}}(U^{1,at}-\max\{L^{1,at},\overline{Y}^{01}\})-\overline{Y}^{10}\geq 0$. Therefore, when $\theta^1\leq 0$, $\forall\tau^1\in[0,UB_a^1]$ in Proposition 2, there exist distributions $F_{Y(1,0)|Z,D}(y_{10}|1,d)$ and $F_{Y(1,1)|Z,D}(y_{11}|1,d)$ defined in equations (25) and (26), where $q_{10}^{11}\in[\frac{p_{1|0}}{p_{1|1}}y^l+\frac{p_{1|1}-p_{1|0}}{p_{1|1}}L^{0,c},\overline{Y}^{11}]$ and $q_{11}^{10}\in[\overline{Y}^{10},y^u-\max\{0,U^{0,nt}-\overline{Y}^{10}\}+\frac{p_{1|0}}{p_{0|1}}(U^{1,at}-\max\{L^{1,at},\overline{Y}^{01}\})]$, and $\Delta(1)=\tau^1$ evaluated under these distributions.

Similarly, for the bounds on $\Delta(0)$ in Proposition 2, the proof is completed based on the two values of UB^0, UB_a^0 and UB_b^0. When deriving these bounds in the Internet Appendix, we

obtain
$UB^0_a - UB^0_b = UB^0_\beta - UB^0_\gamma = p_{1|0}\min\{L^{1,at} - \overline{Y}^{01}, 0\} + p_{0|1}\min\{\overline{Y}^{10} - L^{0,nt}, U^{0,nt} - L^{0,nt}\}$.
Let us denote this difference as θ^0. When $\theta^0 \geq 0$, $UB^0 = UB^0_b$, otherwise $UB^0 = UB^0_a$. In the followings, we first show that when $\theta^0 \geq 0$, $\forall \tau^0 \in [0, UB^0_b]$ in Proposition 2, there exist distributions consistent with the observed data and Assumptions 1 through 5, and $\Delta(0) = \tau^0$ evaluated under such distributions. According to equation (27), we redefine $F_{Y(0,0)|Z,D}(y_{00}|0,d)$ and $F_{Y(0,1)|Z,D}(y_{01}|0,d)$ in equations (28) and (29) by modifying the ranges of q^{01}_{00} and q^{00}_{01}. By the principal stratification under Assumptions 1 through 3, $q^{01}_{00} = E[Y(0,0)|at]$, and $q^{00}_{01} = \frac{\pi_{nt}}{p_{0|0}}E[Y(0,1)|nt] + \frac{\pi_c}{p_{0|0}}E[Y(0,D_1)|c]$. We have derived their bounds under Assumptions 4 and 5: $E[Y(0,0)|at] \in [y^l, \overline{Y}^{01}]$, $E[Y(0,1)|nt] \in [L^{0,nt}, y^u]$, and $E[Y(0,D_1)|c] \in [L^{0,c}, U^{1,c}]$. Thus, the ranges of q^{01}_{00} and q^{00}_{01} under the current assumptions are $q^{01}_{00} \in [y^l, \overline{Y}^{01}]$ and $q^{00}_{01} \in [\frac{p_{0|1}}{p_{0|0}}L^{0,nt} + \frac{p_{0|0}-p_{0|1}}{p_{0|0}}L^{0,c}, \frac{p_{0|1}}{p_{0|0}}y^u + \frac{p_{0|0}-p_{0|1}}{p_{0|0}}U^{1,c}]$. By equation (27), $LB^0 = 0$ is achieved when $q^{01}_{00} = \overline{Y}^{01}$ and $q^{00}_{01} = \overline{Y}^{00}$. Rearranging the terms in UB^0_b by equation (27), we have:

$$\begin{aligned} UB^0_b &= p_{1|0}\overline{Y}^{01} - p_{1|0}y^l + p_{0|1}y^u + (p_{0|0} - p_{0|1})U^{1,c} - p_{0|0}\overline{Y}^{00} \\ &= p_{1|0}\overline{Y}^{01} - p_{1|0}y^l + p_{0|0}(\frac{p_{0|1}}{p_{0|0}}y^u + \frac{p_{0|0} - p_{0|1}}{p_{0|0}}U^{1,c}) - p_{0|0}\overline{Y}^{00}. \end{aligned}$$

Thus, UB^0_b is achieved when $q^{01}_{00} = y^l$ and $q^{00}_{01} = \frac{p_{0|1}}{p_{0|0}}y^u + \frac{p_{0|0}-p_{0|1}}{p_{0|0}}U^{1,c}$. As a result, the defined ranges of q^{01}_{00} and q^{00}_{01} are $q^{01}_{00} \in [y^l, \overline{Y}^{01}]$ and $q^{00}_{01} \in [\overline{Y}^{00}, \frac{p_{0|1}}{p_{0|0}}y^u + \frac{p_{0|0}-p_{0|1}}{p_{0|0}}U^{1,c}]$. Again, these defined ranges should be subsets of their corresponding ranges implied by the assumptions. For q^{01}_{00}, its defined range equals its range under the assumptions. For q^{00}_{01}, its defined range $[\overline{Y}^{00}, \frac{p_{0|1}}{p_{0|0}}y^u + \frac{p_{0|0}-p_{0|1}}{p_{0|0}}U^{1,c}]$ is a subset of $[\frac{p_{0|1}}{p_{0|0}}L^{0,nt} + \frac{p_{0|0}-p_{0|1}}{p_{0|0}}L^{0,c}, \frac{p_{0|1}}{p_{0|0}}y^u + \frac{p_{0|0}-p_{0|1}}{p_{0|0}}U^{1,c}]$ because $\overline{Y}^{00} = \frac{p_{0|1}}{p_{0|0}}U^{0,nt} + \frac{p_{0|0}-p_{0|1}}{p_{0|0}}L^{0,c} \geq \frac{p_{0|1}}{p_{0|0}}L^{0,nt} + \frac{p_{0|0}-p_{0|1}}{p_{0|0}}L^{0,c}$. Then, we show that the defined ranges of q^{01}_{00} and q^{00}_{01} are valid. $y^l \leq \overline{Y}^{01}$is implied by Assumption 4, while $\overline{Y}^{00} \leq \frac{p_{0|1}}{p_{0|0}}y^u + \frac{p_{0|0}-p_{0|1}}{p_{0|0}}U^{1,c}$ is derived by $U^{0,nt} \leq y^u$ and $L^{0,c} \leq U^{1,c}$. Therefore, when $\theta^0 \geq 0$, $\forall \tau^0 \in [0, UB^0_b]$ in Proposition 2, there exist distributions $F_{Y(0,0)|Z,D}(y_{00}|0,d)$ and $F_{Y(0,1)|Z,D}(y_{01}|0,d)$ defined in equations (28) and (29), where $q^{01}_{00} \in [y^l, \overline{Y}^{01}]$ and $q^{00}_{01} \in [\overline{Y}^{00}, \frac{p_{0|1}}{p_{0|0}}y^u + \frac{p_{0|0}-p_{0|1}}{p_{0|0}}U^{1,c}]$, and $\Delta(0) = \tau^0$ evaluated under these distributions.
The next part of the sharpness proof of Proposition 2 is to show that when $\theta^0 \leq 0$, $\forall \tau^0 \in [0, UB^0_a]$, there exist distributions $F_{Y(0,0)|Z,D}(y_{00}|0,d)$ and $F_{Y(0,1)|Z,D}(y_{01}|0,d)$ consistent with the observed data and Assumptions 1 through 5, and $\Delta(0) = \tau^0$ evaluated under these distributions. Since UB^0_a is derived from equation (20), by expending the local net effects of

the different strata, we write $\Delta(0)$ as:

$$\begin{aligned}\Delta(0) &= \pi_{at}(E[Y(0)|at] - E[Y(0,0)|at]) + \pi_{nt}(E[Y(0,1)|nt] - E[Y(0)|nt]) + E[Y|Z=1] \\ &\quad -E[Y|Z=0] - \pi_{at}(E[Y(1)|at] - E[Y(0)|at]) - \pi_{nt}(E[Y(1)|nt] - E[Y(0)|nt]) \\ &\quad -\pi_c(E[Y(1)|c] - E[Y(0,D_1)|c]) \\ &= -\pi_{at}E[Y(0,0)|at] + \pi_c E[Y(0,D_1)|c] + \pi_{nt}E[Y(0,1)|nt] - E[Y|Z=0] \qquad (31) \\ &\quad +2\pi_{at}E[Y(0)|at] \\ &= 2p_{1|0}\overline{Y}^{01} - E[Y|Z=0] - p_{1|0}E[Y(0,0)|Z=0,D=1] + p_{0|0}E[Y(0,1)|Z=0,D=0].\end{aligned}$$

The last equality is implied by the principal stratification under Assumptions 1 through 3. Equation (31) shows that for any value of $\Delta(0)$, it can be written as $\Delta(0) = 2p_{1|0}\overline{Y}^{01} - E[Y|Z=0] - p_{1|0}q_{00}^{01} + p_{0|0}q_{01}^{00}$. According to equations (27) and (31), we redefine $F_{Y(0,0)|Z,D}(y_{00}|0,d)$ and $F_{Y(0,1)|Z,D}(y_{01}|0,d)$ by modifying the ranges of q_{00}^{01} and q_{01}^{00}. By equation (27), $LB^0 = 0$ is achieved when $q_{00}^{01} = \overline{Y}^{01}$ and $q_{01}^{00} = \overline{Y}^{00}$. Rearranging the terms in UB_a^0 according to equation (31), we have:

$$\begin{aligned}UB_a^0 &= p_{1|0}(\overline{Y}^{01} - y^l) + p_{0|1}(y^u - L^{0,nt}) + E[Y|Z=1] - E[Y|Z=0] \\ &\quad -p_{1|0}(\max\{\overline{Y}^{01}, L^{1,at}\} - \overline{Y}^{01}) - p_{0|1}(\overline{Y}^{10} - \min\{\overline{Y}^{10}, U^{0,nt}\}) \\ &\quad -(p_{1|1} - p_{1|0})(U^{1,c} - U^{1,c}) \\ &= 2p_{1|0}\overline{Y}^{01} - E[Y|Z=0] + p_{0|1}y^u + (p_{1|1} - p_{1|0})U^{1,c} \\ &\quad -p_{1|0}(y^l + \max\{0, \overline{Y}^{01} - L^{1,at}\}) + p_{0|1}(\min\{\overline{Y}^{10}, U^{0,nt}\} - L^{0,nt}) \\ &= 2p_{1|0}\overline{Y}^{01} - E[Y|Z=0] + p_{0|0}(\frac{p_{0|1}}{p_{0|0}}y^u + \frac{p_{1|1} - p_{1|0}}{p_{0|0}}U^{1,c}) \\ &\quad -p_{1|0}[y^l + \max\{0, \overline{Y}^{01} - L^{1,at}\} - \frac{p_{0|1}}{p_{1|0}}(\min\{\overline{Y}^{10}, U^{0,nt}\} - L^{0,nt})].\end{aligned}$$

UB_a^0 is achieved when $q_{00}^{01} = y^l + \max\{0, \overline{Y}^{01} - L^{1,at}\} - \frac{p_{0|1}}{p_{1|0}}(\min\{\overline{Y}^{10}, U^{0,nt}\} - L^{0,nt})$ and $q_{01}^{00} = \frac{p_{0|1}}{p_{0|0}}y^u + \frac{p_{1|1}-p_{1|0}}{p_{0|0}}U^{1,c}$. As a result, the defined ranges of q_{00}^{01} and q_{01}^{00} are $q_{00}^{01} \in [y^l + \max\{0, \overline{Y}^{01} - L^{1,at}\} - \frac{p_{0|1}}{p_{1|0}}(\min\{\overline{Y}^{10}, U^{0,nt}\} - L^{0,nt}), \overline{Y}^{01}]$ and $q_{01}^{00} \in [\overline{Y}^{00}, \frac{p_{0|1}}{p_{0|0}}y^u + \frac{p_{0|0}-p_{0|1}}{p_{0|0}}U^{1,c}]$. We have shown that the defined range of q_{01}^{00}, $[\overline{Y}^{00}, \frac{p_{0|1}}{p_{0|0}}y^u + \frac{p_{0|0}-p_{0|1}}{p_{0|0}}U^{1,c}]$, is a subset of $[\frac{p_{0|1}}{p_{0|0}}L^{0,nt} + \frac{p_{0|0}-p_{0|1}}{p_{0|0}}L^{0,c}, \frac{p_{0|1}}{p_{0|0}}y^u + \frac{p_{0|0}-p_{0|1}}{p_{0|0}}U^{1,c}]$. To show the defined range of q_{00}^{01} is a subset of $[y^l, \overline{Y}^{01}]$, we have to show that $y^l + \max\{0, \overline{Y}^{01} - L^{1,at}\} - \frac{p_{0|1}}{p_{1|0}}(\min\{\overline{Y}^{10}, U^{0,nt}\} - L^{0,nt}) \geq y^l$. That is, $\max\{0, \overline{Y}^{01} - L^{1,at}\} \geq \frac{p_{0|1}}{p_{1|0}}(\min\{\overline{Y}^{10}, U^{0,nt}\} - L^{0,nt})$. Here the condition $\theta^0 \leq 0$ suffices to show this inequality, where $\theta^0 = p_{1|0}\min\{L^{1,at} - \overline{Y}^{01}, 0\} + p_{0|1}\min\{\overline{Y}^{10} - L^{0,nt}, U^{0,nt} - L^{0,nt}\}$. $\theta^0 \leq 0$ implies $p_{0|1}(\min\{\overline{Y}^{10}, U^{0,nt}\} - L^{0,nt}) \leq p_{1|0}\max\{0, \overline{Y}^{01} - L^{1,at}\}$. The final step is to show that these defined ranges themselves are valid. The validity of the defined range of q_{01}^{00} is shown in the case of $\theta^0 \geq 0$. To show the defined range of q_{00}^{01} is also valid, we use the condition $\theta^0 \leq 0$. Because $\min\{\overline{Y}^{10}, U^{0,nt}\} - L^{0,nt} \geq 0$ and $\max\{0, \overline{Y}^{01} - L^{1,at}\} \geq 0$, $\theta^0 \leq 0$ implies $\frac{p_{0|1}}{p_{1|0}} \leq \frac{\max\{0,\overline{Y}^{01}-L^{1,at}\}}{\min\{\overline{Y}^{10},U^{0,nt}\}-L^{0,nt}}$. For positive values of π_{at} and π_{nt}, "$\leq$" holds when $\overline{Y}^{01} \geq L^{1,at}$. Thus, $[y^l + \max\{0, \overline{Y}^{01} - L^{1,at}\} - \frac{p_{0|1}}{p_{1|0}}(\min\{\overline{Y}^{10}, U^{0,nt}\} - L^{0,nt})] - \overline{Y}^{01} = y^l + \overline{Y}^{01} - L^{1,at} - \frac{p_{0|1}}{p_{1|0}}(\min\{\overline{Y}^{10}, U^{0,nt}\} - L^{0,nt}) - \overline{Y}^{01} \leq 0$. Therefore, when $\theta^0 \leq 0$, $\forall \tau^0 \in [0, UB_a^0]$ in Proposition 2, there exist distributions $F_{Y(0,0)|Z,D}(y_{00}|0,d)$ and $F_{Y(0,1)|Z,D}(y_{01}|0,d)$ defined in equations (28) and (29), where $q_{00}^{01} \in [y^l + \max\{0, \overline{Y}^{01} - L^{1,at}\} - \frac{p_{0|1}}{p_{1|0}}(\min\{\overline{Y}^{10}, U^{0,nt}\} - L^{0,nt}), \overline{Y}^{01}]$ and $q_{01}^{00} \in [\overline{Y}^{00}, \frac{p_{0|1}}{p_{0|0}}y^u + \frac{p_{0|0}-p_{0|1}}{p_{0|0}}U^{1,c}]$, and $\Delta(0) = \tau^0$ evaluated under these distributions.

Bounds on ATT. For the bounds on the ATT, the proof is completed based on the two values of ub, ub_a and ub_b. When deriving these bounds in the Internet Appendix, we have: if $\overline{Y}^{10} \geq U^{0,nt}$, $ub_b = \frac{w_1}{r_1}ub_\gamma^1 + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - y^l)$; otherwise, $ub_a = \frac{w_1}{r_1}ub_\beta^1 + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - y^l)$. Thus, we first show that when $\overline{Y}^{10} \geq U^{0,nt}$, $\forall \rho \in [0, ub_b]$, there exist distributions consistent with the observed data and Assumptions 1 through 5, and $ATT = \rho$ evaluated under such distributions. According to equation (12), we redefine $F_{Y(1,0)|Z,D}(y_{10}|1,1)$ and $F_{Y(0,0)|Z,D}(y_{00}|0,1)$ in equations (25) and (28) by modifying the ranges of q_{10}^{11} and q_{00}^{01}. As above, we have derived the ranges of q_{00}^{01} and q_{10}^{11} under the Assumptions 1 through 5, which are $q_{00}^{01} \in [y^l, \overline{Y}^{01}]$ and $q_{10}^{11} \in [\frac{p_{1|0}}{p_{1|1}}y^l + \frac{p_{1|1}-p_{1|0}}{p_{1|1}}L^{0,c}, \frac{p_{1|0}}{p_{1|1}}U^{1,at} + \frac{p_{1|1}-p_{0|1}}{p_{1|1}}U^{1,c}]$. By equation (12), $lb = 0$ is achieved when $q_{00}^{01} = \overline{Y}^{01}$ and $q_{10}^{11} = \overline{Y}^{11}$. Rearranging the terms in ub_b according to equation (12), we have:

$$
\begin{aligned}
ub_b &= \frac{w_1}{r_1}ub_\gamma^1 + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - y^l) \\
&= \frac{w_1}{r_1}(p_{1|1}\overline{Y}^{11} - p_{1|0}y^l - (p_{1|1} - p_{1|0})L^{0,c}) + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - y^l) \\
&= E[Y|D=1] - \frac{w_1 p_{1|1}}{r_1}(\frac{p_{1|0}}{p_{1|1}}y^l + \frac{p_{1|1} - p_{1|0}}{p_{1|1}}L^{0,c}) - \frac{w_0 p_{1|0}}{r_1}y^l
\end{aligned}
$$

Thus, ub_b is achieved when $q_{00}^{01} = y^l$ and $q_{10}^{11} = \frac{p_{1|0}}{p_{1|1}}y^l + \frac{p_{1|1}-p_{1|0}}{p_{1|1}}L^{0,c}$. As a result, the defined ranges of q_{00}^{01} and q_{10}^{11} are $q_{00}^{01} \in [y^l, \overline{Y}^{01}]$ and $q_{10}^{11} \in [\frac{p_{1|0}}{p_{1|1}}y^l + \frac{p_{1|1}-p_{1|0}}{p_{1|1}}L^{0,c}, \overline{Y}^{11}]$. Again, these defined ranges should be subsets of their corresponding ranges implied by the assumptions. For q_{00}^{01}, the defined range equals its range under the assumptions. For q_{10}^{11}, its defined range $[\frac{p_{1|0}}{p_{1|1}}y^l + \frac{p_{1|1}-p_{1|0}}{p_{1|1}}L^{0,c}, \overline{Y}^{11}]$ is a subset of $[\frac{p_{1|0}}{p_{1|1}}y^l + \frac{p_{1|1}-p_{1|0}}{p_{1|1}}L^{0,c}, \frac{p_{1|0}}{p_{1|1}}U^{1,at} + \frac{p_{1|1}-p_{0|1}}{p_{1|1}}U^{1,c}]$ because $\overline{Y}^{11} = \frac{p_{1|0}}{p_{1|1}}U^{1,at} + \frac{p_{1|1}-p_{0|1}}{p_{1|1}}L^{1,c} \leq \frac{p_{1|0}}{p_{1|1}}U^{1,at} + \frac{p_{1|1}-p_{0|1}}{p_{1|1}}U^{1,c}$. Then, we show that the defined ranges of q_{00}^{01} and q_{10}^{11} are valid. $y^l \leq \overline{Y}^{01}$ is implied by Assumption 4, while $\frac{p_{1|0}}{p_{1|1}}y^l + \frac{p_{1|1}-p_{1|0}}{p_{1|1}}L^{0,c} \leq \overline{Y}^{11}$ is derived by $y^l \leq L^{1,at}$ and $L^{0,c} \leq U^{1,c}$. Therefore, when $\overline{Y}^{10} \geq U^{0,nt}$, $\forall \rho \in [0, ub_b]$, there exist distributions $F_{Y(1,0)|Z,D}(y_{10}|1,1)$ and $F_{Y(0,0)|Z,D}(y_{00}|0,1)$ defined in equations (25) and (28), where $q_{00}^{01} \in [y^l, \overline{Y}^{01}]$ and $q_{10}^{11} \in [\frac{p_{1|0}}{p_{1|1}}y^l + \frac{p_{1|1}-p_{1|0}}{p_{1|1}}L^{0,c}, \overline{Y}^{11}]$, and $ATT = \rho$ evaluated under these distributions.

The last part of the sharpness proof of Proposition 2 is to show that when $\overline{Y}^{10} \leq U^{0,nt}$, $\forall \rho \in [0, ub_a]$, there exist distributions $F_{Y(0,0)|Z,D}(y_{00}|0,1)$ and $F_{Y(1,0)|Z,D}(y_{10}|1,1)$ consistent with the observed data and Assumptions 1 through 5, and $ATT = \rho$ evaluated under these distributions. Since ub_a is derived by expanding equation (14), we write ATT as:

$$\begin{aligned} ATT &= \frac{w_1 p_{1|1}}{r_1}(\overline{Y}^{11} - E[Y(1,0)|Z=1, D=1]) + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - E[Y(0,0)|Z=0, D=1]) \\ &= \frac{w_1}{r_1}(E[Y|Z=1] - p_{0|1}\overline{Y}^{10} - p_{1|1}q_{10}^{11}) + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - q_{00}^{01}) \end{aligned} \tag{32}$$

The last equality is implied by $p_{0|1}\overline{Y}^{10} + p_{1|1}\overline{Y}^{11} = E[Y|Z=1]$. Equation (32) shows that for any value of ATT, it can be written as a function of q_{00}^{01} and q_{10}^{11}. According to equation (32), we redefine $F_{Y(0,0)|Z,D}(y_{00}|0,1)$ and $F_{Y(1,0)|Z,D}(y_{10}|1,1)$ by modifying the ranges of q_{00}^{01} and q_{10}^{11}. $lb = 0$ is achieved when $q_{00}^{01} = \overline{Y}^{01}$ and $q_{10}^{11} = \overline{Y}^{11}$. Rearranging the terms in ub_a according to equation (32), we have:

$$\begin{aligned} ub_a &= \frac{w_1}{r_1}ub_\beta^1 + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - y^l) \\ &= \frac{w_1}{r_1}(p_{1|0}(\overline{Y}^{01} - y^l) + E[Y|Z=1] - E[Y|Z=0]) + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - y^l) \\ &= \frac{w_1}{r_1}[E[Y|Z=1] - p_{0|1}\overline{Y}^{10} - \frac{p_{1|1}}{p_{1|1}}(p_{1|0}y^l + p_{0|0}\overline{Y}^{00} - p_{0|1}\overline{Y}^{10})] + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - y^l) \end{aligned}$$

Thus, ub_a is achieved when $q_{00}^{01} = y^l$ and $q_{10}^{11} = \frac{p_{1|0}y^l + p_{0|0}\overline{Y}^{00} - p_{0|1}\overline{Y}^{10}}{p_{1|1}}$. As a result, the defined ranges of q_{00}^{01} and q_{10}^{11} are $q_{00}^{01} \in [y^l, \overline{Y}^{01}]$ and $q_{10}^{11} \in [\frac{p_{1|0}y^l + p_{0|0}\overline{Y}^{00} - p_{0|1}\overline{Y}^{10}}{p_{1|1}}, \overline{Y}^{11}]$. Again, these defined ranges should be subsets of their corresponding ranges implied by the assumptions. For q_{00}^{01}, its defined range equals its range under the assumptions. For q_{10}^{11}, we have to show that its defined range $[\frac{p_{1|0}y^l + p_{0|0}\overline{Y}^{00} - p_{0|1}\overline{Y}^{10}}{p_{1|1}}, \overline{Y}^{11}]$ is a subset of $[\frac{p_{1|0}}{p_{1|1}}y^l + \frac{p_{1|1} - p_{1|0}}{p_{1|1}}L^{0,c}, \frac{p_{1|0}}{p_{1|1}}U^{1,at} + \frac{p_{1|1} - p_{0|1}}{p_{1|1}}U^{1,c}]$. Since we have shown that $\overline{Y}^{11} \leq \frac{p_{1|0}}{p_{1|1}}U^{1,at} + \frac{p_{1|1} - p_{0|1}}{p_{1|1}}U^{1,c}$, we only need to show that $\frac{p_{1|0}y^l + p_{0|0}\overline{Y}^{00} - p_{0|1}\overline{Y}^{10}}{p_{1|1}} \geq \frac{p_{1|0}}{p_{1|1}}y^l + \frac{p_{1|1} - p_{1|0}}{p_{1|1}}L^{0,c}$, that is, $p_{0|0}\overline{Y}^{00} - p_{0|1}\overline{Y}^{10} \geq (p_{1|1} - p_{1|0})L^{0,c}$, which is equivalent to show $\pi_{nt}(U^{0,nt} - \overline{Y}^{10}) \geq 0$. The last inequality is satisfied by the condition $\overline{Y}^{10} \leq U^{0,nt}$. Then, we have to show that the defined range of q_{10}^{11} is valid, i.e., $\frac{p_{1|0}y^l + p_{0|0}\overline{Y}^{00} - p_{0|1}\overline{Y}^{10}}{p_{1|1}} \leq \overline{Y}^{11}$, which is equivalent to show that $p_{1|0}y^l + p_{0|0}\overline{Y}^{00} - p_{0|1}\overline{Y}^{10} - p_{1|1}\overline{Y}^{11} \leq 0$.

$LHS = \pi_{at}(y^l - E[Y(1)|at]) + \pi_{nt}(E[Y(0)|nt] - E[Y(1)|nt]) + \pi_c(E[Y(0)|c] - E[Y(1)|c]) \leq 0$ by the assumptions. Therefore, when $\overline{Y}^{10} \leq U^{0,nt}$, $\forall \rho \in [0, ub_a]$, there exist distributions $F_{Y(1,0)|Z,D}(y_{10}|1,1)$ and $F_{Y(0,0)|Z,D}(y_{00}|0,1)$ defined in equations (25) and (28), where

$q_{00}^{01} \in [y^l, \overline{Y}^{01}]$ and $q_{10}^{11} \in [\frac{p_{1|0}y^l + p_{0|0}\overline{Y}^{00} - p_{0|1}\overline{Y}^{10}}{p_{1|1}}, \overline{Y}^{11}]$, and $ATT = \rho$ evaluated under these distributions. Q.D.E.

5.3.3 Proof of Proposition 3

Bounds on $\Delta(z)$. We start by showing that $\forall \tau^1 \in [LB^1, UB^1]$ in Proposition 3, there exist distributions consistent with the observed data and Assumptions 1 through 4, and 6, and $\Delta(1) = \tau^1$ evaluated under such distributions. According to equation (24), we redefine $F_{Y(1,0)|Z,D}(y_{10}|1,d)$ and $F_{Y(1,1)|Z,D}(y_{11}|1,d)$ in equations (25) and (26) by modifying the ranges of q_{10}^{11} and q_{11}^{10} according to the values of LB^1 and UB^1 in Proposition 3. As we have mentioned before, under Assumptions 1 through 3 $q_{10}^{11} = \frac{\pi_{at}}{p_{1|1}} E[Y(1,0)|at] + \frac{\pi_c}{p_{1|1}} E[Y(1,D_0)|c]$, and $q_{11}^{10} = E[Y(1,1)|nt]$. When deriving the bounds in Proposition 3, we have shown that Assumptions 4 and 6 imply $E[Y(1,0)|at] \in [\overline{Y}^{10}, y^u]$, $E[Y(1,D_0)|c] \in [\overline{Y}^{10}, U^{1,at}]$, and $E[Y(1,1)|nt] \in [y^l, \overline{Y}^{11}]$. Thus, the ranges of q_{10}^{11} and q_{11}^{10} are $q_{10}^{11} \in [\overline{Y}^{10}, \frac{p_{1|0}}{p_{1|1}} y^u + \frac{p_{1|1} - p_{1|0}}{p_{1|1}} U^{1,at}]$ and $q_{11}^{10} \in [y^l, \overline{Y}^{11}]$. Again, the ranges for defining $F_{Y(1,0)|Z,D}(y_{10}|1,d)$ and $F_{Y(1,1)|Z,D}(y_{11}|1,d)$ should be subsets of the above ranges. By equation (24), we rearrange the terms in LB^1 and UB^1:

$$
\begin{aligned}
LB^1 &= p_{1|1}\overline{Y}^{11} - p_{1|0}y^u - (p_{1|1} - p_{1|0})U^{1,at} + p_{0|1}y^l - p_{0|1}\overline{Y}^{10} \\
&= p_{1|1}\overline{Y}^{11} - p_{1|1}(\frac{p_{1|0}}{p_{1|1}} y^u + \frac{p_{1|1} - p_{1|0}}{p_{1|1}} U^{1,at}) + p_{0|1}y^l - p_{0|1}\overline{Y}^{10} \\
UB^1 &= p_{1|1}\overline{Y}^{11} - p_{1|0}\overline{Y}^{10} - (p_{1|1} - p_{1|0})\overline{Y}^{10} + p_{0|1}\overline{Y}^{11} - p_{0|1}\overline{Y}^{10} \\
&= p_{1|1}\overline{Y}^{11} - p_{1|1}\overline{Y}^{10} + p_{0|1}\overline{Y}^{11} - p_{0|1}\overline{Y}^{10}.
\end{aligned}
$$

LB^1 is achieved when $q_{10}^{11} = \frac{p_{1|0}}{p_{1|1}} y^u + \frac{p_{1|1} - p_{1|0}}{p_{1|1}} U^{1,at}$ and $q_{11}^{10} = y^l$, while UB^1 is achieved when $q_{10}^{11} = \overline{Y}^{10}$ and $q_{11}^{10} = \overline{Y}^{11}$. Thus, the defined ranges of q_{10}^{11} and q_{11}^{10} are $q_{10}^{11} \in [\overline{Y}^{10}, \frac{p_{1|0}}{p_{1|1}} y^u + \frac{p_{1|1} - p_{1|0}}{p_{1|1}} U^{1,at}]$ and $q_{11}^{10} \in [y^l, \overline{Y}^{11}]$. It turns out that the defined ranges of q_{10}^{11} and q_{11}^{10} equal their respective ranges implied by the assumptions. Therefore, $\forall \tau^1 \in [LB^1, UB^1]$ in Proposition 3, there exist distributions $F_{Y(1,0)|Z,D}(y_{10}|1,d)$ and $F_{Y(1,1)|Z,D}(y_{11}|1,d)$ defined in equations (25) and (26), where $q_{10}^{11} \in [\overline{Y}^{10}, \frac{p_{1|0}}{p_{1|1}} y^u + \frac{p_{1|1} - p_{1|0}}{p_{1|1}} U^{1,at}]$ and $q_{11}^{10} \in [y^l, \overline{Y}^{11}]$, and $\Delta(1) = \tau^1$ evaluated under these distributions.

We now show that $\forall \tau^0 \in [LB^0, UB^0]$ in Proposition 3, there exist distributions consistent with the observed data and Assumptions 1 through 4 and 6, and $\Delta(0) = \tau^0$ evaluated under such distributions. According to equation (27), we redefine $F_{Y(0,0)|Z,D}(y_{00}|0,d)$ and $F_{Y(0,1)|Z,D}(y_{01}|0,d)$ in equations (28) and (29) by modifying the ranges of q_{00}^{01} and q_{01}^{00} according to the values of LB^0 and UB^0. We have shown that under Assumptions 1 through 3, $q_{00}^{01} = E[Y(0,0)|at]$, and $q_{01}^{00} = \frac{\pi_{nt}}{p_{0|0}} E[Y(0,1)|nt] + \frac{\pi_c}{p_{0|0}} E[Y(0,D_1)|c]$ and that Assumptions 4 and 6 imply: $E[Y(0,0)|at] \in [\overline{Y}^{00}, y^u]$, $E[Y(0,1)|nt] \in [y^l, \overline{Y}^{01}]$, and

$E[Y(0,D_1)|c] \in [L^{0,nt}, \overline{Y}^{01}]$. Thus, the ranges of q_{00}^{01} and q_{01}^{00} under the current assumptions are $q_{00}^{01} \in [\overline{Y}^{00}, y^u]$ and $q_{01}^{00} \in [\frac{p_{0|1}}{p_{0|0}} y^l + \frac{p_{0|0}-p_{0|1}}{p_{0|0}} L^{0,nt}, \overline{Y}^{01}]$. Again, the ranges for defining $F_{Y(0,0)|Z,D}(y_{00}|0,d)$ and $F_{Y(0,1)|Z,D}(y_{01}|0,d)$ should be subsets of the above ranges. By equation (27), we rearrange the terms in LB^0 and UB^0:

$$
\begin{aligned}
LB^0 &= p_{1|0}\overline{Y}^{01} - p_{1|0}y^u + p_{0|1}y^l + (p_{1|1} - p_{1|0})L^{0,nt} - p_{0|0}\overline{Y}^{00} \\
&= p_{1|0}\overline{Y}^{01} - p_{1|0}y^u + p_{0|0}(\frac{p_{0|1}}{p_{0|0}}y^l + \frac{p_{0|0} - p_{0|1}}{p_{0|0}}L^{0,nt}) - p_{0|0}\overline{Y}^{00} \\
UB^0 &= p_{1|0}\overline{Y}^{01} - p_{1|0}\overline{Y}^{00} + p_{0|1}\overline{Y}^{01} + (p_{1|1} - p_{1|0})\overline{Y}^{01} - p_{0|0}\overline{Y}^{00} \\
&= p_{1|0}\overline{Y}^{01} - p_{1|0}\overline{Y}^{00} + p_{0|0}\overline{Y}^{01} - p_{0|0}\overline{Y}^{00}.
\end{aligned}
$$

LB^0 is achieved when $q_{00}^{01} = y^u$ and $q_{01}^{00} = \frac{p_{0|1}}{p_{0|0}} y^l + \frac{p_{0|0}-p_{0|1}}{p_{0|0}} L^{0,nt}$, while UB^0 is achieved when $q_{00}^{01} = \overline{Y}^{00}$ and $q_{01}^{00} = \overline{Y}^{01}$. Thus, the defined ranges of q_{00}^{01} and q_{01}^{00} are $q_{00}^{01} \in [\overline{Y}^{00}, y^u]$ and $q_{01}^{00} \in [\frac{p_{0|1}}{p_{0|0}} y^l + \frac{p_{0|0}-p_{0|1}}{p_{0|0}} L^{0,nt}, \overline{Y}^{01}]$. As a result, the defined ranges of q_{00}^{01} and q_{01}^{00} equal their respective ranges under the current assumptions. Therefore, $\forall \tau^0 \in [LB^0, UB^0]$ in Proposition 3, there exist distributions $F_{Y(0,0)|Z,D}(y_{00}|0,d)$ and $F_{Y(0,1)|Z,D}(y_{01}|0,d)$ defined in equations (28) and (29), where $q_{00}^{01} \in [\overline{Y}^{00}, y^u]$ and $q_{01}^{00} \in [\frac{p_{0|1}}{p_{0|0}} y^l + \frac{p_{0|0}-p_{0|1}}{p_{0|0}} L^{0,nt}, \overline{Y}^{01}]$, and $\Delta(0) = \tau^0$ evaluated under these distributions.

Bounds on ***ATT*****.** By equation (12), we redefine the distributions $F_{Y(1,0)|Z,D}(y_{10}|1,1)$ and $F_{Y(0,0)|Z,D}(y_{00}|0,1)$ defined in equations (25) and (28) by modifying the range of q_{10}^{11} and q_{00}^{01}. We have derived the ranges of q_{10}^{11} and q_{00}^{01} under Assumptions 4 and 6: $q_{00}^{01} \in [\overline{Y}^{00}, y^u]$ and $q_{10}^{11} \in [\overline{Y}^{10}, \frac{p_{1|0}}{p_{1|1}} y^u + \frac{p_{1|1}-p_{0|1}}{p_{1|1}} U^{1,at}]$. By equation (12), rearranging the terms in lb and ub, we have:

$$
\begin{aligned}
lb &= \frac{w_1}{r_1}(p_{1|1}\overline{Y}^{11} - p_{1|0}y^u - (p_{1|1} - p_{1|0})U^{1,at}) + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - y^u) \\
&= \frac{w_1 p_{1|1}}{r_1}[\overline{Y}^{11} - (\frac{p_{1|0}}{p_{1|1}}y^u + \frac{p_{1|1} - p_{1|0}}{p_{1|1}}U^{1,at})] + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - y^u) \\
ub &= \frac{w_1}{r_1}(p_{1|1}\overline{Y}^{11} - p_{1|0}\overline{Y}^{10} - (p_{1|1} - p_{1|0})\overline{Y}^{10}) + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - \overline{Y}^{00}) \\
&= \frac{w_1 p_{1|1}}{r_1}(\overline{Y}^{11} - \overline{Y}^{10}) + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - \overline{Y}^{00})
\end{aligned}
$$

Thus, lb is achieved when $q_{00}^{01} = y^u$ and $q_{10}^{11} = \frac{p_{1|0}}{p_{1|1}} y^u + \frac{p_{1|1}-p_{0|1}}{p_{1|1}} U^{1,at}$, while ub is achieved when $q_{00}^{01} = \overline{Y}^{00}$ and $q_{10}^{11} = \overline{Y}^{10}$. As a result, the defined ranges of q_{00}^{01} and q_{10}^{11} are $q_{00}^{01} \in [\overline{Y}^{00}, y^u]$ and $q_{10}^{11} \in [\overline{Y}^{10}, \frac{p_{1|0}}{p_{1|1}} y^u + \frac{p_{1|1}-p_{0|1}}{p_{1|1}} U^{1,at}]$. These defined ranges equal their ranges under the assumptions. Therefore, there exist distributions $F_{Y(1,0)|Z,D}(y_{10}|1,1)$ and $F_{Y(0,0)|Z,D}(y_{00}|0,1)$ defined in equations (25) and (28), where $q_{00}^{01} \in [\overline{Y}^{00}, y^u]$ and $q_{10}^{11} \in [\overline{Y}^{10}, \frac{p_{1|0}}{p_{1|1}} y^u + \frac{p_{1|1}-p_{0|1}}{p_{1|1}} U^{1,at}]$, and $ATT = \rho$ evaluated under these distributions. Q.D.E.

5.3.4 Proof of Proposition 4

Bounds on $\Delta(z)$. Similar to the proof of Proposition 2, since the upper bounds in Proposition 4 involve min/max operators, the sharpness proof is completed conditional on the specific values of the upper bounds. First, we derive the difference between the two values of the upper bound on $\Delta(1)$ in Proposition 4.
$UB_a^1 - UB_b^1 = p_{0|1}\overline{Y}^{10} - E[Y|Z=0] + p_{1|0}(U^{1,at} - \overline{Y}^{00} - \max\{0, \overline{Y}^{11} - \overline{Y}^{01}\}) + p_{1|1}\max\{\overline{Y}^{10}, \overline{Y}^{00}\}$.
We denote this difference as θ^1. When $\theta^1 \geq 0$, $UB^1 = UB_b^1$, otherwise $UB^1 = UB_a^1$.
We start by showing that when $\theta^1 \geq 0$, $\forall \tau^1 \in [0, UB_b^1]$ in Proposition 4, there exist distributions consistent with the observed data and Assumptions 1 through 3, 5 and 6, and $\Delta(1) = \tau^1$ evaluated under such distributions. According to equation (24), we redefine $F_{Y(1,0)|Z,D}(y_{10}|1,d)$ and $F_{Y(1,1)|Z,D}(y_{11}|1,d)$ in equations (25) and (26) by modifying the ranges of q_{10}^{11} and q_{11}^{10} according to the values 0 and UB_b^1 in Proposition 4. We have shown the current assumptions imply $E[Y(1,0)|at] \in [\max\{\overline{Y}^{10}, \overline{Y}^{00}\}, U^{1,at}]$,
$E[Y(1,D_0)|c] \in [\max\{\overline{Y}^{10}, \overline{Y}^{00}\}, \overline{Y}^{11}]$, and $E[Y(1,1)|nt] \in [\overline{Y}^{10}, \overline{Y}^{11}]$. Thus, the ranges of q_{10}^{11} and q_{11}^{10} under the assumptions are $q_{10}^{11} \in [\max\{\overline{Y}^{10}, \overline{Y}^{00}\}, \frac{p_{1|0}}{p_{1|1}}U^{1,at} + \frac{p_{1|1}-p_{1|0}}{p_{1|1}}\overline{Y}^{11}]$ and $q_{11}^{10} \in [\overline{Y}^{10}, \overline{Y}^{11}]$. By equation (24), $LB^1 = 0$ is achieved when $q_{10}^{11} = \overline{Y}^{11}$ and $q_{11}^{10} = \overline{Y}^{10}$.
Rearrange the terms in UB_b^1 by equation (24), we have:

$$\begin{aligned} UB_b^1 &= p_{1|1}\overline{Y}^{11} - p_{1|0}\max\{\overline{Y}^{10}, \overline{Y}^{00}\} - (p_{1|1} - p_{1|0})\max\{\overline{Y}^{10}, \overline{Y}^{00}\} + p_{0|1}\overline{Y}^{11} - p_{0|1}\overline{Y}^{10} \\ &= p_{1|1}\overline{Y}^{11} - p_{1|1}\max\{\overline{Y}^{10}, \overline{Y}^{00}\} + p_{0|1}\overline{Y}^{11} - p_{0|1}\overline{Y}^{10}. \end{aligned}$$

Thus, UB_b^1 is achieved when $q_{10}^{11} = \max\{\overline{Y}^{10}, \overline{Y}^{00}\}$ and $q_{11}^{10} = \overline{Y}^{11}$. As a result, the defined ranges of q_{10}^{11} and q_{11}^{10} are $q_{10}^{11} \in [\max\{\overline{Y}^{10}, \overline{Y}^{00}\}, \overline{Y}^{11}]$ and $q_{11}^{10} \in [\overline{Y}^{10}, \overline{Y}^{11}]$. For q_{10}^{11}, its defined range $[\max\{\overline{Y}^{10}, \overline{Y}^{00}\}, \overline{Y}^{11}]$ is a subset of its range implied by the assumptions because $\overline{Y}^{11} = \frac{p_{1|0}}{p_{1|1}}\overline{Y}^{11} + \frac{p_{1|1}-p_{1|0}}{p_{1|1}}\overline{Y}^{11} \leq \frac{p_{1|0}}{p_{1|1}}U^{1,at} + \frac{p_{1|1}-p_{1|0}}{p_{1|1}}\overline{Y}^{11}$. For q_{11}^{10}, its defined range $[\overline{Y}^{10}, \overline{Y}^{11}]$ equals its range implied by the assumptions. At last, it is straightforward to show that the defined ranges of q_{10}^{11} and q_{11}^{10} are valid by employing Assumptions 5 and 6. Therefore, when $\theta^1 \geq 0$, $\forall \tau^1 \in [0, UB_b^1]$ in Proposition 4, there exist distributions $F_{Y(1,0)|Z,D}(y_{10}|1,d)$ and $F_{Y(1,1)|Z,D}(y_{11}|1,d)$ defined in equations (25) and (26), where $q_{10}^{11} \in [\max\{\overline{Y}^{10}, \overline{Y}^{00}\}, \overline{Y}^{11}]$ and $q_{11}^{10} \in [\overline{Y}^{10}, \overline{Y}^{11}]$, and $\Delta(1) = \tau^1$ evaluated under these distributions.
Now we show that when $\theta^1 \leq 0$, $\forall \tau^1 \in [0, UB_a^1]$ in Proposition 4, there exist distributions $F_{Y(1,0)|Z,D}(y_{10}|1,d)$ and $F_{Y(1,1)|Z,D}(y_{11}|1,d)$ that are consistent with the observed data and Assumptions 1 through 3, 5 and 6, and $\Delta(1) = \tau^1$ evaluated under these distributions.
According to equations (24) and (30), we redefine $F_{Y(1,0)|Z,D}(y_{10}|1,d)$ and $F_{Y(1,1)|Z,D}(y_{11}|1,d)$ by modifying the ranges of q_{10}^{11} and q_{11}^{10} according to the values 0 and UB_a^1. By equation (24), $LB^1 = 0$ is achieved when $q_{10}^{11} = \overline{Y}^{11}$ and $q_{11}^{10} = \overline{Y}^{10}$. Rearranging the terms in UB_a^1 according

to equation (30), we have:

$$\begin{aligned} UB_a^1 &= \overline{Y}^{11} - E[Y|Z=0] + p_{1|0}(U^{1,at} - \overline{Y}^{00} - \max\{0, \overline{Y}^{11} - \overline{Y}^{01}\}) \\ &= E[Y|Z=1] - 2p_{0|1}\overline{Y}^{10} + p_{0|1}\overline{Y}^{10} - E[Y|Z=0] \\ &\quad + p_{1|0}(U^{1,at} - \overline{Y}^{00} - \max\{0, \overline{Y}^{11} - \overline{Y}^{01}\}) + p_{0|1}\overline{Y}^{11} \\ &= E[Y|Z=1] - 2p_{0|1}\overline{Y}^{10} - p_{1|1}(\max\{\overline{Y}^{10}, \overline{Y}^{00}\} - \frac{\theta^1}{p_{1|1}}) + p_{0|1}\overline{Y}^{11} \end{aligned}$$

UB_a^1 is achieved when $q_{10}^{11} = \max\{\overline{Y}^{10}, \overline{Y}^{00}\} - \frac{\theta^1}{p_{1|1}}$, where $\theta^1 = p_{0|1}\overline{Y}^{10} - E[Y|Z=0] + p_{1|0}(U^{1,at} - \overline{Y}^{00} - \max\{0, \overline{Y}^{11} - \overline{Y}^{01}\}) + p_{1|1}\max\{\overline{Y}^{10}, \overline{Y}^{00}\} \leq 0$, and $q_{11}^{10} = \overline{Y}^{11}$. As a result, the defined ranges of q_{10}^{11} and q_{11}^{10} are $q_{10}^{11} \in [\max\{\overline{Y}^{10}, \overline{Y}^{00}\} - \frac{\theta^1}{p_{1|1}}, \overline{Y}^{11}]$ and $q_{11}^{10} \in [\overline{Y}^{10}, \overline{Y}^{11}]$. These defined ranges should be subsets of their ranges under the assumptions derived in the last paragraph. Clearly, the defined range of q_{11}^{10} equals its range implied by the assumptions. For q_{10}^{11}, to show $[\max\{\overline{Y}^{10}, \overline{Y}^{00}\} - \frac{\theta^1}{p_{1|1}}, \overline{Y}^{11}]$ is a subset of $[\max\{\overline{Y}^{10}, \overline{Y}^{00}\}, \frac{p_{1|0}}{p_{1|1}}U^{1,at} + \frac{p_{1|1}-p_{1|0}}{p_{1|1}}\overline{Y}^{11}]$, we have to show that $\max\{\overline{Y}^{10}, \overline{Y}^{00}\} - \frac{\theta^1}{p_{1|1}} \geq \max\{\overline{Y}^{10}, \overline{Y}^{00}\}$ and $\overline{Y}^{11} \leq \frac{p_{1|0}}{p_{1|1}}U^{1,at} + \frac{p_{1|1}-p_{1|0}}{p_{1|1}}\overline{Y}^{11}$. The "$\geq$" holds for $\theta^1 \leq 0$ and the "$\leq$" holds for $\overline{Y}^{11} \leq U^{1,at}$. Finally, we show that the defined range of q_{10}^{11} itself is valid. That is, $\overline{Y}^{11} \geq \max\{\overline{Y}^{10}, \overline{Y}^{00}\} - \frac{\theta^1}{p_{1|1}}$, which is equivalent to show $p_{1|1}\overline{Y}^{11} - p_{1|1}\max\{\overline{Y}^{10}, \overline{Y}^{00}\} + \theta^1 \geq 0$. $LHS = \pi_{at}(E[Y(1)|at] - \overline{Y}^{00} + U^{1,at} - \max\{\overline{Y}^{11}, \overline{Y}^{01}\}) + \pi_c(E[Y(1)|c] - E[Y(0)|c]) + \pi_{nt}(\overline{Y}^{10} - E[Y(0)|nt]) \geq 0$ by the assumptions. Thus, the defined range of q_{10}^{11} itself is valid. Therefore, when $\theta^1 \leq 0$, $\forall \tau^1 \in [0, UB_a^1]$ in Proposition 4, there exist distributions $F_{Y(1,0)|Z,D}(y_{10}|1,d)$ and $F_{Y(1,1)|Z,D}(y_{11}|1,d)$ defined in equations (25) and (26), where $q_{10}^{11} \in [\max\{\overline{Y}^{10}, \overline{Y}^{00}\} - \frac{\theta^1}{p_{1|1}}, \overline{Y}^{11}]$ and $q_{11}^{10} \in [\overline{Y}^{10}, \overline{Y}^{11}]$, and $\Delta(1) = \tau^1$ evaluated under these distributions.

Now we show the case of $\Delta(0)$ in Proposition 4. We derive the difference between the two values of the upper bound.
$UB_a^0 - UB_b^0 = E[Y|Z=1] - p_{0|1}(L^{0,nt} + \max\{0, \overline{Y}^{10} - \overline{Y}^{00}\} - \overline{Y}^{11}) - p_{1|0}\overline{Y}^{01} - p_{0|0}\min\{\overline{Y}^{11}, \overline{Y}^{01}\}$. Let us denote this difference as θ^0. When $\theta^0 \geq 0$, $UB^0 = UB_b^0$, otherwise $UB^0 = UB_a^0$. In the following of the proof, we first show that when $\theta^0 \geq 0$, $\forall \tau^0 \in [0, UB_b^0]$ in Proposition 4, there exist distributions consistent with the observed data and Assumptions 1 through 3, 5 and 6, and $\Delta(0) = \tau^0$ evaluated under such distributions. According to equation (27), we redefine $F_{Y(0,0)|Z,D}(y_{00}|0,d)$ and $F_{Y(0,1)|Z,D}(y_{01}|0,d)$ in equations (28) and (29) by modifying the ranges of q_{00}^{01} and q_{01}^{00}. When deriving the bounds in Proposition 4, we have shown: $E[Y(0,0)|at] \in [\overline{Y}^{00}, \overline{Y}^{01}]$, $E[Y(0,1)|nt] \in [L^{0,nt}, \min\{\overline{Y}^{11}, \overline{Y}^{01}\}]$, and $E[Y(0,D_1)|c] \in [\overline{Y}^{00}, \min\{\overline{Y}^{11}, \overline{Y}^{01}\}]$. Thus, the ranges of q_{00}^{01} and q_{01}^{00} under the current assumptions are $q_{00}^{01} \in [\overline{Y}^{00}, \overline{Y}^{01}]$ and $q_{01}^{00} \in [\frac{p_{0|1}}{p_{0|0}}L^{0,nt} + \frac{p_{0|0}-p_{0|1}}{p_{0|0}}\overline{Y}^{00}, \min\{\overline{Y}^{11}, \overline{Y}^{01}\}]$. By

equation (27), $LB^0 = 0$ is achieved when $q^{01}_{00} = \overline{Y}^{01}$ and $q^{00}_{01} = \overline{Y}^{00}$. Rearranging the terms in UB^0_b by equation (27), we have:

$$\begin{aligned} UB^0_b &= p_{1|0}\overline{Y}^{01} - p_{1|0}\overline{Y}^{00} + p_{0|1}\min\{\overline{Y}^{11}, \overline{Y}^{01}\} + (p_{0|0} - p_{0|1})\min\{\overline{Y}^{11}, \overline{Y}^{01}\} - p_{0|0}\overline{Y}^{00} \\ &= p_{1|0}\overline{Y}^{01} - p_{1|0}\overline{Y}^{00} + p_{0|0}\min\{\overline{Y}^{11}, \overline{Y}^{01}\} - p_{0|0}\overline{Y}^{00}. \end{aligned}$$

Thus, UB^0_b is achieved when $q^{01}_{00} = \overline{Y}^{00}$ and $q^{00}_{01} = \min\{\overline{Y}^{11}, \overline{Y}^{01}\}$. As a result, the defined ranges of q^{01}_{00} and q^{00}_{01} are $q^{01}_{00} \in [\overline{Y}^{00}, \overline{Y}^{01}]$ and $q^{00}_{01} \in [\overline{Y}^{00}, \min\{\overline{Y}^{11}, \overline{Y}^{01}\}]$. For q^{01}_{00}, its defined range equals its range implied by the assumptions. For q^{00}_{01}, its defined range $[\overline{Y}^{00}, \min\{\overline{Y}^{11}, \overline{Y}^{01}\}]$ is a subset of $[\frac{p_{0|1}}{p_{0|0}} L^{0,nt} + \frac{p_{0|0}-p_{0|1}}{p_{0|0}}\overline{Y}^{00}, \min\{\overline{Y}^{11}, \overline{Y}^{01}\}]$ because $\overline{Y}^{00} = \frac{p_{0|1}}{p_{0|0}}\overline{Y}^{00} + \frac{p_{0|0}-p_{0|1}}{p_{0|0}}\overline{Y}^{00} \geq \frac{p_{0|1}}{p_{0|0}} L^{0,nt} + \frac{p_{0|0}-p_{0|1}}{p_{0|0}}\overline{Y}^{00}$. Combination of Assumptions 5 and 6 suffices to show the validity of the defined ranges of q^{01}_{00} and q^{00}_{01}. Therefore, when $\theta^0 \geq 0$, $\forall \tau^0 \in [0, UB^0_b]$ in Proposition 4, there exist distributions $F_{Y(0,0)|Z,D}(y_{00}|0, d)$ and $F_{Y(0,1)|Z,D}(y_{01}|0, d)$ defined in equations (28) and (29), where $q^{01}_{00} \in [\overline{Y}^{00}, \overline{Y}^{01}]$ and $q^{00}_{01} \in [\overline{Y}^{00}, \min\{\overline{Y}^{11}, \overline{Y}^{01}\}]$, and $\Delta(0) = \tau^0$ evaluated under these distributions.
The next part of the sharpness proof of Proposition 4 is to show that when $\theta^0 \leq 0$, $\forall \tau^0 \in [0, UB^0_a]$, there exist distributions $F_{Y(0,0)|Z,D}(y_{00}|0, d)$ and $F_{Y(0,1)|Z,D}(y_{01}|0, d)$ consistent with the observed data and Assumptions 1 through 3, 5 and 6, and $\Delta(0) = \tau^0$ evaluated under these distributions. According to equations (27) and (31), we redefine $F_{Y(0,0)|Z,D}(y_{00}|0, d)$ and $F_{Y(0,1)|Z,D}(y_{01}|0, d)$ by modifying the ranges of q^{01}_{00} and q^{00}_{01}. By equation (27), $LB^0 = 0$ is achieved when $q^{01}_{00} = \overline{Y}^{01}$ and $q^{00}_{01} = \overline{Y}^{00}$. Rearranging the terms in UB^0_a according to equation (31), we have:

$$\begin{aligned} UB^0_a &= E[Y|Z=1] - \overline{Y}^{00} - p_{0|1}(L^{0,nt} + \max\{0, \overline{Y}^{10} - \overline{Y}^{00}\} - \overline{Y}^{11}) \\ &= 2p_{1|0}\overline{Y}^{01} - E[Y|Z=0] - p_{1|0}\overline{Y}^{00} + E[Y|Z=1] \\ &\quad - p_{0|1}(L^{0,nt} + \max\{0, \overline{Y}^{10} - \overline{Y}^{00}\} - \overline{Y}^{11}) - p_{1|0}\overline{Y}^{01} \\ &= 2p_{1|0}\overline{Y}^{01} - E[Y|Z=0] - p_{1|0}\overline{Y}^{00} + p_{0|0}(\min\{\overline{Y}^{11}, \overline{Y}^{01}\} + \frac{\theta^0}{p_{0|0}}). \end{aligned}$$

UB^0_a is achieved when $q^{01}_{00} = \overline{Y}^{00}$ and $q^{00}_{01} = \min\{\overline{Y}^{11}, \overline{Y}^{01}\} + \frac{\theta^0}{p_{0|0}}$, where $\theta^0 = E[Y|Z=1] - p_{0|1}(L^{0,nt} + \max\{0, \overline{Y}^{10} - \overline{Y}^{00}\} - \overline{Y}^{11}) - p_{1|0}\overline{Y}^{01} - p_{0|0}\min\{\overline{Y}^{11}, \overline{Y}^{01}\} \leq 0$. Thus, the defined ranges of q^{01}_{00} and q^{00}_{01} are $q^{01}_{00} \in [\overline{Y}^{00}, \overline{Y}^{01}]$ and $q^{00}_{01} \in [\overline{Y}^{00}, \min\{\overline{Y}^{11}, \overline{Y}^{01}\} + \frac{\theta^0}{p_{0|0}}]$. These defined ranges should be subsets of their respective ranges under the assumptions derived in the previous paragraph. Clearly, the defined range of q^{01}_{00} equals its range implied by the assumptions. For q^{00}_{01}, to show $[\overline{Y}^{00}, \min\{\overline{Y}^{11}, \overline{Y}^{01}\} + \frac{\theta^0}{p_{0|0}}]$ is a subset of $[\frac{p_{0|1}}{p_{0|0}} L^{0,nt} + \frac{p_{0|0}-p_{0|1}}{p_{0|0}}\overline{Y}^{00}, \min\{\overline{Y}^{11}, \overline{Y}^{01}\}]$, we have to show that $\overline{Y}^{00} \geq \frac{p_{0|1}}{p_{0|0}} L^{0,nt} + \frac{p_{0|0}-p_{0|1}}{p_{0|0}}\overline{Y}^{00}$ and $\min\{\overline{Y}^{11}, \overline{Y}^{01}\} + \frac{\theta^0}{p_{0|0}} \leq \min\{\overline{Y}^{11}, \overline{Y}^{01}\}$. The "$\geq$" holds for $\overline{Y}^{00} \geq L^{0,nt}$ and the "$\leq$" holds for $\theta^0 \leq 0$. Finally, we show that the defined range of q^{00}_{01} itself is valid. That is, $\overline{Y}^{00} \leq \min\{\overline{Y}^{11}, \overline{Y}^{01}\} + \frac{\theta^0}{p_{0|0}}$, which is equivalent to show $p_{0|0}\overline{Y}^{00} - p_{0|0}\min\{\overline{Y}^{11}, \overline{Y}^{01}\} - \theta^0 \leq 0$. $LHS = \pi_{nt}(E[Y(0)|nt] - \min\{\overline{Y}^{10}, \overline{Y}^{00}\} + L^{0,nt} - \overline{Y}^{11}) + p_{1|0}(\overline{Y}^{01} - E[Y(1)|at]) + \pi_c(E[Y(0)|c] - E[Y(1)|c]) \leq 0$ by the assumptions. Thus, the defined range of q_{01}^{00} itself is valid. Therefore, when $\theta^0 \leq 0$, $\forall \tau^0 \in [0, UB_a^0]$ in Proposition 4, there exist distributions $F_{Y(0,0)|Z,D}(y_{00}|0,d)$ and $F_{Y(0,1)|Z,D}(y_{01}|0,d)$ defined in equations (28) and (29), where $q_{00}^{01} \in [\overline{Y}^{00}, \overline{Y}^{01}]$ and $q_{01}^{00} \in [\overline{Y}^{00}, \min\{\overline{Y}^{11}, \overline{Y}^{01}\} + \frac{\theta^0}{p_{0|0}}]$, and $\Delta(0) = \tau^0$ evaluated under these distributions.

Bounds on ATT. For the bounds on the ATT, the proof is completed based on the two values of ub, ub_a and ub_b. When deriving these bounds in the Internet Appendix, we have: if $\overline{Y}^{10} \geq \overline{Y}^{00}$, $ub = ub_b = \frac{w_1}{r_1}ub_\gamma^1 + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - \overline{Y}^{00})$, otherwise $ub = ub_a = \frac{w_1}{r_1}ub_\beta^1 + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - \overline{Y}^{00})$. Thus, we first show that when $\overline{Y}^{10} \geq \overline{Y}^{00}$, $\forall \rho \in [0, ub_b]$, there exist distributions consistent with the observed data and Assumptions 1 through 6, and $ATT = \rho$ evaluated under such distributions.

According to equation (12), we redefine $F_{Y(1,0)|Z,D}(y_{10}|1,1)$ and $F_{Y(0,0)|Z,D}(y_{00}|0,1)$ in equations (25) and (28) by modifying the ranges of q_{10}^{11} and q_{00}^{01}. We have derived the ranges of q_{10}^{11} and q_{00}^{01} under the combined assumptions: $q_{10}^{11} \in [\max\{\overline{Y}^{00}, \overline{Y}^{10}\}, \frac{p_{1|0}}{p_{1|1}}U^{1,at} + \frac{p_{1|1}-p_{0|1}}{p_{1|1}}\overline{Y}^{11}]$ and $q_{00}^{01} \in [\overline{Y}^{00}, \overline{Y}^{01}]$. By equation (12), $lb = 0$ is achieved when $q_{00}^{01} = \overline{Y}^{01}$ and $q_{10}^{11} = \overline{Y}^{11}$. Rearranging the terms in ub_b according to equation (12), we have:

$$\begin{aligned} ub_b &= \frac{w_1}{r_1}(p_{1|1}\overline{Y}^{11} - p_{1|1}\overline{Y}^{10}) + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - \overline{Y}^{00}) \\ &= E[Y|D=1] - \frac{w_1 p_{1|1}}{r_1}\overline{Y}^{10} - \frac{w_0 p_{1|0}}{r_1}\overline{Y}^{00} \end{aligned}$$

Thus, ub_b is achieved when $q_{00}^{01} = \overline{Y}^{00}$ and $q_{10}^{11} = \overline{Y}^{10}$. As a result, the defined ranges of q_{00}^{01} and q_{10}^{11} are $q_{00}^{01} \in [\overline{Y}^{00}, \overline{Y}^{01}]$ and $q_{10}^{11} \in [\overline{Y}^{10}, \overline{Y}^{11}]$. For q_{00}^{01}, the defined range equals its range under the assumptions. For q_{10}^{11}, we have to show that its defined range $[\overline{Y}^{10}, \overline{Y}^{11}]$ is a subset of $[\max\{\overline{Y}^{00}, \overline{Y}^{10}\}, \frac{p_{1|0}}{p_{1|1}}U^{1,at} + \frac{p_{1|1}-p_{0|1}}{p_{1|1}}\overline{Y}^{11}]$. When $\overline{Y}^{10} \geq \overline{Y}^{00}$, $\overline{Y}^{10} \geq \max\{\overline{Y}^{00}, \overline{Y}^{10}\}$, and $\overline{Y}^{11} = \frac{p_{1|0}}{p_{1|1}}U^{1,at} + \frac{p_{1|1}-p_{0|1}}{p_{1|1}}L^{1,c} \leq \frac{p_{1|0}}{p_{1|1}}U^{1,at} + \frac{p_{1|1}-p_{0|1}}{p_{1|1}}\overline{Y}^{11}$. Then, we show that the defined range of q_{10}^{11} is valid. $\overline{Y}^{10} \leq \overline{Y}^{11}$ is implied by Assumption 6.2. Therefore, when $\overline{Y}^{10} \geq \overline{Y}^{00}$, $\forall \rho \in [0, ub_b]$, there exist distributions $F_{Y(1,0)|Z,D}(y_{10}|1,1)$ and $F_{Y(0,0)|Z,D}(y_{00}|0,1)$ defined in equations (25) and (28) , where $q_{00}^{01} \in [\overline{Y}^{00}, \overline{Y}^{01}]$ and $q_{10}^{11} \in [\overline{Y}^{10}, \overline{Y}^{11}]$, and $ATT = \rho$ evaluated under these distributions.

The last part of the sharpness proof of Proposition 4 is to show that when $\overline{Y}^{10} \leq \overline{Y}^{00}$, $\forall \rho \in [0, ub_a]$, there exist distributions $F_{Y(0,0)|Z,D}(y_{00}|0,1)$ and $F_{Y(1,0)|Z,D}(y_{10}|1,1)$ consistent with the observed data and Assumptions 1 through 6, and $ATT = \rho$ evaluated under these distributions. According to equation (32), we redefine $F_{Y(0,0)|Z,D}(y_{00}|0,1)$ and $F_{Y(1,0)|Z,D}(y_{10}|1,1)$ by modifying the ranges of q_{00}^{01} and q_{10}^{11}. $lb = 0$ is achieved when $q_{00}^{01} = \overline{Y}^{01}$

and $q_{10}^{11} = \overline{Y}^{11}$. Rearranging the terms in ub_a according to equation (32), we have:

$$\begin{aligned} ub_a &= \frac{w_1}{r_1}(E[Y|Z=1] - \overline{Y}^{00}) + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - \overline{Y}^{00}) \\ &= \frac{w_1}{r_1}[E[Y|Z=1] - p_{0|1}\overline{Y}^{10} - \frac{p_{1|1}}{p_{1|1}}(\overline{Y}^{00} - p_{0|1}\overline{Y}^{10})] + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - \overline{Y}^{00}) \end{aligned}$$

Thus, ub_a is achieved when $q_{00}^{01} = \overline{Y}^{00}$ and $q_{10}^{11} = \frac{\overline{Y}^{00} - p_{0|1}\overline{Y}^{10}}{p_{1|1}}$. As a result, the defined ranges of q_{00}^{01} and q_{10}^{11} are $q_{00}^{01} \in [\overline{Y}^{00}, \overline{Y}^{01}]$ and $q_{10}^{11} \in [\frac{\overline{Y}^{00} - p_{0|1}\overline{Y}^{10}}{p_{1|1}}, \overline{Y}^{11}]$. Again, these defined ranges should be subsets of their corresponding ranges implied by the assumptions. For q_{00}^{01}, its defined range equals its range under the assumptions. For q_{10}^{11}, we have to show that its defined range $[\frac{\overline{Y}^{00} - p_{0|1}\overline{Y}^{10}}{p_{1|1}}, \overline{Y}^{11}]$ is a subset of $[\max\{\overline{Y}^{00}, \overline{Y}^{10}\}, \frac{p_{1|0}}{p_{1|1}}U^{1,at} + \frac{p_{1|1} - p_{0|1}}{p_{1|1}}\overline{Y}^{11}]$ when $\overline{Y}^{10} \leq \overline{Y}^{00}$. Since we have shown that $\overline{Y}^{11} \leq \frac{p_{1|0}}{p_{1|1}}U^{1,at} + \frac{p_{1|1} - p_{0|1}}{p_{1|1}}U^{1,c}$, we only need to show that $\frac{\overline{Y}^{00} - p_{0|1}\overline{Y}^{10}}{p_{1|1}} \geq \max\{\overline{Y}^{00}, \overline{Y}^{10}\} = \overline{Y}^{00}$, that is, $\overline{Y}^{00} - p_{0|1}\overline{Y}^{10} \geq p_{1|1}\overline{Y}^{00}$, which is equivalent to show $\pi_{nt}(\overline{Y}^{00} - \overline{Y}^{10}) \geq 0$. The last inequality is satisfied by the condition $\overline{Y}^{10} \leq \overline{Y}^{00}$. Then, we have to show that the defined range q_{10}^{11} is valid, i.e., $\frac{\overline{Y}^{00} - p_{0|1}\overline{Y}^{10}}{p_{1|1}} \leq \overline{Y}^{11}$, which is equivalent to show that $\overline{Y}^{00} - p_{0|1}\overline{Y}^{10} - p_{1|1}\overline{Y}^{11} \leq 0$.
$LHS = \pi_{at}(\overline{Y}^{00} - E[Y(1)|at]) + \pi_{nt}(E[Y(0)|nt] - E[Y(1)|nt]) + \pi_c(E[Y(0)|c] - E[Y(1)|c]) \leq 0$ by Assumptions 5.2 and 6.2. Therefore, when $\overline{Y}^{10} \leq \overline{Y}^{00}$, $\forall \rho \in [0, ub_a]$, there exist distributions $F_{Y(1,0)|Z,D}(y_{10}|1,1)$ and $F_{Y(0,0)|Z,D}(y_{00}|0,1)$ defined in equations (25) and (28), where $q_{00}^{01} \in [\overline{Y}^{00}, \overline{Y}^{01}]$ and $q_{10}^{11} \in [\frac{\overline{Y}^{00} - p_{0|1}\overline{Y}^{10}}{p_{1|1}}, \overline{Y}^{11}]$, and $ATT = \rho$ evaluated under these distributions. Q.D.E.

5.4 Bounds under the Reversed Weak Monotonicity across Strata

Assumption 6'. (*Weak Monotonicity of Mean Potential Outcomes Across Strata).*

6.1' $E\left[Y\left(z\right)|at\right] \leq E\left[Y\left(z, D_{1-z}\right)|c\right] \leq E\left[Y\left(z\right)|nt\right]$;

6.2' $E\left[Y\left(z\right)|at\right] \leq E\left[Y\left(z\right)|c\right] \leq E\left[Y\left(z\right)|nt\right]$;

6.3' $E\left[Y\left(z,0\right)|at\right] \leq E\left[Y\left(z, D_0\right)|c\right], E\left[Y\left(z, D_1\right)|c\right] \leq E\left[Y\left(z,1\right)|nt\right]$, where $z = 0, 1$.

5.4.1 Bounds on ATE and ATT under the Assumption 6'

Proposition 3' *If Assumptions 1 through 4, and 6' hold, then the bounds* $LB^0 \leq \Delta(0) \leq UB^0$ *and* $LB^1 \leq \Delta(1) \leq UB^1$ *are sharp. And for* $z = 0, 1$,

$$\Pr(Z=0)LB^0 + \Pr(Z=1)LB^1 \leq E[\Delta(z)] \leq \Pr(Z=0)UB^0 + \Pr(Z=1)UB^1,$$

where

$$\begin{aligned}
LB^0 &= \overline{Y}^{01} - \overline{Y}^{00} \\
LB^1 &= \overline{Y}^{11} - \overline{Y}^{10}; \\
UB^0 &= p_{1|0}(\overline{Y}^{01} - y^l) - p_{0|0}\overline{Y}^{00} + p_{0|1}y^u + (p_{1|1} - p_{1|0})U^{0,nt} \\
UB^1 &= p_{1|1}\overline{Y}^{11} + p_{0|1}(y^u - \overline{Y}^{10}) - p_{1|0}y^l - (p_{1|1} - p_{1|0})L^{1,at}
\end{aligned}$$

$$\begin{aligned}
L^{1,at} &= E[Y|Z=1, D=1, Y \leq y^{11}_{(p_{1|0}/p_{1|1})}] \\
U^{0,nt} &= E[Y|Z=0, D=0, Y \geq y^{00}_{1-(p_{0|1}/p_{0|0})}].
\end{aligned}$$

Proof. See the next subsection.

Proposition 4'. *If Assumptions 1 through 3, 5 and 6' hold, then* $0 \leq \Delta(0) \leq \min\{UB^0_a, UB^0_b\}$ *and* $0 \leq \Delta(1) \leq \min\{UB^1_a, UB^1_b\}$ *are sharp. And for* $z = 0, 1$,

$$0 \leq E[\Delta(z)] \leq \Pr(Z=0)\min\{UB^0_a, UB^0_b\} + \Pr(Z=1)\min\{UB^1_a, UB^1_b\},$$

where

$$\begin{aligned}
UB^0_a &= p_{1|0}(\overline{Y}^{01} - y^l - \max\{0, L^{1,at} - \overline{Y}^{01}\}) + E[Y|Z=1] - E[Y|Z=0] \\
&\quad + p_{0|1}(y^u - \overline{Y}^{00} - \max\{0, \overline{Y}^{10} - U^{0,nt}\}) - (p_{1|1} - p_{1|0})\max\{0, \overline{Y}^{11} - U^{0,nt}\} \\
UB^0_b &= p_{1|0}(\overline{Y}^{01} - y^l) - p_{0|0}\overline{Y}^{00} + p_{0|1}y^u + (p_{1|1} - p_{1|0})\min\{\overline{Y}^{10}, U^{0,nt}, U^{1,c}\} \\
UB^1_a &= p_{1|0}(\overline{Y}^{11} - y^l - \max\{0, L^{1,at} - \overline{Y}^{01}\}) + E[Y|Z=1] - E[Y|Z=0] \\
&\quad + p_{0|1}(y^u - \overline{Y}^{10} - \max\{0, \overline{Y}^{10} - U^{0,nt}\}) - (p_{1|1} - p_{1|0})\max\{0, L^{1,at} - \overline{Y}^{00}\} \\
UB^1_b &= p_{1|1}\overline{Y}^{11} + p_{0|1}(y^u - \overline{Y}^{10}) - p_{1|0}y^l - (p_{1|1} - p_{1|0})\max\{\overline{Y}^{01}, L^{0,c}, L^{1,at}\};
\end{aligned}$$

$$\begin{aligned}
L^{1,at} &= E[Y|Z=1, D=1, Y \leq y^{11}_{(p_{1|0}/p_{1|1})}] \\
U^{0,nt} &= E[Y|Z=0, D=0, Y \geq y^{00}_{1-(p_{0|1}/p_{0|0})}] \\
L^{0,c} &= E[Y|Z=0, D=0, Y \leq y^{00}_{1-(p_{0|1}/p_{0|0})}] \\
U^{1,c} &= E[Y|Z=1, D=1, Y \geq y^{00}_{p_{1|0}/p_{1|1})}]
\end{aligned}$$

Proof. See the next subsection.

Under the assumptions in Proposition 3' (A6'), the bounds $lb \leq ATT \leq ub$ are sharp, where

$$\begin{aligned}
lb &= E[Y|D=1] - \frac{w_1 p_{1|1}}{r_1}\overline{Y}^{10} - \frac{w_0 p_{1|0}}{r_1}\overline{Y}^{00} \\
ub &= E[Y|D=1] - \frac{w_1}{r_1}(p_{1|1} - p_{1|0})L^{1,at} - \frac{p_{1|0}}{r_1}y^l.
\end{aligned}$$

Under the assumptions in Proposition 4' (A5 & A6'), the bounds $0 \leq ATT \leq \min\{ub_a, ub_b\}$ are sharp, where

$$\begin{aligned} ub_a &= E[Y|D=1] - \frac{p_{1|0}}{r_1} y^l + \frac{w_1}{r_1} [p_{0|1} (\min\{\overline{Y}^{10}, U^{0,nt}\} - \overline{Y}^{00}) \\ & \quad -(p_{1|1} - p_{1|0}) \max\{L^{1,at}, \overline{Y}^{00}\}] \\ ub_b &= E[Y|D=1] - \frac{p_{1|0}}{r_1} y^l - \frac{w_1}{r_1} (p_{1|1} - p_{1|0}) \max\{\overline{Y}^{01}, L^{0,c}, L^{1,at}\}. \end{aligned}$$

5.4.2 Proof of Proposition 3'

As before, we first derive bounds for the non-point identified mean potential outcomes of the strata, and for all the local net and mechanism average treatment effects.

Bounds for $E[Y(0)|nt]$: A6.2' implies $\overline{Y}^{01} = E[Y(0)|at] \leq E[Y(0)|nt]$. A6.2' and the equation $\pi_{nt} E[Y(0)|nt] + \pi_c E[Y(0)|c] = p_{0|0} \overline{Y}^{00}$ imply $E[Y(0)|nt] \geq \overline{Y}^{00}$. Since $\overline{Y}^{00} \geq \overline{Y}^{01}$ by A6.2', the lower bound is $\overline{Y}^{00}$. A6' does not provide any additional information for an upper bound of $E[Y(0)|nt]$. Thus, $\overline{Y}^{00} \leq E[Y(0)|nt] \leq U^{0,nt}$.

Bounds for $E[Y(1)|at]$: A6.2' implies $E[Y(1)|at] \leq E[Y(1)|nt] = \overline{Y}^{10}$. A6.2' and the equation $\pi_{at} E[Y(1)|at] + \pi_c E[Y(1)|c] = p_{1|1} \overline{Y}^{11}$ yield $E[Y(1)|at] \leq \overline{Y}^{11}$. Since $\overline{Y}^{11} \leq \overline{Y}^{10}$ by A6.2', the upper bound is $\overline{Y}^{11}$. A6' does not provide any additional information for a lower bound of $E[Y(1)|at]$. Thus, $L^{1,at} \leq E[Y(1)|at] \leq \overline{Y}^{11}$.

Bounds for $E[Y(0)|c]$: A6.2' and the equation $\pi_{nt} E[Y(0)|nt] + \pi_c E[Y(0)|c] = p_{0|0} \overline{Y}^{00}$ yield $E[Y(0)|c] \leq \overline{Y}^{00}$. As for the lower bound, A6.2' implies $E[Y(0)|c] \geq E[Y(0)|at] = \overline{Y}^{01}$, which can be greater or less than $L^{0,c}$. Thus, $\max\{L^{0,c}, \overline{Y}^{01}\} \leq E[Y(0)|c] \leq \overline{Y}^{00}$.

Bounds for $E[Y(1)|c]$: A6.2' and the equation $\pi_{at} E[Y(1)|at] + \pi_c E[Y(1)|c] = p_{1|1} \overline{Y}^{11}$ yield $E[Y(1)|c] \geq \overline{Y}^{11}$. As for the upper bound, A6.2' implies $E[Y(1)|c] \leq E[Y(1)|nt] = \overline{Y}^{10}$, which can be greater or less than $U^{1,c}$. Thus, $\overline{Y}^{11} \leq E[Y(1)|c] \leq \min\{U^{1,c}, \overline{Y}^{10}\}$.

Bounds for $E[Y(1, D_0)|c]$: A6.1' implies $E[Y(1, D_0)|c] \leq E[Y(1)|nt] = \overline{Y}^{10}$. Combining with the bounds previously derived for $E[Y(1)|at]$ yields
$E[Y(1, D_0)|c] \geq E[Y(1)|at] \geq L^{1,at}$. Hence, $L^{1,at} \leq E[Y(1, D_0)|c] \leq \overline{Y}^{10}$.

Bounds for $E[Y(0, D_1)|c]$: A6.1' implies $E[Y(0, D_1)|c] \geq E[Y(0)|at] = \overline{Y}^{01}$. Combining with the bounds previously derived for $E[Y(0)|nt]$ yields
$E[Y(0, D_1)|c] \leq E[Y(0)|nt] \leq U^{0,nt}$. Hence, $\overline{Y}^{01} \leq E[Y(0, D_1)|c] \leq U^{0,nt}$.

Bounds for $LNATE_{nt}^z$, for $z = 0, 1$: From (23), $LNATE_{nt}^z = \overline{Y}^{10} - E[Y(0)|nt]$. Using the bounds previously derived for $E[Y(0)|nt]$: $\overline{Y}^{10} - U^{0,nt} \leq LNATE_{nt}^z \leq \overline{Y}^{10} - \overline{Y}^{00}$, for $z = 0, 1$.

Bounds for $LNATE_{at}^z$, for $z = 0, 1$: From (23), $LNATE_{at}^z = E[Y(1)|at] - \overline{Y}^{01}$. Using the bounds previously derived for $E[Y(1)|at]$: $L^{1,at} - \overline{Y}^{01} \leq LNATE_{at}^z \leq \overline{Y}^{11} - \overline{Y}^{01}$, for $z = 0, 1$.

Bounds for $LNATE_c^0$: From (23), $LNATE_c^0 = E[Y(1, D_0)|c] - E[Y(0)|c]$. Using the bounds previously obtained for the components in $LNATE_c^0$, we obtain
$L^{1,at} - \overline{Y}^{00} \leq LNATE_c^0 \leq \overline{Y}^{10} - \max\{L^{0,c}, \overline{Y}^{01}\}$.

Bounds for $LNATE_c^1$: From (23), $LNATE_c^1 = E[Y(1)|c] - E[Y(0, D_1)|c]$. Using the bounds previously derived for the components of $LNATE_c^1$, we have
$\overline{Y}^{11} - U^{0,nt} \leq LNATE_c^1 \leq \min\{U^{1,c}, \overline{Y}^{10}\} - \overline{Y}^{01}$.
Bounds for $LMATE_c^1$: $LMATE_c^1 = E[Y(1)|c] - E[Y(1, D_0)|c]$. Using the bounds previously derived for the components of $LMATE_c^1$, we have
$\overline{Y}^{11} - \overline{Y}^{10} \leq LMATE_c^1 \leq \min\{U^{1,c}, \overline{Y}^{10}\} - L^{1,at}$.
Bounds for $LMATE_c^0$: $LMATE_c^0 = E[Y(0, D_1)|c] - E[Y(0)|c]$. Using the bounds previously derived for the components of $LMATE_c^0$, we have
$\overline{Y}^{01} - \overline{Y}^{00} \leq LMATE_c^0 \leq U^{0,nt} - \max\{L^{0,c}, \overline{Y}^{01}\}$.
Bounds for $E[Y(1,1)|nt]$: A4 and A6.3' imply $E[Y(1)|c] \leq E[Y(1,1)|nt] \leq y^u$. Combining with the bounds previously derived for $E[Y(1)|c]$ yields $\overline{Y}^{11} \leq E[Y(1,1)|nt] \leq y^u$. And thus $E[Y(1,1)|nt] - E[Y(1)|nt]$ in (16) and (17) has the following bounds
$\overline{Y}^{11} - \overline{Y}^{10} \leq E[Y(1,1)|nt] - E[Y(1)|nt] \leq y^u - \overline{Y}^{10}$.
Bounds for $E[Y(0,1)|nt]$: A4 and A6.3' imply $E[Y(0, D_1)|c] \leq E[Y(0,1)|nt] \leq y^u$. Combining with the bounds previously derived for $E[Y(0, D_1)|c]$ yields
$\overline{Y}^{01} \leq E[Y(0,1)|nt] \leq y^u$. Using the bounds previously derived for $E[Y(0)|nt]$, we have $E[Y(0,1)|nt] - E[Y(0)|nt]$ in (19) and (20) has the following bounds
$\overline{Y}^{01} - U^{0,nt} \leq E[Y(0,1)|nt] - E[Y(0)|nt] \leq y^u - \overline{Y}^{00}$.
Bounds for $E[Y(1,0)|at]$: A4 and A6.3' imply $y^l \leq E[Y(1,0)|at] \leq E[Y(1, D_0)|c]$. Combining with the bounds previously derived for $E[Y(1, D_0)|c]$ yields
$y^l \leq E[Y(1,0)|at] \leq \overline{Y}^{10}$. Using the bounds previously derived for $E[Y(1)|at]$, we have $E[Y(1)|at] - E[Y(1,0)|at]$ in (16) and (17) has the following bounds
$L^{1,at} - \overline{Y}^{10} \leq E[Y(1)|at] - E[Y(1,0)|at] \leq \overline{Y}^{11} - y^l$.
Bounds for $E[Y(0,0)|at]$: A4 and A6.3' imply $y^l \leq E[Y(0,0)|at] \leq E[Y(0)|c]$. Combining with the bounds previously derived for $E[Y(0)|c]$ yields $y^l \leq E[Y(0,0)|at] \leq \overline{Y}^{00}$. And thus $E[Y(0)|at] - E[Y(0,0)|at]$ in (19) and (20) has the following bounds
$\overline{Y}^{01} - \overline{Y}^{00} \leq E[Y(0)|at] - E[Y(0,0)|at] \leq \overline{Y}^{01} - y^l$.

Bounds on $\Delta(z)$. We now derive the lower bound of $\Delta(1)$ by the use of the equations (16) through (18). The corresponding three potential lower bounds are:

$$\begin{aligned}
LB_\alpha^1 &= \pi_{at}(L^{1,at} - \overline{Y}^{10}) + \pi_{nt}(\overline{Y}^{11} - \overline{Y}^{10}) + \pi_c(\overline{Y}^{11} - \overline{Y}^{10}) \\
LB_\beta^1 &= \pi_{at}(L^{1,at} - \overline{Y}^{10}) + \pi_{nt}(\overline{Y}^{11} - \overline{Y}^{10}) + E[Y|Z=1] - E[Y|Z=0] \\
&\quad -\pi_{at}(\overline{Y}^{11} - \overline{Y}^{01}) - \pi_{nt}(\overline{Y}^{10} - \overline{Y}^{00}) - \pi_c(\overline{Y}^{10} - \max\{L^{0,c}, \overline{Y}^{01}\}) \\
LB_\gamma^1 &= p_{1|1}\overline{Y}^{11} - p_{0|1}\overline{Y}^{10} - \pi_{at}\overline{Y}^{10} + \pi_{nt}\overline{Y}^{11} - \pi_c\overline{Y}^{10}
\end{aligned}$$

After some algebra, we have $LB_\beta^1 - LB_\alpha^1 = E[Y|Z=1] - E[Y|Z=0] - p_{1|0}(\overline{Y}^{11} - \overline{Y}^{01}) - p_{0|1}(\overline{Y}^{10} - \overline{Y}^{00}) + (p_{1|1} - p_{1|0})(\max\{L^{0,c}, \overline{Y}^{01}\} - \overline{Y}^{11}) = (p_{1|1} - p_{1|0})(\max\{L^{0,c}, \overline{Y}^{01}\} - \overline{Y}^{00}) \leq 0$.

Hence, LB^1_α dominates LB^1_β. $LB^1_\gamma - LB^1_\alpha = p_{1|0}(\overline{Y}^{11} - L^{1,at}) \geq 0$. Hence, LB^1_γ dominates LB^1_α. Therefore, the lower bound for $\Delta(1) = LB^1_\gamma = \overline{Y}^{11} - \overline{Y}^{10}$.

Similarly, we use equations (19) through (21) to derive potential lower bounds for $\Delta(0)$:

$$\begin{aligned}
LB^0_\alpha &= \pi_{at}(\overline{Y}^{01} - \overline{Y}^{00}) + \pi_{nt}(\overline{Y}^{01} - U^{0,nt}) + \pi_c(\overline{Y}^{01} - \overline{Y}^{00}) \\
LB^0_\beta &= \pi_{at}(\overline{Y}^{01} - \overline{Y}^{00}) + \pi_{nt}(\overline{Y}^{01} - U^{0,nt}) + E[Y|Z=1] - E[Y|Z=0] \\
&\quad -\pi_{at}(\overline{Y}^{11} - \overline{Y}^{01}) - \pi_{nt}(\overline{Y}^{10} - \overline{Y}^{00}) - \pi_c(\min\{U^{1,c}, \overline{Y}^{10}\} - \overline{Y}^{01}) \\
LB^0_\gamma &= p_{1|0}\overline{Y}^{01} - p_{0|0}\overline{Y}^{00} - \pi_{at}\overline{Y}^{00} + \pi_{nt}\overline{Y}^{01} + \pi_c\overline{Y}^{01}
\end{aligned}$$

After some algebra, we have $LB^0_\beta - LB^0_\alpha = E[Y|Z=1] - E[Y|Z=0] - p_{1|0}(\overline{Y}^{11} - \overline{Y}^{01}) - p_{0|1}(\overline{Y}^{10} - \overline{Y}^{00}) + (p_{1|1} - p_{1|0})(\overline{Y}^{00} - \min\{U^{1,c}, \overline{Y}^{10}\}) = (p_{1|1} - p_{1|0})(\overline{Y}^{11} - \min\{U^{1,c}, \overline{Y}^{10}\}) \leq 0$. Hence, LB^0_α dominates LB^0_β. $LB^0_\gamma - LB^0_\alpha = p_{0|1}(U^{0,nt} - \overline{Y}^{00}) \geq 0$. Hence, LB^0_γ dominates LB^0_α. Therefore, the lower bound for $\Delta(0) = LB^0_\gamma = \overline{Y}^{01} - \overline{Y}^{00}$.

We now use equations (16) through (18) to derive potential upper bounds for $\Delta(1)$:

$$\begin{aligned}
UB^1_\alpha &= \pi_{at}(\overline{Y}^{11} - y^l) + \pi_{nt}(y^u - \overline{Y}^{10}) + \pi_c(\min\{U^{1,c}, \overline{Y}^{10}\} - L^{1,at}) \\
UB^1_\beta &= \pi_{at}(\overline{Y}^{11} - y^l) + \pi_{nt}(y^u - \overline{Y}^{10}) + E[Y|Z=1] - E[Y|Z=0] \\
&\quad -\pi_{at}(L^{1,at} - \overline{Y}^{01}) - \pi_{nt}(\overline{Y}^{10} - U^{0,nt}) - \pi_c(L^{1,at} - \overline{Y}^{00}) \\
UB^1_\gamma &= p_{1|1}\overline{Y}^{11} - p_{0|1}\overline{Y}^{10} - \pi_{at}y^l + \pi_{nt}y^u - \pi_c L^{1,at}
\end{aligned}$$

$UB^1_\beta - UB^1_\alpha = E[Y|Z=1] - E[Y|Z=0] - p_{1|0}(L^{1,at} - \overline{Y}^{01}) - p_{0|1}(\overline{Y}^{10} - U^{0,nt}) + (p_{1|1} - p_{1|0})(\overline{Y}^{00} - \min\{U^{1,c}, \overline{Y}^{10}\}) = (p_{1|1} - p_{1|0})(\overline{Y}^{00} - L^{0,c} + U^{1,c} - \min\{U^{1,c}, \overline{Y}^{10}\}) \geq 0$. UB^1_α dominates UB^1_β. $UB^1_\gamma - UB^1_a = \pi_c(\overline{Y}^{11} - \min\{U^{1,c}, \overline{Y}^{10}\}) \leq 0$. Thus, UB^1_γ dominates UB^1_a. Therefore, the upper bound for
$\Delta(1) = UB^1_\gamma = p_{1|1}\overline{Y}^{11} + p_{0|1}(y^u - \overline{Y}^{10}) - p_{1|0}y^l - (p_{1|1} - p_{1|0})L^{1,at}$.

We now use equations (19) through (21) to derive potential upper bounds for $\Delta(0)$:

$$\begin{aligned}
UB^0_\alpha &= \pi_{at}(\overline{Y}^{01} - y^l) + \pi_{nt}(y^u - \overline{Y}^{00}) + \pi_c(U^{0,nt} - \max\{L^{0,c}, \overline{Y}^{01}\}) \\
UB^0_\beta &= \pi_{at}(\overline{Y}^{01} - y^l) + \pi_{nt}(y^u - \overline{Y}^{00}) + E[Y|Z=1] - E[Y|Z=0] \\
&\quad -\pi_{at}(L^{1,at} - \overline{Y}^{01}) - \pi_{nt}(\overline{Y}^{10} - U^{0,nt}) - \pi_c(\overline{Y}^{11} - U^{0,nt}) \\
UB^0_\gamma &= p_{1|0}\overline{Y}^{01} - p_{0|0}\overline{Y}^{00} - \pi_{at}y^l + \pi_{nt}y^u + \pi_c U^{0,nt}
\end{aligned}$$

$UB^0_\beta - UB^0_\alpha = E[Y|Z=1] - E[Y|Z=0] - p_{1|0}(L^{1,at} - \overline{Y}^{01}) - p_{0|1}(\overline{Y}^{10} - U^{0,nt}) + (p_{1|1} - p_{1|0})(\max\{L^{0,c}, \overline{Y}^{01}\} - \overline{Y}^{11}) = (p_{1|1} - p_{1|0})(\max\{L^{0,c}, \overline{Y}^{01}\} - L^{0,c} + U^{1,c} - \overline{Y}^{11}) \geq 0$. Thus, UB^0_α dominates UB^0_β. $UB^0_\gamma - UB^0_a = \pi_c(\max\{L^{0,c}, \overline{Y}^{01}\} - \overline{Y}^{00}) \leq 0$. Thus, UB^0_γ dominates UB^0_a. Therefore, the upper bound for
$\Delta(0) = UB^0_\gamma = p_{1|0}(\overline{Y}^{01} - y^l) - p_{0|0}\overline{Y}^{00} + p_{0|1}y^u + (p_{1|1} - p_{1|0})U^{0,nt}$.

Finally, the bounds for $E[\Delta(z)]$ are obtained by directly plugging the corresponding terms into the equation $\Pr(Z=1)\Delta(1) + \Pr(Z=0)\Delta(0)$.

Bounds on ATT**.** Under A4 and A6.3', $p_{1|0}(\overline{Y}^{01} - \overline{Y}^{00}) \leq \Gamma(0) \leq p_{1|0}(\overline{Y}^{01} - y^l)$. The lower bounds on $\Gamma(1)$ are:

$$\begin{aligned} lb_\alpha^1 &= \pi_{at}(L^{1,at} - \overline{Y}^{10}) + \pi_c(\overline{Y}^{11} - \overline{Y}^{10}) \\ lb_\beta^1 &= \pi_{at}(\overline{Y}^{01} - \overline{Y}^{10}) + E[Y|Z=1] - E[Y|Z=0] \\ &\quad -\pi_{nt}(\overline{Y}^{10} - \overline{Y}^{00}) - \pi_c(\overline{Y}^{10} - \max\{L^{0,c}, \overline{Y}^{01}\}) \\ lb_\gamma^1 &= p_{1|1}\overline{Y}^{11} - \pi_{at}\overline{Y}^{10} - \pi_c\overline{Y}^{10} \end{aligned}$$

$lb_\alpha^1 - lb_\gamma^1 = \pi_{at}(L^{1,at} - \overline{Y}^{11}) \leq 0$. $lb_\beta^1 - lb_\gamma^1 = \pi_c(\max\{L^{0,c}, \overline{Y}^{01}\} - \overline{Y}^{00}) \leq 0$. Thus, $lb^1 = lb_\gamma^1 = p_{1|1}(\overline{Y}^{11} - \overline{Y}^{10})$. The upper bounds on $\Gamma(1)$ are:

$$\begin{aligned} ub_\alpha^1 &= \pi_{at}(\overline{Y}^{11} - y^l) + \pi_c(\min\{U^{1,c}, \overline{Y}^{10}\} - L^{1,at}) \\ ub_\beta^1 &= \pi_{at}(\overline{Y}^{01} - y^l) + E[Y|Z=1] - E[Y|Z=0] \\ &\quad -\pi_{nt}(\overline{Y}^{10} - U^{0,nt}) - \pi_c(L^{1,at} - \overline{Y}^{00}) \\ ub_\gamma^1 &= p_{1|1}\overline{Y}^{11} - \pi_{at}y^l - \pi_c L^{1,at} \end{aligned}$$

$ub_\alpha^1 - ub_\gamma^1 = \pi_c(\min\{U^{1,c}, \overline{Y}^{10}\} - \overline{Y}^{11}) \geq 0$. $ub_\beta^1 - ub_\gamma^1 = \pi_{nt}(U^{0,nt} - \overline{Y}^{00}) \geq 0$. Thus, $ub^1 = ub_\gamma^1 = p_{1|1}\overline{Y}^{11} - p_{1|0}y^l - (p_{1|1} - p_{1|0})L^{1,at}$. According to $ATT = \frac{w_1}{r_1}\Gamma(1) + \frac{w_0}{r_1}\Gamma(0)$, we have $lb = \frac{w_1}{r_1}lb_\gamma^1 + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - \overline{Y}^{00})$ and $ub = \frac{w_1}{r_1}ub_\gamma^1 + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - y^l)$. Q.D.E

5.4.3 Proof of Proposition 4'

Bounds for $E[Y(0)|nt]$: A5.2 implies $E[Y(0)|nt] \leq E[Y(1)|nt] = \overline{Y}^{10}$, and $E[Y(0)|nt] \leq U^{0,nt}$; and A6.2' implies $E[Y(0)|nt] \geq \overline{Y}^{00}$. Thus, $\overline{Y}^{00} \leq E[Y(0)|nt] \leq \min\{\overline{Y}^{10}, U^{0,nt}\}$.
Bounds for $E[Y(1)|at]$: A5.2 implies $E[Y(1)|at] \geq E[Y(0)|at] = \overline{Y}^{01}$, and $E[Y(1)|at] \geq L^{1,at}$; and A6.2' implies $E[Y(1)|at] \leq \overline{Y}^{11}$. Thus, $\max\{\overline{Y}^{01}, L^{1,at}\} \leq E[Y(1)|at] \leq \overline{Y}^{11}$.
Bounds for $E[Y(0)|c]$: A6.2' and the equation $\pi_{nt}E[Y(0)|nt] + \pi_c E[Y(0)|c] = p_{0|0}\overline{Y}^{00}$ yield $E[Y(0)|c] \leq \overline{Y}^{00}$. Regarding a lower bound, the trimming procedure implies $E[Y(0)|c] \geq L^{0,c}$. A6.2' implies $E[Y(0)|c] \geq E[Y(0)|at] = \overline{Y}^{01}$. Finally, A5 implies $E[Y(1)|c] \geq E[Y(0)|c]$. Thus, we obtain $\max\{\overline{Y}^{01}, L^{0,c}\} \leq E[Y(0)|c] \leq \min\{\overline{Y}^{00}, U^{1,c}\}$.
Bounds for $E[Y(1)|c]$: A6.2' and the equation $\pi_{at}E[Y(1)|at] + \pi_c E[Y(1)|c] = p_{1|1}\overline{Y}^{11}$ yield $E[Y(1)|c] \geq \overline{Y}^{11}$. Regarding an upper bound, the trimming procedure implies $E[Y(1)|c] \leq U^{1,c}$. A6.2' implies $E[Y(1)|c] \leq E[Y(1)|nt] = \overline{Y}^{10}$. Finally, A5 implies $E[Y(1)|c] \geq E[Y(0)|c]$. Thus, we obtain $\max\{L^{0,c}, \overline{Y}^{11}\} \leq E[Y(1)|c] \leq \min\{\overline{Y}^{10}, U^{1,c}\}$.
Bounds for $E[Y(1, D_0)|c]$: A5.2 implies $E[Y(1, D_0)|c] \geq E[Y(0)|c]$. From above, the lower bound for $E[Y(0)|c]$ equals $\max\{\overline{Y}^{01}, L^{0,c}\}$. A6.1' implies $E[Y(1, D_0)|c] \leq E[Y(1)|nt] = \overline{Y}^{10}$. A5.1 implies $E[Y(1)|c] \geq E[Y(1, D_0)|c]$. From above, the upper bound for $E[Y(1)|c]$ equals

$\min\{\overline{Y}^{10}, U^{1,c}\}$. Note that A6.1' implies $E[Y(1,D_0)|c] \geq E[Y(1)|at] \geq \max\{\overline{Y}^{01}, L^{1,at}\}$. Therefore, $\max\{\overline{Y}^{01}, L^{0,c}, L^{1,at}\} \leq E[Y(1,D_0)|c] \leq \min\{\overline{Y}^{10}, U^{1,c}\}$.

Bounds for $E[Y(0,D_1)|c]$: A5.2 implies $E[Y(0,D_1)|c] \leq E[Y(1)|c]$. From above, the upper bound for $E[Y(1)|c]$ equals $\min\{\overline{Y}^{10}, U^{1,c}\}$. A6.1' implies $E[Y(0,D_1)|c] \geq E[Y(0)|at] = \overline{Y}^{01}$. A5.1 implies $E[Y(0,D_1)|c] \geq E[Y(0)|c]$. From above, the lower bound for $E[Y(0)|c]$ equals $\max\{\overline{Y}^{01}, L^{0,c}\}$. Note that A6.1' implies $E[Y(0,D_1)|c] \leq E[Y(0)|nt] \leq \min\{\overline{Y}^{10}, U^{0,nt}\}$. Therefore, $\max\{\overline{Y}^{01}, L^{0,c}\} \leq E[Y(0,D_1)|c] \leq \min\{\overline{Y}^{10}, U^{0,nt}, U^{1,c}\}$.

Bounds for $LNATE^z_{nt}$, for $z = 0, 1$: From (23), $LNATE^z_{nt} = \overline{Y}^{10} - E[Y(0)|nt]$. Using the bounds previously derived for $E[Y(0)|nt]$ we have:
$\max\{0, \overline{Y}^{10} - U^{0,nt}\} \leq LNATE^z_{nt} \leq \overline{Y}^{10} - \overline{Y}^{00}$, for $z = 0, 1$.

Bounds for $LNATE^z_{at}$, for $z = 0, 1$: From (23), $LNATE^z_{at} = E[Y(1)|at] - \overline{Y}^{01}$. Using the bounds previously derived for $E[Y(1)|at]$ we have:
$\max\{0, L^{1,at} - \overline{Y}^{01}\} \leq LNATE^z_{at} \leq \overline{Y}^{11} - \overline{Y}^{01}$, for $z = 0, 1$.

Bounds for $LNATE^0_c$: From (23), $LNATE^0_c = E[Y(1,D_0)|c] - E[Y(0)|c]$. A5.2 directly implies $LNATE^0_c \geq 0$. We have derived
$\max\{\overline{Y}^{01}, L^{0,c}, L^{1,at}\} \leq E[Y(1,D_0)|c] \leq \min\{\overline{Y}^{10}, U^{1,c}\}$ and
$\max\{\overline{Y}^{01}, L^{0,c}\} \leq E[Y(0)|c] \leq \min\{\overline{Y}^{00}, U^{1,c}\}$. Using the bounds previously obtained for the components of $LNATE^0_c$ we obtain six additional potential lower bounds: $\overline{Y}^{01} - \overline{Y}^{00}$, $\overline{Y}^{01} - U^{1,c}$, $L^{0,c} - \overline{Y}^{00}$, $L^{0,c} - U^{1,c}$, $L^{1,at} - \overline{Y}^{00}$ and $L^{1,at} - U^{1,c}$. Note that: $\overline{Y}^{01} - \overline{Y}^{00} \leq 0$, $\overline{Y}^{01} - U^{1,c} \leq 0$, $L^{0,c} - \overline{Y}^{00} \leq 0$, $L^{0,c} - U^{1,c} \leq 0$, and $L^{1,at} - U^{1,c} \leq 0$. Hence, $LNATE^0_c \geq \max\{0, L^{1,at} - \overline{Y}^{00}\}$. We have the upper bound $\min\{\overline{Y}^{10}, U^{1,c}\} - \max\{\overline{Y}^{01}, L^{0,c}\}$. Thus, $\max\{0, L^{1,at} - \overline{Y}^{00}\} \leq LNATE^0_c \leq \min\{\overline{Y}^{10}, U^{1,c}\} - \max\{\overline{Y}^{01}, L^{0,c}\}$.

Bounds for $LNATE^1_c$: From (23), $LNATE^1_c = E[Y(1)|c] - E[Y(0,D_1)|c]$. A5.2 directly implies $LNATE^1_c \geq 0$. We have derived $\max\{L^{0,c}, \overline{Y}^{11}\} \leq E[Y(1)|c] \leq \min\{\overline{Y}^{10}, U^{1,c}\}$ and $\max\{\overline{Y}^{01}, L^{0,c}\} \leq E[Y(0,D_1)|c] \leq \min\{\overline{Y}^{10}, U^{0,nt}, U^{1,c}\}$. Using the bounds previously obtained for the components of $LNATE^1_c$ we obtain six additional potential lower bounds: $L^{0,c} - \overline{Y}^{10}$, $L^{0,c} - U^{0,nt}$, $L^{0,c} - U^{1,c}$, $\overline{Y}^{11} - \overline{Y}^{10}$, $\overline{Y}^{11} - U^{0,nt}$, and $\overline{Y}^{11} - U^{1,c}$. Note that: $L^{0,c} - \overline{Y}^{10} \leq 0$, $L^{0,c} - U^{0,nt} \leq 0$, $L^{0,c} - U^{1,c} \leq 0$, $\overline{Y}^{11} - \overline{Y}^{10} \leq 0$, and $\overline{Y}^{11} - U^{1,c} \leq 0$. Hence, $LNATE^1_c \geq \max\{0, \overline{Y}^{11} - U^{0,nt}\}$. We have the upper bound $\min\{\overline{Y}^{10}, U^{1,c}\} - \max\{\overline{Y}^{01}, L^{0,c}\}$. Thus, $\max\{0, \overline{Y}^{11} - U^{0,nt}\} \leq LNATE^1_c \leq \min\{\overline{Y}^{10}, U^{1,c}\} - \max\{\overline{Y}^{01}, L^{0,c}\}$.

Bounds for $LMATE^1_c$: $LMATE^1_c = E[Y(1)|c] - E[Y(1,D_0)|c]$. A5.1 directly implies $LMATE^1_c \geq 0$. We have derived that $\max\{L^{0,c}, \overline{Y}^{11}\} \leq E[Y(1)|c] \leq \min\{\overline{Y}^{10}, U^{1,c}\}$ and $\max\{\overline{Y}^{01}, L^{0,c}, L^{1,at}\} \leq E[Y(1,D_0)|c] \leq \min\{\overline{Y}^{10}, U^{1,c}\}$. Using the bounds previously obtained for the components of $LMATE^1_c$ we obtain four additional potential lower bounds: $L^{0,c} - \overline{Y}^{10}, \overline{Y}^{11} - U^{1,c}, \overline{Y}^{11} - \overline{Y}^{10}$, and $L^{0,c} - U^{1,c}$. Each of these four expressions is less than or equal to zero. We have the upper bound $\min\{\overline{Y}^{10}, U^{1,c}\} - \max\{\overline{Y}^{01}, L^{0,c}, L^{1,at}\}$. Thus, $0 \leq LMATE^1_c \leq \min\{\overline{Y}^{10}, U^{1,c}\} - \max\{\overline{Y}^{01}, L^{0,c}, L^{1,at}\}$.

Bounds for $LMATE_c^0$: $LMATE_c^0 = E[Y(0,D_1)|c] - E[Y(0)|c]$. A5.1 directly implies $LMATE_c^0 \geq 0$. We have drived $\max\{\overline{Y}^{01}, L^{0,c}\} \leq E[Y(0,D_1)|c] \leq \min\{\overline{Y}^{10}, U^{0,nt}, U^{1,c}\}$ and $\max\{\overline{Y}^{01}, L^{0,c}\} \leq E[Y(0)|c] \leq \min\{\overline{Y}^{00}, U^{1,c}\}$. Using the bounds previously obtained for the components of $LMATE_c^0$ we obtain four additional potential lower bounds:
$\overline{Y}^{01} - \overline{Y}^{00}, L^{0,c} - \overline{Y}^{00}, \overline{Y}^{01} - U^{1,c}$, and $L^{0,c} - U^{1,c}$. Each of these four expressions is less than or equal to zero. We have the upper bound $\min\{\overline{Y}^{10}, U^{0,nt}, U^{1,c}\} - \max\{\overline{Y}^{01}, L^{0,c}\}$. Thus, $0 \leq LMATE_c^0 \leq \min\{\overline{Y}^{10}, U^{0,nt}, U^{1,c}\} - \max\{\overline{Y}^{01}, L^{0,c}\}$.
Bounds for $E[Y(1,1)|nt]$: A5.3 and A6.3' imply $E[Y(1,1)|nt] \geq E[Y(1)|nt] \geq E[Y(1)|c]$, so $\overline{Y}^{10} \leq E[Y(1,1)|nt] \leq y^u$. And thus $E[Y(1,1)|nt] - E[Y(1)|nt]$ in (16) and (17) has the following bounds $0 \leq E[Y(1,1)|nt] - E[Y(1)|nt] \leq y^u - \overline{Y}^{10}$.
Bounds for $E[Y(0,1)|nt]$: A5.3 and A6.3' imply that $E[Y(0,D_1)|c] \leq E[Y(0,1)|nt]$, and $E[Y(0,1)|nt] \geq E[Y(0)|nt]$. Combining with the bounds derived for $E[Y(0,D_1)|c]$ and $E[Y(0)|nt]$ yields $\overline{Y}^{00} \leq E[Y(0,1)|nt] \leq y^u$. Since $\overline{Y}^{00} \leq E[Y(0)|nt] \leq \min\{\overline{Y}^{10}, U^{0,nt}\}$, we have $E[Y(0,1)|nt] - E[Y(0)|nt]$ in (19) and (20) has the following bounds
$0 \leq E[Y(0,1)|nt] - E[Y(0)|nt] \leq y^u - \overline{Y}^{00}$.
Bounds for $E[Y(1,0)|at]$: A5.3 and A6.3' imply that $E[Y(1)|at] \geq E[Y(1,0)|at]$, and $E[Y(1,D_0)|c] \geq E[Y(1,0)|at]$. Combining with the bounds previously derived for $E[Y(1,D_0)|c]$ and $E[Y(1)|at]$ yields $y^l \leq E[Y(1,0)|at] \leq \overline{Y}^{11}$. Since $\max\{\overline{Y}^{01}, L^{1,at}\} \leq E[Y(1)|at] \leq \overline{Y}^{11}$, we have $E[Y(1)|at] - E[Y(1,0)|at]$ in (16) and (17) has the following bounds $0 \leq E[Y(1)|at] - E[Y(1,0)|at] \leq \overline{Y}^{11} - y^l$.
Bounds for $E[Y(0,0)|at]$: A5.3 and A6.3' imply $E[Y(0)|c] \geq E[Y(0)|at] \geq E[Y(0,0)|at]$. Combining with the bounds previously derived for $E[Y(0)|c]$ yields $y^l \leq E[Y(0,0)|at] \leq \overline{Y}^{01}$. And thus $E[Y(0)|at] - E[Y(0,0)|at]$ in (19) and (20) has the following bounds
$0 \leq E[Y(0)|at] - E[Y(0,0)|at] \leq \overline{Y}^{01} - y^l$.

Bounds on $\Delta(z)$. We now use equations (16) through (18) to derive the lower bounds for $\Delta(1)$. The corresponding three potential lower bounds are:

$$\begin{aligned}
LB_\alpha^1 &= 0 \\
LB_\beta^1 &= E[Y|Z=1] - E[Y|Z=0] - \pi_{at}(\overline{Y}^{11} - \overline{Y}^{01}) \\
&\quad -\pi_{nt}(\overline{Y}^{10} - \overline{Y}^{00}) - \pi_c(\min\{\overline{Y}^{10}, U^{1,c}\} - \max\{\overline{Y}^{01}, L^{0,c}\}) \\
LB_\gamma^1 &= p_{1|1}\overline{Y}^{11} - p_{0|1}\overline{Y}^{10} - \pi_{at}\overline{Y}^{11} + \pi_{nt}\overline{Y}^{10} - \pi_c \min\{\overline{Y}^{10}, U^{1,c}\}
\end{aligned}$$

After some algebra, we have $LB_\beta^1 = \pi_c(\overline{Y}^{11} - \min\{\overline{Y}^{10}, U^{1,c}\} + \max\{\overline{Y}^{01}, L^{0,c}\} - \overline{Y}^{00}) \leq 0$. Thus, $LB_\beta^1 \leq LB_\alpha^1$. $LB_\gamma^1 = \pi_c(\overline{Y}^{11} - \min\{\overline{Y}^{10}, U^{1,c}\}) \leq 0$. Hence, $LB_\gamma^1 \leq LB_\alpha^1$. Therefore, the lower bound for $\Delta(1) = LB_\alpha^1 = 0$.

Similarly, we use equations (19) through (21) to derive potential lower bounds for $\Delta(0)$:

$$
\begin{aligned}
LB^0_\alpha &= 0 \\
LB^0_\beta &= E[Y|Z=1] - E[Y|Z=0] - \pi_{at}(\overline{Y}^{11} - \overline{Y}^{01}) \\
&\quad -\pi_{nt}(\overline{Y}^{10} - \overline{Y}^{00}) - \pi_c(\min\{\overline{Y}^{10}, U^{1,c}\} - \max\{\overline{Y}^{01}, L^{0,c}\}) \\
LB^0_\gamma &= p_{1|0}\overline{Y}^{01} - p_{0|0}\overline{Y}^{00} - \pi_{at}\overline{Y}^{01} + \pi_{nt}\overline{Y}^{00} + \pi_c \max\{\overline{Y}^{01}, L^{0,c}\}
\end{aligned}
$$

After comparison, we have $LB^0_\beta = LB^1_\beta \leq 0$. Thus, $LB^0_\beta \leq LB^0_\alpha$.
$LB^1_\gamma = \pi_c(\max\{\overline{Y}^{01}, L^{0,c}\} - \overline{Y}^{00})$. Hence, $LB^0_\gamma \leq LB^0_\alpha$. Therefore, the lower bound for $\Delta(0) = LB^0_\alpha = 0$.
We now use equations (16) through (18) to derive potential upper bounds for $\Delta(1)$:

$$
\begin{aligned}
UB^1_\alpha &= \pi_{at}(\overline{Y}^{11} - y^l) + \pi_{nt}(y^u - \overline{Y}^{10}) + \pi_c(\min\{\overline{Y}^{10}, U^{1,c}\} - \max\{\overline{Y}^{01}, L^{0,c}, L^{1,at}\}) \\
UB^1_\beta &= \pi_{at}(\overline{Y}^{11} - y^l) + \pi_{nt}(y^u - \overline{Y}^{10}) + E[Y|Z=1] - E[Y|Z=0] \\
&\quad -\pi_{at}\max\{0, L^{1,at} - \overline{Y}^{01}\} - \pi_{nt}\max\{0, \overline{Y}^{10} - U^{0,nt}\} - \pi_c \max\{0, L^{1,at} - \overline{Y}^{00}\} \\
UB^1_\gamma &= p_{1|1}\overline{Y}^{11} - p_{0|1}\overline{Y}^{10} - \pi_{at}y^l + \pi_{nt}y^u - \pi_c \max\{\overline{Y}^{01}, L^{0,c}, L^{1,at}\}
\end{aligned}
$$

To compare, $UB^1_\gamma - UB^1_\alpha = \pi_c(\overline{Y}^{11} - \min\{\overline{Y}^{10}, U^{1,c}\}) \leq 0$. Thus, UB^1_γ dominates UB^1_α.
$UB^1_\beta - UB^1_\gamma = \pi_{at}(\overline{Y}^{11} - \max\{L^{1,at}, \overline{Y}^{01}\}) + \pi_{nt}(\min\{\overline{Y}^{10}, U^{0,nt}\} - \overline{Y}^{00}) +$
$\pi_c(\max\{\overline{Y}^{01}, L^{0,c}, L^{1,at}\} - \max\{L^{1,at}, \overline{Y}^{00}\})$. Because $\overline{Y}^{11} - \max\{L^{1,at}, \overline{Y}^{01}\} \geq 0$,
$\min\{\overline{Y}^{10}, U^{0,nt}\} - \overline{Y}^{00} \geq 0$, and $\max\{\overline{Y}^{01}, L^{0,c}, L^{1,at}\} - \max\{L^{1,at}, \overline{Y}^{00}\} \leq 0$,
$UB^1 = \min\{UB^1_\beta, UB^1_\gamma\}$.
We now use equations (19) through (21) to derive potential upper bounds for $\Delta(0)$:

$$
\begin{aligned}
UB^0_\alpha &= \pi_{at}(\overline{Y}^{01} - y^l) + \pi_{nt}(y^u - \overline{Y}^{00}) + \pi_c(\min\{\overline{Y}^{10}, U^{0,nt}, U^{1,c}\} - \max\{\overline{Y}^{01}, L^{0,c}\}) \\
UB^0_\beta &= \pi_{at}(\overline{Y}^{01} - y^l) + \pi_{nt}(y^u - \overline{Y}^{00}) + E[Y|Z=1] - E[Y|Z=0] \\
&\quad -\pi_{at}\max\{0, L^{1,at} - \overline{Y}^{01}\} - \pi_{nt}\max\{0, \overline{Y}^{10} - U^{0,nt}\} - \pi_c \max\{0, \overline{Y}^{11} - U^{0,nt}\} \\
UB^0_\gamma &= p_{1|0}\overline{Y}^{01} - p_{0|0}\overline{Y}^{00} - \pi_{at}y^l + \pi_{nt}y^u + \pi_c \min\{\overline{Y}^{10}, U^{0,nt}, U^{1,c}\}
\end{aligned}
$$

Similarly, $UB^0_\gamma - UB^0_\alpha = \pi_c(\max\{\overline{Y}^{01}, L^{0,c}\} - \overline{Y}^{00}) \leq 0$. Thus, $UB^0_\gamma \leq UB^0_\alpha$.
$UB^0_\beta - UB^0_\gamma = p_{1|0}(\overline{Y}^{11} - \max\{L^{1,at}, \overline{Y}^{01}\}) + p_{0|1}(\min\{\overline{Y}^{10}, U^{0,nt}\} - \overline{Y}^{00}) + (p_{1|1} -$
$p_{1|0})(\min\{\overline{Y}^{11}, U^{0,nt}\} - \min\{\overline{Y}^{10}, U^{0,nt}, U^{1,c}\})$. Because $\overline{Y}^{11} - \max\{L^{1,at}, \overline{Y}^{01}\} \geq 0$,
$\min\{\overline{Y}^{10}, U^{0,nt}\} - \overline{Y}^{00} \geq 0$, and $\min\{\overline{Y}^{11}, U^{0,nt}\} - \min\{\overline{Y}^{10}, U^{0,nt}, U^{1,c}\} \leq 0$,
$UB^0 = \min\{UB^0_\beta, UB^0_\gamma\}$.
Finally, the bounds for $E[\Delta(z)]$ are obtained by directly plugging the corresponding terms into the equation $\Pr(Z=1)\Delta(1) + \Pr(Z=0)\Delta(0)$.

Bounds on ATT. Under the same set of the assumptions, $0 \leq \Gamma(0) \leq p_{1|0}(\overline{Y}^{01} - y^l)$. The lower bounds on $\Gamma(1)$ are:

$$\begin{aligned} lb^1_\alpha &= 0 \\ lb^1_\beta &= \pi_{at}(\overline{Y}^{01} - \overline{Y}^{11}) + E[Y|Z=1] - E[Y|Z=0] \\ &\quad -\pi_{nt}(\overline{Y}^{10} - \overline{Y}^{00}) - \pi_c(\min\{\overline{Y}^{10}, U^{1,c}\} - \max\{\overline{Y}^{01}, L^{0,c}\}) \\ lb^1_\gamma &= p_{1|1}\overline{Y}^{11} - \pi_{at}\overline{Y}^{11} - \pi_c \min\{\overline{Y}^{10}, U^{1,c}\} \end{aligned}$$

After arrangement, $lb^1_\beta = \pi_c(\overline{Y}^{11} - \min\{\overline{Y}^{10}, U^{1,c}\} + \max\{\overline{Y}^{01}, L^{0,c}\} - \overline{Y}^{00}) \leq 0$.
$lb^1_\gamma = \pi_c(\overline{Y}^{11} - \min\{\overline{Y}^{10}, U^{1,c}\}) \leq 0$. Thus, $lb^1 = lb^1_\alpha = 0$. The upper bounds on $\Gamma(1)$ are:

$$\begin{aligned} ub^1_\alpha &= \pi_{at}(\overline{Y}^{11} - y^l) + \pi_c(\min\{\overline{Y}^{10}, U^{1,c}\} - \max\{\overline{Y}^{01}, L^{0,c}, L^{1,at}\}) \\ ub^1_\beta &= \pi_{at}(\overline{Y}^{01} - y^l) + E[Y|Z=1] - E[Y|Z=0] \\ &\quad -\pi_{nt}\max\{0, \overline{Y}^{10} - U^{0,nt}\} - \pi_c \max\{0, L^{1,at} - \overline{Y}^{00}\} \\ ub^1_\gamma &= p_{1|1}\overline{Y}^{11} - \pi_{at}y^l - \pi_c \max\{\overline{Y}^{01}, L^{0,c}, L^{1,at}\} \end{aligned}$$

$ub^1_\alpha - ub^1_\gamma = \pi_c(\min\{\overline{Y}^{10}, U^{1,c}\} - \overline{Y}^{11}) \geq 0$.
$ub^1_\beta - ub^1_\gamma = \pi_{nt}(\min\{\overline{Y}^{10}, U^{0,nt}\} - \overline{Y}^{00}) + \pi_c(\max\{\overline{Y}^{01}, L^{0,c}, L^{1,at}\} - \max\{L^{1,at}, \overline{Y}^{00}\})$.
$\min\{\overline{Y}^{10}, U^{0,nt}\} - \overline{Y}^{00} \geq 0$, $\max\{\overline{Y}^{01}, L^{0,c}, L^{1,at}\} - \max\{L^{1,at}, \overline{Y}^{00}\} \leq 0$, so
$ub^1 = \min\{ub^1_\beta, ub^1_\gamma\}$. After rearrangement, $ub_a = \frac{w_1}{r_1} ub^1_\beta + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - y^l)$ and
$ub_b = \frac{w_1}{r_1} ub^1_\gamma + \frac{w_0 p_{1|0}}{r_1}(\overline{Y}^{01} - y^l)$. Q.D.E.

5.5 GMM Moment Function

We write the moment functions for average baseline characteristics of all the strata based on the conditional expectation defined by $\{Z, D\}$. Let $\overline{x}_k$ denote the expectation of a scalar baseline variable for a certain stratum k. The moment function for this variable is defined as:

$$g(\{\overline{x}_k\}) = \begin{bmatrix} (x - \overline{x}_{at})(1-Z)D \\ (x - \overline{x}_{nt})Z(1-D) \\ (x - \overline{x}_c \frac{\pi_c}{p_{1|1}} - \overline{x}_a \frac{\pi_{at}}{p_{1|1}})ZD \\ (x - \overline{x}_c \frac{\pi_c}{p_{0|0}} - \overline{x}_n \frac{\pi_{nt}}{p_{0|0}})(1-Z)(1-D) \\ x - \sum_k \pi_k \overline{x}_k \end{bmatrix}$$

where $\{\overline{x}_k\} = \{\overline{x}_{at}, \overline{x}_{nt}, \overline{x}_c\}$. By Law of Iterated Expectation, $E[g(\{\overline{x}_k\})] = 0$ when evaluated at the true value of $\{\overline{x}_k\}$.
Alternatively, we could also write the moment function for the proportions of all the strata and then estimate the model together with the average baseline characteristics simultaneously by GMM. However, such GMM estimators do not behave well in our data. Thus, in our application, we first identify the proportions of all the strata, and then estimate all the average baseline characteristics given the identified proportions. As seen in $g(\{\overline{x}_k\})$, for each

variable, we have 5 equations (4 derived from the conditional expectations defined by $\{Z, D\}$ plus one from the expectation for the entire sample) to identify 3 means, i.e., $\{\overline{x}_k\}$. Since the standard errors obtained from this GMM model do not take into account the fact that the proportions of the strata are also estimated, we employ a 500-repetition bootstrap to calculate the standard errors of the estimated average baseline characteristics.