Lindahl Equilibrium as a Collective Choice Rule

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Motivation

allocate offices to faculty; allocate infrastructure investments among neighborhoods; allocate students to dorm rooms

examples of allocation problems without transfers

goal: find an efficient and equitable mechanism

this paper: analyze collective choice markets with equal budgets

Collective Choice Markets

we study the following mechanism:

Collective Choice Markets

we study the following mechanism:

1. Each group member is given an equal budget of fiat money
2. Each member confronts a price for each of the relevant alternatives under consideration.
3. Chooses an alternative that maximizes utility subject to the budget constraint.
4. The organization acts as an auctioneer and implements an alternative that maximizes revenue.

Motivation...

To get efficiency....

Choices need to be stochastic

Agents choose lotteries over outcomes

Organization picks (and then implements) a lottery

Why this Mechanism?

Generalization of market based mechanisms for allocation problems (Hylland and Zeckhauser (1979), Buddish (2011), Gul, Pesendorfer and Zhang (2019)))

“Works” even if there are externalities or complementarities

Yields (ex ante) Pareto efficient outcomes (unlike deterministic mechanisms)

Main Question and Result

What notion of equity is implied by equilibria of collective choice markets?

Map collective choice problem to an $n-$person bargaining problem

Define **Equitable Solution** for the bargaining problem

Main Result:

1. Every equilibrium of a collective choice market is equitable
2. Every equitable solution is an equilibrium of the collective choice market

Collective Choice Markets

Collective Choice Problem

n agents and k outcomes plus a disagreement outcome that yields zero utility to every agent

a (random) social outcome, q, is an element of the k-dimensional unit simplex

$u_i = (u_i^1, \ldots, u_i^k)$ is i's utility index

$u_i \cdot q$ is i's utility if the outcome is q

u_i is non-negative and not identically zero

The utility profile $u = (u_1, \ldots, u_n)$ defines a collective choice problem

Collective Choice Market: Consumers

$e = (1, \dots 1), q = (q^1, \dots, q^k), p_i = (p_i^1, \dots, p_i^k)$

Consumer i has one unit of fiat money and purchases probability q^j of outcome j at price p_i^j to maximize utility:

$$\begin{aligned} \underset{q}{\text{maximize}} \quad & u_i \cdot q \\ \text{subject to} \quad & p_i \cdot q \le 1, \quad \text{(Budget constraint)}, \\ & e \cdot q \le 1, \quad \text{(Probability constraint)} \end{aligned} \tag{1}$$

A minimal cost solution to the consumer's problem is a solution to the above problem that minimizes the expenditure of fiat money

Collective Choice Market: Firm

The firm chooses the social outcome q to maximize profit:

$$\begin{aligned} \underset{q}{\text{maximize}} \quad & \sum_{i=1}^{n} p_i \cdot q \\ \text{subject to} \quad & e \cdot q = 1, \quad \text{(Probability constraint)} \end{aligned} \tag{2}$$

Collective Choice Market: Equilibrium

The pair (p, q) is a **Lindahl equilibrium** (LE) if q is a minimal-cost solution to every consumer's maximization problem at prices p_i and solves the firm's maximization problem at prices p.

Lemma 1 Every collective choice market has a Lindahl equilibrium; all Lindahl equilibria are Pareto efficient.

Example...

- 3 students must be assigned to 2 rooms; a double and a single room.
- Allocations $k = \{1, 2, 3\}$ where allocation k assigns student $i = k$ to a single room.
- utilities:

$$u_1 = (10, 2, 0)$$
$$u_2 = (5, 10, 0)$$
$$u_3 = (0, 9, 10)$$

- every student prefers the single room; students 1 and 2 do not want to be matched with each other; student 3 would much rather be matched with student 1 than student 2.

...Example

Lindahl Equilibrium:

The lottery $q = (.27, 0.73, 0)$ is a Lindahl equilibrium allocation:

Either student 1 or student 2 gets the single room.

Equilibrium price: p_i is the price student i pays for the allocations:

$$\begin{aligned} \text{single} &= 1 && 2 && 3 \\ p_1 &= 2.42 && 0.49 && 0 \\ p_2 &= 0.58 && 1.15 && 0 \\ p_3 &= 0 && 1.36, && 1.51 \\ \sum p_i &= 3.00 && 3.00 && 1.51 \end{aligned}$$

All 3 students are better off in equilibrium than under a uniform lottery (serial dictatorship)

Bargaining and Equitable Solutions

The Bargaining Problem

Bargaining Problem: full dimensional and comprehensive polytope $B \subset \mathbb{R}^n$

Disagreement point: $d(B)$ is the component-wise minimum of B

Comprehensive: if $y \in B$ then B contains all points $d(B) \leq x \leq y$

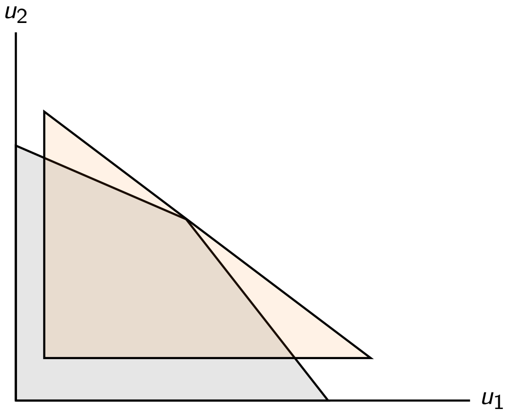
unit simplex: Δ is the $n-$dimensional unit simplex

Let $a \otimes x = (a_1 \cdot z_1, \ldots, a_n \cdot z_n)$;

Simplex: $B = z + a \otimes \Delta$ affine transformation of the unit simplex

Ordering Bargaining Problems

$A \geq B$ if for every $x \in A, y \in B$, there exist $x' \in A$, $y' \in B$ such that $x' \geq y$ and $x \geq y'$.



The Equitable Solution

Fair Outcome: equal division ($u_i = 1/n$) if the bargaining set is the unit simplex;

vNM utilities are unique only up to positive affine transformations, therefore:

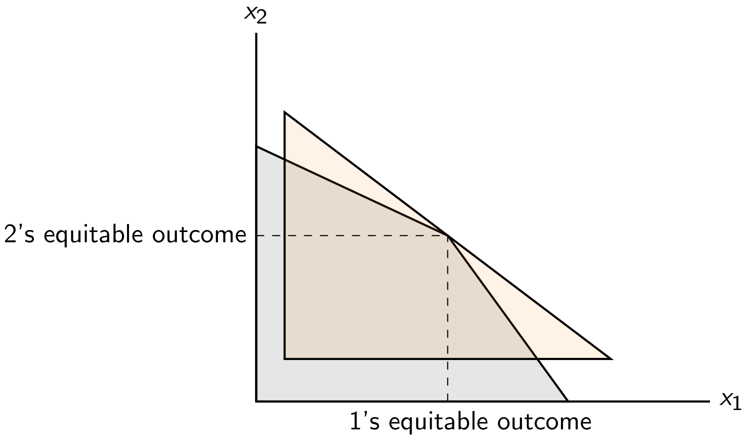
$x_i = a_i/n + z_i$ is the fair outcome of simplex $B = a \otimes \Delta + z$

Fair outcomes are only defined for simplices: $F(A)$ is the fair outcome if A is a simplex; $F(A)$ is empty otherwise.

Equitable Solution: outcomes of B that coincide with the fair outcome of a simplex $A \geq B$:

$$E(B) := \{x \in B \cap F(A) \,|\, A \geq B\}$$

Equitable Solution



Representation Theorem

S is a set valued solution to the bargaining problem if $S(B) \subset B$.

Scale Invariance $S(a \otimes B + z) = a \otimes S(B) + z$

Symmetry $S(\Delta) = \{\frac{1}{n} \cdot e\}$.

Consistency $B \leq A$ implies $S(A) \cap B \subset S(B)$.

Justifiability $x \in S(B)$ implies $B \leq A$ and $\{x\} = S(A)$ for some A.

Theorem 1 S satisfies the four axioms above if and only if it is the equitable solution.

Justifying Justifiability

Ann and Bob must divide a peanut butter cake and a chocolate cake. Bob is allergic to peanuts while Ann likes all cakes equally.

Equitable solution #1: pb-cake to Ann; ch-cake 50-50.

Ann: *"Since Bob has no use for the peanut butter cake, from his perspective the situation is as if we only had the chocolate cake and in that situation it's obviously equitable to divide the chocolate cake equally."* Perles and Maschler (1981) provide an axiomatic foundation for Ann's argument.

Equitable solution #2: pb-cake to Ann; ch-cake to Bob.

Bob: *"If I were not allergic to peanuts we would each get one cake. Since Ann is indifferent between the chocolate cake and the peanut butter cake it makes sense that she gets the peanut butter cake and I get the chocolate cake."* Nash (1951) provides an axiomatic foundation for Bob's argument.

Other equitable solutions: in-between; e.g., pb-cake to Ann, ch-cake 1/3-2/3 (Kalai-Smorodinsky)

Relationship to other Bargaining Solutions

- the equitable solution always contains the Nash bargaining solution
- for two players, the equitable solution contains all standard Bargaining solutions that satisfy scale invariance.

Main Result

Mapping Collective Choice Problems to Bargaining Problems

The utility profile of outcome j is $u^j = (u^j_1, \ldots, u^j_n)$

The convex and comprehensive hull of these utility profiles and the origin forms the bargaining problem B_u

Therefore, **for each u we get a bargaining problem B_u**

Conversely, let B be any bargaining problem with the origin as disagreement point

The utility profile at an extreme point corresponds to the utility profile of an outcome j

Therefore, **for each B we get a collective choice problem u**

Main Result

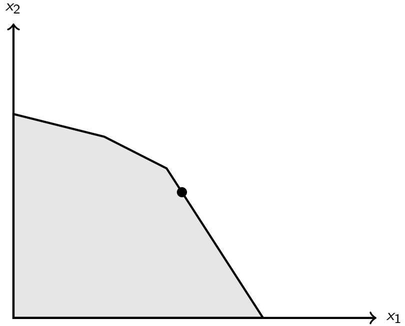
Theorem 2 The set of Lindahl equilibrium utilities coincides with the equitable solution of the corresponding bargaining problem.

Argument: Justifying a Lindahl Equilibrium

$$\begin{aligned} \underset{q}{\text{maximize}} \quad & u_i \cdot q \\ \text{subject to} \quad & e \cdot q \le 1 \quad (c_i), \\ & p_i \cdot q \le 1 \quad (a_i) \end{aligned}$$

$$\begin{aligned} \underset{c_i, a_i \ge 0}{\text{minimize}} \quad & c_i + a_i \\ \text{subject to} \quad & c_i e + a_i p_i \ge u_i \end{aligned}$$

Lindahl equilibrium: (q, c_i, a_i) solves consumer i's problem; q maximizes the auctioneer's profit



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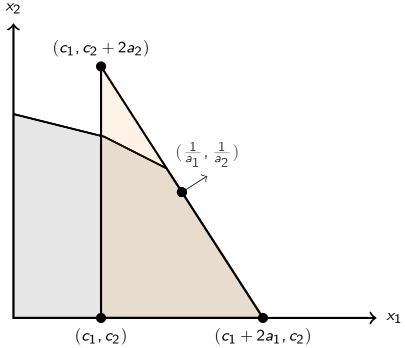


Explaining the graph...

$$\begin{aligned} \underset{q}{\text{maximize}} \quad & u_i \cdot q \\ \text{subject to} \quad & e \cdot q \leq 1, \\ & p_i \cdot q \leq 1 \end{aligned}$$

$$\begin{aligned} \underset{c_i, a_i \geq 0}{\text{minimize}} \quad & c_i + a_i \\ \text{subject to} \quad & c_i e + a_i p_i \geq u_i \end{aligned}$$

Complementary Slackness: $q \cdot (c_i e + a_i p_i - u_i) = 0$

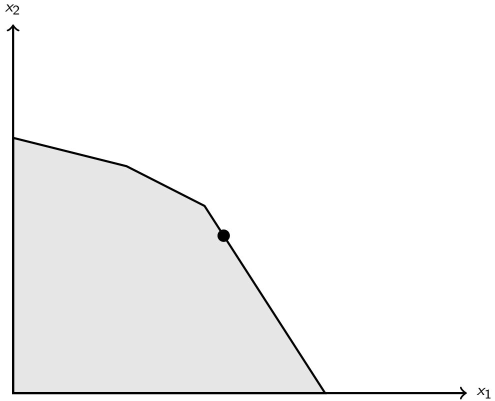
Therefore, if $q^j > 0$:

$$p_i^j = \frac{1}{a_i}(u_i^j - c_i)$$

... Explaining the Graph

value of dual $= c_i + a_i = u_i \cdot q$

Profit maximization implies that $(\frac{1}{a_1}, \frac{1}{a_2})$ must be perpendicular to B_u



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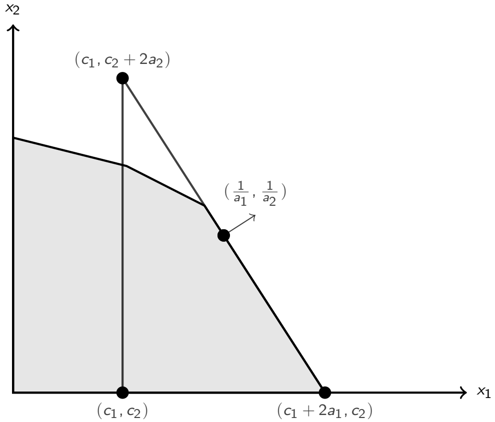

Lindahl equilibria and Nash Bargaining

Nash allocation:

$$\begin{aligned} \arg\max_{q} \quad & \sum_i \log(w_i \cdot q) \\ \text{subject to} \quad & e \cdot q \leq 1 \end{aligned}$$

Define

$$\bar{u}_i^j(c_i) := \max\{u_i^j - c_i, 0\}$$

Admissible Let q be the Nash allocation for $w_i = \bar{u}_i(c_i)$. Then, $c = (c_1, \ldots, c_n)$ is admissible if $u_i^j \geq c_i$ for all j such that $q^j > 0$ and for all i.

Lindahl equilibria and Nash Bargaining

Theorem 2 If q is the Nash allocation for $\bar{u}(c)$ and c is admissible, then (p, q) such that $p_i = \bar{u}_i(c_i)/(\bar{u}_i(c_i) \cdot q)$ is a Lindahl equilibrium for u.

Conversely, if q is a Lindahl allocation for u, then there is an admissible c such that q is a Nash allocation for $\bar{u}(c)$.

why admissibility?

- For q to be a Lindahl equilibrium, $u_i \cdot q = \bar{u}_i(c) \cdot q + c_i$ must hold
- In general, $\bar{u}_i(c) \cdot q + c_i \geq u_i \cdot q$
- Equality holds if and only if c is admissible.

Applications 1: Matching

Application: Matching

A group of agents must decide who matches with whom as

A matching, is a bijection j from the set of all agents to itself such that $j(j(i)) = i$ for all i. If $j(i) = i$, then i is said to be unmatched.

w_i^m is the utility of agent i when she matches with agent m.

Walrasian Market:

- Each agent has one unit of fiat money, must pay price π_i^m for matching with agent m
- Agents choose lotteries (over partners) that maximize their utilities subject to the budget constraint.
- Feasibility: there is a lottery over allocations that implements all the chosen lotteries

Matching continued

- Walrasian equilibrium: consumers specify demands for private goods (individual match)
- Lindahl equilibrium: consumers specify demand for collective goods (matches for everyone)

Theorem 3 Lindahl equilibrium allocations coincide with Walrasian equilibrium allocations

Corollary: Matching is equitable if and only if it can be implemented via a Walrasian economy with equal budgets

Application 2: Allocation Problems

Office Allocation Example

Three agents must decide on an office allocation:

$$v_1(1) = 10, v_1(2) = 4, v_1(3) = 2$$
$$v_2(1) = 10, v_2(2) = 7, v_2(3) = 3$$
$$v_3(1) = 10, v_3(2) = 5, v_3(3) = 1$$

- office 1 has a premium desk
- office 2 has a good desk
- office 3 has the standard desk

Walrasian economy #1

3 goods = 3 offices

$$v_1(1) = 10, v_1(2) = 4, v_1(3) = 2$$
$$v_2(1) = 10, v_2(2) = 7, v_2(3) = 3$$
$$v_3(1) = 10, v_3(2) = 5, v_3(3) = 1$$

Economy has a unique Walrasian equilibrium in which:

- Good 2 is allocated to agent 2
- Agents 1 and 3 each have an equal chance at getting good 1 or 3.

Walrasian economy #2

- Agents have designated offices but can be assigned different desks;
- Each agent can derive utility only if she gets her designated office; otherwise her utility is 0.
- 5 goods: the three offices (goods 1, 2 and 3) and the two premium desks (good 4, the good desk and good 5, the premium desk).

$$\hat{v}_1(\{1,5\}) = 10, \hat{v}_1(\{1,4\}) = 4, \hat{v}_1(1) = 2$$
$$\hat{v}_2(\{2,5\}) = 10, \hat{v}_2(\{2,4\}) = 7, \hat{v}_2(2) = 3$$
$$\hat{v}_3(\{3,5\}) = 10, \hat{v}_3(\{3,4\}) = 5, \hat{v}_3(3) = 1$$

Walrasian economy #2 cont'd

- Economy #2 has multiple equilibria; Equilibrium allocations coincide with the Lindahl equilibria
- Walrasian outcomes depend on **commodification**; Lindahl outcomes do not

Walrasian economies

Finite set of goods $H = \{1, \dots, r\}$

Agent's i utility for bundle M is $v_i(M)$; $v_i(\emptyset) = 0$ and $v_i(L) \leq v_i(M)$ whenever $L \subset M$.

$p : 2^H \to \boldsymbol{R}_+$ is a (possibly non-additive) price

Consumers choose random consumption θ_i to solve:

$$\begin{aligned} \underset{\theta_i}{\text{maximize}} \quad & \sum_M v_i(M)\theta_i(M) \\ \text{subject to} \quad & \sum_M p(M)\theta_i(M) \leq 1 \end{aligned}$$

A (deterministic) allocation is feasible if it is a partition of H; a random allocation is feasible if every allocation in the support is feasible.

Walrasian Equilibrium

Definition A random allocation and a price are a Walrasian equilibrium if the allocation

(i) is feasible,

(ii) yields a least-cost utility maximal consumption lottery for every consumer;

(iii) maximizes the auctioneer's revenue.

Existence of Walrasian equilibrium requires restrictions on utility functions

If utilities satisfy the gross substitutes property (Gul, Pesendorfer and Zhang (2019)), equilibria exist

Commodification Theorem

Commodification: utility function $v = (v_1, \ldots, v_n)$ defined on a finite set of (private) goods so that $B_v = B$ (where B_v is the Bargaining problem for v).

Theorem 4

1. Every Walrasian equilibrium allocation is a Lindahl equilibrium allocation
2. For every bargaining problem B there is a commodification v such that the set of Walrasian allocations coincide with the set of Lindahl allocations.

Lindahl vs Walrasian equilibria

- Lindahl equilibria depend only on the bargaining game
- two exchange economies that yield the same bargaining game may have two different sets of Walrasian equilibria.
- Walrasian equilibria are typically simpler than Lindahl equilibria because the former involve many fewer prices. This is so because the number of allocations typically exceeds the number of goods and because Lindahl prices are personal while Walrasian prices are not.
- If the commodity space is rich enough, as is the commodity space we construct in the proof of Theorem 4, the distinction between Lindahl equilibrium and Walrasian equilibrium disappears.

Related Literature

Bargaining: axiomatic treatment closely related to Nash (1950); Kalai and Smorodinsky (1975); Perles-Maschler solution (Perles and Maschler (1981); survey by Thomson (1994).

Lindahl Allocations and Bargaining: In a public goods setting with linear costs and transfers, Fain, Guel and Munagala (2016) show that the Nash bargaining solution coincides with the Lindahl equilibrium.

Walrasian Equilibria as Allocation Mechanisms: Hylland and Zeckhauser (1979) propose Walrasian equilibria as solutions to stochastic allocation problems. Gul, Pesendorfer and Zhang (2020) extend Hylland and Zeckhauser from unit demand preferences to general gross-substitutes preferences. Collective choice markets allow for arbitrary preferences, public goods and externalities.

Fairness and Equilibrium: Foley (1967), Schmeidler and Vind (1972) and Varian (1974) associate equity with envy-freeness. Walrasian equilibria with equal budgets are envy free. The equitable solution is a notion of fairness adapted to collective choice markets.