# Optimal Dynamic Fiscal Policy with Endogenous Debt Limits

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**Abstract**

VERY PRELIMINARY. PLEASE DO NOT CITE OR CIRCULATE.

Governments use debt to smooth revenues relative to spending. An important concern of economic policy is the debt capacity of an economy. We study this problem using a common model but with important changes. First, we assume that government spending is flexible. Second, we solve for the government's dynamic problem and explicitly account for possible nonconvexities and binding constraints. Third, our numerical results can be verified. Fourth, we compute the endogenous limit, not imposing any artificial limit. These new features lead to substantially different results. First, there is no tendency to accumulating a war chest large enough to allow taxation to disappear. Second, there are multiple stationary points in the long-run distribution of debt. Third, allowing for flexibility in government spending fundamentally changes an economy's debt capacity. In fact, US historical experience is inconsistent with debt for many reasonable preference specifications if spending is exogenous.

# 1 Introduction

Since the financial crisis of 2008 and increased government debt levels worldwide, fiscal austerity has been a focal point in public debates. Central to these debates is the natural debt limit, i.e. the level of public debt that's sustainable in the long run, and the design of fiscal

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policy that is consistent with that limit. In much of the earlier work on dynamic fiscal policy, governments are not allowed to lend, and the upper limit on debt is determined in an ad-hoc manner. Aiyagari et al. (2002)'s (AMSS) seminal paper on fiscal policy in incomplete markets relaxed the lending assumption and revisited earlier work of Barro (1979) and Lucas and Stokey (1983) to study the implications on tax policy. Their results implied that taxes should roughly follow a random walk. They also presented examples where the long-run tax rate is zero, and any spending is financed out of its asset income (i.e., government holds debt of the people). However, their approach had some weaknesses. First, it imposed an artificial limit on government debt and therefore did not address the question of a natural debt limit. Second, it assumed, as much of the literature prior to it did, government spending to be exogenous. Third, it ignored the potential nonconvexities in the government's dynamic optimization problem. Fourth, it assumed that government policy could be well approximated with a low-degree polynomial. We relax the assumptions on debt and spending, and we use computational methods that do not rely only on local optimality. While we focus on the models examined in AMSS, we present a framework that can address fiscal policy issues in a self-consistent manner. In particular, we derive the endogenous limits on debt and allow for endogenous government spending.

Our approach involves recasting the policy problem as an infinite horizon dynamic programming problem (see, e.g., Judd (1998)). The government's value function may not be concave and it can also very high curvature, particularly as debt approaches its endogenous limit. In dynamic taxation problems, the government's problem is a mathematical program with complementary constraints (MPCC). We explicitly use the MPCC formulation, which is essential in order to do the necessary global optimization analysis of the government's problem. Our MPCC approach uses the computational algorithms that were developed only in the past twenty years, and allows us to solve the problem reliably and accurately. The resulting dynamic programming problem is challenging in terms of both formulation and its use of computational resources. We have embedded our model into the scalable and flexible high-performance computing (HPC) value function iteration code of Brumm and Scheidegger (2013), Brumm et al. (2014), exploiting the sparse grids literature, which also is a relatively recent set of multivariate approximation methods. Using our combination of computational tools and more general economic assumptions, we re-address questions regarding optimal taxation and debt management in a more realistic setup. These tools allow us to determine debt limits implied by assumptions on the primitives of the economic environment and to assess how the level of debt affects both tax policy and general economic performance, and the time series properties of tax rates and debt levels.

Our results show that under the more general framework of endogenous government debt

limits and spending has substantially different implications than earlier analyses have suggested. First, the behavior of optimal policy is, over long horizons (e.g., 1000 years), is much more complex than simpler models imply. In particular, the long-run distribution of debt is multimodal, and the long-run level of debt is history-dependent. If initial debt is low enough and government spending is not hit with large shocks, then the government will accumulate a "war chest" which allows long-run tax rates to be zero. However, if, in the same model, initial debt is high and/or the government gets hit with a long series of bad spending shocks, then debt will rise to a high level and will not fall even if the government is not hit with bad spending shocks. In the second case, governments with large debt levels will avoid default by reducing spending and use taxes to finance a persistently high debt.

We examine the case of fixed government spending and find that the results are dramatically affected. In particular, we illustrate a case where if spending shocks are of moderate size (less than US historical experience) no positive level of debt is feasible. That is, if a government begins with positive debt then there is a sequence of spending shocks such that there is no feasible tax and borrowing policy to finance those expenditures. In such cases, exogenous spending assumptions imply that governments must have their endowed war chests in the beginning and cannot with probability one build up its war chest. These examples illustrate clearly that any analysis of fiscal policy that wants examine historical fiscal policy must consider making spending flexible.

The application of our methodology is not limited to optimal tax problems. Optimal macroeconomic policy problems, as well as social insurance design typically involve solving high-dimensional dynamic programming problems. Solving such problems is a complicated, but very important task, as the policy recommendations depend crucially on the accuracy of the numerical results. In much of the optimal macroeconomic policy and social insurance literature, accuracy of the numerical solutions is unclear. Additionally, most solution approaches ignore feasibility issues and impose ad-hoc limits on state variables such as government debt. An accurate approach to solving dynamic policy models requires the ability to handle the high-dimensional nature of the problems as well as the unknown, feasible state space. The methodology offered in this paper can be used for computing high-dimensional dynamic policy problems with unknown state spaces.

## 1.1   Related Fiscal Policy Literature

Many OLG models in 1970 use dynamic optimization to analyse labor versus consumption versus capital taxation. Turnovsky and Brock (1980) applies standard control theory for optimal taxation and monetary policies and treats the consumer primal and dual variables as states, but ignores feasibility. Kydland and Prescott (1980) show that the time inconsistency

of an optimal taxation plan precludes the use of standard control theory. They formulate the optimal tax problem, same way as Turnovsky and Brock (1980) and extend it to uncertainty. They note that the feasible set of state variables is unknown, acknowledge that it's a hard problem to solve, but offer no concrete solutions. Our environment is similar to AMSS. We assume that the government can only issue one-period risk-free debt. The government can use a flat labor tax rate to extract resources from the representative agent. However, unlike AMSS, we do not assume that government spending is exogenous. We also do not put ad-hoc limits on the government debt, but endogeneously determine the sustainable amount of debt.

# 2  The Economy

## 2.1  Environment

The economy is inhabited by a government and a continuum of identical households, all are assumed to be infinitely-lived. Consumers are endowed with one unit of time in each period, provide labor $\ell$ to produce market consumption goods $c$ or public goods $g$. Time not spent in formal labor activities, $1 - \ell$, can be spent at home, dedicated to leisure activities or home production.

## 2.2  Consumers

Utility for each consumer is a function of personal consumption, $c$, labor, $\ell$, and government consumption, $g$. Furthermore, the utility of government consumption depends on a taste shock $z \in R$ which follows a finite Markov process, with transition matrix $\pi(z'|z)$. We can handle the more general case, but we will follow the literature and assume utility is additively separable in consumption, government spending and labor and takes the form:

$$u(c, \ell, g, z) = uc(c) + u\ell(\ell) + ug(g, z). \tag{1}$$

We typically use the following specification for the utility function:

$$u(c, \ell, g, z) = \frac{(c - cb)^{1-\sigma_1}}{1 - \sigma_1} + \eta \frac{(1 - \ell)^{1-\sigma_2}}{1 - \sigma_2} - \theta(g - z)^{\sigma_3}. \tag{2}$$

where $\theta$, $\eta$, and $cb$ are positive parameters. Assuming $cb > 0$ implies that the revenue-mazimizing tax rate is strictly less than unity. The last term in the utility specification reflects the disutility households get from government spending that deviates from the taste shock $z$. With this utility specification, marginal utility is finite when $c$=0. This allows for labor supply to be zero in response to high taxes and ensures the existence of a Laffer curve. The case where

$g$ is set (exogenously) exactly equal to $z$ can be thought of as assuming $\theta = \infty$, however we will see that the case of exogenous $g$ is not the limit as $\theta$ increases without bound. Consumers evaluate the dynamic consumption, labor and government spending streams according to

$$E_0 \sum_{t=0}^{\infty} \beta^t\, u(c_t, \ell_t, g_t, z_t), \tag{3}$$

where $\beta \in (0,1)$ and $E_0$ denotes the mathematical expectation operator conditioned on time 0 information.

The only assets in the economy are non-contingent, real, risk-free 1-period government bonds which are assumed to have a net supply of 0. One bond at time $t$ promises to deliver one unit of consumption at $t+1$, and has price $p_{b,t}$. We use $b_t$ for payouts of debt at beginning of current period. When $b > 0$, the government is in debt. When $b < 0$, the consumers are in debt to the government. Consumers also pay time-varying flat rate tax $\tau_t$ on their labor income. All consumers receive a transfer payment $tr_t \geq 0$ in each period. The time $t$ budget constraint of a consumer is given by

$$(c_t + p_{b,t}b_{t+1}) - (b_t + tr_t + (1 - \tau_t)\ell_t) \leq 0, \tag{4}$$

For each consumer, the fiscal policy $\{\tau, tr, g\}$, and taste shock $z$ are exogenous variables and the only endogenous state variable is the consumer's bond holdings. We let $\Phi$ represent the fiscal state set $\Phi$ . Consumers maximize their expected discounted payoff

$$\max_{\{b_{t+1}, c_t, \ell_t\}} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t, g_t, z_t) | \Phi_t \right\} \tag{5}$$

given fiscal policy $\{\tau, tr, g\}$ and taste shock $z$, subject to the sequence of intertemporal budget constraints given by Equation 4, and the non-negativity constraint on consumption, $c_t \geq 0$.

The labor supply-consumption and debt decisions, $\ell, c$ and $b$, of the consumer depend not only upon the current state of the economy, $z$, and current fiscal policy $\Phi$, but also upon future fiscal policy. Until the sequences of fiscal policies are specified, the current equilibrium decisions of the consumers cannot be determined. Suppose the optimal policy sequence chosen at time 0, $\{\Phi_t^0\}_{t=0}^{\infty}$, exists and is unique. The optimal policy will be time inconsistent in the sense that the policy $\{\Phi_t^0\}_{t=s}^{\infty}$ will not be optimal at time $s > 0$. The reason it is not optimal is because current equilibrium decisions of the consumer are functions of the current state, current policy decisions, and anticipated future policy actions. The time inconsistency severely complicates the computation of the optimal policy. Standard recursive methods are no longer applicable. In what follows we outline a possible computational procedure, and

point out the difficulties involved. To obtain restrictions imposed by the rational expectations equilibrium assumption, we formulate the Lagrangean for the consumer

$$L = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t, \ell_t, g_t, z_t) - \lambda_t(c_t + p_{b,t}b_{t+1} - b_t - tr_t - (1 - \tau_t)\ell_t) + \mu_t c_t \mid \Phi_t \right\}$$

where $\lambda_t$ is the multiplier on the t-period budget constraint and $\mu_t$ is the multiplier on $t$-period consumption. The first-order conditions from the consumer problem are:

$$\text{FOC}c: \quad -\lambda_t + uc'(c_t) + \mu_t = 0, \tag{6}$$

$$\text{FOC}\ell: \quad (1 - \tau_t)\lambda_t + u\ell'(\ell_t) = 0, \tag{7}$$

$$\text{Euler}: \quad \beta \sum_{z_{t+1}} \lambda_{t+1} \pi(z_{t+1}|z_t) - p_{b,t}\lambda_t = 0, \tag{8}$$

$$\text{Budget}: \quad -b_t + c_t - tr_t - \ell_t(1 - \tau_t) + p_{b,t}b_{t+1} = 0, \tag{9}$$

$$\text{KKT}c: \quad \mu_t c_t = 0, \tag{10}$$

$$\mu_t \geq 0, \; c_t \geq 0. \tag{11}$$

Given our assumptions on the utility function, the consumer's problem is concave, so the first-order conditions are both necessary and sufficient.

The constraint given by Equation 8 has both the current shadow price $\lambda_t$ and the next period shadow price $\lambda_{t+1}$. Future policies affect $\lambda_{t+1}$, which in turn affect consumers' choices in period $t$. When the government decides its policies, it must take into consideration the effect of its future policies on consumers' behavior in earlier periods.

## 2.3   Government

Each period $t$, the government collects labor tax revenue, pays off its old debt, $b_t$, issues new debt, $b_{t+1}$, spends $g_t$, and makes lump-sum transfers, $tr_t$. Its period $t$ budget constraint is:

$$(\tau_t l_t + p_{b,t}b_{t+1}) - (b_t + tr + g_t) = 0, \tag{12}$$

We use $b_t$ for payouts of debt at the beginning of period $t$. The government's plan at time $t$ for debt payments at $t+1$ is denoted $b_{t+1}$ and $b_{t+1}^*$ represents the period demand for (and purchases of) real debt. In equilibrium, the two will be equal: $b_{t+1} = b_{t+1}^*$.

We assume a linear technology for consumption goods $c$ and government goods $g$. With linear technology, the real wage $w$ is normalized to 1. The economy-wide resource constraint

at $t$ is given by

$$(1 - \ell_t) + c_t + g_t = 1. \tag{13}$$

The timing of moves is as follows. After the realization of the current spending shock $z$, the government makes its tax and spending decisions $\tau, tr, g$, chooses market price for bonds $p_b$, the shadow price for the next period, and recommends allocations for the consumer $c, \ell, b^*$. The consumers solve their own optimization problem, given the fiscal policy choice and will pick the same allocation as suggested by the government, if it's in their interest to do so. To ensure that consumers follow through with his plan, the government chooses fiscal policy and consumer allocations that are consistent with consumers' optimal choices of consumption and labor. Additionally, these policies must deliver the shadow price $\lambda^*$ and debt $b^*$ from the consumer's problem. The shock $z$, is an exogenous state variable, and the endogenous state variables for the government problem are $b$ and $\lambda$. In the recursive formulation of the government's problem, we drop the $t$ subscripts, and use superscript $+$ to denote next period's variables. The dynamic programming problem the government solves is the following.

$$V(b, \lambda, z) = max \ u(c, \ell, g, z) + \beta EV(b^+, \lambda^+, z^+) \tag{14}$$

subject to its budget constraint

$$(\tau\ell + p_b b^*) - (b + tr + g) = 0,$$

aggregate resource constraint

$$\ell = c + g,$$

promise-keeping constraints for $\lambda$ and $b$

$$\lambda^* = \lambda$$
$$b^* = b^+,$$

and the first-order conditions from the household's problem,

$$-\lambda^* + uc'(c) + \mu = 0$$
$$(1-\tau)\lambda^* - u\ell'(1-\ell) = 0$$
$$\beta \sum_{z^+} \lambda^+ \pi(z^+|z) - p_b \lambda^* = 0$$
$$-b + c + tr - \ell(1-\tau) + p_b b^* = 0$$
$$\mu c = 0,$$

non-negativity constraints,

$$c, \ell, g, \tau, p_b, \lambda^*, \lambda^+, \mu, tr \geq 0,$$

and the government feasibility constraint

$$(b^+, \lambda^+) \in \Omega(z^+),$$

where $\Omega(z^+)$ is the set of $(b, \lambda)$ for which there exists a policy sequence with an equilibrium in state $z^+$.

The constrained optimization problem defined by Equation 14 is not a standard dynamic programming problem. First and foremost, the set of feasible $b$ and $\lambda$ combinations, $\Omega$, is unknown. Second, the constraint set is not convex. Third, the value function may not be convex. These issues present nontrivial computational challenges which we address next.

## 3 Model relaxations

Solving the dynamic policy problem of the government presents many computational challenges. Before we can approximate the government's value function, we must deal with the unknown domain. More precisely, we must identify the combinations of $b$ and $\lambda$ that are economically feasible. Kydland and Prescott (1980) suggest an iterative procedure that embeds the dynamic programming problem of the government into a fixed point problem for finding $\Omega(z)$, but do not give an actual algorithm. They acknowledge that "This formulation leads to unusual constraints, however, and the problem of actually computing an optimal policy would appear quite formidable even for relatively simple parametric structures." One contribution of this paper is to develop an approach to computing the optimal policy.

Even if we were able to determine the set $\Omega(z)$, it may not be rectangular, which would further complicate the approximation of the value function. The value function is likely to

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be badly behaved along the boundaries. In particular, the marginal value of debt will be close to the marginal social cost of revenue, which will diverge as the revenue approaches the maximum possible. The approximation procedure for the value function must be flexible enough to handle that.

We follow the common strategy of "relaxation"; that is, we replace our inequality constraints with penalty terms. Any state will be feasible in the relaxed dynamic programming problem but the value function will be very low at the previously infeasible states. One way to think about the value function in the original problem is that it is negative infinity at the infeasible states, making it defined mathematically on the extended real line everywhere. One could imagine a relaxation that essentially approximates negative infinity, but finite precision arithmetic prevents us from just using a very large negative number to represent infeasible states.

We have found two relaxations that work for us. We introduce two instruments for the government, $manna \geq 0$ and lump-sum taxes, $LS$.[1] These two instruments provide resources for society and alter both the consumers' and the government's budget constraints. Lump sum taxes are directly taken out of the consumers budget whereas $manna$ appears only in the government's budget constraint. The modified budget constraints for the consumers and the government are given by

$$\text{ConsBud}: \ (c_t + p_{b,t}b^*_{t+1} + LS_t) - (b_t + tr_t + (1 - \tau_t)\ell_t) = 0,$$
$$\text{GovBud}: \ (\tau_t\ell_t + p_{b,t}b^*_{t+1} + manna_t) - (b_t + tr_t - LS_t + g_t) = 0.$$

The lump sum tax $LS \geq 0$ gives the government a way to decrease its debt, and manna makes it easier for the government to meet its $\lambda$ promise. These two instruments relax the government's problem, allowing it to be numerically well-defined over a larger set of $(b, \lambda)$ pairs.

We add a penalty term to the government's payoff function to punish it for using these instruments. The new problem of the government is [2]

$$V(b, \lambda, z) \ = \ \max \ u(c, \ell, g, z) + \beta EV(b^+, \lambda^+, z^+) - \rho\sqrt{LS^2 + manna^2}, \qquad (15)$$

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[1]As in "manna from heaven", an unexpected resource.

[2]Sometimes we consider the "period-0 problem" where the government has fewer constraints, representing the case of a new government. In some cases, the new government is constrained by outstanding debt $b$ but not by any $\lambda$ promise. In other cases, the new government can ignore outstanding debt as well as any previous government's promises. In any case, the subsequent governments inherit both the debt $b$ and the shadow price $\lambda$ as state variables. This paper focuses on the nature of policy in those states.

where $\rho$ is the penalty parameter and will typically be a large number. The penalty function is essentially a L1 penalty since it is linear along any ray from the origin in (LS,manna) space. If there is a feasible solution to the original problem, then there will be a finite $\rho$ such that the relaxed problem has the same solution as the original problem. In our case, there may be states in which there is no solution to the unrelaxed problem. The relaxed problem will always have a solution but with positive values for LS and/or manna.

With the relaxed problem, we can determine the economically feasible/infeasible regions and we will use the following definition of infeasibility to classify $(b, \lambda)$ pairs.

**Definition 1.** Given the policy functions, a pair $(\hat{b}, \hat{\lambda})$ is *economically infeasible* if there is a positive probability that the government uses lump-sum taxes at some time $t$ if it starts at $(\hat{b}, \hat{\lambda})$.

# 4    Computational Program

Our method of choice to tackle the numerical issues that arise from the unknown state space and the repeated construction and evaluation of multivariate value and policy functions are adaptive sparse grids. In this section, we summarize their basics. "Classical" sparse grids of Bungartz and Griebel (2004) have been used extensively in the numerics literature to alleviate the curse of dimensionality by reducing the number of grid points substantially with only slightly deteriorated accuracy compared to Cartesian grids if the underlying function is sufficiently smooth.[3] *Adaptive* sparse grids introduce an extra layer of sparsity on top of sparse grids and thus can reduce the approximation error much more efficiently.[4] This is possible as grid points are placed only where high resolution is needed while just a few points are put in areas where the value functions that have to be approximated vary little. This feature of adaptive sparse grids is especially useful for our tax policy problem since the value functions have a high degree of curvature near the feasible/infeasible boundary.

We initially set our computational domain to

$$\underline{\lambda} = uc'(1), \quad \overline{\lambda} = uc'(0)$$
$$\underline{b} = -\max_{z \in Z}\{z\}/(1-\beta), \quad \overline{b} = 10.$$

The latter has then to be mapped onto the adaptive sparse grid. The endogenous domain is contained within this set, and as we go through the value function iteration and discover the

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[3]See for example Brumm and Scheidegger (2013); Brumm et al. (2015) for applications to economic problems.

[4]For thorough derivations, we point the reader, e.g., to Bungartz and Griebel (2004); Ma and Zabaras (2009); Pflüger (2010).

parts of the state space that is infeasible, we adjust the bounds accordingly.

# 5 Results

In this section we display the computational results from a variety of parameterizations of our model. The utility function for the consumers, as mentioned before, take the form

$$u(c, \ell, g, z) = \frac{(c - cb)^{1-\sigma_1}}{1 - \sigma_1} + \eta \frac{(1 - \ell)^{1-\sigma_2}}{1 - \sigma_2} - \theta(g - z)^{\sigma_3}. \tag{16}$$

For all of the cases reported, we set $\eta = 1$ and $\theta = 100$. The discount factor $\beta$ is fixed at 0.96. Table 1 reports the specifics of the four cases we compute. The debt, $b$ bounds and shadow price $\lambda$ bounds are the state spaces that include both the economically feasible and economically infeasible, but computationally feasible points. For each case we display the endogenously determined, economically feasible/infeasible regions inside these sets. In cases 1 and 3, government spending $g$ is a choice variable. In cases 2 and 4, government spending is constrained to equal the exogenous shocks and these cases provide a direct a comparison to the results in AMSS.

Table 1: Computed Cases

| Case # | $\sigma_1$ | $\sigma_2$ | g fixed | g shocks | b bounds | $\lambda$ bounds |
|--------|------------|------------|---------|-----------|----------|------------------|
| 1 | 1.0 | 1.0 | No | {0.09, 0.27} | [-5, 8] | [1.25,10] |
| 2 | 1.0 | 1.0 | Yes | {0.09, 0.27} | [-5, 8] | [1.67,10] |
| 3 | 0.5 | 0.5 | No | {0.09, 0.27} | [-5, 5] | [12.91,31.62] |
| 4 | 2.0 | 2.0 | Yes | {0.10, 0.20} | [-5, 10] | [2.78,100] |

Figure 1 displays the feasible/infeasible regions for Case 1, which features log preferences and endogenous $g$. The feasible region is non-rectangular; at high $\lambda$, higher debt levels become infeasible. Figure 2 shows that lump-sum taxes are non-zero at positive debt and high $\lambda$ levels. Given the policy functions, if a simulated path from a $(b, \lambda)$ ventures into this positive LS region at some point along the path, then it is considered infeasible and marked with a red dot in Figure 1.

Figure 3 shows the state of the economy after 51,101 and 1000 periods of simulated $g$ shocks, starting from a war state and from $(b, \lambda)$ points plotted in Period 1. By period 51, the set of states collapse to small area along low $\lambda$. By period 101, a handful of areas of mass start to emerge. By period 1000, there are roughly two areas with mass points. Very low $\lambda$ and negative $b$, low $\lambda$ and large, positive $b$. These results suggest that the economy ends
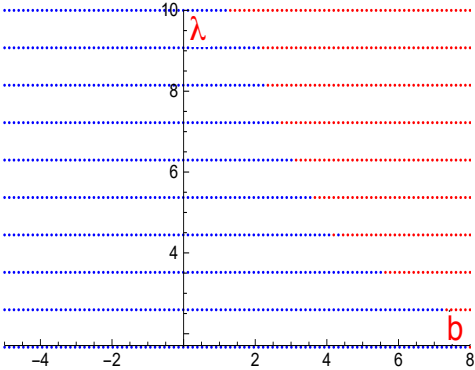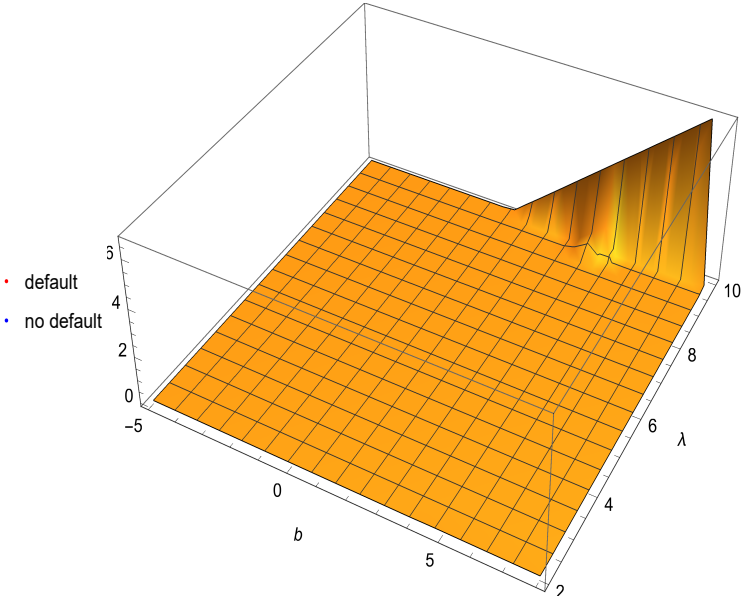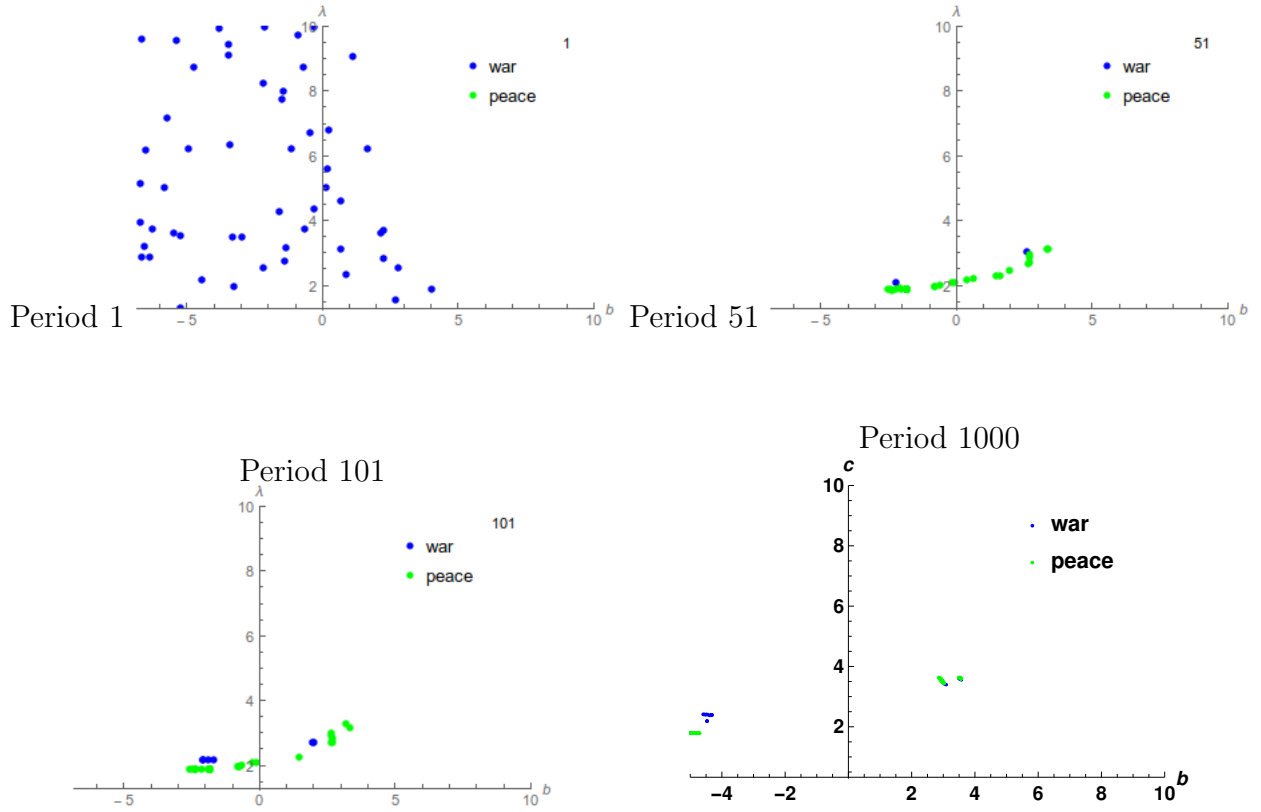
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Figure 1: Case 1: Feasible/infeasible region  Figure 2: Case 1: Lump-sum tax



up in one of the two long-run positions: with a war chest, or with a persistently high, but sustainable level of debt.

Figure 3: Case 1: Long-run debt and $\lambda$ position



Period 1



Period 51

Period 101



Period 1000



Figures 4 and 5 show the GDP, debt and tax rates from 2 such long-run simulations. For Figure 4, the government initially has with a low, but positive debt level. It experiences long periods of peace, during which it accumulates a war chest, output stays high and taxes very low. In the second simulation shown in Figure 5, there are periods of protracted wars, the government maintains a high, but stable level of debt and taxes more. Production is about 1/3 of the economy shown in Figure 4.

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Figure 4: Endogenous g, Long-lived peace     Figure 5: Endogenous g, Intermittent war
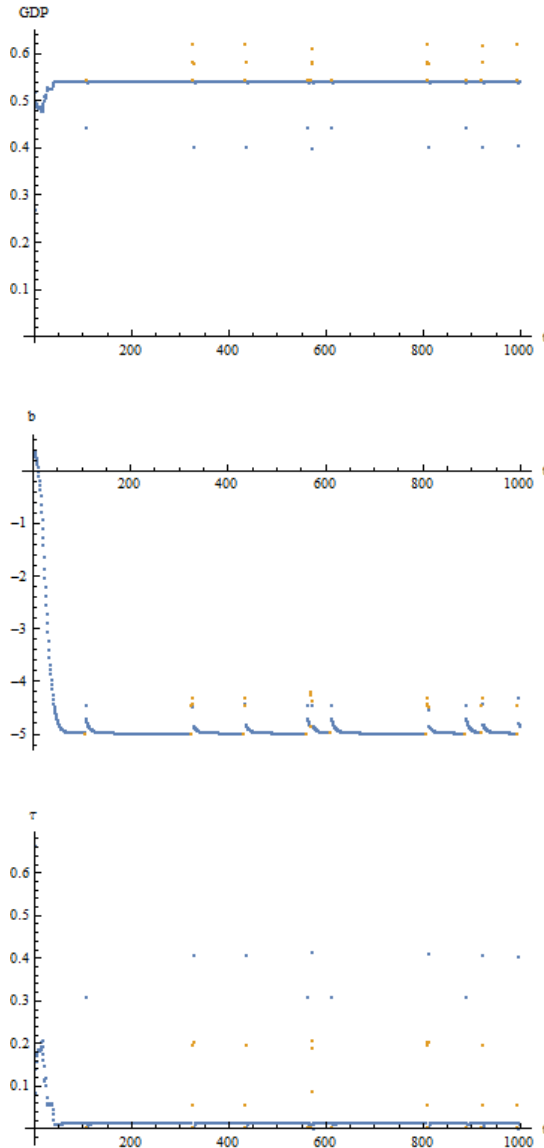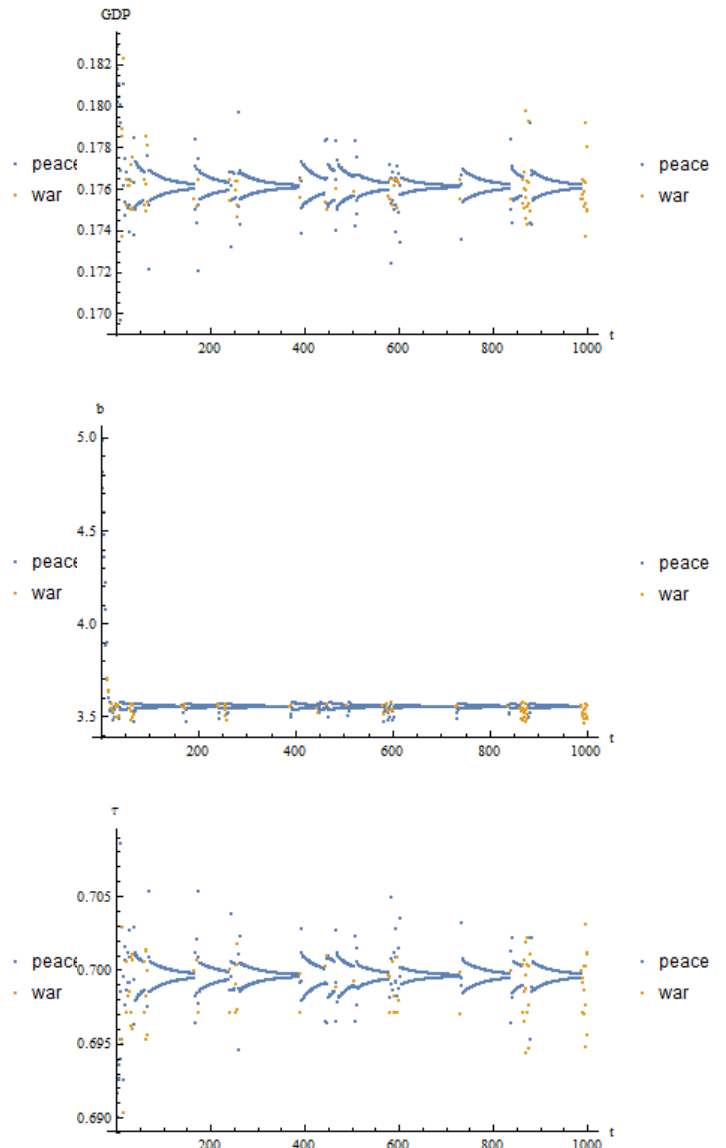


We next display results from the exogenous spending cases and show the results are significantly altered in this case. We examine the case of fixed government spending and find that the results are dramatically affected. In particular, we illustrate two cases. One in which the spending shocks are the same as in Case 1, and another in which the spending shocks are of moderate size. In both cases, as shown by Figures 6 and 7 no positive level of debt is feasible. That is, if a government begins with positive debt then there is a sequence of spending shocks such that there is no feasible tax and borrowing policy to finance those expenditures. In such cases, exogenous spending assumptions imply that governments must

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have their endowed war chests in the beginning and cannot with probability one build up its war chest.

Figure 6: Case 2: Exogenous, high variance $g$ Figure 7: Case 2: Exogenous, low variance $g$
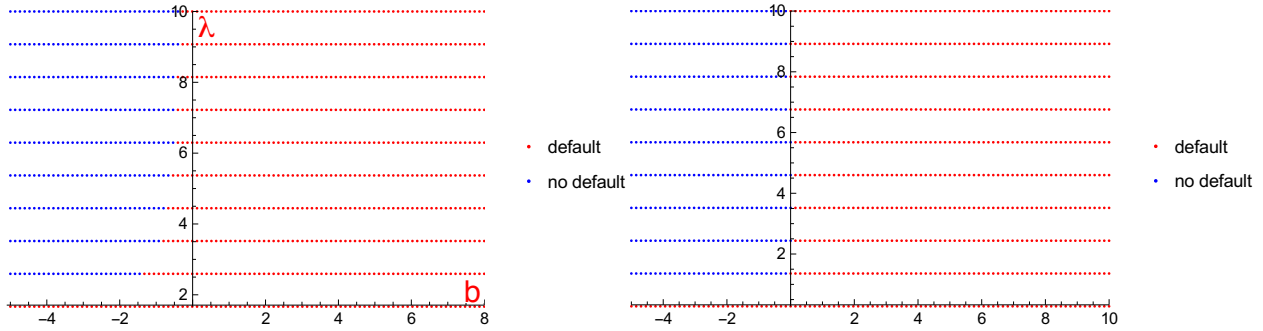


Figures 8 and 9 show GDP, debt and tax rates for two simulated paths from Case 2. The path shown in Figure 8 is unsustainable. Although the government starts with a low debt level in period 1, after about 800 periods, the debt level begins to blow up, tax rates are significantly increased and GDP tanks. It becomes impossible for the government to manage its debt, even in the presence of long, peaceful periods of low spending, without resorting to lump-sum taxes.

The second simulation, shown in Figure 9, on the other hand, has the government endowed with a war chest in period 1. Despite getting hit by periodically high spending shocks, the government is able to maintain a war chest in the long run. Although the tax rates are low, they don't converge to 0 even after 1000 periods, both GDP and tax rates show some variation.

Figure 8: Case 2, Unsustainable path            Figure 9: Case 2, Sustainable path
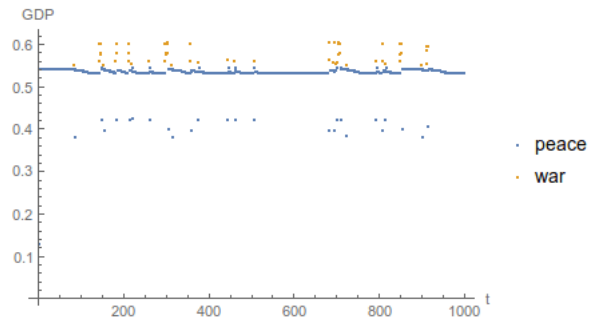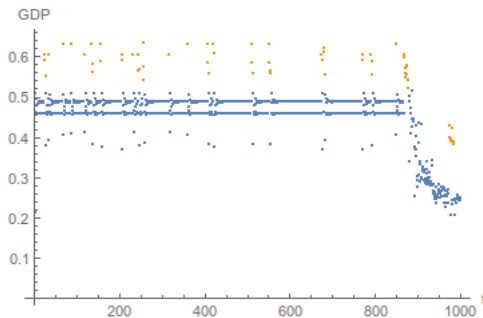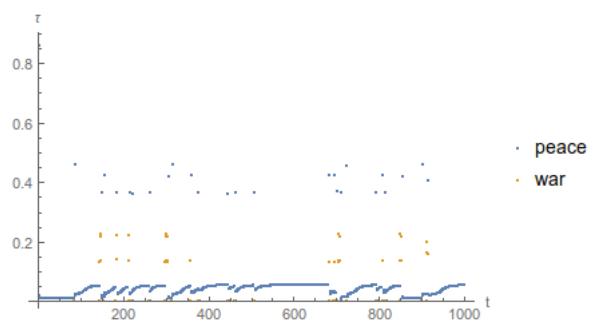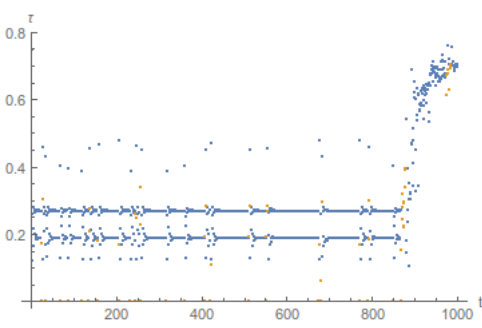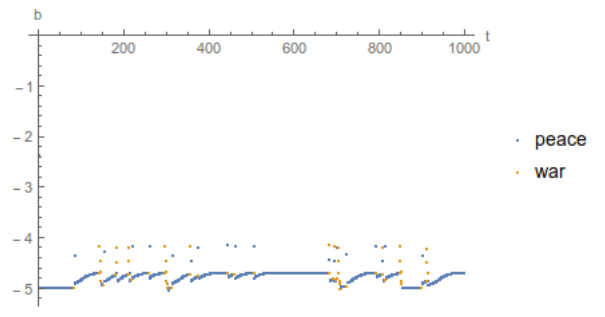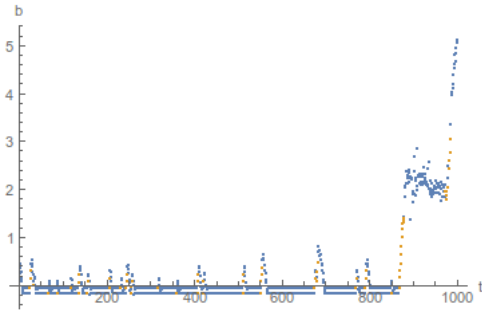
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# Appendix

We apply the adaptive sparse grid approximation method to value function iteration. In one dimension, these are just the piecewise linear functions use to compute linear splines. When we go to higher dimensions, we do not take the simple tensor product basis but carefully choose products of the one-dimensional basis functions which do well in approximating moderately smooth multidimensional functions. Furthermore, we can choose which basis functions to use and what points to use for interpolation, refining the approximation in those regions where it appears that refinement will help significantly. The resulting approximation substantially alleviates the curse of dimensionality that can arise in multidimensional approximation. For thorough derivations, we point the reader, e.g., to Bungartz and Griebel (2004); Ma and Zabaras (2009); Pflüger (2010).

We consider the representation of a piecewise $d$-linear function $f : \Omega \to \mathbb{R}$ for a certain mesh width $h_n = 2^{1-n}$ with some discretization level $n \in \mathbb{N}$. As we aim to discretize $\Omega$, we restrict our domain of interest to the compact subvolume $\Omega = [0, 1]^d$, where $d$ in our case is the dimensionality of the state space. This situation can be achieved for most other domains by rescaling and possibly carefully truncating the original domain. In order to generate an approximation $u$ of $f$, we construct an expansion

$$f(\vec{x}) \approx u(\vec{x}) := \sum_{j=1}^{N} \alpha_j \phi_j(\vec{x}) \tag{17}$$

with $N$ basis functions $\phi_j$ and coefficients $\alpha_j$. We use one-dimensional hat functions

$$\phi_{l,i}(x) = \begin{cases} 1, & l = i = 1, \\ \max(1 - 2^{l-1} \cdot |x - x_{l,i}|, 0), & i = 0, ..., 2^{l-1}, \, l > 1, \end{cases}$$

which depend on a level $l \in \mathbb{N}$ and index $i \in \mathbb{N}$. The corresponding grid points are distributed as

$$x_{l,i} = \begin{cases} 0.5, & l = i = 1, \\ i \cdot 2^{1-l}, & i = 0, ..., 2^{l-1}, \, l > 1, \end{cases} \tag{18}$$

and are depicted in Figure 10. We use a sparse grid interpolation method that is based on a hierarchical decomposition of the underlying approximation space. Hence, we introduce

next, hierarchical index sets $I_l$:

$$
I_l := \begin{cases} \{i = 1\}, \\ \{0 \le i \le 2, \, i \text{ even}\}, \\ \{0 \le i \le 2^{l_t - 1}, \, i \text{ odd}\} & \text{else,} \end{cases} \tag{19}
$$

that lead to hierarchical subspaces $W_l$ spanned by the corresponding basis $\phi_l := \{\phi_{l,i}(x), i \in I_l\}$. See Figure 10 for the basis functions up to level 3.

Tensor product operations is one way to extend the hierarchical basis functions to the multivariate:

$$
\phi_{\vec{l},\vec{i}}(\vec{x}) := \prod_{t=1}^{d} \phi_{l_t, i_t}(x_t), \tag{20}
$$

where $\vec{l}$ and $\vec{i}$ are multi-indices, uniquely indicating level and index of the underlying one-dimensional hat functions for each dimension. They span the multivariate subspaces by

$$
W_{\vec{l}} := \text{span}\{\phi_{\vec{l},\vec{i}} : \vec{i} \in I_{\vec{l}}\} \tag{21}
$$

with the index set $I_{\vec{l}}$ given by a multidimensional extension to (19):

$$
I_{\vec{l}} := \begin{cases} \{\vec{i} : i_t = 1, 1 \le t \le d\} & \text{if } l = 1, \\ \{\vec{i} : 0 \le i_t \le 2, \, i_t \text{ even}, 1 \le t \le d\} & \text{if } l = 2, \\ \{\vec{i} : 0 \le i_t \le 2^{l_t - 1}, \, i_t \text{ odd}, 1 \le t \le d\} & \text{else.} \end{cases} \tag{22}
$$

The space of piecewise linear functions $V_n$ on a Cartesian grid with mesh size $h_n$ for a given level $n$ is then defined by the direct sum of the increment spaces (cf. Eq. (21)):

$$
V_n := \bigoplus_{|l|_\infty \le n} W_{\vec{l}}, \quad |l|_\infty := \max_{1 \le t \le d} l_t. \tag{23}
$$

The interpolant of $f$, namely, $u(\vec{x}) \in V_n$, can now uniquely be represented by

$$
f(\vec{x}) \approx u(\vec{x}) = \sum_{|l|_\infty \le n} \sum_{\vec{i} \in I_{\vec{l}}} \alpha_{\vec{l},\vec{i}} \cdot \phi_{\vec{l},\vec{i}}(\vec{x}). \tag{24}
$$

Note that the coefficients $\alpha_{\vec{l},\vec{i}} \in \mathbb{R}$ are commonly termed hierarchical surpluses Zenger (1991); Bungartz and Griebel (2004). They are simply the difference between the function values at the current and the previous interpolation levels. For a sufficiently smooth function $f$ the asymptotic error decays as $\mathcal{O}(h_n^2)$ but at the cost of spending $\mathcal{O}(h_n^{-d}) = \mathcal{O}(2^{nd})$ grid points

Bungartz and Griebel (2004), thus suffering the curse of dimensionality Bellman and Bellman (1961).

As a consequence, the question that needs to be answered is how we can construct discrete approximation spaces that are better than $V_n$ in the sense that the same number of invested grid points leads to a higher order of accuracy. Luckily, for functions with bounded second mixed derivatives, it can be shown that the hierarchical coefficients rapidly decay, namely, $|\alpha_{\vec{l},\vec{i}}| = \mathcal{O}\left(2^{-2|\vec{l}|_1}\right)$ (see, e.g., Bungartz and Griebel (2004)) Hence, the hierarchical subspace splitting allows us to select those $W_{\vec{l}}$ that contribute most to the overall approximation. This can be done by an a priori selection Bungartz and Griebel (2004); Garcke and Griebel (2012), resulting in the sparse grid space $V_n^S$ of level $n$, defined by

$$V_n^S := \bigoplus_{|\vec{l}|_1 \leq n+d-1} W_{\vec{l}}, \quad |\vec{l}|_1 = \sum_{i=1}^{d} l_t. \tag{25}$$

In Figure 10 we depict its construction for $n = 3$ in two dimensions. $V_3^S$ there consists of the hierarchical increment spaces $W_{(l_1,l_2)}$ for $1 \leq l_1, l_2 \leq n = 3$.

The number of grid points required by the space $V_n^S$ is now of order $\mathcal{O}\left(2^n \cdot n^{d-1}\right)$ (cf. Bungartz and Griebel (2004); Garcke and Griebel (2012)), which is a significant reduction of the number of grid points, and thus of the computational and storage requirements compared to the Cartesian grid space. In analogy to (24), a function $f \in V_n^S \subset V_n$ can now be expanded by

$$f(\vec{x}) \approx u(\vec{x}) = \sum_{|l|_1 \leq n+d-1} \sum_{\vec{i} \in I_{\vec{l}}} \alpha_{\vec{l},\vec{i}} \cdot \phi_{\vec{l},\vec{i}}(\vec{x}), \tag{26}$$

which contains substantially fewer terms. The classical sparse grid construction introduced in (25) defines an a priori selection of grid points that is optimal for functions with bounded second-order mixed derivatives.
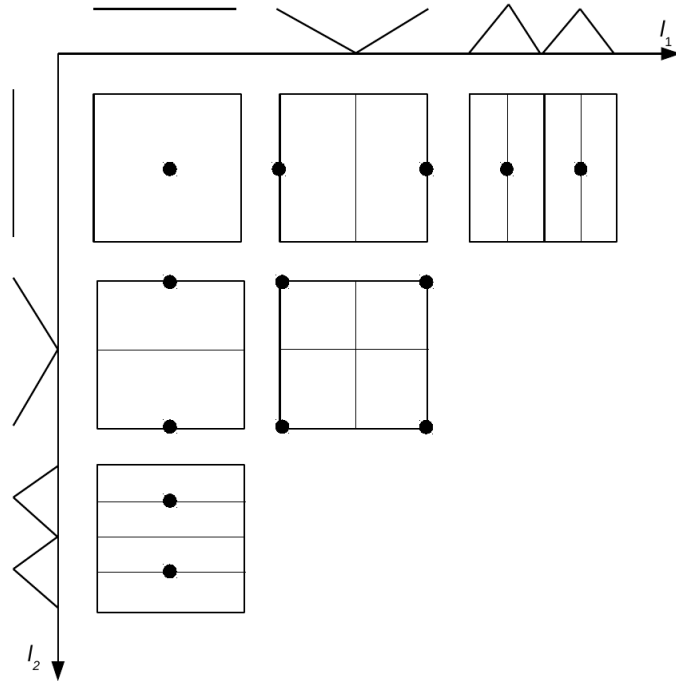
Figure 10: Hierarchical increment spaces $W_{(l_1,l_2)}$ for $1 \leq l_1, l_2 \leq n = 3$ with its corresponding one-dimensional piecewise linear basis functions of levels 1, 2, and 3 (cf. (18)).
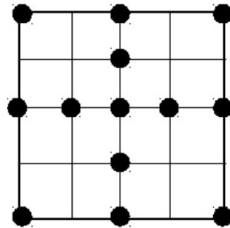


Figure 11: Construction of a classical sparse grid $V_3^S$ (see Eq. 25).

In the case that functions do not meet certain smoothness requirements, they can still be tackled efficiently with sparse grids if spatial adaptivity is used.