Macro Qualifier Exam

Penn State: August, 2022

- You have 3.5 hours. There are four (multi-part) questions, one for each quarter of the first-year macro sequence. Each question is worth 45 points, so if you progress at a rate of one point per minute, you will be able to complete the exam with some time to spare.
- Neither books nor notes are permitted.
- If you make any assumptions beyond what's in the text of the question, please state those assumptions clearly.
- If you need more space, please ask for additional sheets of paper. If you use more sheets, please number the pages, write your identifying number in lieu of your name, and label clearly which question you are answering.
- Please write clearly. Show intermediate work for partial credit. Unannotated scratch work will receive no credit.
- Good luck!

Quarter 1

Consider a two-period overlapping generation model of Diamond (1965) with production. The population of each generation is constant N. Agents in each generation are identical: each has 1 unit of labor when young, which he supplies to the labor market inelastically. He can not work when old. A young agent saves part of his labor earnings, which turns into equal amount of capital when he is old. He supplies his capital holdings to the capital market and earns rent. Each initial old agent is endowed with $k_1 > 0$ units of capital at $t = 1$, and consumes the income from his capital endowment. A generation-t agent's preference over two-period consumption, c_t^t when young and c_{t+1}^t when old, is given by

$$u(c_t^t, c_{t+1}^t) = \ln(c_t^t) + \beta \ln(c_{t+1}^t)$$

where $\beta \in (0, 1)$. The production function in an arbitrary period t is given by

$$F(K_t, L_t) = K_t^\alpha L_t^{1-\alpha}$$

where $\alpha \in (0, 1)$, and K_t and L_t are amount of capital and labor employed in production at t, respectively. Capital depreciates fully each period after being used. All markets (goods, capital, and labor) in each period are perfectly competitive.

1. (8 Points) Formulate the planner's problem of choosing the stationary allocation to maximize a representative two-period-lived agent's welfare (stationary allocation: $c_t^t = c_0$, $c_{t+1}^t = c_1$ for all t). Solve for the optimal stationary allocation as function of the parameters of the model.

2. (22 Points) Competitive equilibrium.

   (a) Formulate the individual optimization problem of a representative generation-t agent, derive the agent's optimal saving function.

   (b) Define a competitive equilibrium. Explicitly specify the set of equations that a competitive equilibrium must satisfy.

   (c) Solve explicitly a stationary competitive equilibrium (allocation is stationary over time) in terms of the parameters of the model.

3. (6 Points) Compare the stationary competitive equilibrium you solved in (2.c) with the planner's solution in (1). What are the conditions in terms of the parameters of the model such that the stationary competitive equilibrium is *not* Pareto optimal?

4. (9 Points) Suppose the conditions in (3) hold. Can an infinitely-lived benevolent government correct the inefficiency by implementing some government policy? If the answer is yes, outline the policy. If the answer is no, explain why.

Quarter 2

Consider a steady-state economy, in continuous time, where firms and workers are ex-ante homogeneous and discount future at a rate r. The match-specific output y is a random draw from a continuous distribution G supported on $\mathbb{R}_+$. The matching rate for a firm is $q(x)$, the matching rate for a worker is $xq(x)$, where x is the number of vacancy per unemployed worker. The function q has the usual properties and, in particular, q is decreasing and $\lim_{x \to \infty} q(x) = 0$. Separation is exogenous and occurs at rate s. Assume that the vacancy cost is c and the flow income of an unemployed worker is z.

Let us assume that the output of a match is publicly known, and the bargaining wage depends on the output. Denote the bargained wage as $w(y)$, the value of the employed worker with output y as $J_e(y)$, the value of an unemployed worker as J_u, the value of a matched firm with output y as $J_f(y)$, and the value of an unmatched firm as J_v. Competitive entry pushed the value of an unmatched firm to zero.

1. (9 Points) Fix productivity level y. Write down the HJB equations for the value of a matched firm $J_f(y)$ and for the value of an employed worker $J_e(y)$. Use the HJB equations to solve for $J_f(y)$ and $J_e(y)$ as functions of J_v, J_u and parameters of the model.

2. (9 Points) Let $S(y)$ be the joint matching surplus, $S(y) = J_f(y) - J_v + J_e(y) - J_u$. Express $S(y)$ in terms of in terms of y, J_u, J_v. Show that $S'(y) = \frac{1}{r+s}$. Let y_0 be the solution to $S(y) = 0$, then $S(y) = \frac{1}{r+s}(y - y_0)$. What does y_0 represent?

3. (9 Points) When $y \geq y_0$, assume that the bargained wage $w(y)$ solves the following problem:

$$\max_w \left(J_f(y) - J_v\right)^{1-b} \left(J_e(y) - J_u\right)^b$$

   where b is the bargaining weight for workers, and the outside options J_v and J_u are taken as given at the bargaining stage. Show that at the optimum, we have $J_f(y) - J_v = (1-b)S(y)$ and $J_e(y) - J_u = bS(y)$.

4. (9 Points) Write down the HJB equations for the value of a unmatched firm J_v and for the value of an unemployed worker J_u. Show that $rJ_v = -c + q(x)(1-b)\int_{y_0}^{\infty} S(y)\,dG(y)$ and $rJ_u = z + xq(x)\,b\int_{y_0}^{\infty} S(y)\,dG(y)$.

5. (9 Points) Show that the endogenous variables y_0 and x have to satisfy the following conditions: $\frac{q(x)(1-b)}{r+s}\left[\int_{y_0}^{\infty}(y - y_0)dG(y)\right] = c$ and $y_0 = z + \frac{b}{1-b}cx$. Explain the intuition of the two equations and draw them in the (x, y_0)-plane. Does the equilibrium exist and is it unique?

Quarter 3

There is measure-1 of ex-ante identical agents, and three periods, $t = 0,\ 1,\ 2$. No consumption or production takes place at $t = 0$. There is one good at each of the remaining two periods. At period t, an agent receives an endowment $w_t \in \{w^l,\ w^h\}$, $0 < w^l < w^h$, $\text{Prob}\{w_t = w^i\} = 1/2$, $\forall\, i = l,\ h$, $\forall\, t = 1,\ 2$. The idiosyncratic endowment shock w_t is i.i.d. across time and agents.

Agents have identical preferences, represented by lifetime utility function $u(c_1) + \beta u(c_2)$, where c_t is an agent's date-t consumption, $u(c) = -e^{-\alpha c}$, $\alpha > 0$ and $\beta \in (0, 1)$. There is a storage technology that can transform 1 unit of $t = 1$ good into β^{-1} units of $t = 2$ good.

1. (10 Points) Formulate the planner's problem when agents' idiosyncratic endowment realizations are observable. Solve the first-best allocation.

2. (20 Points) Suppose that there is a debt security market at $t = 1$ where agents can borrow and lend from one another at the market-clearing interest rate risk-free. That is, all loans made at $t = 1$ have to be fully repaid at $t = 2$, no default is allowed. Characterize fully the equilibrium (allocations and interest rate) that can be achieved in this economy.

3. (15 Points) Assume that the realizations of each agent's endowment is his/her private information. Formulate the planner's problem that solves for constrained efficient allocation when

   (a) agents can not save secretly.

   (b) agents can save without being detected.

   Note: no need to solve them.

Quarter 4

This problem will have you solve a real business cycle model where workers accumulate human capital from their past labor market experience. Time is discrete with an infinite horizon. There is a representative household and a representative firm. Markets are perfectly competitive. There is no money in this economy, so all variables are real, not nominal.

1. (7 Points) The representative household's utility in period t is $u(c_t) - v(\ell_t)$, where c_t is consumption, ℓ_t is hours of work, $u(\cdot)$ is increasing and concave, and $v(\cdot)$ is increasing and weakly convex. The household's discount factor is β. The household's period-t labor income is $w_t x_t \ell_t$, where x_t is the household's stock of human capital, and w_t is the real wage. Notice that the real wage is compensation for each effective unit of labor supplied, not compensation per hour. For simplicity, assume that the household has no non-labor income, and there's no borrowing nor saving. The budget constraint is therefore:

$$c_t = w_t x_t \ell_t. \tag{1}$$

The household's stock of human capital depends on its past labor-market experience. Specifically, assume that:

$$x_{t+1} = x_t^{1-\phi} \ell_t, \tag{2}$$

where $\phi \in (0,1]$. The household takes wages as given, so they treat w_t as a stationary and exogenous Markov process. What are the household's state variables? Write the household's Bellman equation, and denote the value function $f(\cdot)$.

2. (8 Points) What are the first-order and envelope conditions for the household? (You don't have to prove that $f(\cdot)$ is differentiable. Also, don't worry about any non-negativity constraints, and assume that the solution is interior.)

3. (7 Points) Combine the first-order and envelope conditions to eliminate any terms that depend on the derivative of $f(\cdot)$. Your answer should give you an expectational difference equation.

4. (7 Points) The representative firm is assumed to maximize profits period-by-period. The firm's period-t output is $z_t g(h_t)$, where z_t is an exogenous productivity process, and h_t is the effective quantity of labor that the firm employs. (In an equilibrium, h_t will have to equal $x_t \ell_t$, but the firm only cares about the effective quantity of labor h_t, not x_t and ℓ_t separately.) The firm also takes w_t as given, and its period-t cost of production is $w_t h_t$. Assume that $g(\cdot)$ is increasing and weakly concave. Write down the firm's optimization problem. What is the first-order condition?

5. (8 Points) Given a stochastic process for z_t, the equilibrium is characterized by equations (1) and (2), your answers to questions 3 and 4, and the market-clearing condition for human capital:

$$h_t = x_t \ell_t. \tag{3}$$

For the remainder of the exam, assume the following functional forms:

$$\begin{aligned}
\log(z_t) &= (1-\rho)\log(\bar{z}) + \rho \log(z_{t-1}) + \epsilon_t + \psi \epsilon_{t-1}, \quad \epsilon_t \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, \sigma^2) && (4)\\
u(c_t) &= \frac{c_t^{1-\gamma}}{1-\gamma} && (5)\\
v(\ell_t) &= \ell_t && (6)\\
g(h_t) &= h_t, && (7)
\end{aligned}$$

where γ, ρ, σ, and $\bar{z}$ are exogenous parameters. As in class, let $(\bar{c}, \bar{\ell}, \bar{x}, \bar{h}, \bar{w}, \bar{z})$ denote the values of $(c_t, \ell_t, x_t, h_t, w_t, z_t)$ that would prevail in a steady state, i.e., an equilibrium in which ϵ_t is zero and all other variables are constant. You don't have to solve for the steady-state equilibrium; instead, we'll analyze fluctuations around the steady state. As in class, let "hats" denote log deviations from steady-state:

$$\hat{c}_t \equiv \log(c_t) - \log(\bar{c}), \tag{8}$$

with $\left(\hat{\ell}_t, \hat{x}_t, \hat{h}_t, \hat{w}_t, \hat{z}_t\right)$ defined analogously. For the remainder of the exam, assume that $\beta = 0$, meaning households are myopic. (You don't have to re-solve the household's problem; just set β to zero in the conditions that you've already derived.) Using the functional forms above, log-linearize equations (1)-(4) and your answers to questions 3 and 4 under the assumption $\beta = 0$.

6. (8 Points) Consolidate the log-linearized equations to obtain a difference equation that contains only $\hat{x}_t$ and $\hat{z}_t$. Show that, in an equilibrium, $\hat{x}_t$ follows a finite-order ARMA(p, q) process. What are the values of p and q? What conditions must be satisfied for $\hat{x}_t$ to be stationary? What conditions must be satisfied for $\hat{x}_t$ to be invertible? State your answer in terms of the parameters of the equilibrium model, not just the general conditions for stationarity and invertibility.
Hint: Use lag polynomials. You will likely get an expression that looks like $A(L)\,\hat{x}_{t+1} = B(L)\,\epsilon_t$, where $A(L)$ and $B(L)$ are polynomials with non-negative powers of L. Don't be distracted by the fact that $\hat{x}_{t+1}$, rather than $\hat{x}_t$, is being multiplied by the lag polynomial $A(L)$. For the purposes of showing that $\hat{x}_t$ is an ARMA process, all that matters is that the polynomials contain non-negative powers of L and that $\{\epsilon_t\}$ is white noise.