School choice tiebreaking: How competition guides design

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joint work with Itai Ashlagi

School choice mechanisms and tiebreaking rules

- School choice mechanisms have been adopted in NYC, Boston, Amsterdam, ...

- In practice, many students belong to the same priority group and schools ration supply using lotteries (tiebreaking rules)

- What are the welfare and equity effects of tiebreaking rules?

Applying for High Schools in NYC

	High School 700+ Programs at 400+ High Schools
EARLY FALL	Get Round 1 application from school counselor
LATE FALL	Complete application by listing 12 programs in your true order of preference
MARCH	1 Offer
	Round 2 (if necessary)

80,000 students participate in NYC School Assignment

Factors involved in admissions

- Priority groups
- Eligibility
- Selection Criteria of the schools

A central mechanism takes students' and schools' preference lists as input and generates an assignment

School choice programs based on the Deferred Acceptance (DA) algorithm

Gale-Shapley's Deferred Acceptance (DA) Algorithm (since 2003):

Repeat:

1. Unassigned students apply to their next top choice
2. Schools tentatively accept if a seat is available; otherwise reject the least preferred students

- Student-proposing DA finds a stable assignment
  - Stability: if a student s prefers a school c to her current assignment, then school c is full and prefers any of the students assigned to it to student s
- Used in NYC, Boston, Amsterdam,...

Ties and Tiebreaking rules

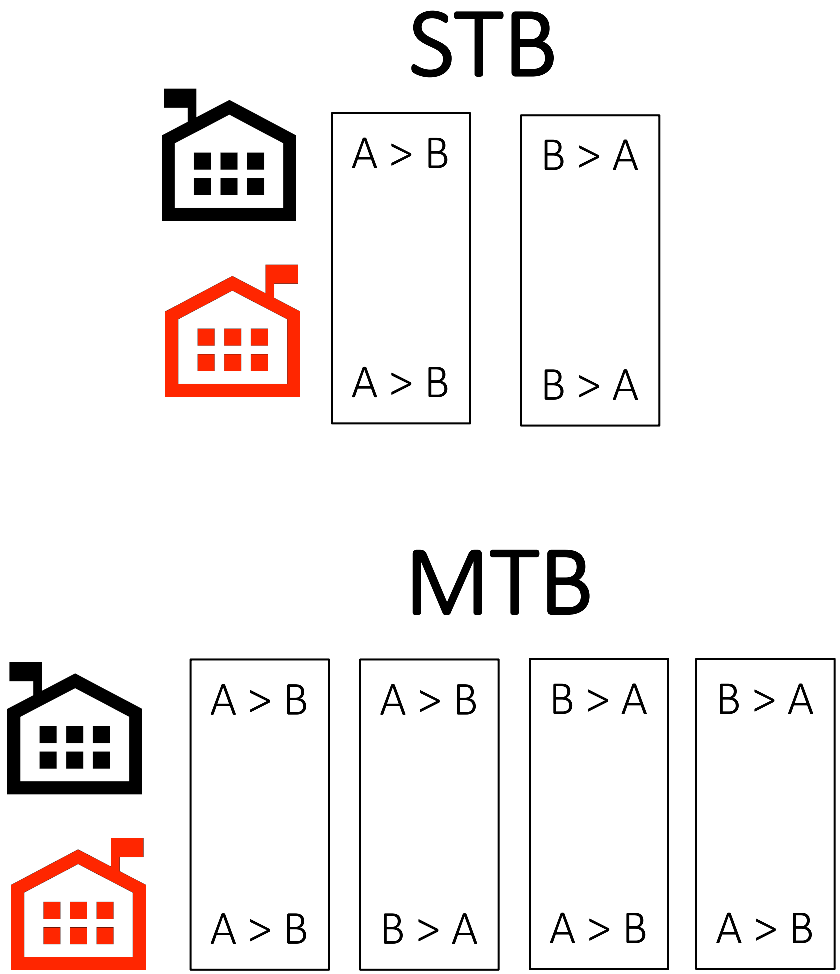
- Schools often have coarse preferences over students
- Tiebreaking rules are used to resolve the indifferences and ration supply

School choice and Tiebreakings

- Two common tiebreaking rules are:
  - Single Tiebreaking rule (STB)
  - Multiple Tiebreaking rule (MTB)
  - Boston - STB, New York - STB, Amsterdam - MTB in their first year
- STB and MTB do not harm students' incentives
  - Intuitively because they do not depend on students' preferences

STB vs MTB example

- Two students: A,B
- Two schools, indifferent between A,B



Differences between STB and MTB

- Why MTB?
  - “It is more fair”
  - “In MTB, the pain [of not getting the top choice] is distributed more equitably” [“School Fight in Amsterdam”, http://www.kennislink.nl/]
- Why STB?
  - “Using [STB] might be a better idea [...] since this practice eliminates part of the inefficiency” [Abdulkadirouglu and Sonmez, AER’03]

- Confusion about tiebreakings in NYC, Amsterdam, etc.

This work: revisits these findings

- The welfare and equity effects of tiebreaking rules depend on the market supply and demand
  - Different effects in “popular” and “non-popular” schools
  - E.g., **STB stochastically dominates MTB in popular schools**
- The aggregated rank distribution misses important information
- Towards refined guidelines by studying submarkets

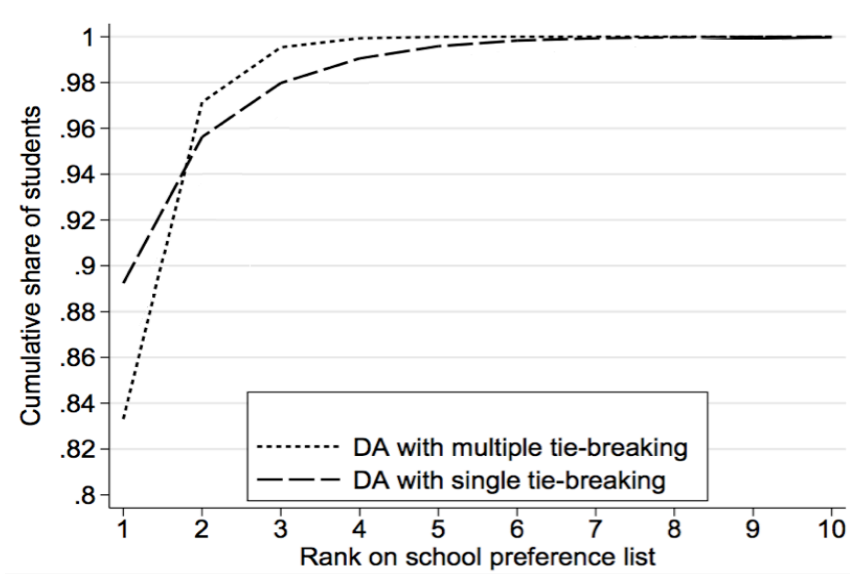
Empirical work on STB and MTB

- NYC: Abdulkadiroglu, Pathak, Roth (AER 09)
- Amsterdam: De Haan, Pgautier, Oosterbeek, Klaauw (15)

NYC

Choice	DA-STB (1)	DA-MTB (2)
1	32,105.3	29,849.9
2	14,296.0	14,562.3
3	9,279.4	9,859.7
4	6,112.8	6,653.3
5	3,988.2	4,386.8
6	2,628.8	2,910.1
7	1,732.7	1,919.1
8	1,099.1	1,212.2
9	761.9	817.1
10	526.4	548.4
11	348.0	353.2
12	236.0	229.3
Unassigned	5,613.4	5,426.7

Amsterdam



No stochastic dominance between rank distributions

Outline

- Related work
- Model and results
- Intuition and proof ideas
- Robustness check: school choice data from NYC

Related work

- Empirical studies
  - Abdulkadiroglu, Pathak, Roth (AER 09), De Haan, PGautier, Hessel Oosterbeek, Klaauw (15)
- Sophisticated tiebreaking rules:
  - Erdil, Ergin (AER 08), Che, Tercieux (15), Kesten and Unver (TE 15)
- Properties of the rank distribution in random matching markets:
  - Ashlagi, Kanoria, Leshno (JPE 15), Ashlagi, Nikzad, Romm (EC’15), Arnosti (EC’15)

Motivating example

- s students
- c schools with unit capacities
- p “popular” schools, c-p “non-popular” schools
  - Students prefer any popular school to any non-popular school
- Students have private strict preferences over schools
  - They rank *popular* schools uniformly at random
  - They rank *non-popular* schools uniformly at random
- Schools are indifferent between students
- There are enough seats for all students, but not in popular schools ($p < s < c$)

Motivating example

Define two new markets:

- Over-demanded market
  - #students=s, #schools=p (i.e. shortage of seats)
  - Students rank schools uniformly at random
- Under-demanded market
  - #students=s-p, #schools=c-p (i.e. excess of seats)
  - Students rank schools uniformly at random

- The (expected) rank distribution in the over-demanded market is identical to the rank distribution in popular schools
  - The same holds for the under-demanded market and non-popular schools

The model

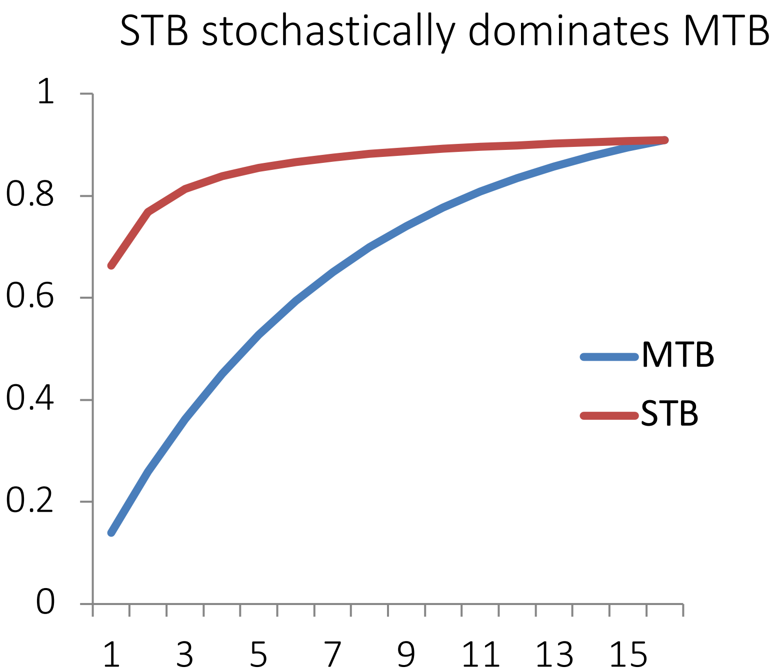
- *n* students, *m* schools
- Students rank schools independently and uniformly at random
  - A student's preference list is a random permutation of schools
- Schools are indifferent between students

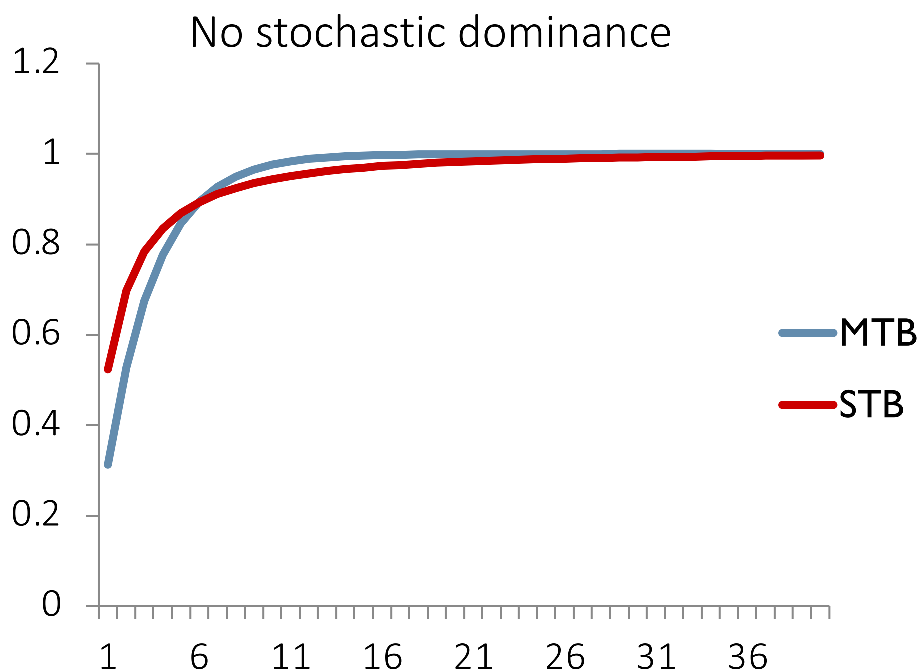
- We compare STB and MTB when
  - $n>m$: over-demanded market ("popular" schools)
  - $n<m$: under-demanded market ("non-popular" schools)

Notions of comparison

- Rank efficiency
- Number of *Pareto-improving pairs*
- Equity

Rank efficiency





Pareto-improving Pairs

A pair of students *(s,t)* form a *Pareto-improving pair* under the matching μ if

s and t prefer to swap positions under μ

- Under STB there are no Pareto-improving pairs
- Does MTB generate many Pareto-improving pairs?

- Amsterdam 2015: families who wanted to swap schools filed a law suit

Equity

- Variation in students' ranks
- Let μ be a matching of students to schools
- Let r_s denote the rank that student s gets in μ
- Let r be the average rank of assigned students:

$$r = \frac{1}{\#Assigned} \cdot \sum_{s} r_s$$

- Define the *social inequity in* μ as

$$Si(\mu) = \frac{1}{\#Assigned} \cdot \sum_{s} (r - r_s)^2$$

- We compare <u>expected social inequity</u> under STB and MTB
  - The expectation is taken over students' preferences and tiebreaking
- A measure for (ex-post) equal treatment of (ex-ante) equals

Notions of comparison

- Rank efficiency (stochastic dominance)
- Number of *Pareto-improving pairs*
- Equity (variance of the rank)

The imbalance between demand and supply is the determinant factor

Overview of main findings

In an over-demanded market

- STB “dominates” MTB in terms of rank efficiency
- MTB has many Pareto-improving pairs
- STB has a lower social inequity (variance) than MTB

- In an under-demanded market
  - No stochastic dominance relation between STB and MTB
  - MTB has almost no Pareto-improving pairs
  - STB has a higher social inequity (variance) than MTB

Results

Theorem (Over-demanded market)

When n = m+1,

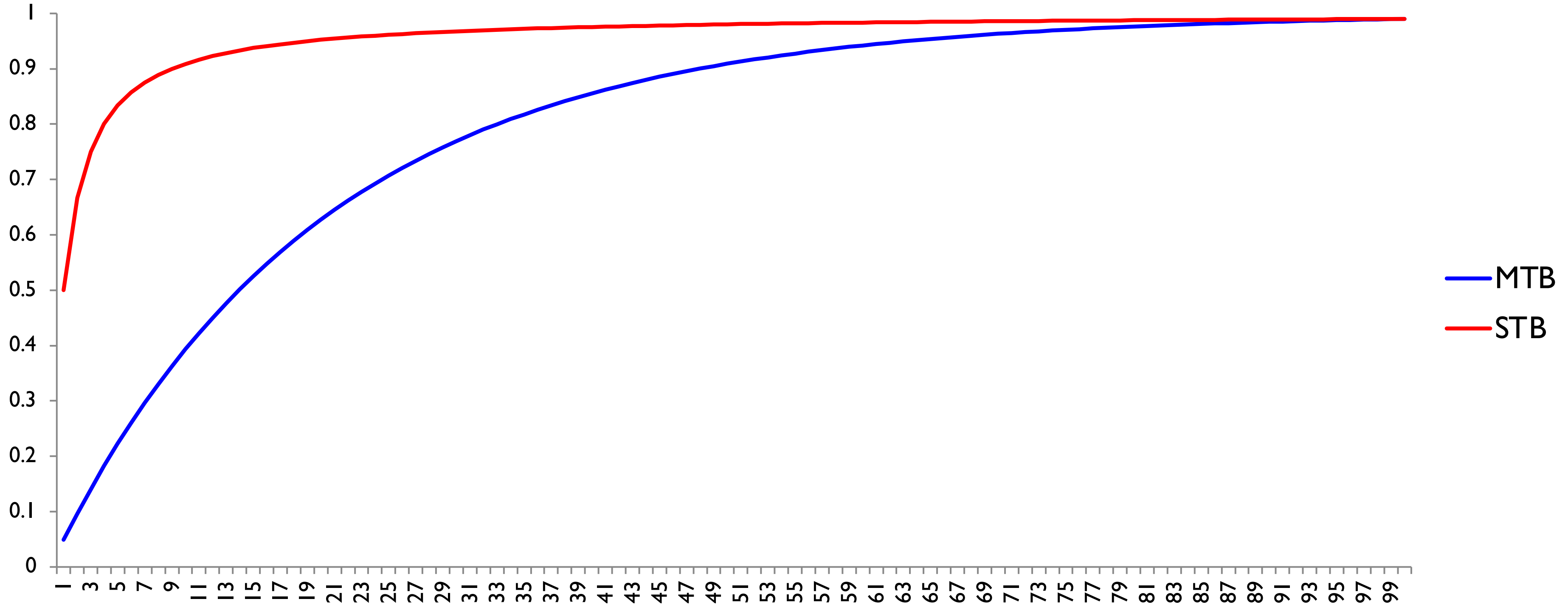
i. STB “almost stochastically” dominates MTB

Almost stochastic domination

Conjecture:
STB stochastically dominates MTB when *n=m+1*

We prove that the conjecture "almost holds":
It holds when the bottom *log n* students are removed

#Students=100
#Seats=99
Capacity=1



Results

Theorem (Over-demanded market)

When $n = m+1$,

i. STB "almost stochastically" dominates MTB

ii. The number of Pareto-improving pairs that contain a fixed student is of the order of $n / \ln^{2+\varepsilon} n$, whp.

Results

Theorem (Over-demanded market)

When n = m+1,

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ii. The number of Pareto-improving pairs that contain a fixed student is of the order of $n / ln^{2+\varepsilon} n$, whp.
iii. Expected social inequity under STB is of the order of n, and under MTB is of the order of $n^2 / ln^2 n$

Results

Theorem (Over-demanded market)

When n = m+1,

i. STB "almost stochastically" dominates MTB
ii. The number of Pareto-improving pairs that contain a fixed student is of the order of $n / ln^{2+\varepsilon} n$, whp.
iii. Expected social inequity under STB is of the order of n, and under MTB is of the order of $n^2 / ln^2 n$

Theorem (Under-demanded market)

When n = m-1,

i. Neither STB or MTB stochastically dominate the other.
ii. The number of Pareto-improving pairs that contain a fixed student is 0, whp.
iii. Expected social inequity under STB is of the order of n, and under MTB is of the order of $ln^2 n$

Implications from the main results

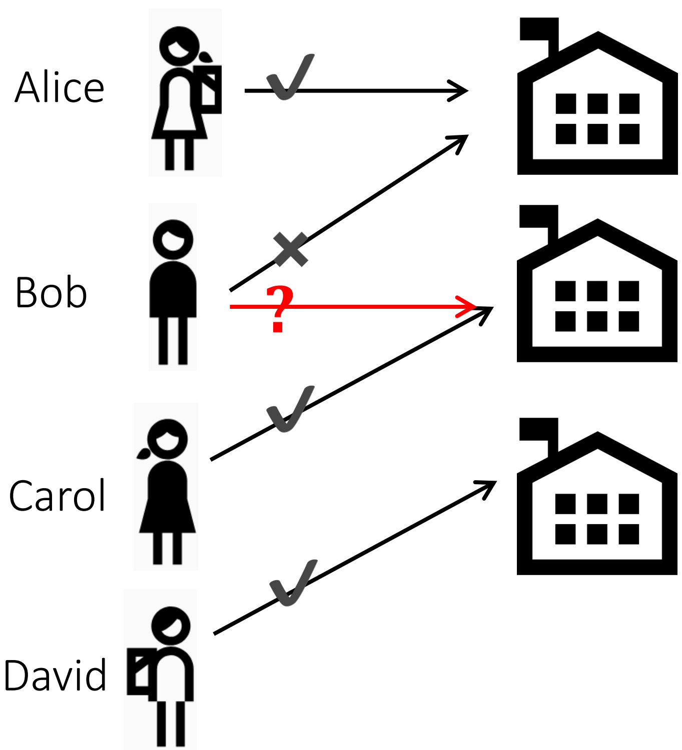
- There is no tradeoff in an over-demanded market
- If MTB is used in popular schools:
  - Students will be assigned to worse ranks in popular schools
  - There will be many Pareto-improving pairs (e.g. in Amsterdam)
- Using STB in popular schools resolves the above issues
  - We need to identify popular schools
  - Amsterdam has four "over-subscribed" schools [de Haan et al]
- If MTB is used in non-popular schools:
  - There will be very few Pareto-improving pairs
  - There will be lower social inequity
- Possible use of a hybrid tiebreaking rule

Outline

- Related work
- Model and results
- Intuition and proof ideas
- Robustness check: school choice data from NYC

Intuition: over-demanded market

- STB “dominates” MTB in terms of rank efficiency
- Example: the first two rounds of Deferred Acceptance



Key fact:

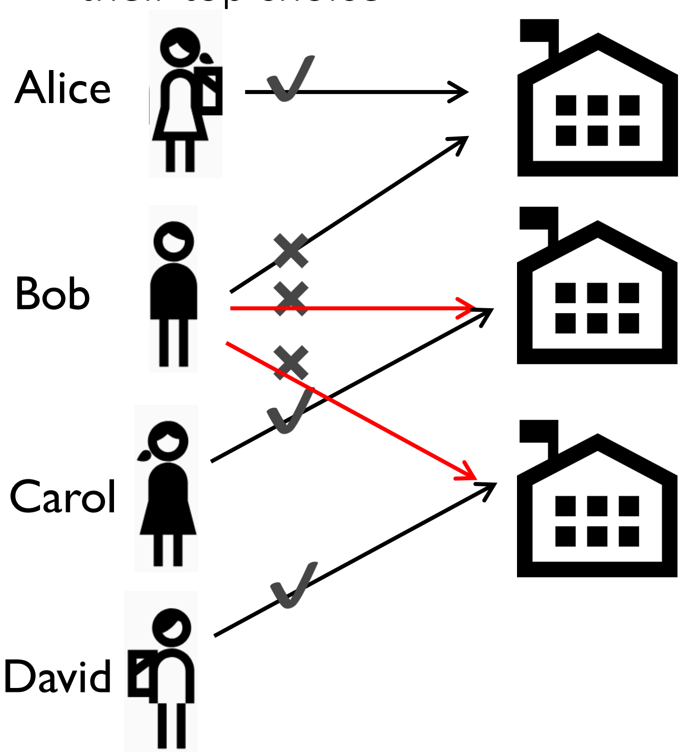
Bob is more likely to be rejected under STB than under MTB

Intuition: over-demanded market

- STB "dominates" MTB in terms of rank efficiency

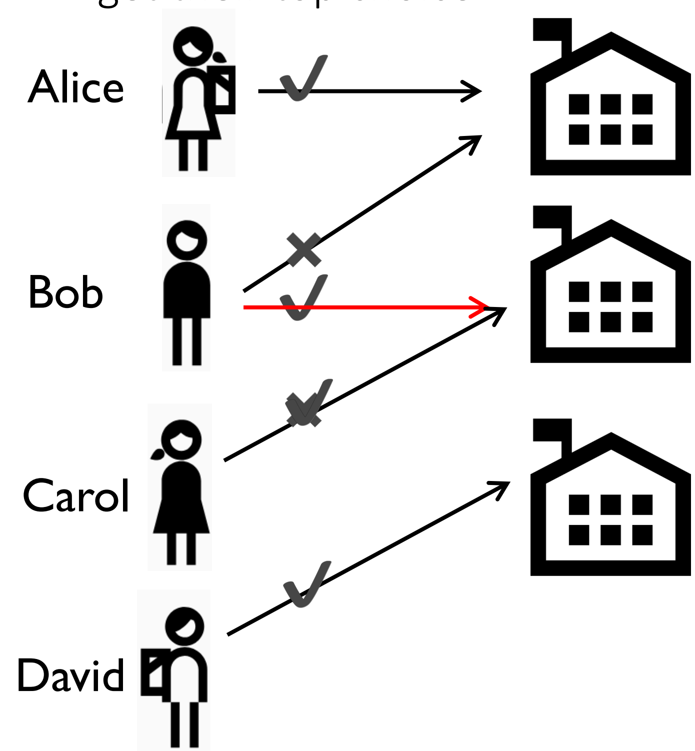
STB:

- All assigned students get their top choice



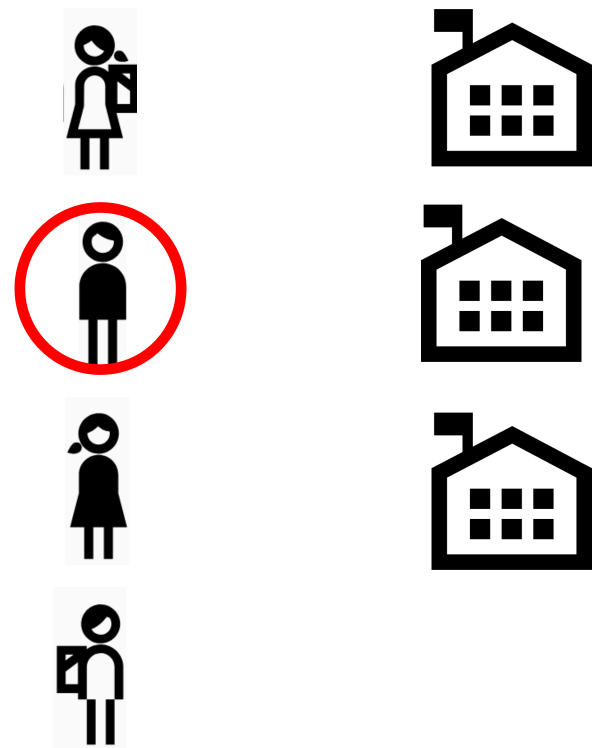
MTB:

- Not all assigned students get their top choice



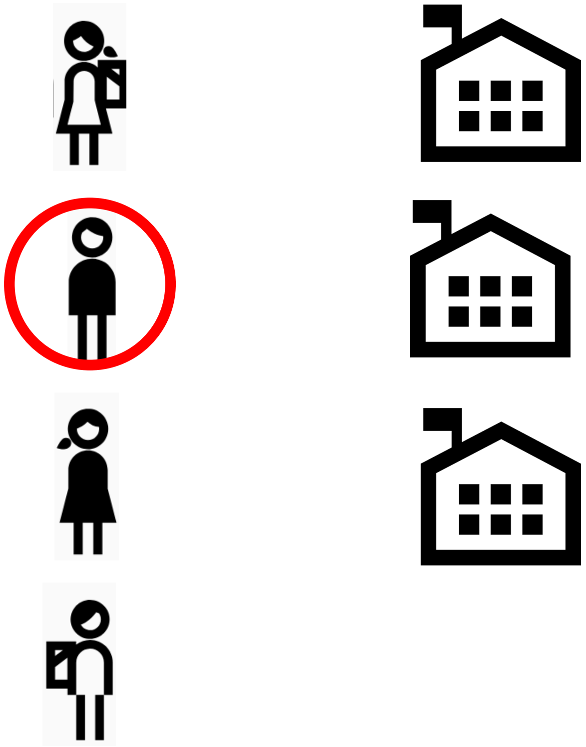
Intuition: over-demanded market

- STB "dominates" MTB in terms of rank efficiency
- In the course of Deferred Acceptance
  - There is always an unassigned student
  - The unassigned student is more likely to be rejected under STB, because her lottery number is not redrawn
  - Under STB, student with the worst lottery number keeps getting rejected
  - Under MTB, there is no such student. Competition for not being unassigned is harsher: the resulting rejections are endured collectively by all students



Intuition: over-demanded market

- STB "dominates" MTB in terms of rank efficiency
  - STB is an early coin-flip
  - MTB is a delayed coin-flip



Intuition: under-demanded market

- Stochastic dominance does not happen
  - MTB assigns fewer students to their bottom ranks, but also fewer to their top ranks.

Why?

- Under STB, students with low lottery numbers get bad ranks
- They get better ranks under MTB by displacing other students (with their redrawn lottery numbers)
- The “rejection chain” under MTB ends sooner in an under-demanded market, so stochastic dominance does not happen

Expected social inequity: idea

Theorem (Over-demanded market)

When $n = m + 1$,

iii. Expected social inequity under STB is of the order of *n*, and under MTB is of the order of $n^2 / ln^2 n$

Steps:

μ - student-optimal stable matching,

η - school optimal stable matching

① E[*Si(μ)*] = Var[*rank of s in μ*], for any student *s*

② Var[*rank of s in μ*] ≈ Var[*rank of s in η*] (core is "small" [AKL 2015])

③ Var[*rank of s in η*] ≈ $n^2/ln^2 n$

Expected social inequity: proof sketch

③ Var[*rank of s in η*] ≈ $n^2/ln^2 n$

Proof:

1) **Claim A.** Whp, any student receives at most $d \approx ln\ n$ proposals from schools

2) Rank of a student is the first order stochastic of the proposals that it receives:

$$rank\ of\ s\ in\ \eta = \min\{ X_1,\ldots, X_d \}$$

3) X_i's are "almost independent and uniformly distributed"

4) Var[*rank of s in η*] ≈ $(n/d)^2$

$\approx n^2/ln^2 n$

Expected social inequity: proof sketch

Claim A.

Whp, any student receives at most $d \approx \ln n$ proposals

Proof Idea:

- Coupling the DA process with a simpler stochastic process, P.
- In the process P, about $n \ln n$ coins with success probabilities carefully chosen
- The coupling (DA, P) is defined such that the number of successful coin-flips becomes an upper bound on d, in almost all sample paths
- It is easy to compute the number of successful coin flips

Expected social inequity: takeaways

Why variance is “large” in over-demanded markets?

- Each student receives “few” proposals in school-proposing DA

Why variance is “small” in under-demanded markets?

- Each student receives “many” proposals in school-proposing DA

Experiments with New York city data

- Year 2007-2008
- 73000 students
- 670 programs
- Students' preference lists include at most 12 schools

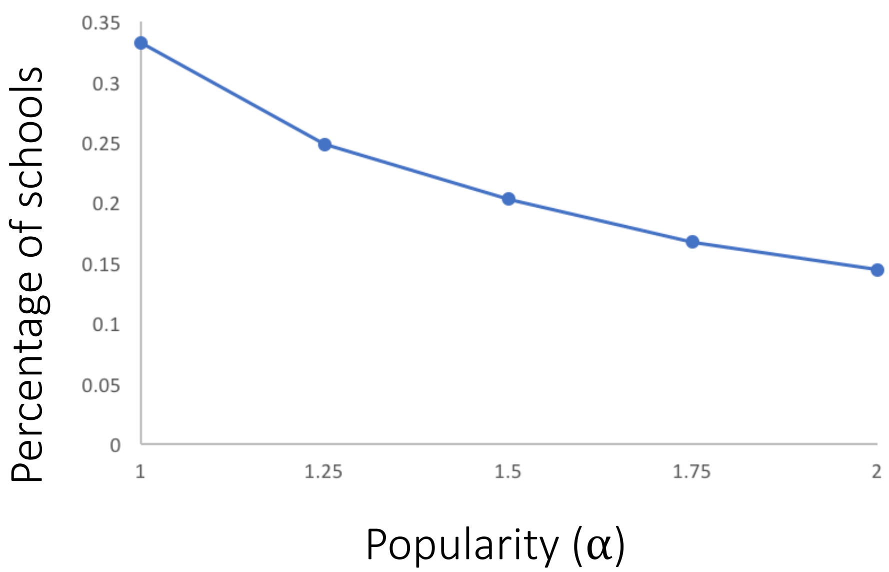
- Assumption: submitted lists are the true preferences

Experiments with New York city data

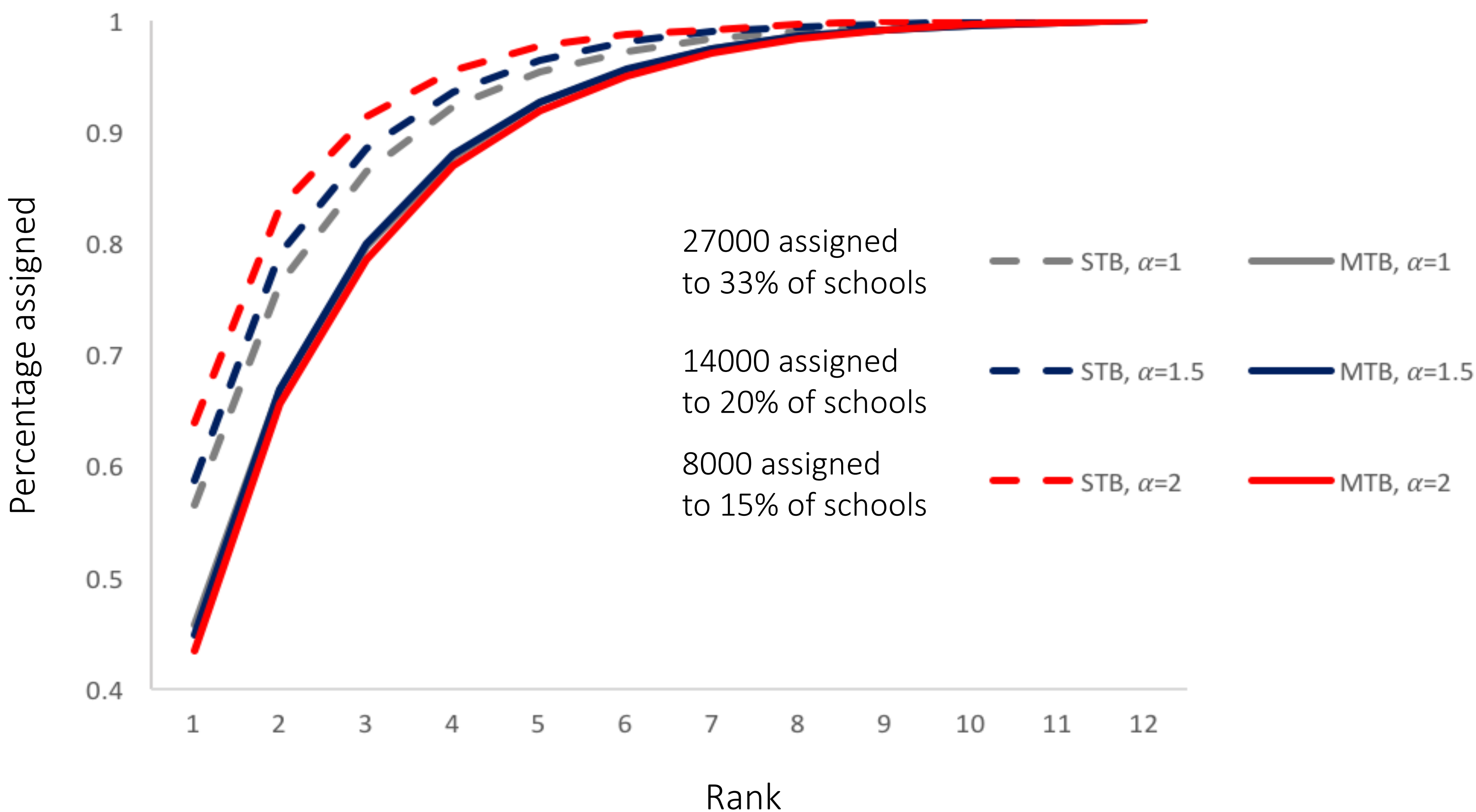
Define the *popularity* of school s to be:

$$\alpha_s = \frac{\#\text{students listing } s \text{ as first choice}}{\text{capacity of } s}$$

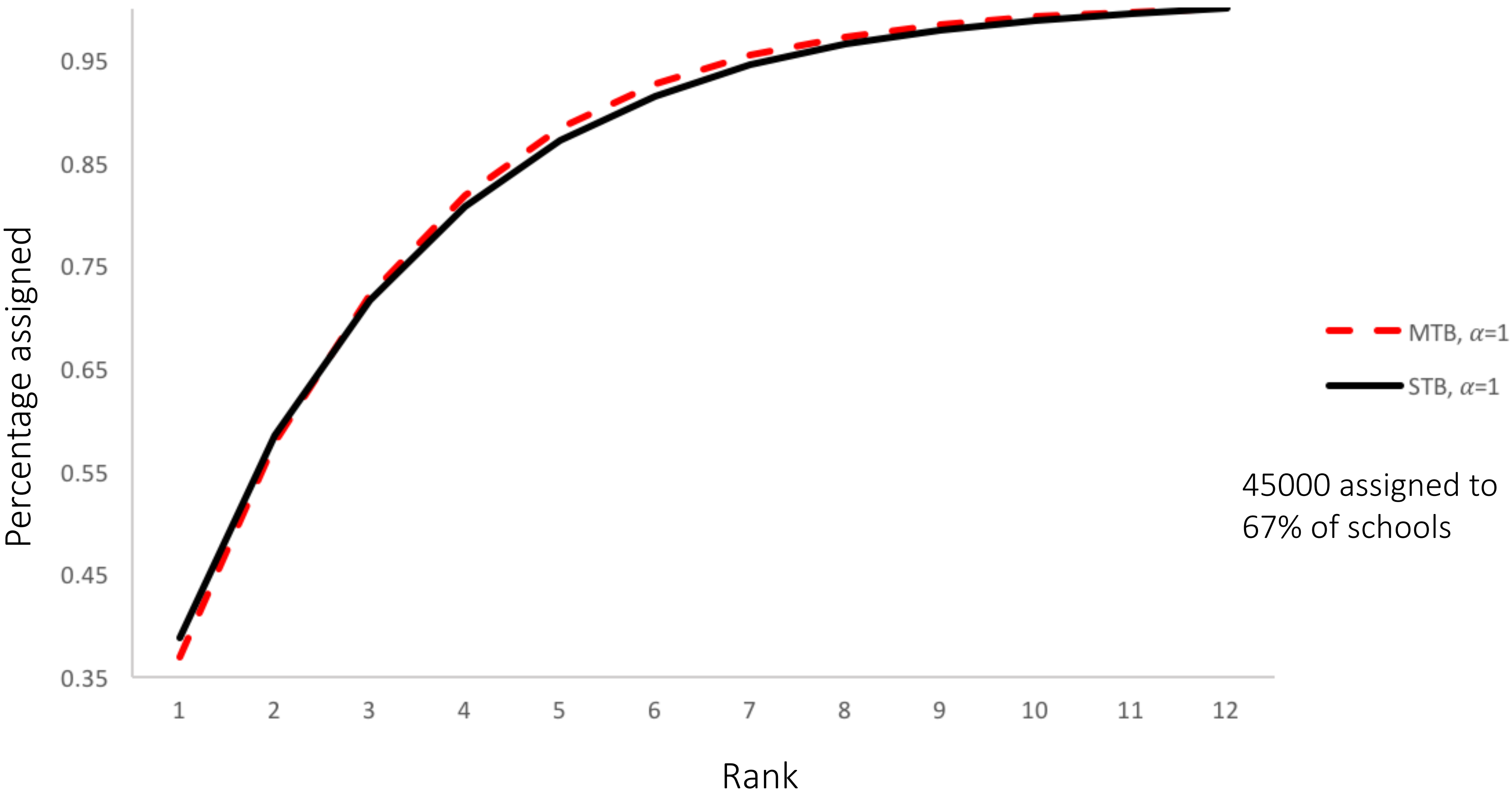
- "Popular schools": schools with *popularity* above a fixed α



STB vs MTB in NYC in schools with popularity level at least α



STB vs MTB in NYC in schools with popularity level *at most α*



45000 assigned to 67% of schools

Pareto improving pairs

Popularity level (α)	% of students in Pareto improving pairs	
	in popular schools	in non-popular schools
1	20	0.9

Pareto improving pairs

Popularity level (α)	% of students in Pareto improving pairs	
	in popular schools	in non-popular schools
1	20	0.9
1.25	21.5	1.9
1.5	23.7	2.6

Expected social inequity

Popular schools

α		STB	MTB
1	Social inequity	2.10	2.99
	Average rank	1.83	2.21
1.5	Social inequity	1.47	2.87
	Average rank	1.65	2.18
2	Social inequity	1.27	2.99
	Average rank	1.59	2.19
2.5	Social inequity	1.09	2.81
	Average rank	1.51	2.21

Non-Popular schools

α		STB	MTB
1	Social inequity	4.22	3.69
	Average rank	2.52	2.50
1.5	Social inequity	3.90	3.58
	Average rank	2.41	2.44
2	Social inequity	3.76	3.52
	Average rank	2.34	2.42
2.5	Social inequity	3.64	3.47
	Average rank	2.31	2.40

Another measure for popularity

Define the *popularity* of school s to be:

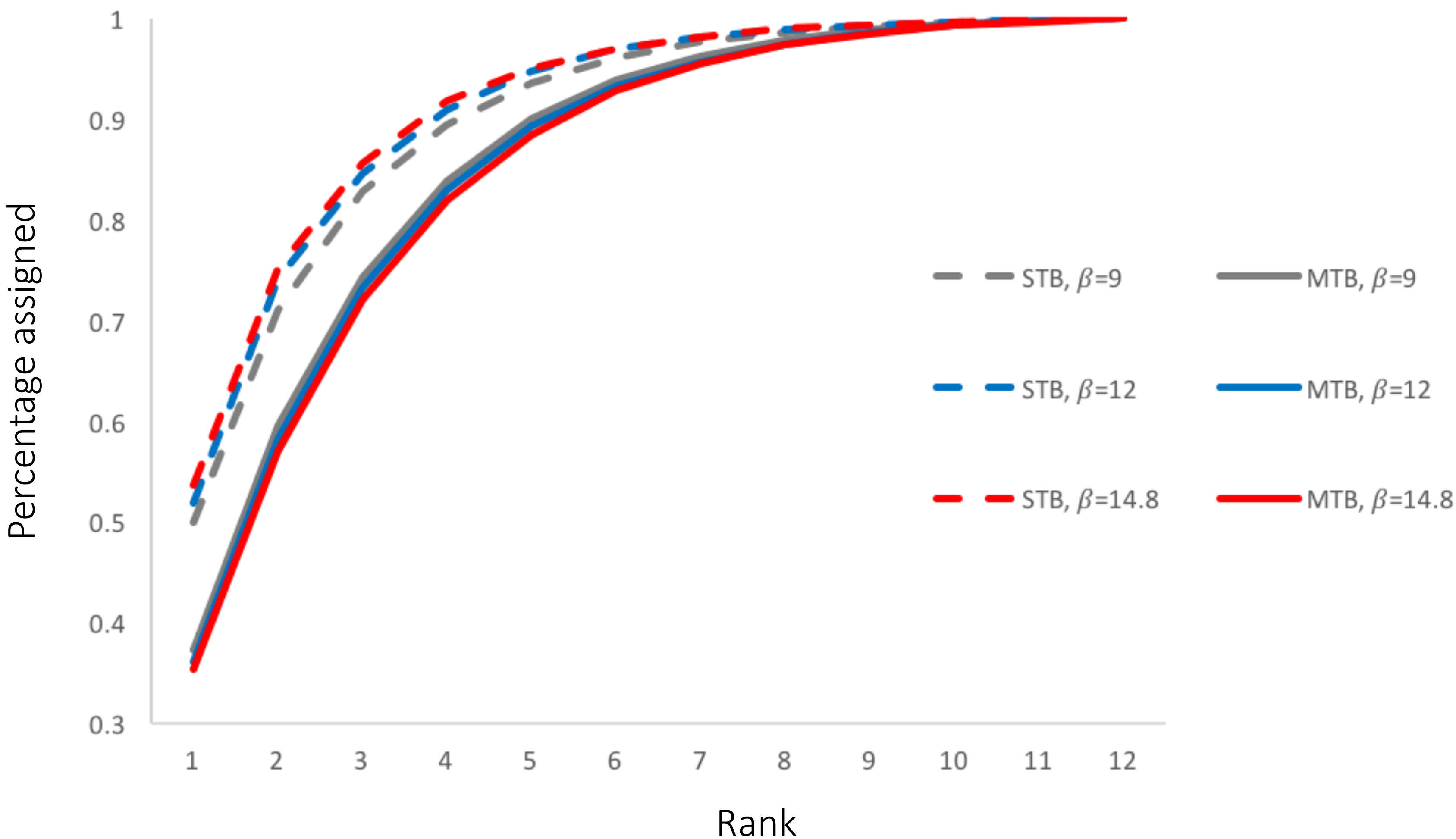
$$\beta_s = \frac{\#\text{students listing } s}{\text{capacity of } s}$$

% of popular schools	α	β	% of popular schools common between the two measures
33%	1	9	75%
20%	1.5	12	70%
14%	2	14.8	65%

NYC High school directory:

- Applicant-seat ratio > 9: high demand
- Applicant-seat ratio < 9: average or low demand

Stochastic dominance in popular schools



Summary

- The balance between demand and supply plays a crucial role in the comparison between the tiebreaking rules

- Over-demanded markets (popular schools)
  - No tradeoff: STB is “better” than MTB under all measures
- Under-demanded markets (non-popular schools):
  - A real tradeoff between MTB and STB

- Possible use of a hybrid rule
  - More to be done on defining popular schools…

Appendix

Hybrid tiebreaking rule with $\alpha = 2$ compared to STB and MTB