# Selling Consumer Data for Profit: Optimal Market-Segmentation Design and its Consequences

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# Introduction

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This paper studies the **sale of consumer data (market segmentation)** and its implications.

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**Model outline**

- A unit mass of **consumers** with unit demand.
- A **producer** sells a product to the consumers at a **constant marginal cost**.
- The marginal cost is **private information**.
- **A data broker** can sell consumer data to the producer using **any selling mechanism** (but cannot contract on how the data are used).

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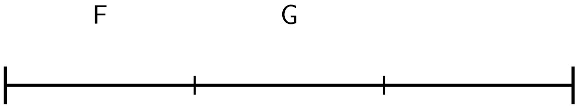

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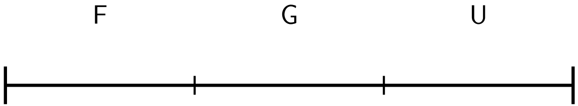

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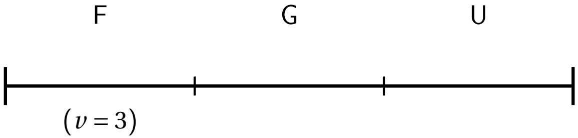

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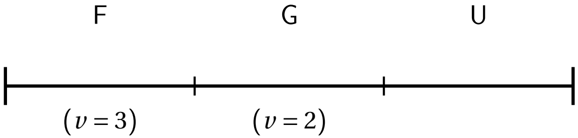

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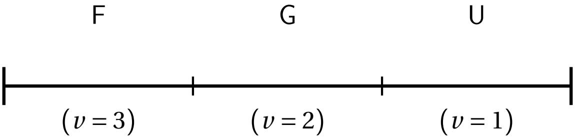

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$(v = 3)$ $(v = 2)$ $(v = 1)$

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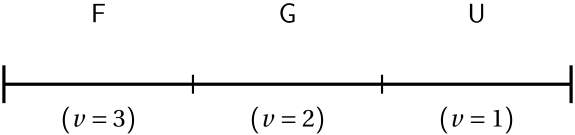

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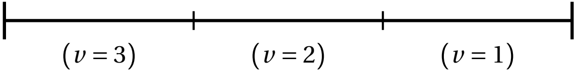

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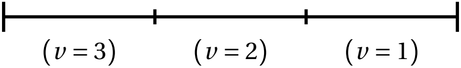

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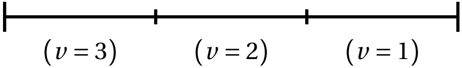

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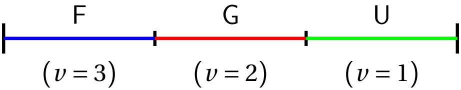

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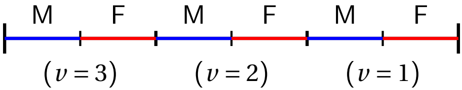

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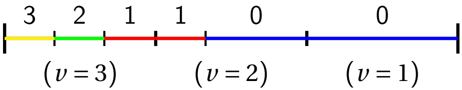

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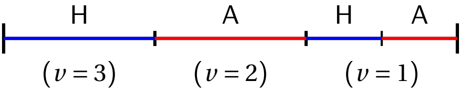

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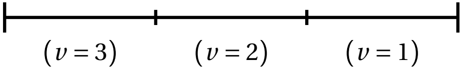

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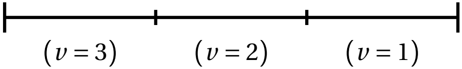

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How should the data broker sell these data?

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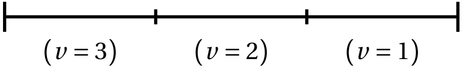

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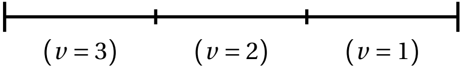

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**Relaxed problem**: Suppose that the broker can also contract on quantity.

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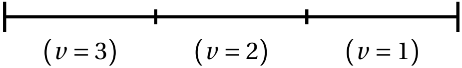

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$\Big($data (partition of the line), $q$ (upper-bound for quantity sold), $\tau$ (payment)$\Big)$

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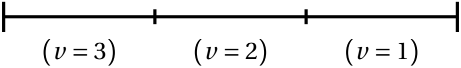

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The publisher purchases one item. Otherwise, she obtains her optimal uniform pricing profit: $\max\left\{(1-c), \frac{2}{3}(2-c), \frac{1}{3}(3-c)\right\}$.

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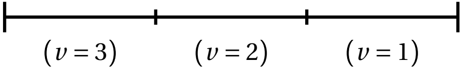

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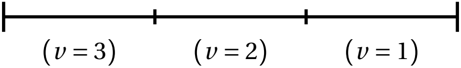

Optimal menu:

$$\left\{\left(\text{value-revealing data}, q=1, \tau=\frac{2}{3}\right), \left(\text{value-revealing data}, q=\frac{2}{3}, \tau=\frac{1}{3}\right)\right\}.$$

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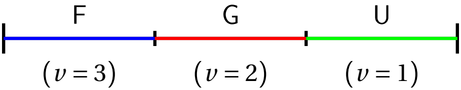

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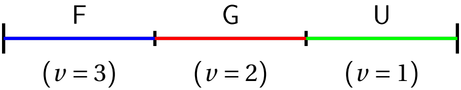

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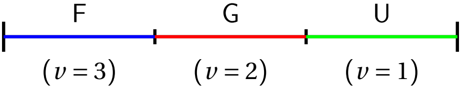

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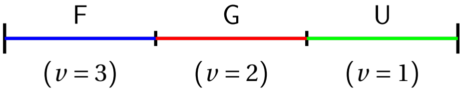

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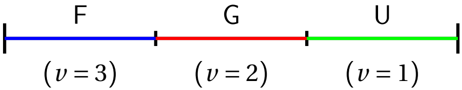

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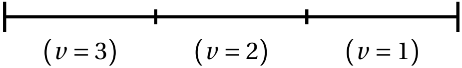

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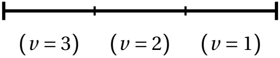

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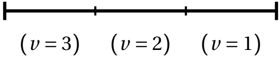

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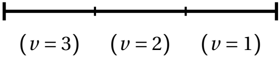

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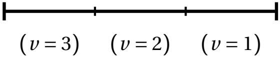

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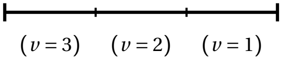

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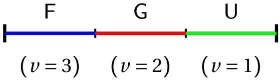

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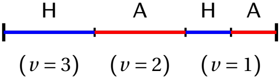

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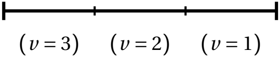

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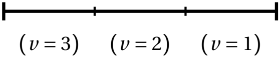

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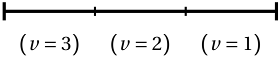

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value-revealing data + $\left\{\left(q=1,\tau=\frac{2}{3}\right),\left(q=\frac{2}{3},\tau=\frac{1}{3}\right)\right\}$.

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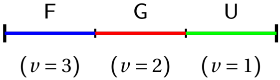

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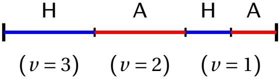

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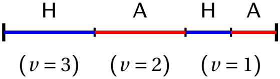

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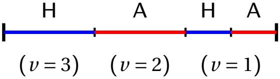

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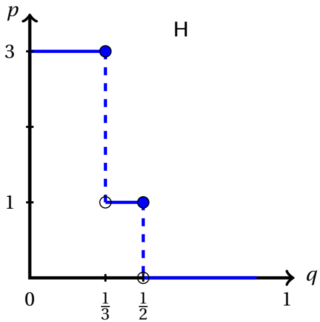

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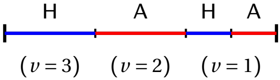

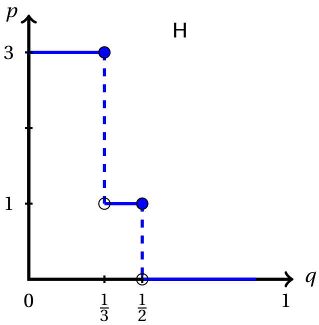

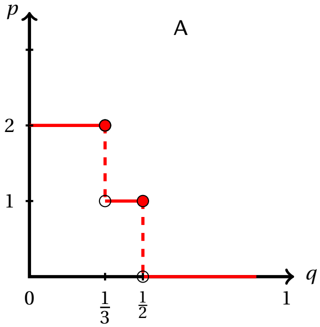

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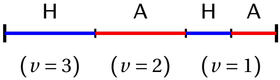

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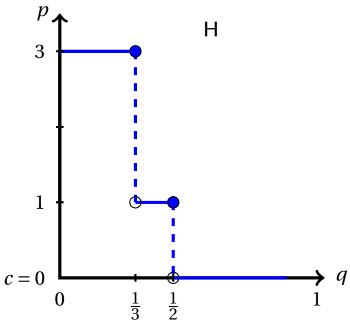

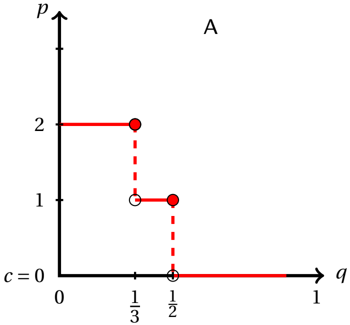

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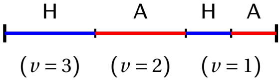

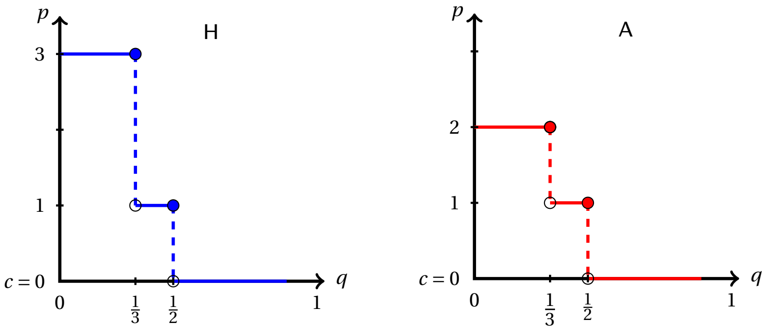

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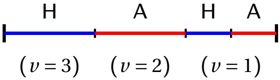

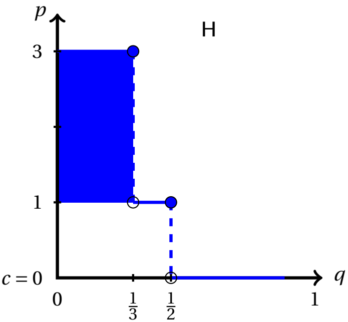

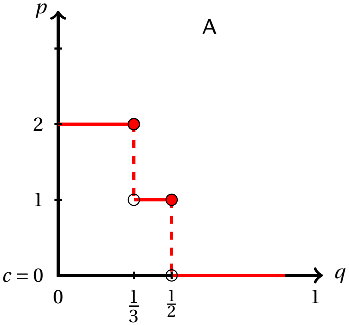

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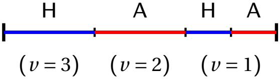

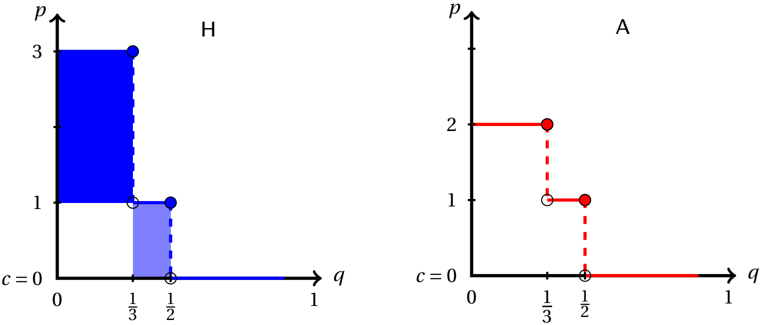

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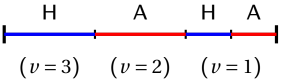

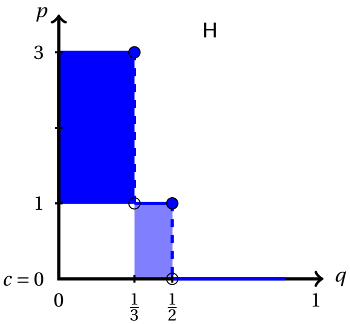

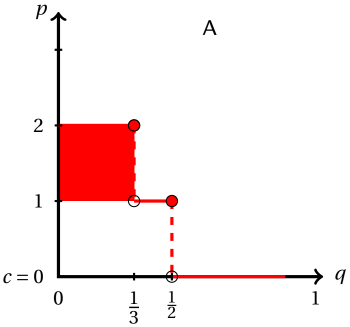

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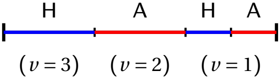

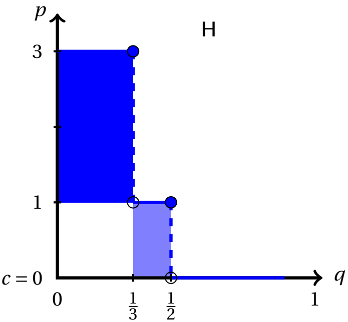

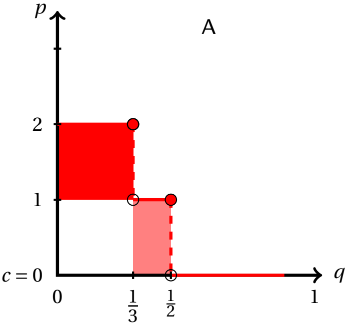

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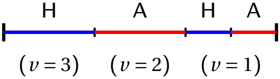

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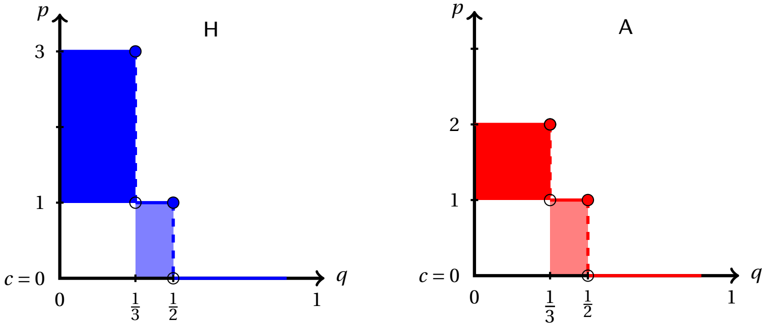

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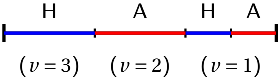

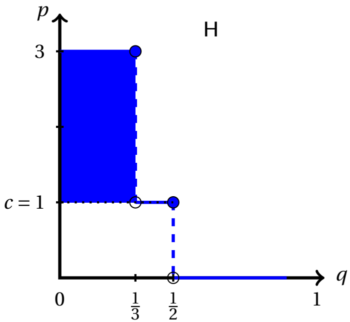

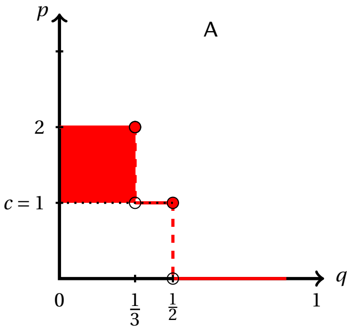

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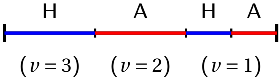

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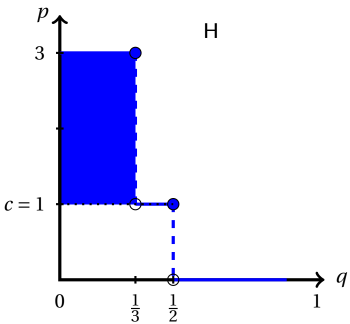

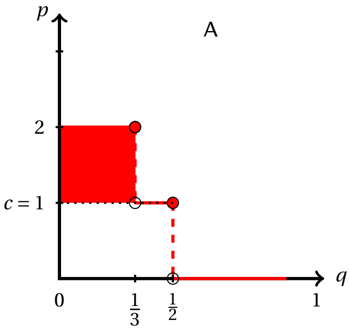

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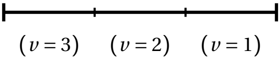

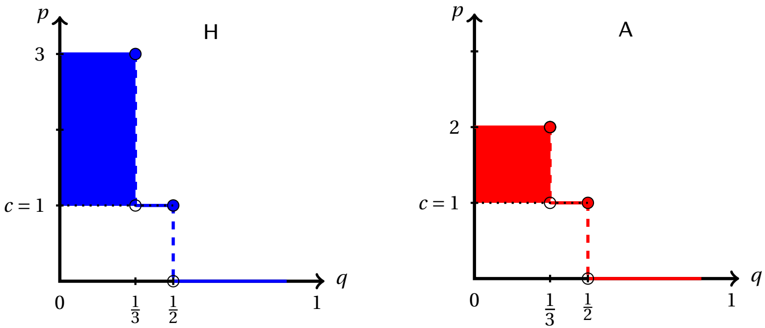

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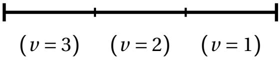

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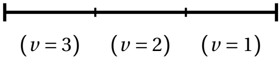

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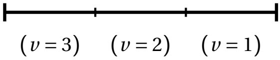

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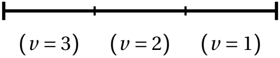

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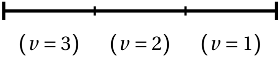

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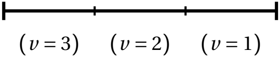

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$\mathscr{M}^*$ replicates the outcome even if the broker cannot contract on quantity.

# Some Remarks

$\mathscr{M}^*$ is optimal.

It separates the high-value consumers while pooling the low-value consumers with them.
⇒ Pooling low-value consumers to discourage trade.

$\mathscr{M}^*$ remains optimal even when $c \in \{\varepsilon, 1-\varepsilon\}$ for $\varepsilon$ small enough.
⇒ Consumers with $v = 1$ may not be served even if there are gains from trade.

The data broker's optimal revenue is the same even if he can contract on quantity.

These features continue to hold in a more general model.

# Model

# Model

Single product, one producer (she),
a unit mass of consumers, and a *data broker* (he).

Consumers: Unit demand, values $v \in V = [\underline{v}, \overline{v}] \subset \mathbb{R}_+$,
$D_M$ (market demand) describes the value distribution
(i.e., $D_M(p)$: share of consumers with $v \geq p$ for all $p \in V$).

$D_M$ is nonincreasing, u.s.c., $D_M(\underline{v}) = 1$, $D_M(\overline{v}^+) = 0$.

Assume: $D_M$ is regular (i.e., $D_M$ is decreasing, differentiable and the marginal revenue of $D_M$ is decreasing)

# Model

$\mathscr{D}$: collection of demand functions that are of the same sizes as $D_M$:

$$\mathscr{D} := \{D : V \to [0,1] | D \text{ nonincreasing, u.s.c., } D(\underline{v}) = 1, D(\overline{v}^+) = 0\}.$$

# Model

$\mathscr{D}$: collection of demand functions that are of the same sizes as $D_M$:

$$\mathscr{D} := \{D : V \to [0,1] | D \text{ nonincreasing, u.s.c., } D(\underline{v}) = 1, D(\overline{v}^+) = 0\}.$$

A **market segmentation** is a way to split the market demand $D_M$. i.e., $s \in \Delta(\mathscr{D})$ s.t.

$$\int_{\mathscr{D}} D(p)s(\mathrm{d}D) = D_M(p), \forall p.$$

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The data broker can sell to the producer any market segmentation.

- Can arbitrarily segment the consumers according to their values.
- Can always reveal the value.
- Equivalent to selling any Blackwell experiment.

# Model

The producer:

- Sells the product to consumers.
- Has private marginal cost of production $c \in C = [\underline{c}, \bar{c}] \subset \mathbb{R}_+$.
- $c \sim G$, $G$ has density $g > 0$ on $C$.
- Let $\phi_G(c) := c + G(c)/g(c)$ denote the virtual cost.
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Given market segmentation $s \in \Delta(\mathscr{D})$, the producer with cost $c$ solves

$$\max_{p \geq 0}(p - c)D(p),$$

for all $D$ in the support of $s$.

# Model

A mechanism is a pair $(\sigma, \tau)$ that specifies, for each reported cost $c$,

- a market segmentation $\sigma(c) \in \Delta(\mathscr{D})$
- and a transfer $\tau(c) \in \mathbb{R}$ from the producer to the data broker.

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Comparison to standard monopolistic pricing models:

- Large (infinite-dimensional) allocation space.
- Type-dependent outside option.

# Model

For any $c \in C$ and for and $D \in \mathscr{D}$, let

$$\pi_D(c) := \max_{p \geq 0}(p - c)D(p).$$

A mechanism $(\sigma, \tau)$ is:

- **incentive compatible** if for any $c, c' \in C$

$$\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \tau(c) \geq \int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c') - \tau(c');$$

- **individually rational** if for any $c \in C$,

$$\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \tau(c) \geq \pi_{D_M}(c).$$

# Optimal Mechanism

# Revenue Equivalence Formula

For any $c \in C$ and for any $D \in \mathscr{D}$, let $\boldsymbol{p}_D(c)$ be the largest solution of

$$\max_{p \geq 0} (p - c) D(p),$$

Recall:

$$\phi_G(c) := c + \frac{G(c)}{g(c)}$$

is the virtual cost induced by $G$.

# Revenue Equivalence Formula

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*A mechanism $(\sigma, \tau)$ is incentive compatible if and only if*

1. *Revenue equivalence*
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2. $\int_c^{c'} \int_{\mathcal{D}} D(\boldsymbol{p}_D(z))(\sigma(\mathrm{d}D|z) - \sigma(\mathrm{d}D|c))\,\mathrm{d}z \geq 0$, *for all* $c, c'$

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The expected revenue under any IC mechanism $(\sigma,\tau)$ can be written as

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(maximize profit w.r.t. $\phi_G(c)$ pointwise)

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(profit ≤ total surplus)

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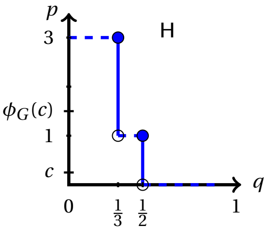

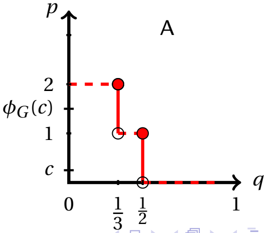

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# Quasi-Perfect Price Discrimination

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$$\boxed{=} \int_C \left( \int_{\{v \geq \phi_G(c)\}} (v - \phi_G(c)) D_M(\mathrm{d}v) \right) G(\mathrm{d}c) - \pi_{D_M}(\overline{c})$$

**Definition.**

$s \in \Delta(\mathscr{D})$ induces **quasi-perfect price discrimination** for $c$ if the producer with cost $c$, when operates under $s$,

All the consumers with $v \geq \phi_G(c)$ buy and pay their values;
All the consumers with $v < \phi_G(c)$ do not buy

**Remark.**

Suppose that there is an IC & IR mechanism $(\sigma, \tau)$ such that for all $c$, $\sigma(c)$ induces quasi-perfect price discrimination for $c$.
$\Rightarrow$ $(\sigma, \tau)$ is optimal!

# Constructing an Optimal Mechanism

Assume (for this talk): $\phi_G(c) \leq \boldsymbol{p}_{D_M}(c)$ for all $c \in C$.

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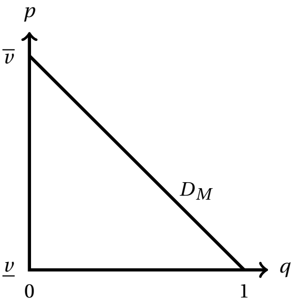

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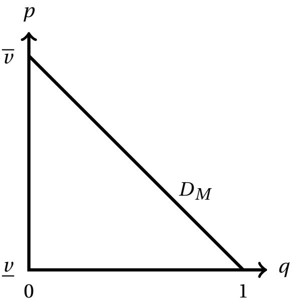

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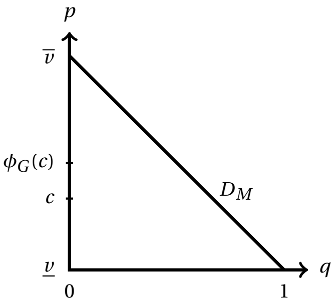

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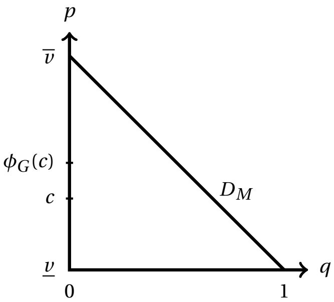

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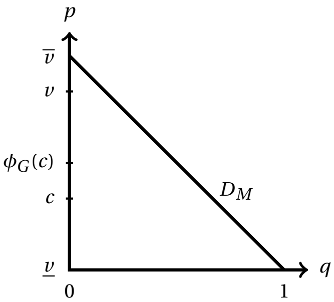

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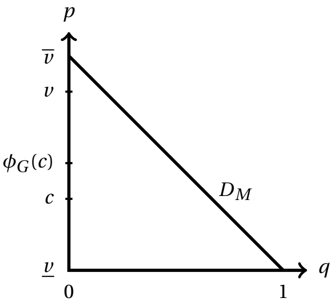

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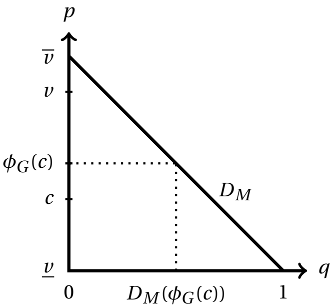

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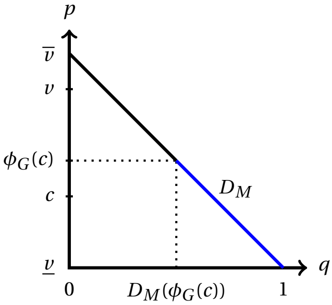

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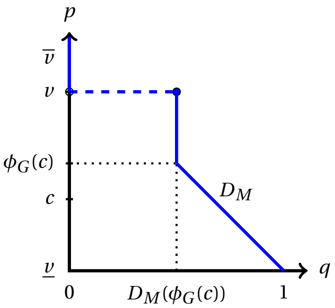

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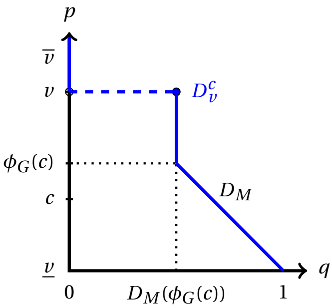

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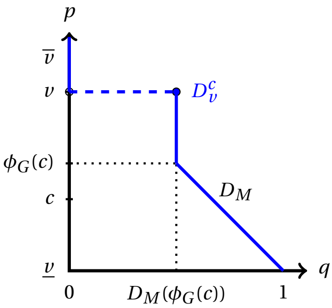

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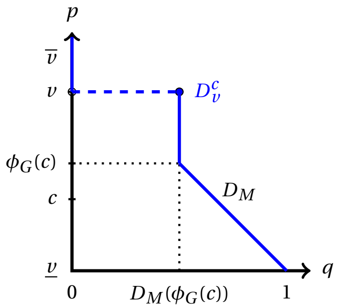

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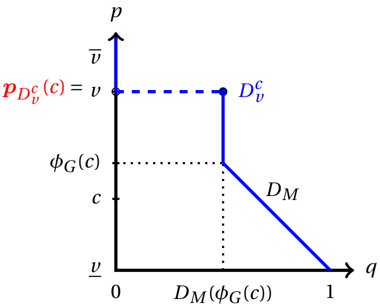

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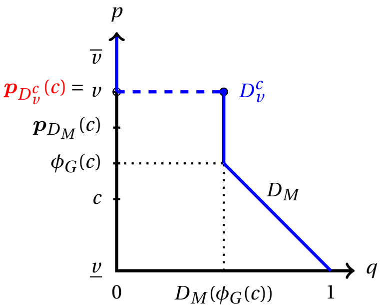

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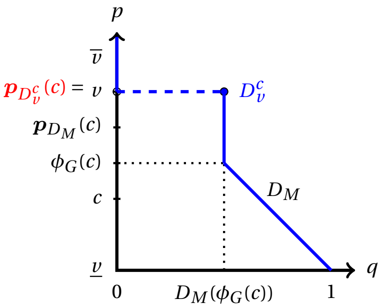

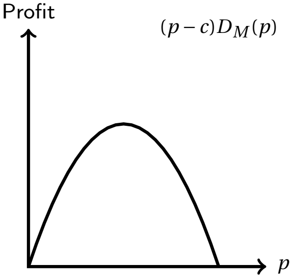

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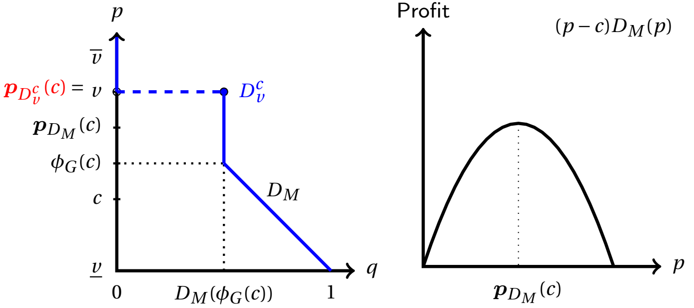

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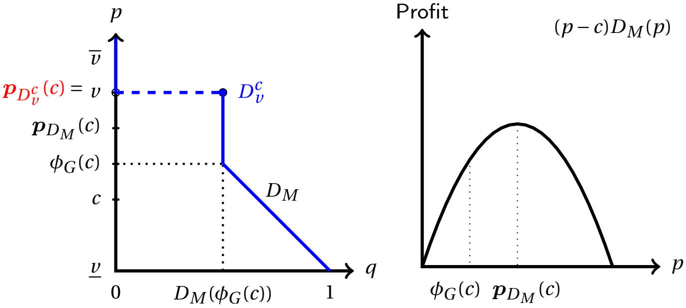

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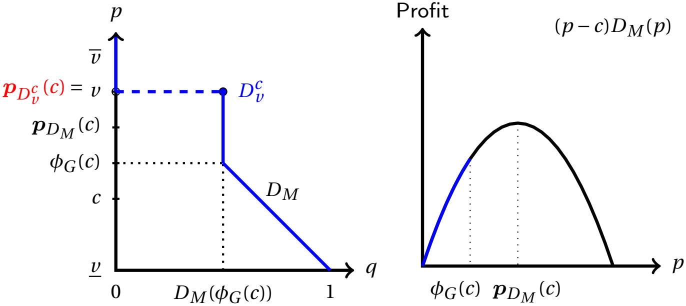

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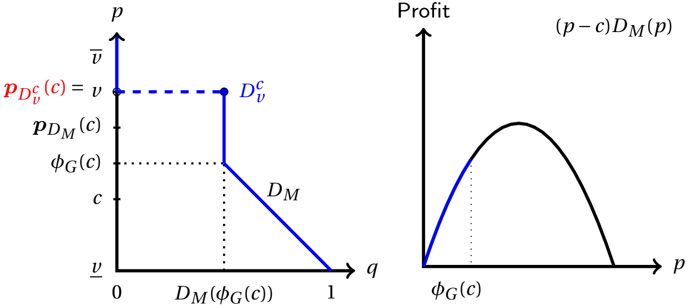

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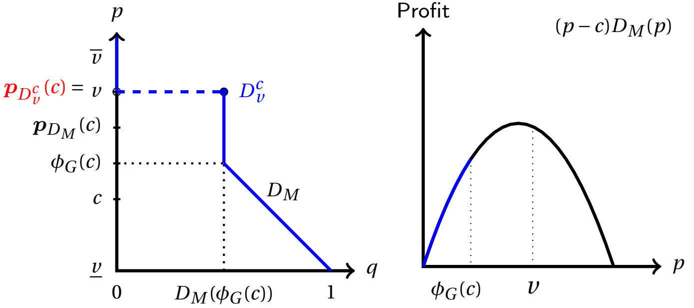

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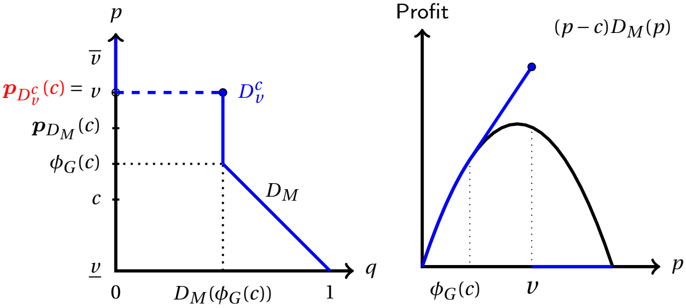

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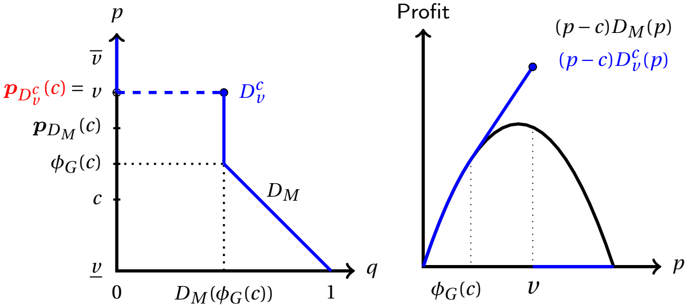

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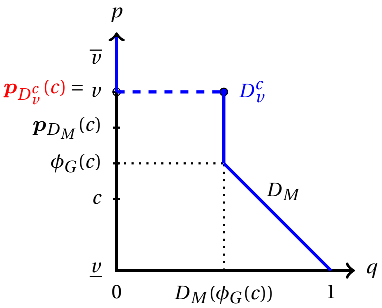

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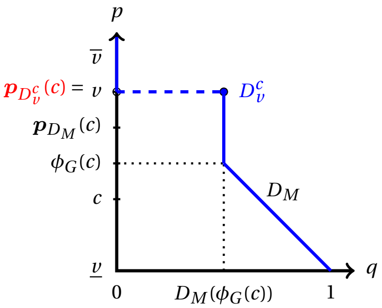

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**Theorem (Optimal Mechanism)**

$(\sigma^*, \tau^*)$ *is an optimal mechanism. Furthermore, for any optimal mechanism* $(\sigma, \tau)$ *and for any* $c$, $\sigma(c)$ *induces quasi-perfect price discrimination for* $c$.

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**Need to show:**

- For any $c$, $\sigma^*(c)$ induces quasi-perfect price discrimination for $c$.
- $(\sigma^*,\tau^*)$ is IC & IR .

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**Need to show:**

- For any $c$, $\sigma^*(c)$ induces quasi-perfect price discrimination for $c$.✓
- $(\sigma^*, \tau^*)$ is IC & IR .

# Constructing an Optimal Mechanism

$\sigma^*(c)$ has support $\{D_v^c\}_{v \in [\phi_G(c),\overline{v}]}$ and assigns density $\frac{|D'_M(v)|}{D_M(\phi_G(c))}$ to $D_v^c$.

For all $c \in C$, let

$$\tau^*(c) := \int_{\mathscr{D}} \pi_D(c)\sigma^*(\mathrm{d}D|c) - \int_c^{\overline{c}} \left( \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma^*(\mathrm{d}D|z) \right) \mathrm{d}z - \pi_{D_M}(\overline{c})$$

**Theorem (Optimal Mechanism)**

*$(\sigma^*,\tau^*)$ is an optimal mechanism. Furthermore, for any optimal mechanism $(\sigma,\tau)$ and for any $c$, $\sigma(c)$ induces quasi-perfect price discrimination for $c$.*

**Need to show:**

- For any $c$, $\sigma^*(c)$ induces quasi-perfect price discrimination for $c$.✓
- $(\sigma^*,\tau^*)$ is IC & IR (later if time permits).

# Optimal Mechanism: Some Remarks

Screening cost ⇒ Data broker has a higher marginal cost than the producer (i.e., $\phi_G(c) \geq c$)

Optimal mechanism pools low-value consumers (i.e., $v \in [c, \phi_G(c)]$) with thew high-values: Preventing the producer from selling at prices below $\phi_G(c)$.

Data broker's revenue is the same even if he can contract on prices.

Allocation is inefficient: Some consumers with $v \geq c$ do not buy.

# Implications

# Surplus Extraction

**Theorem (Surplus Extraction)**

*Consumer surplus is zero under any optimal mechanism*

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**Theorem (Surplus Extraction)**

*Consumer surplus is zero under any optimal mechanism*

**Proof.**

Quasi-perfect price discrimination

⇒ Conditional on purchasing, every consumer pays their value.
Consumer surplus is zero ■

Separating the ownership of consumer data and the ownership of production technology does not benefit the consumers.

⇒ Better to make $c$ common knowledge.

# Outcome Equivalence

There are other natural market regimes under which the data broker can profit from the consumer data he owns.

- Exclusive retail.
- Price-controlling data brokership.

# Exclusive Retail

**Exclusive retail:**
$c$ is private information.

The data broker purchases the product from the producer as a monopsony.

Then the broker sells the purchased product to the consumers **exclusively**, via **perfect price discrimination**.

If the producer does not sell to the broker, she sells to the consumers without data and receives $\pi_{D_M}(c)$.

# Price-Controlling Data Brokership

**Price-Controlling Data Brokership:**
$c$ is private information. The data broker designs a mechanism $(\sigma, \tau, \gamma)$.

For each report $c \in C$,

- $\sigma(c) \in \Delta(\mathscr{D})$: segmentation provided to the producer.
- $\tau(c) \in \mathbb{R}$: payment from the producer to the data broker.
- $\gamma(\cdot|D,c) \in \Delta(\mathbb{R}_+)$: distribution from which price charged in segmet $D$ is drawn.

# Outcome Equivalence

**Theorem (Outcome Equivalence)**

*Exclusive retail, price-controlling data brokership and data brokership are outcome-equivalent.*

# Outcome Equivalence

**Implications**

- Data brokers have no incentives to play a more active role in the product market.
- No concerns even if a data broker gains control over the product market.
- The ability to create and sell market segmentations makes the data broker influential in the product market.

# Discussion and Extension

# Further Discussions

Technical Assumption: Can be relaxed for most of the results.

Data broker's ability to create *any* market segmentation $s \in \Delta(\mathscr{D})$.

- The ability to reveal the value ⇒ Can be extended.
- The ability to split $D_M$ arbitrarily

⇒ Can be interpreted as partitioning an abstract characteristic space.

Comparison with uniform pricing & Consumers' property right over data.

Private information about the market.

Can allow targeting marketing.

Thank you!

# Proof of the Main Theorem

# Sketch of Proof

**Theorem (Optimal Mechanism)**

*$(\sigma^*, \tau^*)$ is an optimal mechanism. Furthermore, for any optimal mechanism $(\sigma, \tau)$ and for any $c$, $\sigma(c)$ must induce quasi-perfect price discrimination for $c$.*

Suffices to show:

- For any $c$, $\sigma^*(c)$ induces quasi-perfect price discrimination for $c$.
- $(\sigma^*, \tau^*)$ is IC & IR.

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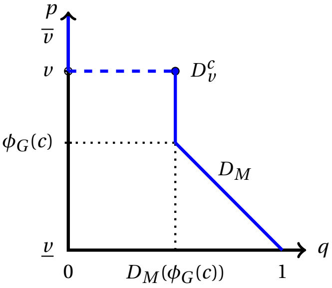
Suffices to show:

- For any $c$, $\sigma^*(c)$ induces quasi-perfect price discrimination for $c$.✓
- $(\sigma^*, \tau^*)$ is IC & IR.

# Sketch of Proof

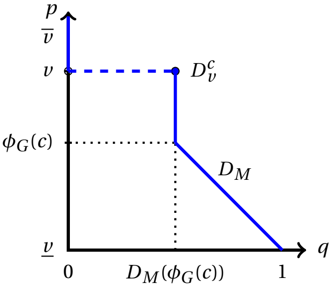
For any $c \in C$ and for any $v \geq \phi_G(c)$, define $D_v^c$ as

$$D_v^c(p) := \begin{cases} D_M(p) & \text{if } p \in [\underline{v}, \phi_G(c)] \\ D_M(\phi_G(c)) & \text{if } p \in (\phi_G(c), v] \\ 0, & \text{if } p \in (v, \overline{v}] \end{cases} .$$



$\sigma^*(c)$ has support $\{D_v^c\}_{v \in [\phi_G(c), \overline{v}]}$ and assigns size $\frac{|D_M'(v)|}{D_M(\phi_G(c)}$ to $D_v^c$.

# Sketch of Proof



$\sigma^*(c)$ has support $\{D_v^c\}_{v \in [\phi_G(c), \overline{v}]}$ and assigns size $\frac{|D'_M(v)|}{D_M(\phi_G(c)}$ to $D_v^c$.

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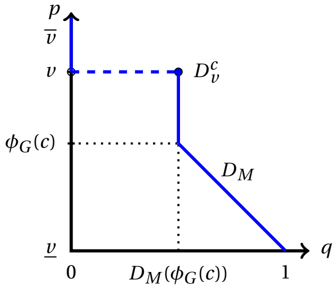

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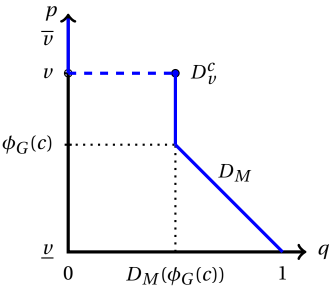


$\sigma^*(c)$ has support $\{D_v^c\}_{v\in[\phi_G(c),\overline{v}]}$ and assigns size $|D'_M(v)|$ to $D_v^c$.

**Lemma**

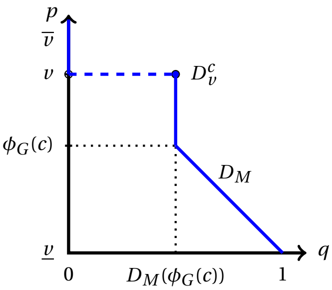
*For any* $c \in C$ *and for any* $D \in \mathrm{supp}(\sigma^*(c))$,

$$\phi_G(z) \leq \boldsymbol{p}_D(z), \tag{*}$$

*for all* $z \in [\underline{c}, c]$.

# Sketch of Proof

Need to show: $\phi_G(z) \le \boldsymbol{p}_{D_v^c}(z)$ for all $z \le c$.



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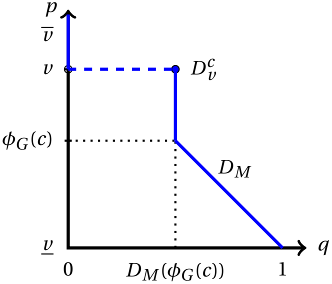

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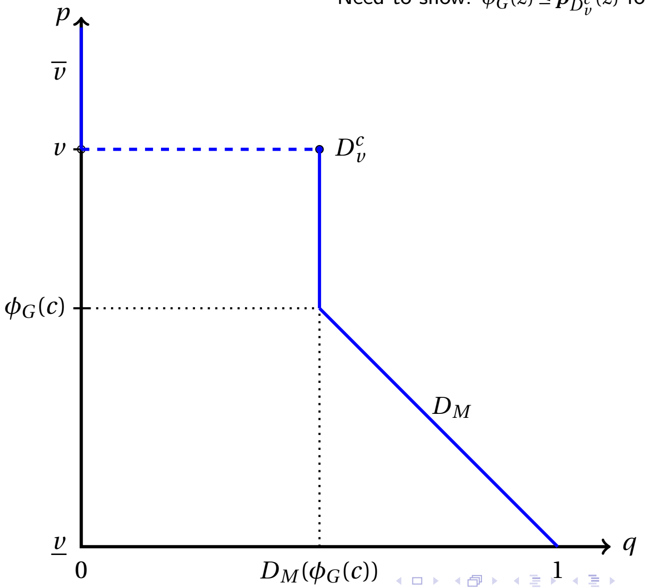

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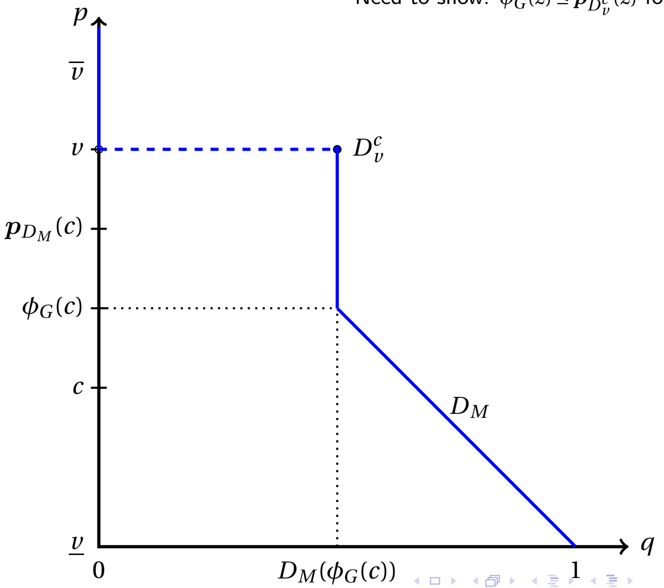

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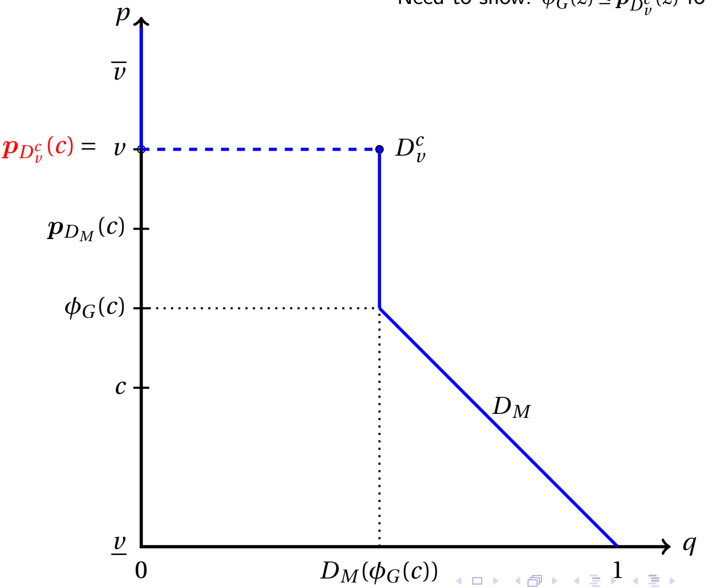

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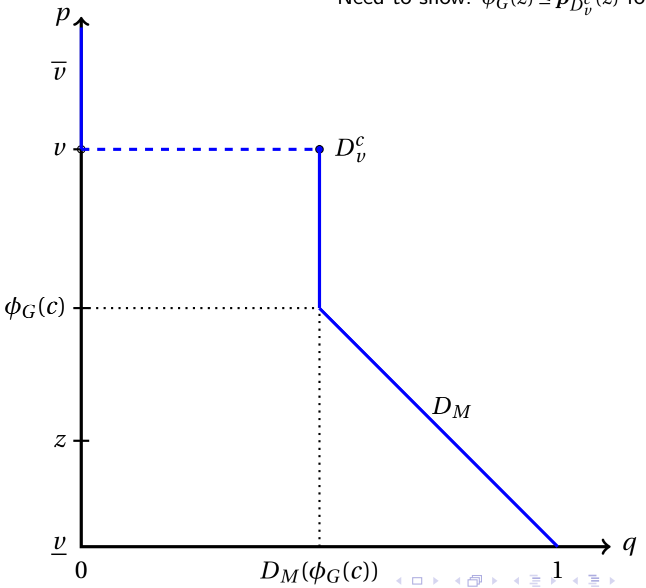

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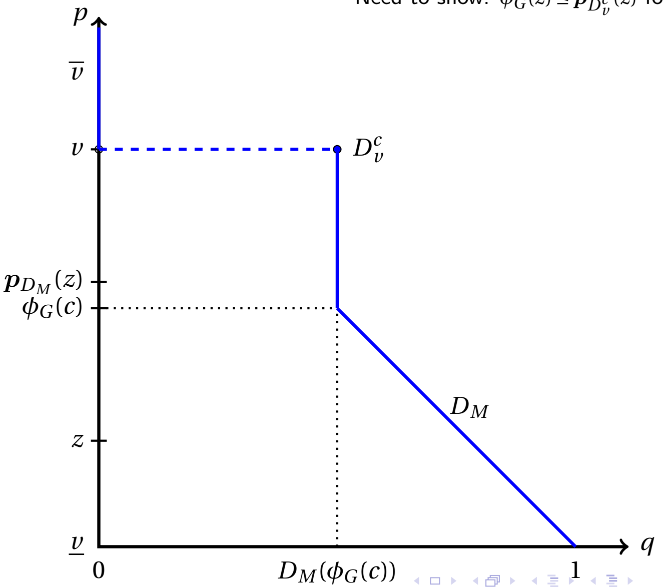

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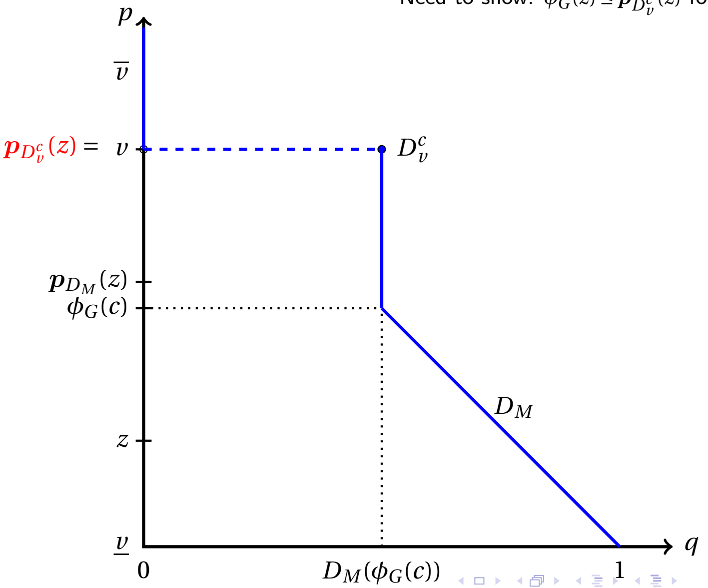

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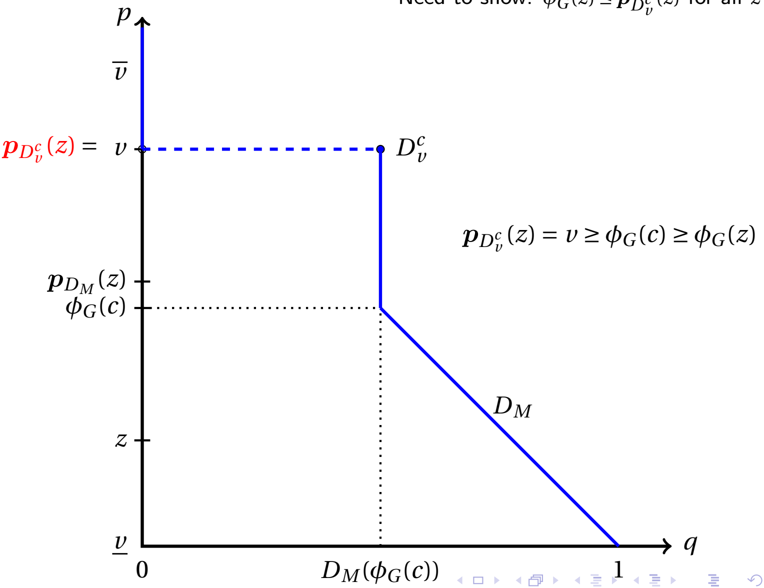

$$\boldsymbol{p}_{D_v^c}(z) = v \geq \phi_G(c) \geq \phi_G(z)$$

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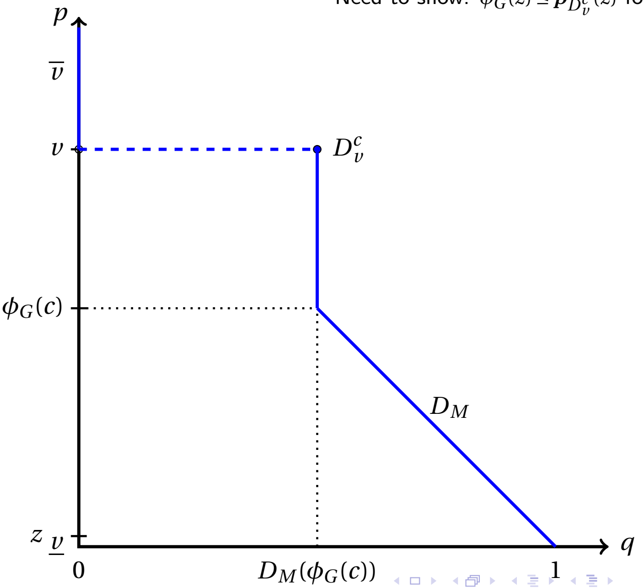

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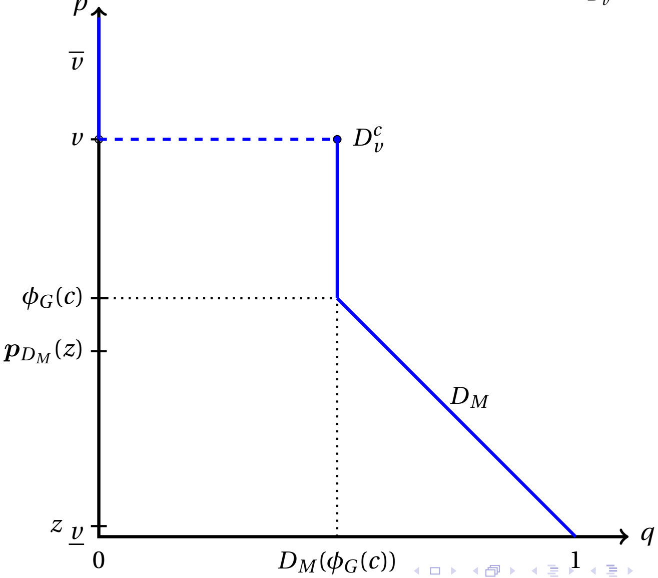

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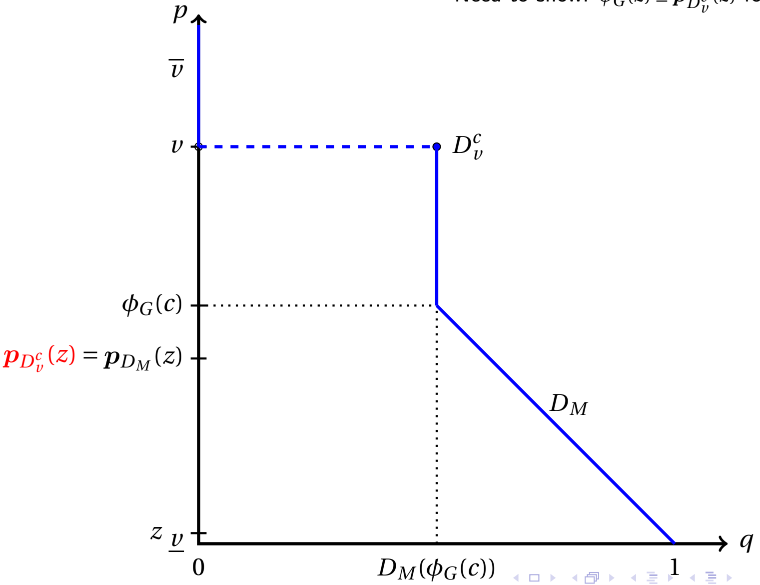

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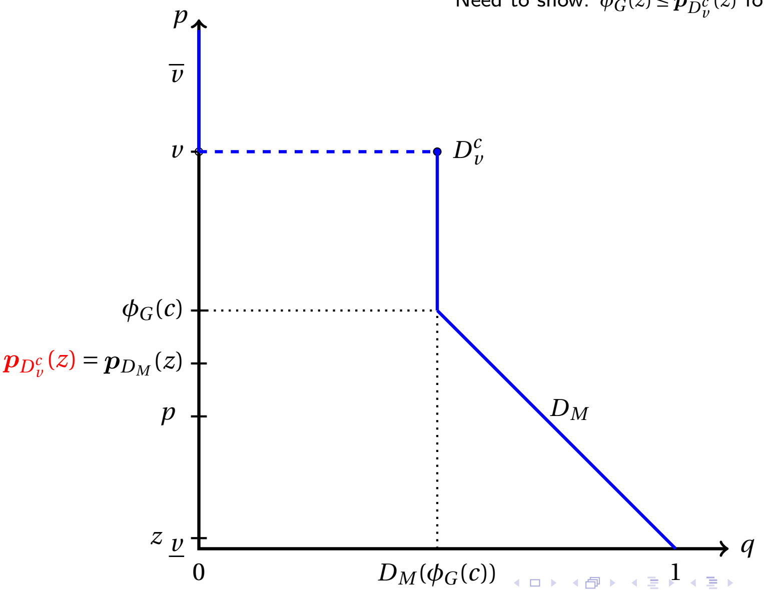

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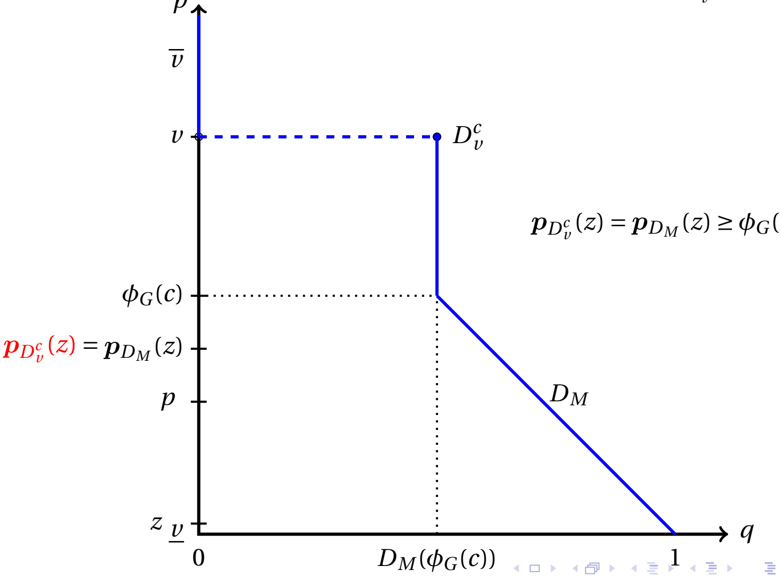

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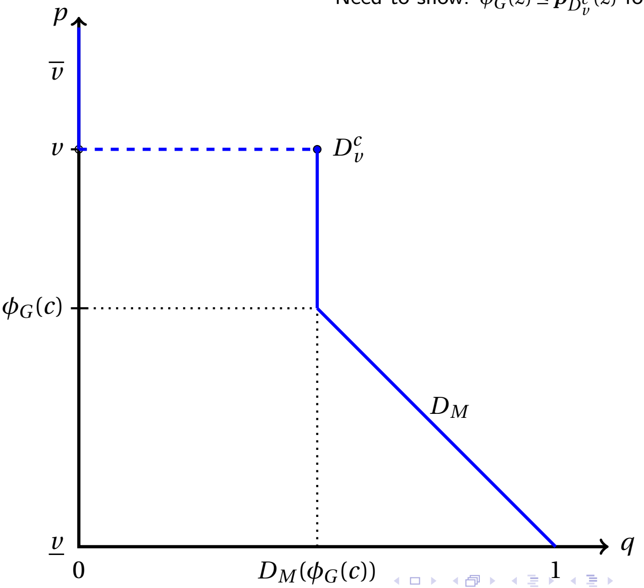

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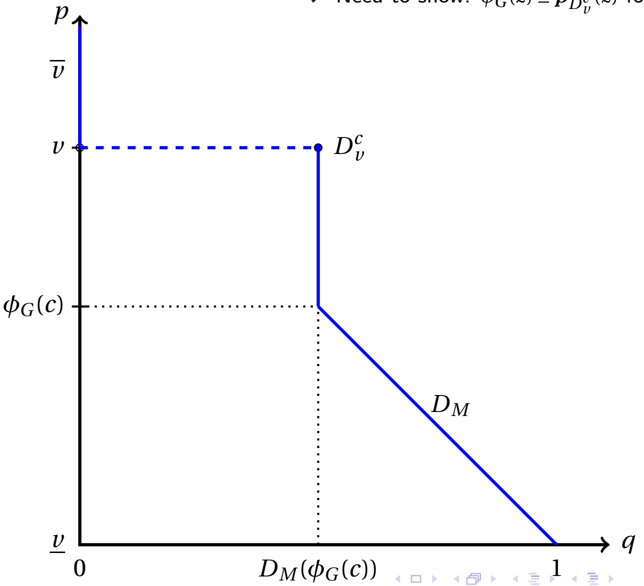

# Sketch of Proof

✓ Need to show: $\phi_G(z) \leq \boldsymbol{p}_{D_v^c}(z)$ for all $z \leq c$.

# Sketch of Proof

**Lemma**

*For any* $c \in C$ *and for any* $D \in \mathrm{supp}(\sigma^*(c))$,

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For any $c \in C$ and for any $D \in \mathrm{supp}(\sigma^*(c))$,
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For any $c \in C$ and for any $D \in \operatorname{supp}(\sigma^*(c))$,
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## Lemma (Revenue Equivalence Lemma)

*A mechanism $(\sigma, \tau)$ is incentive compatible if and only if*

1. *For any $c \in C$,*

$$\tau(c) = \int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \int_c^{\overline{c}} \left( \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma(\mathrm{d}D|z) \right) \mathrm{d}z + U(\overline{c}).$$

2. *For any $c, c' \in C$,*

$$\int_{c'}^{c} \left( \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma(\mathrm{d}D|z) - \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma(\mathrm{d}D|c) \right) \mathrm{d}z \ge 0.$$

▸ Derivation

# Sketch of Proof

For any $c \in C$ and for any $D \in \mathrm{supp}(\sigma^*(c))$,
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For any $c \in C$ and for any $D \in \text{supp}(\sigma^*(c))$,
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For any $c \in C$ and for any $D \in \operatorname{supp}(\sigma^*(c))$,
$\phi_G(z) \leq \boldsymbol{p}_D(z)$, $\forall z \in [\underline{c}, c]$.

For any $c, c'$ with $c' < c$,

# Sketch of Proof

For any $c \in C$ and for any $D \in \operatorname{supp}(\sigma^*(c))$,
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$$= \int_{c'}^{c} \left( D_M(\phi_G(z)) - \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma^*(\mathrm{d}D|c) \right) \mathrm{d}z$$

# Sketch of Proof

For any $c \in C$ and for any $D \in \text{supp}(\sigma^*(c))$,
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\end{aligned}$$

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For any $c \in C$ and for any $D \in \text{supp}(\sigma^*(c))$,
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&= \int_{c'}^{c} \left( D_M(\phi_G(z)) - D_M(\phi_G(z)) \right) \mathrm{d}z
\end{aligned}$$

# Sketch of Proof

For any $c \in C$ and for any $D \in \operatorname{supp}(\sigma^*(c))$,
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=&0.
\end{aligned}$$

# Sketch of Proof

For any $c \in C$ and for any $D \in \operatorname{supp}(\sigma^*(c))$,
$\phi_G(z) \leq \boldsymbol{p}_D(z)$, $\forall z \in [\underline{c}, c]$.

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=& \int_{c'}^{c} \left( D_M(\phi_G(z)) - D_M(\phi_G(z)) \right) \mathrm{d}z \\
=& 0.
\end{aligned}$$

▸ IR

# Sketch of Proof

For any $c \in C$ and for any $D \in \text{supp}(\sigma^*(c))$,
$\phi_G(z) \leq \boldsymbol{p}_D(z)$, $\forall z \in [\underline{c}, c]$.

# Sketch of Proof

For any $c \in C$ and for any $D \in \text{supp}(\sigma^*(c))$,
$\phi_G(z) \leq \boldsymbol{p}_D(z)$, $\forall z \in [\underline{c}, c]$.

For any $c, c'$ with $c' > c$,

# Sketch of Proof

For any $c \in C$ and for any $D \in \text{supp}(\sigma^*(c))$,
$\phi_G(z) \leq \boldsymbol{p}_D(z)$, $\forall z \in [\underline{c}, c]$.

For any $c, c'$ with $c' > c$,

$$\int_c^{c'} \left( \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma^*(\mathrm{d}D|c) - \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma^*(\mathrm{d}D|z) \right) \mathrm{d}z$$

# Sketch of Proof

For any $c \in C$ and for any $D \in \operatorname{supp}(\sigma^*(c))$,
$\phi_G(z) \leq \boldsymbol{p}_D(z),\ \forall z \in [\underline{c}, c]$.

For any $c, c'$ with $c' > c$,

$$\int_c^{c'} \left( \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma^*(\mathrm{d}D|c) - \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma^*(\mathrm{d}D|z) \right) \mathrm{d}z$$

$$= \int_c^{c'} \left( \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma^*(\mathrm{d}D|c) - D_M(\phi_G(z)) \right) \mathrm{d}z$$

# Sketch of Proof

For any $c \in C$ and for any $D \in \mathrm{supp}(\sigma^*(c))$,
$\phi_G(z) \le \boldsymbol{p}_D(z)$, $\forall z \in [\underline{c}, c]$.

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$$= \int_c^{c'} \left( \min\{D_M(\phi_G(c)), D_M(z)\} - D_M(\phi_G(z)) \right) \mathrm{d}z$$

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=& \int_c^{c'} \left( \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma^*(\mathrm{d}D|c) - D_M(\phi_G(z)) \right) \mathrm{d}z \\
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=&0.
\end{aligned}$$

# Sketch of Proof

By the revenue equivalence formula, $(\sigma^*, \tau^*)$ is IC.

Moreover, the producer's indirect utility is

$$\pi_{D_M}(\overline{c}) + \int_{c}^{\overline{c}} \left( \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma^*(\mathrm{d}D|z) \right) \mathrm{d}z$$

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# Sketch of Proof

Recall:

$\phi_G(c) \le \boldsymbol{p}_{D_M}(c)$ for all $c$.

$\pi_D(c) = \max_p (p-c)D(p) \Rightarrow \pi'_D(c) = -D(\boldsymbol{p}_D(c))$.

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\end{aligned}$$

Hence, $(\sigma^*, \tau^*)$ is IR. ■

# Surplus Extraction

**Theorem (Surplus Extraction)**

*Consumer surplus is zero under any optimal mechanism.*

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*Consumer surplus is zero under any optimal mechanism.*

**Proof.**

Every optimal mechanism induces quasi-perfect price discrimination.

For (almost) every $c$, conditional on buying, all consumer pay their values. ■

# Surplus Extraction

**Remarks**

- Consumers surplus is zero regardless of whether the broker is also the owner of production technology.
- Therefore, separation the owners of consumer data from the owners of production technology does not benefit the consumers.

# Comparison with Uniform Pricing

**Proposition**

*The data broker's optimal revenue is greater than the consumer surplus under uniform pricing.*

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**Proposition**

*The data broker's optimal revenue is greater than the consumer surplus under uniform pricing.*

**Two corollaries:**

- Data brokership increases total surplus (compared with uniform pricing).
- If the data broker has to purchase data from the consumers (**before** they learn their values), then data brokership is Pareto improving in the ex-ante sense.

Note: $\mathscr{D}$ is bijective to $\Delta(V)$.

For any $D \in \mathscr{D}$, let $m^D \in \Delta(V)$ be the probability measure associated with $D$.

For any measurable function $h : V \to \mathbb{R}$, define

$$\int_V h(p) D(\mathrm{d}p) := \int_V h(p) m^D(\mathrm{d}p).$$

# Relaxing the Technical Assumptions

Regularity of $G$: Replace $\phi_G$ by the ironed virtual cost.

Regularity of $D_M$: Same outcome, different way to pool the low value consumers, more complicated proof (see paper).

$\phi_G(c) \leq \boldsymbol{p}_{D_M}(c)$ for all $c$: Two weaker conditions:

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1) $\max\{g(c)(\phi_G(c) - \boldsymbol{p}_{D_M}(c)), 0\}$ is nondecreasing.

- Implied by $\phi_G(c) \leq \boldsymbol{p}_{D_M}(c)$ for all $c \in C$
- Admits many commonly seen examples.

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- Implied by $\phi_G(c) \leq \boldsymbol{p}_{D_M}(c)$ for all $c \in C$
- Admits many commonly seen examples.

2) $D_M$ is continuous.

The surplus extraction result does not require any assumptions on $(D_M, G)$.

# Relaxing the Technical Assumptions

Examples where $\max\{g(c)(\phi_G(c) - \boldsymbol{p}_{D_M}(c)), 0\}$ is nondecreasing.

- Linear $D_M$ and uniform $G$;
- $D_M(p) = (1-p)^\beta$, $G(c) = c^\alpha$, for all $\alpha, \beta > 0$;
- Both $D_M$ and $G$ are (truncated) exponential;
- Any mix of the above.

▸ Back

# A Relaxed Problem

The data broker's revenue maximization problem:

$$\max_{(\sigma,\tau)} \int_C \tau(c)G(\mathrm{d}c)$$

$$\text{s.t.} \int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \tau(c) \geq \pi_{D_M}(c), \forall c \in C,$$

$$\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \tau(c) \geq \int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c') - \tau(c'), \qquad \forall c, c' \in C$$

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&\forall c, c' \in C.
\end{aligned}
$$

# A Relaxed Problem

Relaxed problem:

$$\max_{(\sigma,\tau,\gamma)} \int_C \tau(c)G(\mathrm{d}c)$$

$$\text{s.t.} \int_{\mathscr{D}\times\mathbb{R}_+} (p-c)D(p)\gamma(\mathrm{d}p|D,c)\sigma(\mathrm{d}D|c) - \tau(c)$$

$$\geq \int_{\mathscr{D}\times\mathbb{R}_+} (p-c)D(p)\gamma(\mathrm{d}p|D,c')\sigma(\mathrm{d}D|c') - \tau(c'), \tag{R-IC}$$

$$\int_{\mathscr{D}\times\mathbb{R}_+} (p-c)D(p)\gamma(\mathrm{d}p|D,c)\sigma(\mathrm{d}D|c) - \tau(c) \geq \pi_{D_M}(c), \tag{R-IR}$$

$$\forall c, c' \in C$$

▸ Back

# Private Information about the Market

Suppose that a consumer's value is $f(\theta, v)$.

All the consumers, as well as the producer, know $\theta \in [\underline{\theta}, \overline{\theta}] = \Theta$. Data broker does not know the realization of $\theta$. Cost is common knowledge, normalized to zero

$v$ is distributed according to $D_M$ across the consumers; the data broker can create any market segmentation w.r.t $v$. Two parameterized cases:

- Additive case: $f(\theta, v) = v - \theta$, $\underline{\theta} = 0, \overline{\theta} = \underline{v} > 0$.
- Multiplicative case: $f(\theta, v) = \theta \cdot v$, $\underline{\theta} > 0$.

# Additive Case

Suppose that $f(\theta, v) = v - \theta$.

Given any posted price $p$, a consumer buys $v - \theta \geq p$.

Given any market segment $D \in \mathscr{D}$, the producer's pricing problem is

$$\max_{\tilde{p} \geq 0} \tilde{p} D(\tilde{p} + \theta).$$

Let $p = \tilde{p} + \theta$, the seller's problem becomes

$$\max_{p \geq 0} (p - \theta) D(p),$$

which is the same as the pricing problem with cost $\theta$.

Additive case is equivalent to the private cost model.

# Multiplicative Case

Suppose that $f(\theta, v) = \theta \cdot v$.

Given any posted price $p$, a consumer buys iff $v\theta \geq p$.

Given any market segment $D \in \mathscr{D}$, the producer's pricing problem is

$$\max_{\tilde{p} \geq 0} \tilde{p} D\left(\frac{\tilde{p}}{\theta}\right),$$

which, by letting $p = \tilde{p}/\theta$ can be written as

$$\theta \cdot \max_{p \geq 0} pD(p)$$

Seller's pricing problem is independent of type
⇒ Outcome equivalence follows mechanically.

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Partition $V$ into finitely adjacent intervals.

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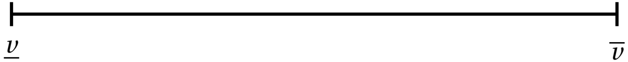

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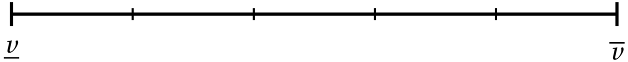

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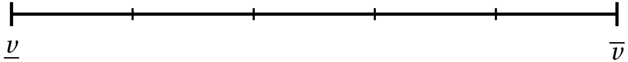
The most predictive data can only identify consumer-values by these intervals.



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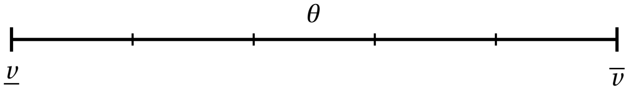

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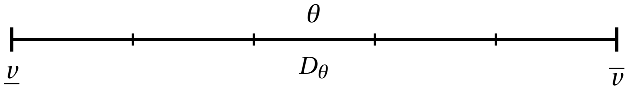

# Extension: Consumers' Private Information

$\Theta$: Finite partition of $V$, $\theta \in \Theta$: Interval.

$\theta \sim \beta_M$, $D_\theta$ demand conditional on $\theta$, market demand $D_M$, where

$$D_M(p) = \sum_{\theta \in \Theta} D_\theta(p)\beta_M(\theta),$$

for all $p \in V$,

$s \in \Delta(\Delta(\Theta))$ is a market segmentation if

$$\int_{\Delta(\Theta)} \beta(\theta)\, s(\mathrm{d}\beta) = \beta_M(\theta),$$

for all $\theta \in \Theta$.

Full disclosing segmentation: $\bar{s}$, where

$$\bar{s}(\delta_{\{\theta\}}) = \beta_M(\theta),$$

for all $\theta \in \Theta$.

# Extension: Consumers' Private Information

**Theorem**

*The consumer surplus under any optimal mechanism of the data broker is lower than that under the full disclosing segmentation.*

**Implications:**

- Separation between the ownership of consumer data and the production technology harms the consumers.
- Vertical integration increases total surplus **and** benefits the consumers.

# Extension: Consumers' Private Information

An optimal mechanism can also be characterized.

Let $u(\theta)$ be the upper bound of interval $\theta$.

For any $c \in C$ and $\theta \in \Theta$ such that $u(\theta) \geq \phi_G(c)$, define $\beta_\theta$ as

$$\beta_\theta(\theta') := \begin{cases} \beta_0(\theta'), & \text{if} u(\theta') < \phi_G(c) \\ \sum_{\{\hat{\theta}: u(\hat{\theta}) \geq \phi_G(c)\}} \beta_0(\hat{\theta}), & \text{if } \theta' = \theta \\ 0, & \text{otherwise} \end{cases}$$

and let

$$\sigma^*(\beta_\theta | c) := \frac{\beta_0(\theta)}{\sum_{\{\hat{\theta}: u(\hat{\theta}\}) \geq \phi_G(c)} \beta_0(\hat{\theta})},$$

for all $\theta \in \Theta$ such that $u(\theta) \geq \phi_G(c)$. Also, let

$$\tau^*(c) := \int_{\Delta(\Theta)} \pi_{D_\beta}(c) \sigma^*(\mathrm{d}\beta | c) - \int_c^{\overline{c}} D_\beta(p_{D_\beta}(z)) \sigma^*(\mathrm{d}\beta | z) \, \mathrm{d}z - \pi_{D_M}(\overline{c}).$$

# Extension: Consumer's Private Information

**Theorem**

$(\sigma^*, \tau^*)$ *is an optimal mechanism.*

Back

# Extension: Targeted Marketing

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$c_j$ is private information to $j$,
$\{c_j\}_{j=1}^J$ indp., $c_j \sim G_j$ for all $j$, $G_j$ admits a density $g_j > 0$ for all $j$.
$C := \prod_{j=1}^J C_j$, $G := \prod_{j=1}^J G_j$.

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A mechanism is $(\sigma_{ij}, \tau_j, q_{ij})_{i\in\mathscr{I}, j\in\mathscr{J}}$. For any report $c \in C$,

- $\sigma_{ij}(c) \in \mathscr{S}_{D_M^{ij}}$ is the segmentation of group $i$ provided to $j$.
- $\tau_j(c) \in \mathbb{R}$ is the amount of payments producer $j$ pays.
- $q_{ij}(c)$ is the fraction of group $i$ that sees $j$, where $\sum_i q_{ij}(c) \le 1$.

# Extension: Targeted Marketing

**Theorem (Surplus Extraction with Targeting)**

*For any* $\{D_0^{ij}\}_{i\in\mathscr{I},j\in\mathscr{J}}\subset\mathscr{D}$ *and any distributions of marginal costs* $\{G_j\}_{j\in\mathscr{J}}$*, there exists an incentive feasible mechanism that maximizes the data broker's revenue. Moreover, under any revenue-maximizing mechanism, consumers retain zero surplus.*

**Theorem (Outcome Equivalence with Targeting)**

*For any* $\{D_0^{ij}\}_{i\in\mathscr{I},j\in\mathscr{J}}\subset\mathscr{D}$ *such that* $\{D_0^{ij}\}_{i\in\mathscr{I}}$ *is ordered by the pointwise ordering for each* $j\in\mathscr{J}$*, and for any regular distributions of marginal costs* $\{G_j\}_{j\in\mathscr{J}}$*, suppose that for any* $i\in\mathscr{I}$ *and any* $j\in\mathscr{J}$*,* $\mathbf{p}_{D_M^{ij}}\geq\phi_{G_j}$ *and* $\mathbf{p}_{D_M^{j}}\geq\phi_{G_j}$*. Then data brokership and price-controlling data brokership are outcome equivalent.*

# Extension: Targeted Marketing

Data broker extracts all the additional surplus created by targeting:

- Target product $j$ to the most profitable group.
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Targeted marketing does not benefit the consumers.

Outcome equivalence still holds even with targeting.

▸ Back

# Revenue Equivalence Formula: Derivation

Recall: $\pi_D(c) = \max_p (p-c)D(p)$.

Define: $U(c) := \int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \tau(c)$.

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$$= \int_C \left( \int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) \right) G(\mathrm{d}c) - \int_C \left( \int_{\mathscr{D}} D(\boldsymbol{p}_D(c)) \frac{G(c)}{g(c)} \sigma(\mathrm{d}D|c) \right) G(\mathrm{d}c) - U(\overline{c})$$

$$= \int_C \left[ \int_{\mathscr{D}} \left( \pi_D(c) - D(\boldsymbol{p}_D(c)) \frac{G(c)}{g(c)} \right) \sigma(\mathrm{d}D|c) \right] G(\mathrm{d}c) - U(\overline{c})$$

$$= \int_C \left[ \int_{\mathscr{D}} (\boldsymbol{p}_D(c) - c) D(\boldsymbol{p}_D(c)) - D(\boldsymbol{p}_D(c)) \frac{G(c)}{g(c)} \right] G(\mathrm{d}c) - U(\overline{c})$$

# Revenue Equivalence Formula: Derivation

$$\tau(c) = \int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \int_c^{\overline{c}} \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma(\mathrm{d}D|z)\,\mathrm{d}z - U(\overline{c})$$

$$\begin{aligned}
&\Rightarrow \mathbb{E}[\tau(c)] \\
&= \int_C \left[\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \int_c^{\overline{c}} \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma(\mathrm{d}D|z)\,\mathrm{d}z\right] G(\mathrm{d}c) - U(\overline{c}) \\
&= \int_C \left(\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c)\right) G(\mathrm{d}c) - \int_C \left(\int_{\mathscr{D}} D(\boldsymbol{p}_D(c))\frac{G(c)}{g(c)}\sigma(\mathrm{d}D|c)\right) G(\mathrm{d}c) - U(\overline{c}) \\
&= \int_C \left[\int_{\mathscr{D}} \left(\pi_D(c) - D(\boldsymbol{p}_D(c))\frac{G(c)}{g(c)}\right)\sigma(\mathrm{d}D|c)\right] G(\mathrm{d}c) - U(\overline{c}) \\
&= \int_C \left[\int_{\mathscr{D}} (\boldsymbol{p}_D(c) - c)D(\boldsymbol{p}_D(c)) - D(\boldsymbol{p}_D(c))\frac{G(c)}{g(c)}\right] G(\mathrm{d}c) - U(\overline{c})
\end{aligned}$$

# Revenue Equivalence Formula: Derivation

$$\tau(c) = \int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \int_c^{\bar{c}} \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma(\mathrm{d}D|z)\,\mathrm{d}z - U(\bar{c})$$

$$
\begin{aligned}
&\Rightarrow \mathbb{E}[\tau(c)] \\
&= \int_C \left[ \int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \int_c^{\bar{c}} \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma(\mathrm{d}D|z)\,\mathrm{d}z \right] G(\mathrm{d}c) - U(\bar{c}) \\
&= \int_C \left( \int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) \right) G(\mathrm{d}c) - \int_C \left( \int_{\mathscr{D}} D(\boldsymbol{p}_D(c)) \frac{G(c)}{g(c)} \sigma(\mathrm{d}D|c) \right) G(\mathrm{d}c) - U(\bar{c}) \\
&= \int_C \left[ \int_{\mathscr{D}} \left( \pi_D(c) - D(\boldsymbol{p}_D(c)) \frac{G(c)}{g(c)} \right) \sigma(\mathrm{d}D|c) \right] G(\mathrm{d}c) - U(\bar{c}) \\
&= \int_C \left[ \int_{\mathscr{D}} (\boldsymbol{p}_D(c) - c) D(\boldsymbol{p}_D(c)) - D(\boldsymbol{p}_D(c)) \frac{G(c)}{g(c)} \right] G(\mathrm{d}c) - U(\bar{c}) \\
&= \int_C \left[ \int_{\mathscr{D}} \left( \boldsymbol{p}_D(c) - \left( c + \frac{G(c)}{g(c)} \right) \right) D(\boldsymbol{p}_D(c))\sigma(\mathrm{d}D|c) \right] G(\mathrm{d}c) - U(\bar{c})
\end{aligned}
$$

# Revenue Equivalence Formula: Derivation

$$\tau(c) = \int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \int_c^{\overline{c}} \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma(\mathrm{d}D|z)\,\mathrm{d}z - U(\overline{c})$$

$$\Rightarrow \mathbb{E}[\tau(c)]$$

$$= \int_C \left[ \int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \int_c^{\overline{c}} \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma(\mathrm{d}D|z)\,\mathrm{d}z \right] G(\mathrm{d}c) - U(\overline{c})$$

$$= \int_C \left( \int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) \right) G(\mathrm{d}c) - \int_C \left( \int_{\mathscr{D}} D(\boldsymbol{p}_D(c)) \frac{G(c)}{g(c)} \sigma(\mathrm{d}D|c) \right) G(\mathrm{d}c) - U(\overline{c})$$

$$= \int_C \left[ \int_{\mathscr{D}} \left( \pi_D(c) - D(\boldsymbol{p}_D(c)) \frac{G(c)}{g(c)} \right) \sigma(\mathrm{d}D|c) \right] G(\mathrm{d}c) - U(\overline{c})$$

$$= \int_C \left[ \int_{\mathscr{D}} (\boldsymbol{p}_D(c) - c) D(\boldsymbol{p}_D(c)) - D(\boldsymbol{p}_D(c)) \frac{G(c)}{g(c)} \right] G(\mathrm{d}c) - U(\overline{c})$$

$$= \int_C \left[ \int_{\mathscr{D}} \left( \boldsymbol{p}_D(c) - \left( c + \frac{G(c)}{g(c)} \right) \right) D(\boldsymbol{p}_D(c))\sigma(\mathrm{d}D|c) \right] G(\mathrm{d}c) - U(\overline{c})$$

$$= \int_C \left( \int_{\mathscr{D}} (\boldsymbol{p}_D(c) - \phi_G(c)) D(\boldsymbol{p}_D(c))\sigma(\mathrm{d}D|c) \right) G(\mathrm{d}c) - U(\overline{c}).$$

# Revenue Equivalence Formula: Derivation

Recall:

$$\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \tau(c) = U(c) = U(\overline{c}) + \int_c^{\overline{c}} \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma(\mathrm{d}D|z)\,\mathrm{d}z$$

# Revenue Equivalence Formula: Derivation

Recall:

$$\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \tau(c) = U(c) = U(\overline{c}) + \int_c^{\overline{c}} \int_{\mathscr{D}} D(p_D(z))\sigma(\mathrm{d}D|z)\,\mathrm{d}z$$

For any $c, c' \in C$,

# Revenue Equivalence Formula: Derivation

Recall:

$$\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \tau(c) = U(c) = U(\overline{c}) + \int_c^{\overline{c}} \int_{\mathscr{D}} D(p_D(z))\sigma(\mathrm{d}D|z)\,\mathrm{d}z$$

For any $c, c' \in C$,

$$0 \leq U(c) - \left[ \int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c') - \tau(c') \right]$$

# Revenue Equivalence Formula: Derivation

Recall:

$$\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \tau(c) = U(c) = U(\overline{c}) + \int_c^{\overline{c}} \int_{\mathscr{D}} D(p_D(z))\sigma(\mathrm{d}D|z)\,\mathrm{d}z$$

For any $c, c' \in C$,

$$\begin{aligned} 0 \leq & U(c) - \left[\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c') - \tau(c')\right] \\ = & U(c) \qquad\qquad - \left[\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c') - \tau(c')\right] \end{aligned}$$

# Revenue Equivalence Formula: Derivation

Recall:

$$\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \tau(c) = U(c) = U(\overline{c}) + \int_c^{\overline{c}} \int_{\mathscr{D}} D(p_D(z))\sigma(\mathrm{d}D|z)\,\mathrm{d}z$$

For any $c, c' \in C$,

$$\begin{aligned} 0 \leq & U(c) - \left[\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c') - \tau(c')\right] \\ = & U(c) - \int_{\mathscr{D}} \pi_D(c')\sigma(\mathrm{d}D|c') + \int_{\mathscr{D}} \pi_D(c')\sigma(\mathrm{d}D|c') - \left[\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c') - \tau(c')\right] \end{aligned}$$

# Revenue Equivalence Formula: Derivation

Recall:

$$\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \tau(c) = U(c) = U(\overline{c}) + \int_c^{\overline{c}} \int_{\mathscr{D}} D(p_D(z))\sigma(\mathrm{d}D|z)\,\mathrm{d}z$$

For any $c, c' \in C$,

$$\begin{aligned} 0 \leq & U(c) - \left[\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c') - \tau(c')\right] \\ = & U(c) - \int_{\mathscr{D}} \pi_D(c')\sigma(\mathrm{d}D|c') + \int_{\mathscr{D}} \pi_D(c')\sigma(\mathrm{d}D|c') - \left[\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c') - \tau(c')\right] \end{aligned}$$

# Revenue Equivalence Formula: Derivation

Recall:

$$\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \tau(c) = U(c) = U(\bar{c}) + \int_c^{\bar{c}} \int_{\mathscr{D}} D(p_D(z))\sigma(\mathrm{d}D|z)\,\mathrm{d}z$$

For any $c, c' \in C$,

$$\begin{aligned}
0 \leq & U(c) - \left[\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c') - \tau(c')\right] \\
= & U(c) - \int_{\mathscr{D}} \pi_D(c')\sigma(\mathrm{d}D|c') + \int_{\mathscr{D}} \pi_D(c')\sigma(\mathrm{d}D|c') - \left[\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c') - \tau(c')\right] \\
= & (U(c) - U(c')) + \int_{\mathscr{D}} (\pi_D(c') - \pi_D(c))\sigma(\mathrm{d}D|c')
\end{aligned}$$

# Revenue Equivalence Formula: Derivation

Recall:

$$\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \tau(c) = U(c) = U(\overline{c}) + \int_c^{\overline{c}} \int_{\mathscr{D}} D(p_D(z))\sigma(\mathrm{d}D|z)\,\mathrm{d}z$$

For any $c, c' \in C$,

$$\begin{aligned}
0 \leq & U(c) - \left[\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c') - \tau(c')\right] \\
= & U(c) - \int_{\mathscr{D}} \pi_D(c')\sigma(\mathrm{d}D|c') + \int_{\mathscr{D}} \pi_D(c')\sigma(\mathrm{d}D|c') - \left[\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c') - \tau(c')\right] \\
= & (U(c) - U(c')) + \int_{\mathscr{D}} (\pi_D(c') - \pi_D(c))\sigma(\mathrm{d}D|c')
\end{aligned}$$

# Revenue Equivalence Formula: Derivation

Recall:

$$\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \tau(c) = U(c) = U(\overline{c}) + \int_c^{\overline{c}} \int_{\mathscr{D}} D(p_D(z))\sigma(\mathrm{d}D|z)\,\mathrm{d}z$$

For any $c, c' \in C$,

$$\begin{aligned} 0 \leq & U(c) - \left[\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c') - \tau(c')\right] \\ = & U(c) - \int_{\mathscr{D}} \pi_D(c')\sigma(\mathrm{d}D|c') + \int_{\mathscr{D}} \pi_D(c')\sigma(\mathrm{d}D|c') - \left[\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c') - \tau(c')\right] \\ = & (U(c) - U(c')) + \int_{\mathscr{D}} (\pi_D(c') - \pi_D(c))\sigma(\mathrm{d}D|c') \\ = & \int_c^{c'} -U'(z)\,\mathrm{d}z + \int_c^{c'} \left(\int_{\mathscr{D}} \pi'_D(z)\sigma(\mathrm{d}D|c')\right)\mathrm{d}z \end{aligned}$$

# Revenue Equivalence Formula: Derivation

Recall:

$$\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \tau(c) = U(c) = U(\overline{c}) + \int_c^{\overline{c}} \int_{\mathscr{D}} D(p_D(z))\sigma(\mathrm{d}D|z)\,\mathrm{d}z$$

For any $c, c' \in C$,

$$\begin{aligned}
0 \leq & U(c) - \left[\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c') - \tau(c')\right] \\
= & U(c) - \int_{\mathscr{D}} \pi_D(c')\sigma(\mathrm{d}D|c') + \int_{\mathscr{D}} \pi_D(c')\sigma(\mathrm{d}D|c') - \left[\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c') - \tau(c')\right] \\
= & (U(c) - U(c')) + \int_{\mathscr{D}} (\pi_D(c') - \pi_D(c))\sigma(\mathrm{d}D|c') \\
= & \int_c^{c'} -U'(z)\,\mathrm{d}z + \int_c^{c'} \left(\int_{\mathscr{D}} \pi'_D(z)\sigma(\mathrm{d}D|c')\right)\mathrm{d}z \\
= & \int_c^{c'} \int_{\mathscr{D}} D(p_D(z))\sigma(\mathrm{d}D|c)\,\mathrm{d}z - \int_c^{c'} \int_{\mathscr{D}} D(p_D(z))\sigma(\mathrm{d}D|c')\,\mathrm{d}z
\end{aligned}$$

# Revenue Equivalence Formula: Derivation

Recall:

$$\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \tau(c) = U(c) = U(\overline{c}) + \int_c^{\overline{c}} \int_{\mathscr{D}} D(p_D(z))\sigma(\mathrm{d}D|z)\,\mathrm{d}z$$

For any $c, c' \in C$,

$$\begin{aligned}
0 \leq & U(c) - \left[\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c') - \tau(c')\right] \\
= & U(c) - \int_{\mathscr{D}} \pi_D(c')\sigma(\mathrm{d}D|c') + \int_{\mathscr{D}} \pi_D(c')\sigma(\mathrm{d}D|c') - \left[\int_{\mathscr{D}} \pi_D(c)\sigma(\mathrm{d}D|c') - \tau(c')\right] \\
= & (U(c) - U(c')) + \int_{\mathscr{D}} (\pi_D(c') - \pi_D(c))\sigma(\mathrm{d}D|c') \\
= & \int_c^{c'} -U'(z)\,\mathrm{d}z + \int_c^{c'} \left(\int_{\mathscr{D}} \pi'_D(z)\sigma(\mathrm{d}D|c')\right)\mathrm{d}z \\
= & \int_c^{c'} \int_{\mathscr{D}} D(p_D(z))\sigma(\mathrm{d}D|c)\,\mathrm{d}z - \int_c^{c'} \int_{\mathscr{D}} D(p_D(z))\sigma(\mathrm{d}D|c')\,\mathrm{d}z
\end{aligned}$$

▸ Back

# Sketch of Proof

For any $D \in \mathscr{D}$, let

$$D^{-1}(q) := \sup\{p \in V : D(p) \geq q\}.$$

**Lemma**

*The price-controlling data broker's problem has a solution. Moreover, the optimal revenue is*

$$R^* = \max_{\boldsymbol{q} \in \mathscr{Q}} \int_C \left( \int_0^{\boldsymbol{q}(c)} (D_M^{-1}(q) - \phi_G(c))\, \mathrm{d}q \right) G(\mathrm{d}c) - \pi_{D_M}(\overline{c})$$

$$\text{s.t. } \bar{\pi} + \int_c^{\overline{c}} \boldsymbol{q}(z)\, \mathrm{d}z \geq \bar{\pi} + \int_c^{\overline{c}} D_M(\boldsymbol{p}_{D_M}(z))\, \mathrm{d}z,$$

*where* $\mathscr{Q}$ *is the collection of all nonincreasing functions from* $C$ *to* $[0,1]$.

▸ Sketch of Proof

# Sketch of Proof

For any $\boldsymbol{q} \in \mathcal{Q}$ let $R(\boldsymbol{q})$ be the price-controlling data broker revenue when choosing $\boldsymbol{q} \in \mathcal{Q}$.

$$R(\boldsymbol{q}) := \int_C \left( \int_0^{\boldsymbol{q}(c)} (D_M^{-1}(q) - \phi_G(c)) \, \mathrm{d}q \right) G(\mathrm{d}c) - \bar{\pi}$$

Consider the dual: For any Borel measure $\mu$, let

$$d(\mu) := \sup_{\boldsymbol{q} \in \mathcal{Q}} \left[ R(\boldsymbol{q}) + \int_C \left( \int_c^{\bar{c}} (\boldsymbol{q}(z) - D_M(\boldsymbol{p}_{D_M}(z))) \, \mathrm{d}z \right) \mu(\mathrm{d}c) \right]$$

and let

$$d^* := \inf_{\mu} d(\mu).$$

# Sketch of Proof

By weak duality, it suffices to find $\mu^*$ so that

- $D_M \circ \phi_G$ is feasible in the primal problem.
- $D_M \circ \phi_G$ solves the dual problem $d(\mu^*)$.
- the complementary slackness condition is satisfied. i.e.,

$$\int_C \left( \int_c^{\bar{c}} (D_M(\phi_G(z)) - D_M(\boldsymbol{p}_{D_M}(z)))\, \mathrm{d}z \right) \mu^*(\mathrm{d}c) = 0$$

# Sketch of Proof

Define

$$M^*(c) := \lim_{z \downarrow c} g(z)(\phi_G(z) - \boldsymbol{p}_{D_M}(z))^+, \forall c \in C$$

By assumption: $M^*$ is nondecreasing and right-continuous.

Let $\mu^*$ be the Borel measure induced by $M^*$.

Can show that $D_M \circ \phi_G$ solves $d(\mu^*)$ and that the complementary slackness condition is satisfied.

# Sketch of Proof

Also, since $\phi_G \leq \boldsymbol{p}_{D_M}$,

$$\bar{\pi} \int_c^{\bar{c}} D_M(\phi_G(z))\,\mathrm{d}z \geq \bar{\pi} + \int_c^{\bar{c}} D_M(\boldsymbol{p}_{D_M}(z))\,\mathrm{d}z, \forall c \in C$$

and thus $D_M \circ \phi_G$ is feasible in the primal problem.

$D_M \circ \phi_G$ solves the primal problem.

By definition,

$$\int_0^{D_M(\phi_G(c))} (D_M^{-1}(q) - \phi_G(c))\,\mathrm{d}q = \int_{\{v \geq \phi_G(c)\}} (v - \phi_G(c)) D_M(\mathrm{d}v).$$

▸ Back

# Implementation by Partitioning Characteristics

A market segmentation can also be thought of as a partition of consumers' characteristics.

Any segmentation $s \in \Delta(\mathscr{D})$ can be generated by partitioning the characteristics, as long as they are rich enough.

$(\Theta, \mathscr{F}, \mathbb{P})$: probability space (characteristics).

$\mathbf{V} : \Theta \to V$, measurable ($\mathbf{V}(\theta)$ is a consumer's value when their characteristic is $\theta$).

# Implementation by Partitioning Characteristics

**Theorem (Generating Countable Segmentation)**

*Suppose that* $(\Theta, \mathscr{F}, \mathbb{P})$ *is nonatomic. Then for any segmentation* $s$ *with* $\mathrm{supp}(s)$ *being countable, there exists a countable partition* $\mathscr{P}$ *of* $\Theta$ *such that for any* $D \in \mathrm{supp}(s)$*, there exists* $F \in \mathscr{P}$ *such that*

$$\mathbb{P}(F \cap \mathbf{V}^{-1}([p, \overline{v}])) = D(p)s(D),$$

*for all* $p \in V$*.*

# Implementation by Partitioning Characteristics

## Definition

Say that $(\Theta, \mathscr{F}, \mathbb{P})$ is rich relative to $\mathbf{V}$ if for any $p \in V$, $(\mathbf{V}^{-1}([p, \overline{v}]), \mathscr{F}|_{\mathbf{V}^{-1}([p,\overline{v}])}, \tilde{\mathbb{P}}_p)$ is isomorphic to $(I, \mathscr{B}([0,1]), L)$ modulo zero for some interval $I \subseteq [0,1]$, where

$$\mathscr{F}|_{\mathbf{V}^{-1}([p,\overline{v}])} := \{F \in \mathscr{F} : F \subseteq \mathbf{V}^{-1}([p, \overline{v}])\},$$

$$\tilde{\mathbb{P}}_p(F) := \mathbb{P}(F \cap \mathbf{V}^{-1}([p, \overline{v}])),$$

for any $F \in \mathscr{F}|_{\mathbf{V}^{-1}([p,\overline{v}])}$ and $L$ is the Lebesgue measure.

Example: $\Theta \subseteq \mathbb{R}^n$, $n \geq 2$; $\mathscr{F}$: Borel $\sigma$-algebre; $\mathbb{P}$ absolutely continuous w.r.t the Lebesgue measure;

$$\{\theta \in \Theta | \mathbf{V}(\theta) = v\}$$

has Hausdorff dimension $\geq 1$ for all $v \in V$.

# Implementation by Partitioning Characteristics

**Theorem (Generating Arbitrary Segmentation)**

*Suppose that* $(\Theta, \mathscr{F}, \mathbb{P})$ *is rich relative to* $\mathbf{V}$*. Then for any segmentation* $s$*, there exists a random variable* $\mathbf{D} : \Theta \to \mathscr{D}$ *such that*

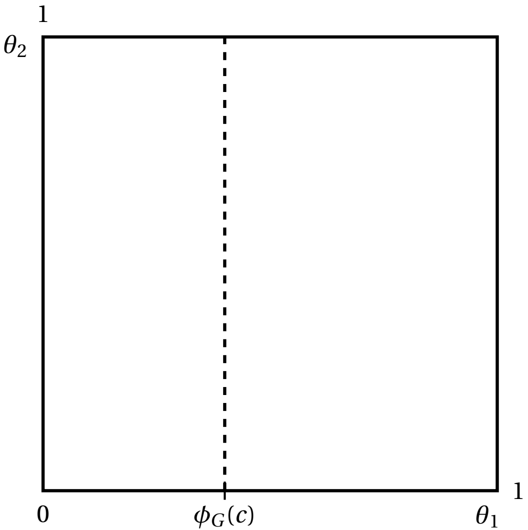
$$\mathbb{P}(\mathbf{D}^{-1}(B) \cap \mathbf{V}^{-1}([p, \overline{v}])) = \int_B D(p) s(\mathrm{d}D),$$

*for all* $p \in V$ *and for any measurable* $B \subseteq \mathscr{D}$*.*

▸ Back

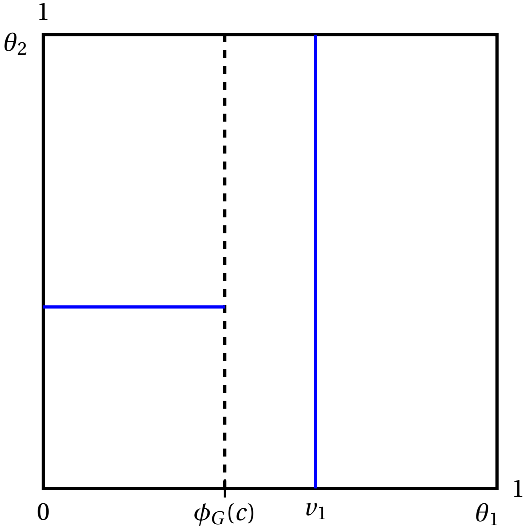
# An Example

$\Theta = [0,1]^2$, $\mathscr{F} = \mathscr{B}([0,1])$, $\mathbb{P}$: Lebesgue measure, $\mathbf{V}(\theta_1,\theta_2) = \theta_1$.



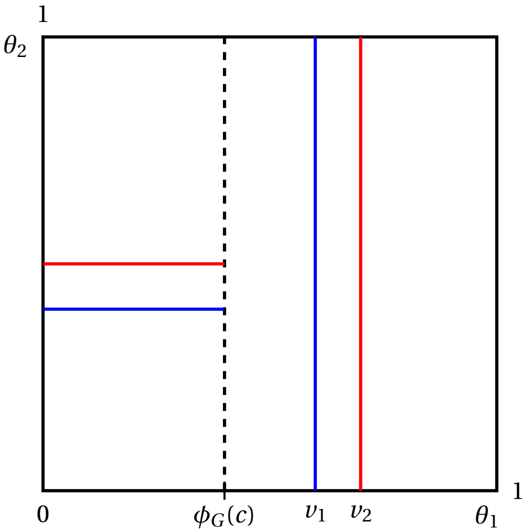
# An Example

$\Theta = [0,1]^2$, $\mathscr{F} = \mathscr{B}([0,1])$, $\mathbb{P}$: Lebesgue measure, $\mathbf{V}(\theta_1,\theta_2) = \theta_1$.



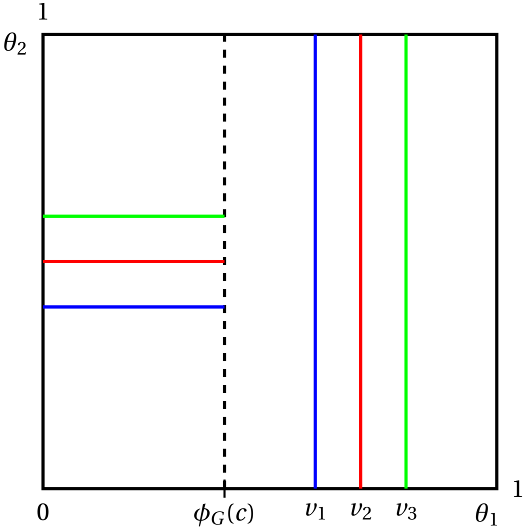
# An Example

$\Theta = [0,1]^2$, $\mathscr{F} = \mathscr{B}([0,1])$, $\mathbb{P}$: Lebesgue measure, $\mathbf{V}(\theta_1, \theta_2) = \theta_1$.



# An Example

$\Theta = [0,1]^2$, $\mathscr{F} = \mathscr{B}([0,1])$, $\mathbb{P}$: Lebesgue measure, $\mathbf{V}(\theta_1, \theta_2) = \theta_1$.



# Exclusive retail

**Exclusive retail:**
$c$ is private information.

The data broker offers a menu consisting of items $(q, t)$. For each item $(q, t)$,

- The data broker pays $t$ to the producer.
- The producer produces $q$ units for the data broker and forfeits the right to sell the product.
- The data broker can sell at most $q$ units to the consumers (via perfect price discrimination).

Then the broker sells the purchased product to the consumers **exclusively**, via **perfect price discrimination**.

If the producer does not choose any item, she sells to the consumers without data and receives $\pi_{D_M}(c)$.

# Sketch of Proof

**Lemma (Decomposition Lemma)**

*For any nondecreasing* $\psi : C \to \mathbb{R}_+$ *with* $\psi(c) \geq c$ *for all* $c \in C$, *there exists* $\sigma^* : C \to \Delta(\mathscr{D})$ *such that for all* $c \in C$, $\sigma^*(c)$ *is a segmentation that induces quasi-perfect price discrimination with cutoff* $\psi(c)$ *for* $c$ *and that*

$$\psi(z) \leq p_D(z), \tag{**}$$

*for any* $z \in [\underline{c}, c]$ *and for any* $D \in \operatorname{supp}(\sigma(c))$.

▸ Back

# Sketch of Proof

Consider first the case where $D_M$ is a step function with finitely many steps.

Let

$$\hat{c} := \inf\{z \in C | \boldsymbol{p}_{D_M}(z) \geq \psi(c)\}.$$

$\psi(c) \leq \boldsymbol{p}_{D_M}(c) \Rightarrow \hat{c} \in [\underline{c}, c]$.

$\boldsymbol{p}_{D_M}$ is nondegreasing $\Rightarrow$ $\boldsymbol{p}_{D_M}(z) \geq \psi(c)$ iff $z \geq \hat{c}$.

If $\hat{c} > \underline{c}$, then it must be that $\underline{\boldsymbol{p}}_0(\hat{c}) < \psi(c) \leq \boldsymbol{p}_{D_M}(\hat{c})$.

# Sketch of Proof ($\hat{c} > \underline{c}$)

| | | $\underline{\boldsymbol{p}}_0(\hat{c})$ | | $\psi(c)$ | | $\boldsymbol{p}_{D_M}(\hat{c})$ |
|---|---|---|---|---|---|---|
| $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ |
| $m^0(v_1)$ | $m^0(v_2)$ | $m^0(v_3)$ | $m^0(v_4)$ | $m^0(v_5)$ | $m^0(v_6)$ | $m^0(v_7)$ |
| $\alpha_{v_1}^{v_7} m^0(v_1)$ | $\alpha_{v_2}^{v_7} m^0(v_2)$ | $\alpha_{v_3}^{v_7} m^0(v_3)$ | $\alpha_{v_4}^{v_7} m^0(v_4)$ | $0$ | $0$ | $m^0(v_7)$ |
| $\alpha_{v_1}^{v_6} m^0(v_1)$ | $\alpha_{v_2}^{v_6} m^0(v_2)$ | $\alpha_{v_3}^{v_6} m^0(v_3)$ | $\alpha_{v_4}^{v_6} m^0(v_4)$ | $0$ | $m^0(v_6)$ | $0$ |
| $\alpha_{v_1}^{v_5} m^0(v_1)$ | $\alpha_{v_2}^{v_5} m^0(v_2)$ | $\alpha_{v_3}^{v_5} m^0(v_3)$ | $\alpha_{v_4}^{v_5} m^0(v_4)$ | $m^0(v_5)$ | $0$ | $0$ |

# Sketch of Proof ($\hat{c} > \underline{c}$)

| $v_1$ | $v_2$ | $\underline{\boldsymbol{p}}_0(\hat{c})$<br>$v_3$ | $v_4$ | $\psi(c)$<br>$v_5$ | $v_6$ | $\hat{\psi}(\hat{c})$<br>$\boldsymbol{p}_{D_M}(\hat{c})$<br>$v_7$ |
|---|---|---|---|---|---|---|
| $m^0(v_1)$ | $m^0(v_2)$ | $m^0(v_3)$ | $m^0(v_4)$ | $m^0(v_5)$ | $m^0(v_6)$ | $m^0(v_7)$ |
| $\alpha_{v_1}^{v_7} m^0(v_1)$ | $\alpha_{v_2}^{v_7} m^0(v_2)$ | $\alpha_{v_3}^{v_7} m^0(v_3)$ | $\alpha_{v_4}^{v_7} m^0(v_4)$ | $0$ | $0$ | $m^0(v_7)$ |
| $\alpha_{v_1}^{v_6} m^0(v_1)$ | $\alpha_{v_2}^{v_6} m^0(v_2)$ | $\alpha_{v_3}^{v_6} m^0(v_3)$ | $\alpha_{v_4}^{v_6} m^0(v_4)$ | $0$ | $m^0(v_6)$ | $0$ |
| $\alpha_{v_1}^{v_5} m^0(v_1)$ | $\alpha_{v_2}^{v_5} m^0(v_2)$ | $\alpha_{v_3}^{v_5} m^0(v_3)$ | $\alpha_{v_4}^{v_5} m^0(v_4)$ | $m^0(v_5)$ | $0$ | $0$ |

# Sketch of Proof ($\hat{c} > \underline{c}$)

BBM cutoff

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | | $\boxed{\hat{\psi}(\hat{c})}$ |
| | | $\underline{\boldsymbol{p}}_0(\hat{c})$ | | $\psi(c)$ | | $\boldsymbol{p}_{D_M}(\hat{c})$ |
| $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ |
| $m^0(v_1)$ | $m^0(v_2)$ | $m^0(v_3)$ | $m^0(v_4)$ | $m^0(v_5)$ | $m^0(v_6)$ | $m^0(v_7)$ |
| $\alpha_{v_1}^{v_7} m^0(v_1)$ | $\alpha_{v_2}^{v_7} m^0(v_2)$ | $\alpha_{v_3}^{v_7} m^0(v_3)$ | $\alpha_{v_4}^{v_7} m^0(v_4)$ | $0$ | $0$ | $m^0(v_7)$ |
| $\alpha_{v_1}^{v_6} m^0(v_1)$ | $\alpha_{v_2}^{v_6} m^0(v_2)$ | $\alpha_{v_3}^{v_6} m^0(v_3)$ | $\alpha_{v_4}^{v_6} m^0(v_4)$ | $0$ | $m^0(v_6)$ | $0$ |
| $\alpha_{v_1}^{v_5} m^0(v_1)$ | $\alpha_{v_2}^{v_5} m^0(v_2)$ | $\alpha_{v_3}^{v_5} m^0(v_3)$ | $\alpha_{v_4}^{v_5} m^0(v_4)$ | $m^0(v_5)$ | $0$ | $0$ |

# Sketch of Proof ($\hat{c} > \underline{c}$)

| | | $\underline{p}_0(\hat{c})$ | | $\psi(c)$ | | $p_{D_M}(\hat{c})$ |
|---|---|---|---|---|---|---|
| $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ |
| $m^0(v_1)$ | $m^0(v_2)$ | $m^0(v_3)$ | $m^0(v_4)$ | $m^0(v_5)$ | $m^0(v_6)$ | $m^0(v_7)$ |
| $\alpha^{v_7} m^0(v_1)$ | $\alpha^{v_7} m^0(v_2)$ | $\beta_{v_3}^{v_7} m^0(v_3)$ | $\beta_{v_4}^{v_7} m^0(v_4)$ | $0$ | $0$ | $m^0(v_7)$ |
| $\alpha^{v_6} m^0(v_1)$ | $\alpha^{v_6} m^0(v_2)$ | $\beta_{v_3}^{v_6} m^0(v_3)$ | $\beta_{v_4}^{v_6} m^0(v_4)$ | $0$ | $m^0(v_6)$ | $0$ |
| $\alpha^{v_5} m^0(v_1)$ | $\alpha^{v_5} m^0(v_2)$ | $\beta_{v_3}^{v_5} m^0(v_3)$ | $\beta_{v_4}^{v_5} m^0(v_4)$ | $m^0(v_5)$ | $0$ | $0$ |

# Sketch of Proof ($\hat{c} > \underline{c}$)

$$\begin{array}{ccccccc}
 & & \underline{p}_0(\hat{c}) & & \psi(c) & & p_{D_M}(\hat{c}) \\
v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\
\hline
m^0(v_1) & m^0(v_2) & m^0(v_3) & m^0(v_4) & m^0(v_5) & m^0(v_6) & m^0(v_7) \\
\begin{array}{c} \alpha^{v_7} m^0(v_1) \\ \alpha^{v_6} m^0(v_1) \\ \alpha^{v_5} m^0(v_1) \end{array} &
\begin{array}{c} \alpha^{v_7} m^0(v_2) \\ \alpha^{v_6} m^0(v_2) \\ \alpha^{v_5} m^0(v_2) \end{array} &
\multicolumn{2}{c}{\underbrace{\boxed{\begin{array}{cc} \beta^{v_7}_{v_3} m^0(v_3) & \beta^{v_7}_{v_4} m^0(v_4) \\ \beta^{v_6}_{v_3} m^0(v_3) & \beta^{v_6}_{v_4} m^0(v_4) \\ \beta^{v_5}_{v_3} m^0(v_3) & \beta^{v_5}_{v_4} m^0(v_4) \end{array}}}_{\text{BBM weights}}} &
\begin{array}{c} 0 \\ 0 \\ m^0(v_5) \end{array} &
\begin{array}{c} 0 \\ m^0(v_6) \\ 0 \end{array} &
\begin{array}{c} m^0(v_7) \\ 0 \\ 0 \end{array}
\end{array}$$

# Sketch of Proof ($\hat{c} > \underline{c}$)

$$
\begin{array}{ccccccc}
 & & \underline{p}_0(\hat{c}) & & \psi(c) & & \boldsymbol{p}_{D_M}(\hat{c}) \\
v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\
\hline
m^0(v_1) & m^0(v_2) & m^0(v_3) & m^0(v_4) & m^0(v_5) & m^0(v_6) & m^0(v_7) \\
\alpha^{v_7} m^0(v_1) & \alpha^{v_7} m^0(v_2) & \beta_{v_3}^{v_7} m^0(v_3) & \beta_{v_4}^{v_7} m^0(v_4) & 0 & 0 & m^0(v_7) \\
\alpha^{v_6} m^0(v_1) & \alpha^{v_6} m^0(v_2) & \beta_{v_3}^{v_6} m^0(v_3) & \beta_{v_4}^{v_6} m^0(v_4) & 0 & m^0(v_6) & 0 \\
\alpha^{v_5} m^0(v_1) & \alpha^{v_5} m^0(v_2) & \beta_{v_3}^{v_5} m^0(v_3) & \beta_{v_4}^{v_5} m^0(v_4) & m^0(v_5) & 0 & 0 \\
\multicolumn{2}{c}{\underbrace{\hphantom{\alpha^{v_5} m^0(v_1) \quad \alpha^{v_5} m^0(v_2)}}_{\text{regular weights}}} & \multicolumn{2}{c}{\underbrace{\hphantom{\beta_{v_3}^{v_5} m^0(v_3) \quad \beta_{v_4}^{v_5} m^0(v_4)}}_{\text{BBM weights}}} & & &
\end{array}
$$

# Sketch of Proof ($\hat{c} > \underline{c}$)

$$
\begin{array}{ccccccc}
 & & \underline{p}_0(\hat{c}) & & \psi(c) & & p_{D_M}(\hat{c}) \\
v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\
\hline
m^0(v_1) & m^0(v_2) & m^0(v_3) & m^0(v_4) & m^0(v_5) & m^0(v_6) & m^0(v_7) \\
\alpha^{v_7} m^0(v_1) & \alpha^{v_7} m^0(v_2) & \beta_{v_3}^{v_7} m^0(v_3) & \beta_{v_4}^{v_7} m^0(v_4) & 0 & 0 & m^0(v_7) \\
\alpha^{v_6} m^0(v_1) & \alpha^{v_6} m^0(v_2) & \beta_{v_3}^{v_6} m^0(v_3) & \beta_{v_4}^{v_6} m^0(v_4) & 0 & m^0(v_6) & 0 \\
\alpha^{v_5} m^0(v_1) & \alpha^{v_5} m^0(v_2) & \beta_{v_3}^{v_5} m^0(v_3) & \beta_{v_4}^{v_5} m^0(v_4) & m^0(v_5) & 0 & 0 \\
\multicolumn{2}{c}{\underbrace{\hspace{6em}}_{\text{regular weights}}} & \multicolumn{2}{c}{\underbrace{\hspace{6em}}_{\text{BBM weights}}} & & &
\end{array}
$$

# Sketch of Proof ($\hat{c} = \underline{c}$)

$$\begin{array}{ccccccc}
 & & \psi(c) & & & p_{D_M}(\underline{c}) & \\
v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \\
\hline
m^0(v_1) & m^0(v_2) & m^0(v_3) & m^0(v_4) & m^0(v_5) & m^0(v_6) & m^0(v_7) \\
\alpha^{v_7} m^0(v_1) & \alpha^{v_7} m^0(v_2) & 0 & 0 & 0 & 0 & m^0(v_7) \\
\alpha^{v_6} m^0(v_1) & \alpha^{v_6} m^0(v_2) & 0 & 0 & 0 & m^0(v_6) & 0 \\
0 & 0 & 0 & 0 & m^0(v_5) & 0 & 0 \\
0 & 0 & 0 & m^0(v_4) & 0 & 0 & 0 \\
0 & 0 & m^0(v_3) & 0 & 0 & 0 & 0
\end{array}$$

# Sketch of Proof

**Lemma**

*Consider any function $\psi \in \mathbb{R}_+^C$ with $c \leq \psi(c)$ for all $c \in C$. Given any $\{D_n\} \subset \mathscr{D}$ and $\{\sigma_n\} \subset \mathscr{S}_{D_n}^C$. Suppose that $\{\sigma_n\} \to \sigma$ pointwisely and $\{D_n\} \to D_M$ for some $\sigma \in \Delta(\mathscr{D})^C$ and $D_M \in \mathscr{D}$. Then $\sigma \in \mathscr{S}^C$. Moreover, suppose further that $\sigma_n$ is a $\psi$-quasi-perfect scheme for all $n \in \mathbb{N}$. Then $\sigma$ is a $\psi$-quasi-perfect scheme.*

# Sketch of Proof

For any $D_M \in \mathscr{D}$, take a sequence of step functions $\{D_n\} \subseteq \mathscr{D}$ such that $\{D_n\} \to D_M$ and that

$$c \leq \psi(c) \leq \boldsymbol{p}_{D_n}(c), \forall c \in C$$

There exists $\{\sigma_n\} \to \sigma^*$ such that $\sigma_n : C \to \mathscr{S}_{D_n}$ is a $\psi$-quasi-perfect scheme satisfying (*).

$\sigma^*$ is as desired.

▸ Back

# Relaxing the Technical Assumptions

Assuming $\max\{g(c)(\phi_G(c) - \boldsymbol{p}_{D_M}(c)), 0\}$ is nondecreasing:

Let $\overline{\varphi}_G(c) := \min\{\varphi_G(c), \boldsymbol{p}_{D_M}(c)\}$ for all $c \in C$, where $\varphi_G$ is the ironed virtual cost.

Can construct an optimal mechanism $(\sigma^{**}, \tau^{**})$.

For any optimal mechanism $(\sigma, \tau)$ and for any $c$, $\sigma(c)$ induces quasi-perfect price discrimination with cutoff $\overline{\varphi}_G(c)$ for $c$.

If, in addition, $D_M$ is regular, then $\sigma^{**} \equiv \sigma^*$ and $\tau^{**} \equiv \tau^*$, with $\phi_G$ being replaced by $\overline{\varphi}_G$.

All the other results remain true.

▸ Sketch of Proof ▸ Back

# Relaxing the Technical Assumptions

Assuming $D_M$ is continuous.

Can construct an optimal mechanism $(\bar{\sigma}, \bar{\tau})$.

For any optimal mechanism $(\sigma, \tau)$ and for any $c$, $\sigma(c)$ induces quasi-perfect price discrimination with cutoff $\varphi^*(c)$ for $c$.

$\varphi^*$ is a nondecreasing function such that $\varphi^*(c) > c$ for all $c > \underline{c}$.

$\varphi^*$ does not have a closed form, the paper (appendix) provides a partial characterization.

Consumer surplus is zero under and optimal mechanism.

Vertical integration is Pareto improving.

Exclusive retail and price-controlling data brokership Pareto dominates data brokership.

Back

# Relaxing the Technical Assumptions

**Theorem (Surplus Extraction)**

*For any* $(D_M, G)$, *there exists an IC & IR mechanism that maximizes the data broker's revenue. Furthermore, under any revenue-maximizing mechanism for the data broker, the consumers retain zero surplus.*

▸ Back

# Sketch of Proof

**Lemma (Decomposition Lemma)**

*For any nondecreasing* $\psi : C \to \mathbb{R}_+$ *with* $\psi(c) \geq c$ *for all* $c \in C$*, there exists* $\sigma^* : C \to \Delta(\mathscr{D})$ *such that for all* $c \in C$*,* $\sigma^*(c)$ *is a segmentation that induces quasi-perfect price discrimination with cutoff* $\psi(c)$ *for* $c$ *and that*

$$\psi(z) \leq p_D(z), \tag{**}$$

*for any* $z \in [\underline{c}, c]$ *and for any* $D \in \operatorname{supp}(\sigma(c))$*.*

# Sketch of Proof

Existence: By the continuity lemmas.

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Consider any IC & IR mechanism $(\sigma, \tau)$ such that the consumers retain positive surplus.

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Consider any IC & IR mechanism $(\sigma, \tau)$ such that the consumers retain positive surplus.

Clearly, $c \leq \boldsymbol{p}_D(c) \leq \boldsymbol{p}_D(c)$ for all $D \in \operatorname{supp}(\sigma(c))$ for all $c \in C$.

# Sketch of Proof

Existence: By the continuity lemmas.

Consider any IC & IR mechanism $(\sigma, \tau)$ such that the consumers retain positive surplus.

Clearly, $c \leq \boldsymbol{p}_D(c) \leq \boldsymbol{p}_D(c)$ for all $D \in \text{supp}(\sigma(c))$ for all $c \in C$.

For all $c \in C$ and for all $D \in \text{supp}(\sigma(c))$, apply the decomposition lemma on $D$ and obtain $\boldsymbol{p}_D$-quasi-perfect scheme, say $\sigma^D$, satisfying (**)

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This induces another segmentation scheme $\hat{\sigma}$.

# Sketch of Proof

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For all $c \in C$ and for all $D \in \text{supp}(\sigma(c))$, apply the decomposition lemma on $D$ and obtain $\boldsymbol{p}_D$-quasi-perfect scheme, say $\sigma^D$, satisfying (**)

This induces another segmentation scheme $\hat{\sigma}$.

Consumer surplus $> 0$ under $(\sigma, \tau) \Rightarrow \hat{\sigma}$ extracts more surplus than $\sigma$.

# Splitting $D$ by $\sigma^D(c)$

| | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ |
|---|---|---|---|---|---|---|
| $D_M:$ | $m^0(v_1)$ | $m^0(v_2)$ | $m^0(v_3)$ | $m^0(v_4)$ | $m^0(v_5)$ | $m^0(v_6)$ |
| $D_1:$ | $m^{D_1}(v_1)$ | $m^{D_1}(v_2)$ | $m^{D_1}(v_3)$ | $0$ | $0$ | $m^{D_1}(v_6)$ |
| $D_2:$ | $m^{D_2}(v_1)$ | $m^{D_2}(v_2)$ | $m^{D_2}(v_3)$ | $0$ | $m^{D_2}(v_5)$ | $0$ |
| $D:$ | $m^D(v_1)$ | $m^D(v_2)$ | $m^D(v_3)$ | $m^D(v_4)$ | $m^D(v_5)$ | $m^D(v_6)$ |

Note: Blue mark indicates the optimal price for producer $c$ under each segment.

# Splitting $D$ by $\sigma^D(c)$

| | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ |
|---|---|---|---|---|---|---|
| $D_M:$ | $m^0(v_1)$ | $m^0(v_2)$ | $m^0(v_3)$ | $m^0(v_4)$ | $m^0(v_5)$ | $m^0(v_6)$ |
| $D_1:$ | $m^{D_1}(v_1)$ | $m^{D_1}(v_2)$ | $m^{D_1}(v_3)$ | $0$ | $0$ | $m^{D_1}(v_6)$ |
| $D_2:$ | $m^{D_2}(v_1)$ | $m^{D_2}(v_2)$ | $m^{D_2}(v_3)$ | $0$ | $m^{D_2}(v_5)$ | $0$ |
| $D:$ | $m^D(v_1)$ | $m^D(v_2)$ | $m^D(v_3)$ | $m^D(v_4)$ | $m^D(v_5)$ | $m^D(v_6)$ |

# Splitting $D$ by $\sigma^D(c)$

| | $p_D(c)$ | | | | | |
|---|---|---|---|---|---|---|
| | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ |
| $D:$ | $m^D(v_1)$ | $m^D(v_2)$ | $m^D(v_3)$ | $m^D(v_4)$ | $m^D(v_5)$ | $m^D(v_6)$ |
| $\hat{D}_{v_6}:$ | $\hat{m}^{v_6}(v_1)$ | $\hat{m}^{v_6}(v_2)$ | $\hat{m}^{v_6}(v_3)$ | $0$ | $0$ | $m^D(v_6)$ |
| $\hat{D}_{v_5}:$ | $\hat{m}^{v_5}(v_1)$ | $\hat{m}^{v_5}(v_2)$ | $\hat{m}^{v_5}(v_3)$ | $0$ | $m^D(v_5)$ | $0$ |
| $\hat{D}^{v_4}:$ | $\hat{m}^{v_4}(v_1)$ | $\hat{m}^{v_4}(v_2)$ | $\hat{m}^{v_4}(v_3)$ | $m^D(v_4)$ | $0$ | $0$ |

# Splitting $D$ by $\sigma^D(c)$

| | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ |
|---|---|---|---|---|---|---|
| $D_M$ : | $m^0(v_1)$ | $m^0(v_2)$ | $m^0(v_3)$ | $m^0(v_4)$ | $m^0(v_5)$ | $m^0(v_6)$ |
| $D_1$ : | $m^{D_1}(v_1)$ | $m^{D_1}(v_2)$ | $m^{D_1}(v_3)$ | $0$ | $0$ | $m^{D_1}(v_6)$ |
| $D_2$ : | $m^{D_2}(v_1)$ | $m^{D_2}(v_2)$ | $m^{D_2}(v_3)$ | $0$ | $m^{D_2}(v_5)$ | $0$ |
| $D$ : | $m^D(v_1)$ | $m^D(v_2)$ | $m^D(v_3)$ | $m^D(v_4)$ | $m^D(v_5)$ | $m^D(v_6)$ |

# Splitting $D$ by $\sigma^D(c)$

| | | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ |
|---|---|---|---|---|---|---|---|
| | $D_M:$ | $m^0(v_1)$ | $m^0(v_2)$ | $m^0(v_3)$ | $m^0(v_4)$ | $m^0(v_5)$ | $m^0(v_6)$ |
| | $D_1:$ | $m^{D_1}(v_1)$ | $m^{D_1}(v_2)$ | $m^{D_1}(v_3)$ | 0 | 0 | $m^{D_1}(v_6)$ |
| | $m^{D_2}:$ | $m^{D_2}(v_1)$ | $m^{D_2}(v_2)$ | $m^{D_2}(v_3)$ | 0 | $m^{D_2}(v_5)$ | 0 |
| $D$ | $\hat{D}_{v_6}:$ | $\hat{m}^{v_6}(v_1)$ | $\hat{m}^{v_6}(v_2)$ | $\hat{m}^{v_6}(v_3)$ | 0 | 0 | $m^D(v_6)$ |
| | $\hat{D}^{v_5}:$ | $\hat{m}^{v_5}(v_1)$ | $\hat{m}^{v_5}(v_2)$ | $\hat{m}^{v_5}(v_3)$ | 0 | $m^D(v_5)$ | 0 |
| | $\hat{D}^{v_4}:$ | $\hat{m}^{v_4}(v_1)$ | $\hat{m}^{v_4}(v_2)$ | $\hat{m}^{v_4}(v_3)$ | $m^D(v_4)$ | 0 | 0 |

# Sketch of Proof

(**) ⇔ For any $c \in C$, any $D \in \text{supp}(\sigma(c))$ and any $D' \in \text{supp}(\sigma^D)$,

$$\boldsymbol{p}_{D'}(z) \geq \boldsymbol{p}_D(z), \forall z \in [\underline{c}, c]$$

# Sketch of Proof

(**) $\Leftrightarrow$ For any $c \in C$, any $D \in \operatorname{supp}(\sigma(c))$ and any $D' \in \operatorname{supp}(\sigma^D)$,

$$\boldsymbol{p}_{D'}(z) \geq \boldsymbol{p}_D(z), \forall z \in [\underline{c}, c]$$

Therefore, for all $c, c' \in C$ with $c' < c$,

# Sketch of Proof

(**) $\Leftrightarrow$ For any $c \in C$, any $D \in \operatorname{supp}(\sigma(c))$ and any $D' \in \operatorname{supp}(\sigma^D)$,

$$\boldsymbol{p}_{D'}(z) \geq \boldsymbol{p}_D(z), \forall z \in [\underline{c}, c]$$

Therefore, for all $c, c' \in C$ with $c' < c$,

$$\int_{c'}^{c} \left( \int_{\mathscr{D}} Q_{D'}(\boldsymbol{p}_{D'}(z))\hat{\sigma}(\mathrm{d}D'|z) - \int_{\mathscr{D}} D'(\boldsymbol{p}_{D'}(z))\hat{\sigma}(\mathrm{d}D'|c) \right) \mathrm{d}z$$

# Sketch of Proof

(**) ⇔ For any $c \in C$, any $D \in \mathrm{supp}(\sigma(c))$ and any $D' \in \mathrm{supp}(\sigma^D)$,

$$\boldsymbol{p}_{D'}(z) \geq \boldsymbol{p}_D(z), \forall z \in [\underline{c}, c]$$

Therefore, for all $c, c' \in C$ with $c' < c$,

$$\int_{c'}^{c} \left( \int_{\mathscr{D}} Q_{D'}(\boldsymbol{p}_{D'}(z))\hat{\sigma}(\mathrm{d}D'|z) - \int_{\mathscr{D}} D'(\boldsymbol{p}_{D'}(z))\hat{\sigma}(\mathrm{d}D'|c) \right) \mathrm{d}z$$
$$= \int_{c'}^{c} \left( \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma(\mathrm{d}D|z) - \int_{\mathscr{D}} D'(\boldsymbol{p}_{D'}(z))\hat{\sigma}(\mathrm{d}D'|c) \right) \mathrm{d}z$$

# Sketch of Proof

(**) $\Leftrightarrow$ For any $c \in C$, any $D \in \operatorname{supp}(\sigma(c))$ and any $D' \in \operatorname{supp}(\sigma^D)$,

$$\boldsymbol{p}_{D'}(z) \geq \boldsymbol{p}_D(z), \forall z \in [\underline{c}, c]$$

Therefore, for all $c, c' \in C$ with $c' < c$,

$$\begin{aligned}
&\int_{c'}^{c} \left( \int_{\mathscr{D}} Q_{D'}(\boldsymbol{p}_{D'}(z))\hat{\sigma}(\mathrm{d}D'|z) - \int_{\mathscr{D}} D'(\boldsymbol{p}_{D'}(z))\hat{\sigma}(\mathrm{d}D'|c) \right) \mathrm{d}z \\
= &\int_{c'}^{c} \left( \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma(\mathrm{d}D|z) - \int_{\mathscr{D}} D'(\boldsymbol{p}_{D'}(z))\hat{\sigma}(\mathrm{d}D'|c) \right) \mathrm{d}z \\
\geq &\int_{c'}^{c} \left( \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma(\mathrm{d}D|z) - \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma(\mathrm{d}D|c) \right) \mathrm{d}z
\end{aligned}$$

# Sketch of Proof

(**) ⇔ For any $c \in C$, any $D \in \mathrm{supp}(\sigma(c))$ and any $D' \in \mathrm{supp}(\sigma^D)$,

$$\boldsymbol{p}_{D'}(z) \geq \boldsymbol{p}_D(z), \forall z \in [\underline{c}, c]$$

Therefore, for all $c, c' \in C$ with $c' < c$,

$$\begin{aligned}
&\int_{c'}^{c} \left( \int_{\mathscr{D}} Q_{D'}(\boldsymbol{p}_{D'}(z))\hat{\sigma}(\mathrm{d}D'|z) - \int_{\mathscr{D}} D'(\boldsymbol{p}_{D'}(z))\hat{\sigma}(\mathrm{d}D'|c) \right) \mathrm{d}z \\
= &\int_{c'}^{c} \left( \int_{\mathscr{D}} D(\boldsymbol{p}_{D}(z))\sigma(\mathrm{d}D|z) - \int_{\mathscr{D}} D'(\boldsymbol{p}_{D'}(z))\hat{\sigma}(\mathrm{d}D'|c) \right) \mathrm{d}z \\
\geq &\int_{c'}^{c} \left( \int_{\mathscr{D}} D(\boldsymbol{p}_{D}(z))\sigma(\mathrm{d}D|z) - \int_{\mathscr{D}} D(\boldsymbol{p}_{D}(z))\sigma(\mathrm{d}D|c) \right) \mathrm{d}z \\
\geq &0
\end{aligned}$$

# Sketch of Proof

(**) ⇔ For any $c \in C$, any $D \in \mathrm{supp}(\sigma(c))$ and any $D' \in \mathrm{supp}(\sigma^D)$,

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=& \int_{c'}^{c} \left( \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma(\mathrm{d}D|z) - \int_{\mathscr{D}} D'(\boldsymbol{p}_{D'}(z))\hat{\sigma}(\mathrm{d}D'|c) \right) \mathrm{d}z \\
\geq& \int_{c'}^{c} \left( \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma(\mathrm{d}D|z) - \int_{\mathscr{D}} D(\boldsymbol{p}_D(z))\sigma(\mathrm{d}D|c) \right) \mathrm{d}z \\
\geq& 0
\end{aligned}$$

⇒ IC is relaxed.

# Sketch of Proof

Can show that IR is also relaxed.

Revenue equivalence formula
⇒ There exists a mechanism $(\hat{\sigma}, \hat{\tau}, \boldsymbol{p})$ that strictly improves the revenue.

▸ Back

# Sketch of Proof

**Step 1:** Finding an upper bound for revenue (the price-controlling data broker's revenue).

**Step 2:** Constructing a feasible mechanism that attains this upper bound.

Back

# Step 1: An Upper Bound for Revenue

The price-controlling data broker's optimal revenue is an upper bound for the data broker's revenue.

A mechanism (of the price-controlling data broker) $(\sigma, \tau, \gamma)$ is:

- incentive compatible if for any $c, c' \in C$,

$$\int_{\mathscr{D} \times \mathbb{R}_+} (p-c)D(p)\gamma(\mathrm{d}p|D,c)\sigma(\mathrm{d}D|c) - \tau(c) \\ \geq \int_{\mathscr{D} \times \mathbb{R}_+} (p-c)D(p)\gamma(\mathrm{d}p|D,c')\sigma(\mathrm{d}D|c') - \tau(c')$$

- individually rational if for any $c \in C$,

$$\int_{\mathscr{D} \times \mathbb{R}_+} (p-c)D(p)\gamma(\mathrm{d}p|D,c)\sigma(\mathrm{d}D|c) - \tau(c) \geq \pi_{D_M}(c)$$

▸ Details

# Solving the Price-Controlling Data Broker's Problem

Prices are contractable ⇒ Can discourage trade by prices.

Can restrict attention to the following mechanisms: For any report $c$, commit to a cutoff $\psi(c)$ so that

- Sell to all consumers with $v \geq \psi(c)$ by charging their values.
- Not sell to the rest of consumers

Choice of mechanism is reduced to a (one-dimensional) cutoff function $\psi$ and transfer scheme $\tau$.

Standard revenue-equivalence formula
⇒ Choosing nondecreasing $\psi$ to maximize virtual profit s.t. IR constraints.

# Solving the Price-Controlling Data Broker's Problem

Recall that

$$\phi_G(c) := c + \frac{G(c)}{g(c)}$$

is the virtual marginal cost and that

$$\overline{\phi}_G(c) := \min\{\phi_G(c), \boldsymbol{p}_{D_M}(c)\}.$$

**Proposition**

*The price-controlling data broker's optimal cutoff function is $\overline{\phi}_G$ and the optimal revenue is*

$$R^* = \int_C \left( \int_{\{v \geq \overline{\phi}_G(c)\}} (v - \phi_G(c)) D_M(\mathrm{d}v) \right) G(\mathrm{d}c) - \pi_{D_M}(\overline{c}).$$

*Furthermore, any optimal mechanism of the price-controlling data broker induces $\bar{\phi}_G(c)$-quasi-perfect price discrimination for $G$-almost all $c \in C$.*

▸ Sketch of Proof ▸ Back

# Sketch of Proof

Using integration by parts, for any $\boldsymbol{q} \in \mathcal{Q}$,

$$\begin{aligned} &\int_C \left( \int_c^{\bar{c}} (\boldsymbol{q}(z) - D_M(\boldsymbol{p}_{D_M}(z))) \, \mathrm{d}z \right) \mu^*(\mathrm{d}c) \\ = &\int_C M^*(c)(\boldsymbol{q}(c) - D_M(\boldsymbol{p}_{D_M}(c))) \, \mathrm{d}c \end{aligned}$$

# Sketch of Proof

Therefore, for any $\boldsymbol{q} \in \mathcal{Q}$,

$$\begin{aligned}
&R(\boldsymbol{q}) + \int_C \left( \int_c^{\bar{c}} (\boldsymbol{q}(z) - D_M(\boldsymbol{p}_{D_M}(z)))\,\mathrm{d}z \right) \mu^*(\mathrm{d}c) \\
&= \int_C \left( \int_0^{\boldsymbol{q}(c)} (D_M^{-1}(y) - \phi_G(c))\,\mathrm{d}y \right) G(\mathrm{d}c) - \bar{\pi} \\
&\quad + \int_C M^*(c)(\boldsymbol{q}(c) - D_M(\boldsymbol{p}_{D_M}(c)))\,\mathrm{d}c \\
&= \int_C \left( \int_0^{\boldsymbol{q}(c)} (D_M^{-1}(y) - \phi_G(c))\,\mathrm{d}y \right) G(\mathrm{d}c) \\
&\quad - \bar{\pi} - \int_C M^*(c) D_M(\boldsymbol{p}_{D_M}(c))\,\mathrm{d}c,
\end{aligned}$$

# Sketch of Proof

Thus, the dual is equivalent to

$$\sup_{\boldsymbol{q}\in\mathcal{Q}} \int_C \left( \int_0^{\boldsymbol{q}(c)} (v - \phi_G(c))\, \mathrm{d}y \right) G(\mathrm{d}c),$$

which has a solution $D_M \circ \phi_G$.

Also, since $\phi_G(c) = \boldsymbol{p}_{D_M}(c)$ for all $c$ such that $M^*(c) > 0$, the complementary slackness condition is also satisfied. That is

$$\begin{aligned}
&\int_C M^*(c)(D_M(\phi_G(c)) - D_M(\boldsymbol{p}_{D_M}(c)))\, \mathrm{d}c \\
=&\int_{c^*}^{\bar{c}} M^*(c)(D_M(\boldsymbol{p}_{D_M}(c)) - D_M(\boldsymbol{p}_{D_M}(c)))\, \mathrm{d}c \\
=&0.
\end{aligned}$$

Back

# Sketch of Proof

Bergemann et al. (2013) construct an output minimizing segmentation.

Given $m^0 \in \Delta^f(V)$ and $\hat{c}$, $\hat{\psi}(\hat{c})$ is the smallest $\hat{\psi}$ such that

$$\pi_0(\hat{c}) \leq \sum_{v \geq \hat{\psi}} (v - \hat{c}) m^0(v).$$

Notice that $\hat{\psi}(\hat{c}) \geq \boldsymbol{p}_{D_M}(\hat{c}) \geq \psi(c)$.

▸ Back

# Sketch of Proof

For each $v \geq \psi(c)$, define $\beta^{v}_{v'}$ recursively by

$$\beta^{v}_{v'} := \frac{(v-\hat{c})m^{0}(v) - (v'-\hat{c})\sum_{\hat{v}>v'} \hat{m}^{v}(\hat{v})}{\sum_{v \geq \psi(v)} \left[(v-\hat{c})m^{0}(v) - (v'-\hat{c})\sum_{\hat{v}>v'} \hat{m}^{v}(\hat{v})\right]}, \forall \underline{p}_{m^{0}}(\hat{c}) \leq v' < \psi(c).$$

Also, let

$$\alpha^{v} := \frac{\sum_{\hat{v} \geq \underline{p}_{m^{0}}(\hat{c})} \hat{m}^{v}(\hat{v})}{\sum_{\hat{v} \geq \underline{p}_{m^{0}}(\hat{c})} m(\hat{v})}, \forall v' < \underline{p}_{m^{0}}(\hat{c}).$$

Then define

$$\hat{m}^{v}(v') := \begin{cases} m^{0}(v), & \text{if } v' = v \\ 0, & \text{if } v' \geq \psi(c),\, v' \neq v \\ \beta^{v}_{v'} m^{0}(v'), & \text{if } \underline{p}_{m^{0}}(\hat{c}) \leq v' < \psi(c) \\ \alpha^{v} m^{0}(v'), & \text{if } v' < \underline{p}_{m^{0}}(\hat{c}) \end{cases},$$

for all $v \geq \psi(c)$ and for all $v'$.

# Sketch of Proof

Can be verified that $\beta^{v}_{v'} \in [0,1]$, $\alpha^{v} \in [0,1]$ and

$$\sum_{v \geq \psi(c)} \beta^{v}_{v'} = \sum_{v \geq \psi(c)} \alpha^{v} = 1, \forall v' < \psi(c)$$

Furthermore, $v \in \boldsymbol{P}_{\hat{m}^{v}}(z)$ for all $z \geq \hat{c}$ and $\boldsymbol{p}_{\hat{m}^{v}}(z) \geq \boldsymbol{p}_{D_M}(z)$ for all $z < \hat{c}$.

▸ Back

# Sketch of Proof

Consider the price-controlling data broker's problem

$$\max_{(\sigma,\tau,\gamma)} \int_C \tau(c)G(\mathrm{d}c)$$

$$\text{s.t.} \int_{\mathscr{D}\times\mathbb{R}_+} (p-c)D(p)\gamma(\mathrm{d}p|D,c)\sigma(\mathrm{d}D|c) - \tau(c)$$

$$\geq \int_{\mathscr{D}\times\mathbb{R}_+} (p-c)D(p)\gamma(\mathrm{d}p|D,c')\sigma(\mathrm{d}D|c') - \tau(c'), \qquad \text{(IC*)}$$

$$\int_{\mathscr{D}\times\mathbb{R}_+} (p-c)D(p)\gamma(\mathrm{d}p|D,c)\sigma(\mathrm{d}D|c) - \tau(c) \geq \pi_{D_M}(c), \qquad \text{(IR*)}$$

$$\forall c, c' \in C$$

Consider any $(\sigma,\tau,\gamma)$ satisfying (IC*) and (IR*).

# Sketch of Proof

Let

$$\boldsymbol{q}(c) := \int_{\mathscr{D}} D(p)\gamma(\mathrm{d}p|D,c)\sigma(\mathrm{d}D|c).$$

# Sketch of Proof

Let

$$\boldsymbol{q}(c) := \int_{\mathscr{D}} D(p)\gamma(\mathrm{d}p|D,c)\sigma(\mathrm{d}D|c).$$

$(\bar{\sigma},\bar{\gamma})$: perfectly price discriminating all consumers with values above the $(1-\boldsymbol{q}(c))$-th percentile for all $c$.

# Sketch of Proof

Let

$$\boldsymbol{q}(c) := \int_{\mathscr{D}} D(p)\gamma(\mathrm{d}p|D,c)\sigma(\mathrm{d}D|c).$$

$(\bar{\sigma},\bar{\gamma})$: perfectly price discriminating all consumers with values above the $(1-\boldsymbol{q}(c))$-th percentile for all $c$.

Let

$$\bar{\tau}(c) := \int_{\mathscr{D}\times\mathbb{R}_+} pD(p)(\bar{\gamma}(\mathrm{d}p|D,c)\bar{\sigma}(\mathrm{d}D|c) - \gamma(\mathrm{d}p|D)\sigma(\mathrm{d}D|c)) + \tau(c).$$

# Sketch of Proof

Let

$$\boldsymbol{q}(c) := \int_{\mathscr{D}} D(p)\gamma(\mathrm{d}p|D,c)\sigma(\mathrm{d}D|c).$$

$(\bar{\sigma},\bar{\gamma})$: perfectly price discriminating all consumers with values above the $(1-\boldsymbol{q}(c))$-th percentile for all $c$.

Let

$$\bar{\tau}(c) := \int_{\mathscr{D}\times\mathbb{R}_+} pD(p)(\bar{\gamma}(\mathrm{d}p|D,c)\bar{\sigma}(\mathrm{d}D|c) - \gamma(\mathrm{d}p|D)\sigma(\mathrm{d}D|c)) + \tau(c).$$

Then, $\bar{\tau}(c) \geq \tau(c)$ and $(\bar{\sigma},\bar{\tau},\bar{\gamma})$ satisfies (R-IC) and (R-IR).

# Sketch of Proof

By the revenue equivalence formula,

$$\mathbb{E}_G[\bar{\tau}(c)] = \int_C \left( \int_0^{\boldsymbol{q}(c)} (D_M^{-1}(q) - \phi_G(c))\,\mathrm{d}q \right) G(\mathrm{d}c) - \pi_{D_M}(\overline{c}),$$

as desired.

▸ Back

# A Relaxed Problem

For any $c, c' \in C$,

$$\int_{\mathscr{D}\times\mathbb{R}_+} (p-c)D(p)\bar{\gamma}(\mathrm{d}p|D,c')\bar{\sigma}(\mathrm{d}D|c') - \bar{\tau}(c')$$

# A Relaxed Problem

For any $c, c' \in C$,

$$\begin{aligned}&\int_{\mathscr{D}\times\mathbb{R}_+}(p-c)D(p)\bar{\gamma}(\mathrm{d}p|D,c')\bar{\sigma}(\mathrm{d}D|c')-\bar{\tau}(c')\\=&\int_{\mathscr{D}\times\mathbb{R}_+}pD(p)\bar{\gamma}(\mathrm{d}p|D,c')\bar{\sigma}(\mathrm{d}D|c')-\bar{\tau}(c')-cq(c')\end{aligned}$$

# A Relaxed Problem

For any $c, c' \in C$,

$$
\begin{aligned}
&\int_{\mathscr{D}\times\mathbb{R}_+} (p-c)D(p)\bar{\gamma}(\mathrm{d}p|D,c')\bar{\sigma}(\mathrm{d}D|c') - \bar{\tau}(c') \\
=&\int_{\mathscr{D}\times\mathbb{R}_+} pD(p)\bar{\gamma}(\mathrm{d}p|D,c')\bar{\sigma}(\mathrm{d}D|c') - \bar{\tau}(c') - cq(c') \\
=&\int_{\mathscr{D}\times\mathbb{R}_+} pD(p)\gamma(\mathrm{d}p|D,c')\sigma(\mathrm{d}D|c') - \tau(c') - cq(c')
\end{aligned}
$$

# A Relaxed Problem

For any $c, c' \in C$,

$$\begin{aligned}
&\int_{\mathscr{D}\times\mathbb{R}_+} (p-c)D(p)\bar{\gamma}(\mathrm{d}p|D,c')\bar{\sigma}(\mathrm{d}D|c') - \bar{\tau}(c') \\
=&\int_{\mathscr{D}\times\mathbb{R}_+} pD(p)\bar{\gamma}(\mathrm{d}p|D,c')\bar{\sigma}(\mathrm{d}D|c') - \bar{\tau}(c') - cq(c') \\
=&\int_{\mathscr{D}\times\mathbb{R}_+} pD(p)\gamma(\mathrm{d}p|D,c')\sigma(\mathrm{d}D|c') - \tau(c') - cq(c') \\
=&\int_{\mathscr{D}\times\mathbb{R}_+} (p-c)D(p)\gamma(\mathrm{d}p|D,c')\sigma(\mathrm{d}D|c') - \tau(c')
\end{aligned}$$

Therefore, $(\sigma, \tau, \gamma)$ satisfies (R-IC) & (R-IR)
$\Rightarrow$ $(\bar{\sigma}, \bar{\tau}, \bar{\gamma})$ satisfies (R-IC) & (R-IR).