Population Growth and Firm Dynamics*

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Abstract

Population growth has declined markedly in almost all major economies since the 1970s. We argue that this trend has important consequences for the process of firm dynamics and aggregate growth. We show analytically that a decline in the rate of population growth reduces creative destruction, increases average firm size and market concentration, raises market power and misallocation, and lowers aggregate growth in a rich model of firm dynamics. Quantitatively, we find that the slowdown in labor force growth in the U.S. since the 1980s can account for the decline in entry and the increase in firm size. It also generates quantitatively significant changes in markups and the aggregate growth rate.

Keywords: Dynamism, Growth, Firm Dynamics, Markups

JEL Codes: O40, D22, J11, O47

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1 Introduction

Almost all major economies experienced a substantial fall in population growth in recent decades. Figure 1 shows historical population growth for a group of major world economies from 1960 to 2020. Despite different political systems, cultures and levels of development, a clear downward trend is evident for all of them. Moreover, this trend is projected to continue for at least the first half of the twenty-first century. A world of low and falling population growth looks like it is here to stay.

In this paper we show that this phenomenon is likely to have important implications for the process of firm dynamics and aggregate economic performance. In particular, we argue that falling population growth reduces creative destruction and entry, increases concentration and average firm size, raises market power and lowers aggregate productivity growth.

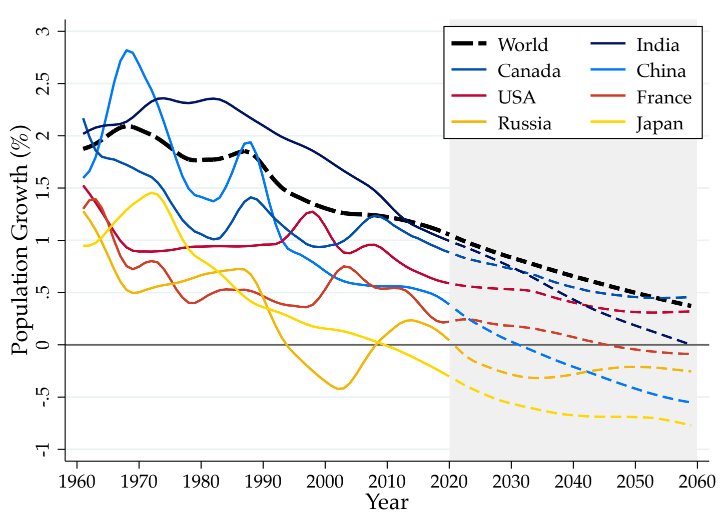
The experience of the U.S. economy since the 1980s is a case in point. The start-up rate has steadily declined, measures of job reallocation have fallen substantially and creative destruction seems to be less potent then it used to be (Haltiwanger et al., 2015; Pugsley and Şahin, 2015). At the same time. market concentration, whether measured by sales or employment, has increased markedly, as have measures of markups (Autor et al., 2017; De Loecker and Eeckhout, 2017). Lastly, save for the I.T.-fueled boom of the late 1990s, productivity growth has been sluggish (Fernald, 2015). While there are surely other contributing forces to these phenomena, they all occurred within an environment of declining population growth, and are key implications of the theory we propose.

The main mechanism of our theory is simple. Along a balanced growth path, the number of firms has to grow at the same rate as the labor force. Hence, a falling rate of labor force growth translates into a fall in the net entry rate of new firms. Because entry is an important component of creative destruction, its decline ripples through the economy. Incumbent firms face less competition from new firms and find it easier to expand and to raise their prices. Average firm size and concentration increases as a result, the pace of job reallocation slows and markups rise. At the same time, the decline in creative destruction also reduces productivity growth at the aggregate level.

To make this intuition precise, we propose a new model of firm growth that is rich enough to rationalize many first-order features of the process of firm dynamics. As in Garcia-Macia et al. (2016), our model allows for creative destruction (by both entrants and incumbents), the creation of new varieties (again by both entrants and incumbents) and own-innovation, where firms improve the quality of the products they own. In order to study the implications for market power, we embed this model in a framework of imperfect product markets, in which firms compete a la Bertrand and markups are determined endogenously.

We begin by presenting a simplified version of our model where we abstract from endogenous market power. This simplified version is similar to the basic model of Klette and Kortum (2004), augmented by the possibility of population growth, new variety creation, own-innovation and a demand elasticity that exceeds unity. We show that this model has a full analytic solution and we can express the process of firm dynamics (meaning the firm size distribution, firm exit rates and life-cycle growth)

Figure 1: Population Growth Across Major Economies



Notes: Solid lines plots historical population growth from the UN World Population Prospects 2019 for several major economies. Dashed lines plots the UN projections for population growth in the "Medium" scenario out to to 2060.

and the aggregate growth rate directly as a function of population growth.

We derive three key results. First, we show that a decline in population growth reduces the equilibrium of entry and increases economic concentration. The reason is the following. Declining population growth reduces creative destruction by lowering firms' incentives to engage in product innovation. Moreover, this decline in creative destruction is only accommodated through a decline in entry. Existing firms' rate of product creation is unaffected by changes in population growth. This change in the composition of product creation has the implication that lower population growth increases firms growth conditional on survival and reduces incumbent firms' exit hazards. As a consequence, concentration and firm size rises and the entry rate falls.

Second, our theory makes clear predictions about the relationship between population growth and income per capita growth. Because declining population growth reduces creative destruction and the rate at which new varieties are created, the long-run rate of income per capita growth is declining in the rate of population growth.

Third, we show that there is an important countervailing effect that makes the relationship between population growth and welfare ambiguous. By reducing creative destruction and hence the rate of firm-exit, lower population increases' corporate valuations because future profits are discounted at a lower rate. Free entry therefore implies an increase in the economy-wide level of varieties to increase competition. This increases income per capita because of variety gains. The welfare consequences of declining population growth therefore hinge on the relative importance of these static variety gains relative to the dynamic losses due to lower growth.

We then show that these results are robust to a variety of changes in the environment. Most importantly, we extend our model to a setting where firms compete a la Bertrand and market power is endogenous. Declining population growth interacts with firms' ability to charge high markups in an interesting way. In our theory, more productive firms post higher markup and productivity increases over the firms' life-cycle. Because creative destruction reduces firms' chances of survival, it hinders incumbents from accumulating market power and hence prevents the emergence of dominant producers. In short: creative destruction is pro-competitive. Declining population growth, by lowering creative destruction, therefore reduces competition and increases markups and misallocation.

To quantify the strength of this mechanism, we calibrate our model to data for the population of US firms. Crucially, by linking firm-level information on sales to the U.S. Census, we can measure firm-level markups in a consistent way for all firms in the US, and hence explicitly target the life-cycle profile of markups. Exploiting information on the evolution of both markups and size at the firm-level is an important aspect of our empirical methodology and is crucial to identify the model.

Our model, despite parsimoniously parametrized, matches many important aspects of the process of firm dynamics in the US remarkably well. Most importantly, our model generates a Pareto tail of both the sales and the employment distribution and captures the declining exit hazard by both age and size. Allowing for population growth is important for the model to successfully replicate these patterns. First, without population growth, the cross-sectional distribution of the number of products firms has has a thin tail as in Klette and Kortum (2004). Second, the rate of population growth emerges as they key determinant of the exit hazard and the tail index of the size distribution.

With the calibrated model in hand, we ask a simple question: what are the implications of the observed decline in the rate of labor force growth since 1980? Empirically, the labor force growth almost halved from 2% to 1%. We find that this decline has quantitatively large effects. In particular, our model can explain almost the entirety of the decline in the entry rate and the increase in average firm size and the degree of concentration. We also find that it predicts a quantitatively significant rise in markups across firms, and a slowdown in aggregate growth.

Throughout the paper, we will often speak of population growth and labor force growth interchangeably. For this paper, we take movements in the size of the labor force to be exogenous to market concentration and firm dynamics. Across the developed world, falls in fertility in the 1960s and 1970s have manifested in slower rates of growth in the labor force in the 1980s and 1990s. In the U.S. in particular, slowing labor force growth also reflects an end to increasing female participation, and declining prime-age male participation. While a declining labor share and rising market power may itself have implications for worker participation, here the simplicity of taking these movements as given yields substantial insight into the changing patterns of firm dynamics we see in the data.

Related Literature. We are not the first to connect the decline in the growth rate of the labor force to changes in firm dynamics. Karahan et al. (2016) and Hathaway and Litan (2014) are early contributions that use geographic variation in the age structure of the population of the U.S. to provide direct support that a lower rate of population growth reduces the start-up rate. Recently, Hopenhayn et al. (2018) document the relationship between changes in demographics and firm dynamics in a quantitative model. Both Karahan et al. (2016) and Hopenhayn et al. (2018) perform their analysis in a model in the spirit of Hopenhayn (1992), where firm productivity and aggregate growth is exogenous and markets are competitive. Engbom (2017, 2020) studies the implications of population aging in the context of a search model. In contrast, our theory builds on models with endogenous firm dynamics and highlights that a declining rate of population growth also affects the extent of market power and aggregate productivity growth, and hence has potentially a much broader macroeconomic impact.

On the theory side, we build on firm-based models of Schumpeterian growth in the tradition of Aghion and Howitt (1992) and Klette and Kortum (2004). We augment these models by allowing for efficiency improvements of existing firms as in Atkeson and Burstein (2010), Luttmer (2007), Akcigit and Kerr (2015) or Cao et al. (2017), the creation of new varieties as in Young (1998), and endogenous markups through Bertrand competition as in Peters (2018) or Acemoglu and Akcigit (2012). Departing from Peters (2018), we consider elasticities of substitution greater than unity, which requires consideration of the full joint distribution of efficiency and markups. Our model is thus a version of Garcia-Macia et al. (2016), augmented by endogenous markups and endogenous innovation choices, and incorporating long run growth in the labor force.

There is a growing literature on the decline of dynamism in the US. On the empirical side this literature shows that the entry rate has fallen substantially (Karahan et al., 2015; Alon et al., 2018; Decker et al., 2014), that broad measures of reallocation have declined since the 80s and 90s (Haltiwanger et al., 2015; Davis and Haltiwanger, 2014), that industries are becoming more concentrated (Kehrig and Vincent, 2017; Autor et al., 2017) and that markups are rising (Edmond et al., 2018; De Loecker and Eeckhout, 2017). See also Akcigit and Ates (2019a) for a summary.

In terms of explanations for these phenomena, Aghion et al. (2019) and Lashkari et al. (2019) argue that improvements in IT technology raised the returns to scale and induced firms with high productivity and high markups to expand. In a related paper, De Ridder (2019) argues that increasing use of intangibles has increased the ratio of fixed to variable costs in production, with the most productive firms in intangible use employing their advantage to raise markups. Akcigit and Ates (2019b) focus on changes in the process of knowledge diffusion. Our paper is complementary to these studies by highlighting that the secular decline in population growth might be a key factor explaining the low-frequency trends of concentration, markups and growth.

The remainder of our paper is structured as follows. In Section 2 we present our baseline model and derive our main results. In Section 3 we extend this framework by allowing for endogenous market power. In Section 4 we calibrate our theory to data for the population of US firms. In Section 5 we quantify the role of population growth for the process of firm dynamics and growth. Section 6 concludes. An Online Appendix contains the formal derivations of our theoretical results.

2 The Baseline Model

We now present our theory to analyze the link between population growth and firm dynamics. We start with a baseline version of our model, where markups are constant and the productivity of firms' existing products grows exogenously. This version of the model has an analytical solution and allows for a tight characterization how population growth affects entrants' and incumbents' firms incentives to engage in creative destruction and the creation of new products. Below we extend our analysis by explicitly allowing for endogenous markups and endogenous own-innovation.

2.1 Environment

Time is continuous. There is a mass L_t of identical individuals, each supplying one unit of labour inelastically. This mass grows at rate η_t, such that $\dot{L}_t / L_t = \eta_t$. The rate of population growth η_t, which we take as exogenous, is the crucial parameter of this paper.[1] Households have preferences over a final consumption good c_t, which are given by

$$U = \int_0^\infty e^{-\rho t} \ln (c_t) \, dt.$$

Production and Market Structure. The final consumption is composed of differentiated varieties. As in Klette and Kortum (2004), we model these varieties as differentiated product lines, which may be produced by multiple firms. The production of the final good takes place in a competitive final sector, that combines the differentiated varieties according to

$$Y_t = \left(\int_0^{N_t} \left(\sum_{f \in S_{it}} y_{fit} \right)^{\frac{\sigma-1}{\sigma}} di \right)^{\frac{\sigma}{\sigma-1}}. \tag{1}$$

Here N_t is the number of active product lines, where these product lines are indexed by i. This number evolves endogenously with the creation and destruction of new products. S_{it} is the set of firms with the knowledge to produce product i, which likewise evolves endogenously.

Firms can be active in multiple product markets. Each firm f is characterized by a set of the products they produce, denoted by Θ_f, and the productivity of these products, indexed by $\{q_{fi}\}_{i \in \Theta_f}$. We denote the number of products firm f produces by n_f. Production of each good uses only production labour, and is given by

$$y_{fi} = q_{fi} l_{fi},$$

where l_{fi} is the amount of labour hired by firm f to produce product i, and q_{fi} denotes the efficiency of firm f in producing product i.

Suppose to begin with that the producing firm charges a constant markup over marginal cost $\mu = \frac{\sigma}{\sigma-1}$.[2] Below we explicitly allow for imperfect competition which gives rise to heterogenous markups.

[1] Empirically, the primary driver of the global decline in population growth has been sustained falls in fertility, occurring in rich and poor countries alike. See Section B.1 in the Appendix for more details.

[2] This can either be the case the relative productivity advantage over the next best firm exceeds $\mu = \frac{\sigma}{\sigma-1}$ or if firms have to pay an infinitesimal fixed cost before producing, in which case the least productive firm will not enter (see Garcia-Macia et al. (2016))

With constant markups, aggregate output Y_t and equilibrium wages w_t are given by

$$Y_t = Q_t N_t^{\frac{1}{\sigma-1}} L_t^P \quad \text{and} \quad w_t = \mu^{-1} Y_t / L_t^P, \tag{2}$$

where $Q_t \equiv \left(\int q_i^{\sigma-1} dF_t(q) \right)^{\frac{1}{\sigma-1}}$ is a measure of average efficiency, F_t is the distribution of product efficiency and L_t^P is the total amount of labor devoted to the production of goods.

Entry, Innovation and Aggregate Growth. Both firms' productivities and the products they sell evolve endogenously. As in Garcia-Macia et al. (2016), our theory allows for three sources of firm dynamics. First we allow for *creative destruction* by incumbents and entrants (as in Klette and Kortum (2004)). Creative destruction occurs when either an existing firm or a new firm improves the productivity of a product i, which is currently produced by another producer. Because the output of firms producing the same product i is considered to be perfectly substitutable (see (1)), such productivity improvements result in churning, whereby the old producer gets replaced. Second we allow for *own-innovation*, whereby firms improve the efficiency q of the products they are currently producing (see Atkeson and Burstein (2010) or Luttmer (2007)). Third, we allow for the endogenous entry of *new varieties*. This margin is the source of variety gains, whereby firms can generate product varieties which are entirely new to the economy. As in Young (1998), it is this margin which implies that the model does not suffer from strong scale effects, i.e. the growth rate, while still endogenous, is independent of the level of the population (see Jones (1995)). Allowing for variety creation is essential to ensure that the model has a stationary firm size distribution in the presence of population growth.

We formalize these decisions in the following way. Existing firms increase the efficiency with which they produce their existing products deterministically at rate I, such that the efficiency with which product i is produced, q_i, evolves according to

$$\frac{\dot{q}_{it}}{q_{it}} = I.$$

To focus on the main economic mechanism how population growth affects firm dynamics we start by assuming that I is exogenous and constant over time. Below we show how to endogenize this rate and the implications of doing so.

Firms can also expand into new product lines. To do so, they choose the Poisson rate X at which the knowledge for how to produce a product new to them is created. Such expansion activities are costly, and we denote these costs (in units of labor) as

$$c_t^X(X, n) = \frac{1}{\varphi_x} X^{\zeta} n^{1-\zeta} = \frac{1}{\varphi_x} x^{\zeta} n, \tag{3}$$

where n denotes the number of products the firm is currently producing and $x = X/n$ is the firms' innovation intensity.[3]

[3]The particular functional form of the innovation cost function in (3) is not essential. All our results equally apply as long as $c_t^X(X, n)$ is homogeneous of degree one in both arguments.

Conditional on successfully creating a new product, this product can either be a new variety to the aggregate economy, or it can improve upon an already existing product from another firm. We assume that innovation is "undirected", such that the firm cannot target new or existing varieties. With probability α the new product represents a technological advance over a (randomly selected) incumbent firm's product, increases the product's efficiency by a factor $\lambda > 1$ and forcing the current producer to exit ("incumbent creative destruction"). With the complementary productivity $1-\alpha$, the product will be new to society as a whole, i.e. the mass of available products N_t grows.[4]

We assume that the production efficiency of new products is given by $q = \omega Q_t$, where ω is drawn from a fixed distribution $\Gamma(\omega)$. Hence, as in Buera and Oberfield (2016), the productivity of new varieties is determined both by the existing knowledge embedded in Q_t and by novel ideas. It is useful to define $\overline{\omega} \equiv \left(\int \omega^{\sigma-1} d\Gamma(\omega)\right)^{\frac{1}{\sigma-1}}$, which we also refer to as the mean productivity of new products (appropriately scaled). As we show below, the equilibrium allocations only depend on $\overline{\omega}$ and not on $\Gamma(\omega)$.

Entrants have the same opportunities as incumbent firms. While they naturally do not own any products they could improve on, they also engage in creative destruction and new variety creation. As with incumbent firms, the share of innovations which result in creative destruction is exogenous and given by α. Entrants have access to a linear entry technology, where each worker they hire for research generates a flow of φ_E ideas.[5]

Let Z_t denote the aggregate flow of entry and $z_t = Z_t/N_t$ the entry intensity per product. Similarly, let x_t denote the average expansion intensity of incumbent firms $x_t = \frac{1}{N_t}\int x_{it} di$. The rate of creative destruction, i.e. the rate at which the producer of a given product is replaced by another producer, is therefore given by

$$\tau_t = \alpha \left(x_t + z_t\right).$$

Similarly, the rate of variety creation is given by

$$g_t^N = \frac{\dot{N}_t}{N_t} = (1-\alpha)\left(x_t + z_t\right) = \frac{1-\alpha}{\alpha}\tau_t. \tag{4}$$

The rate of efficiency growth g^Q is given by

$$g_t^Q = \frac{\dot{Q}_t}{Q_t} = \frac{\lambda^{\sigma-1}-1}{\sigma-1}\tau_t + \frac{\overline{\omega}^{\sigma-1}-1}{\sigma-1}g_t^N + I$$

Because $\lambda > 1$, creative destruction is a source of aggregate productivity growth. Similarly, firms' own-innovation efforts raise the productivity of individual products and hence also the aggregate productivity index Q_t at rate I. Finally, the creation of new varieties also affects the growth rate of the average efficiency of production. If new products are on average as productive as existing

[4] In Section A.2 in the Appendix we generalize our analysis to allows for innovations to be directed and α but can be chosen directly by the firm.

[5] In Section 2.5 below we study the case where entry features decreasing returns at the aggregate level.

products, i.e. $\overline{\omega} = 1$, the growth rate of average efficiency Q_t is independent of the rate of product creation g_N. If new products are an average worse, $\overline{\omega} < 1$, faster product creation is an adverse source of efficiency growth. Finally, the overall growth of labor productivity Y_t/L_t^P is given by (see (2))[6]

$$g_t^{LP} = \frac{d}{dt}\ln\left(Q_t N_t^{\frac{1}{\sigma-1}}\frac{L_t^P}{L}\right) = g_t^Q + \frac{1}{\sigma-1}g_t^N = \frac{\lambda^{\sigma-1}-1}{\sigma-1}\tau_t + I + \frac{\overline{\omega}^{\sigma-1}}{\sigma-1}g_N.$$

In the quantitative section of the paper we also assume that product lines die at an exogenous rate of δ. This can be interpreted as a taste shock in which consumers no longer value a product line for exogenous reasons. Doing so helps ensure stationarity at low or negatives levels of population growth. For the theory below, we set $\delta = 0$ for expositional simplicity, but all our results can be modified to incorporate positive values of δ.

2.2 Optimal Product Creation and Entry

Firms' expansion decisions are forward-looking. The state variables at the firm-level are $\{q_{fi}\}_{i\in\Theta_f}$, which we for simplicity denote as $[q_i]$. The value function of a firm is given by the HJB equation

$$r_t V_t([q_i]) - \dot{V}_t([q_i]) = \underbrace{\sum_{i=1}^{n}\pi_t([q_i])}_{\text{Profits}} + \sum_{i=1}^{n}\underbrace{\tau_t\left[V_t\left([q_j]_{j\neq i}\right) - V([q_i])\right]}_{\text{Creative Destruction}} + I\sum_{i=1}^{n}\underbrace{\frac{\partial V_t([q_i])}{\partial q_i}q_i}_{\text{Own innovation}} + \Xi_t([q_i]), \quad (5)$$

where Ξ_t is the option value of product creation that is given by

$$\Xi_t([q_i]) = \max_X\left\{X\left(\underbrace{\alpha\int V_t([q_i],\lambda q)\,dF_t(q)}_{\text{Replacing an existing firm}} + \underbrace{(1-\alpha)\int V_t([q_i],\omega Q_t)\,d\Gamma(\omega)}_{\text{New variety}} - V_t([q_i])\right) - \frac{1}{\varphi_x}X^{\zeta}n^{1-\zeta}w_t\right\}.$$

The value of a firm consists of multiple additively separable parts. First, the value of the firm is increasing in the current flow profits. Second, the firm might lose any of its products to another firm, which happens at the endogenous rate of creative destruction τ. Third, own-innovation raises the efficiency q_i of each product, and hence profitability. Finally, the firm has the option to obtain a product outside its current portfolio. With probability α it replaces a randomly selected product, with probability $1-\alpha$, the firm creates a new variety, whose efficiency is given by ωQ_t.

Along a balanced growth path (BGP), this value function can be solved explicitly. On a BGP, the interest rate r_t is constant and output grows at a constant rate g^Y. This implies that the rate of product creation g^N and the rate of creative destruction τ are also constant. Moreover, the consumer Euler equation requires that $r - g^Y = \rho$. This implies that we can solve for $V_t([q_i])$ explicitly on a BGP.

Proposition 1. *Consider the value function $V_t([q_i])$ given in (5). $V_t([q_i])$ is given by ,*

$$V_t([q_i]) = \sum_{i=1}^{n} V_t(q_i) \quad \textit{where} \quad V_t(q) = \frac{\pi_t(q)}{\rho+\tau+(g^Q - I)(\sigma-1)} + \frac{\frac{\zeta-1}{\varphi_x}x^{\zeta}w_t}{\rho+\tau},$$

[6] Along a BGP, where the share of production workers L_t^P/L_t is constant, income per capita also grows at rate g_t^{LP}.

with $\pi_t(q) = (\mu - 1)\left(\frac{q_i}{Q_t}\right)^{\sigma-1} \frac{L_t^P}{N_t} w_t$,

$$x = \left(\frac{\varphi_x}{\zeta}\right)^{\frac{1}{\zeta-1}} \left(\alpha \frac{V_t^{CD}}{w_t} + (1-\alpha)\frac{V_t^{NV}}{w_t}\right)^{\frac{1}{\zeta-1}} \tag{6}$$

and

$$V_t^{CD} = \int V_t(\lambda q)\, dF_t(q) = V_t(\lambda Q_t) \text{ and } V_t^{NV} = \int V_t(\omega Q_t)\, d\Gamma(\omega) = V_t(\overline{\omega} Q_t).$$

Proof. See Section B.1 in the Appendix. □

Proposition 1 contains four important results. First, the value function $V_t([q_i])$ is additive, so we can focus on the value of a single product $V_t(q)$. Second, $V_t(q)$ is itself the sum of two components. The first part is the present discounted value of flow profits, the second the present discounted option value of innovation, which equals the inframarginal rents of the innovation technology. Note that the flow profits $\pi_t(q)$ are discounted at the higher rate of $\left(g^Q - I\right)(\sigma - 1)$ than the option value of innovation. This extra discounting reflects the evolution of the relative competitiveness of the firm's product, as the relative efficiency of a product $(q/Q_t)^{\sigma-1}$ changes at rate $\left(I - g^Q\right)(\sigma - 1)$. Hence, if Q_t grows faster than q, the product's profitability declines. Third, the optimal innovation rate x is constant and determined by the average of the creative destruction value V_t^{CD} and the value of new variety creation V_t^{NV} (both relative to the wage). Hence, the link between population growth η and firms' innovation rate x operates via V_t^{CD} and V_t^{NV}. Fourth, these values are in turn simply the value of a single product evaluated at the creative destruction entry point λQ_t or the initial efficiency of a new variety ωQ_t. Note that V_t^{CD} and V_t^{NV} grow at the rate of the wage w_t along a BGP so that x is indeed constant.

Entry. Now consider the behavior of entrants. Free entry requires that the expected value of a successfully created new product (which, like for incumbents, with probability α, stems from an existing firm and with probability $1 - \alpha$ is entirely new to society) does not exceed the cost of entry, i.e.

$$V_t^{Entry} \equiv \alpha V_t^{CD} + (1-\alpha) V_t^{NV} \leq \frac{1}{\varphi_E} w_t. \tag{7}$$

For the remainder of this paper we focus on allocations where the flow of entry is positive and equation (7) holds with equality.

The free entry condition in (7) is a crucial equation in our theory. Most importantly, it implies that the rate of product creation by incumbent firms is a function of technology only. Combining (7) with (6) yields

$$x = \left(\frac{1}{\zeta}\frac{\varphi_x}{\varphi_E}\right)^{\frac{1}{\zeta-1}}. \tag{8}$$

Hence, incumbent product creation is independent of *any* general equilibrium variables. In particular, it does not depend on the rate of population growth η.

This property plays an important role in our analysis and allows for a precise characterization of the role of population growth. It follows from the fact that incumbents' innovation technology has decreasing returns at the firm-level, while entry - that operates at the aggregate level - has constant returns.[7] Hence, the free entry condition pins down the value of product creation (relative to the wage) and incumbent firms optimally chose the rate of product creation to equalize the marginal cost and the marginal benefits. In Section 2.5 below we generalize our results to the case where the entry process has decreasing returns in the aggregate. In that case, x also depends on general equilibrium variables and is affected by population growth.

2.3 Equilibrium

To close the model in general equilibrium, define the two aggregate statistics

$$\ell_t \equiv \frac{L_t}{N_t} \quad \text{and} \quad s_t^P \equiv \frac{L_t^P}{L_t}.$$

We will refer to ℓ_t as the economy's labor intensity and to s_t^P as the production share. Note first that labor market clearing implies that[8]

$$L_t = L_t^P + L_t^R = L_t^P + N_t \left(\frac{1}{\varphi_E} z_t + \frac{1}{\varphi_x} x^\zeta \right).$$

Using that $z_t = \frac{1}{1-\alpha} g_N - x$ and the optimal rate of incumbent expansion given in (8), labor market clearing requires that

$$\ell_t \left(1 - s_t^P\right) = \frac{1}{\varphi_E} \left(\frac{g_N}{1-\alpha} - \frac{\zeta - 1}{\zeta} x \right). \tag{9}$$

Holding the labor intensity ℓ_t constant, a higher production share s_t^P reduces variety growth g_N as less resources are allocated towards research. Conversely, for a given production share, variety growth is increasing in the labor intensity as the amount of research effort per existing variety is higher. Equation (9) is the first key equation to characterize the equilibrium.

The second key equation is the free entry condition in (7). As we show in Section B.1.3 in the Appendix, along a BGP, the free entry condition can be written as

$$\frac{1}{\varphi_E} = \frac{\bar{q}\,(\mu - 1)\,\ell_t s_t^P}{\rho + \frac{\alpha}{1-\alpha} g_N - I\,(\sigma - 1)} + \frac{\frac{\zeta-1}{\varphi_x} x^\zeta}{\rho + \frac{\alpha}{1-\alpha} g_N}, \tag{10}$$

where $\bar{q} = \alpha \lambda^{\sigma-1} + (1-\alpha)\,\overline{\omega}^{\sigma-1}$ is the average quality increase of product creation. Holding ℓ_t fixed, the production share s_t^P and the rate of variety growth g_N are positively related. Intuitively: a

[7] Note that incumbent product creation also has constant return in the aggregate: if the number of incumbent firms were to double, the rate of aggregate product creation performed by incumbents would also double.

[8] In our baseline model we assume that labor is perfectly substitutable between production and research for product creation. In Section A.3 in the Appendix we extend our analysis to the case where labor is not perfectly substitutable between these two activities.

higher rate of variety growth reduces the value of existing firms through two channels: first, a higher number of firms increases competition for each individual producer. Second, under our assumptions, the rate of creative destruction is linked to the rate of variety growth so that faster variety reduces the expected life-span of a product. Both channels therefore increase in the effective discount rate of firms. Free entry therefore requires that the size of the market for each firm $\ell_t s_t^P = L_t^P / N_t$ increases.

Along a BGP, the equilibrium growth rate is constant. This implies that g_N grows at a constant rate. (9) and (10) therefore require that ℓ_t and s_t^P are constant. This has the important implication that the number of varieties N_t has to grow at the rate of population growth as

$$\frac{\dot{\ell}_t}{\ell_t} = \eta - g_N = 0.$$

And given that creative destruction and variety creation are related from (4) as $\tau = \frac{\alpha}{1-\alpha} g_N$, we can explicitly determine the sources of growth as a function of population growth.

Proposition 2. *On a BGP, the following holds:*

1. *The rate of creative destruction τ, the rate of incumbent product creation x and the rate of entry z are given by*

$$\tau = \frac{\alpha}{1-\alpha}\eta \qquad x = \left(\frac{1}{\zeta}\frac{\varphi_x}{\varphi_E}\right)^{\frac{1}{\zeta-1}} \qquad z = \frac{\eta}{1-\alpha} - x. \tag{11}$$

   A decline in population growth reduces creative destruction, $\frac{\partial \tau}{\partial \eta} > 0$, reduces the flow rate of entry, $\frac{\partial z}{\partial \eta} > 0$, but leaves incumbent expansion unchanged $\frac{\partial x}{\partial \eta} = 0$.

2. *Aggregate growth g^y and the growth rate of efficiency g^Q are given by*

$$g^y = g^Q + \frac{1}{\sigma-1}\eta \quad \text{and} \quad g^Q = \frac{1}{\sigma-1}\left(\frac{\alpha\left(\lambda^{\sigma-1}-1\right) + (1-\alpha)(\overline{\omega}^{\sigma-1}-1)}{1-\alpha}\right)\eta + I. \tag{12}$$

   A decline in population growth reduces aggregate growth, $\frac{\partial g^y}{\partial \eta} > 0$ and reduces the growth rate of efficiency if and only if $(1-\alpha)\,\overline{\omega}^{\sigma-1} + \alpha\lambda^{\sigma-1} > 1$.

3. *The production share s^P and the labor intensity ℓ are uniquely determined by the equations*

$$\ell\left(1-s^P\right) = \frac{1}{\varphi_E}\left(\frac{\eta}{1-\alpha} - \frac{\zeta-1}{\zeta}x\right)$$
$$1 = \frac{\varphi_E \overline{q}\left(\mu-1\right)\ell s^P}{\rho + \frac{\alpha}{1-\alpha}\eta - I\left(\sigma-1\right)} + \frac{\zeta-1}{\zeta}\frac{x}{\rho+\frac{\alpha}{1-\alpha}\eta}.$$

   A decline in population growth reduces the labor intensity, $\frac{\partial \ell}{\partial \eta} > 0$.

Proof. See Section B.1.4 in the Appendix. □

Proposition 2 contains three key theoretical results of this paper. First, a decline in population growth reduces creative destruction. Moreover, the entirety of the decline is absorbed by the economy's extensive margin - entrants do all the work. Hence, even though our model allows for incumbents' incentives to engage in product creation to respond, in equilibrium free entry implies that incumbents' rate of product creation is insulated from demographics.

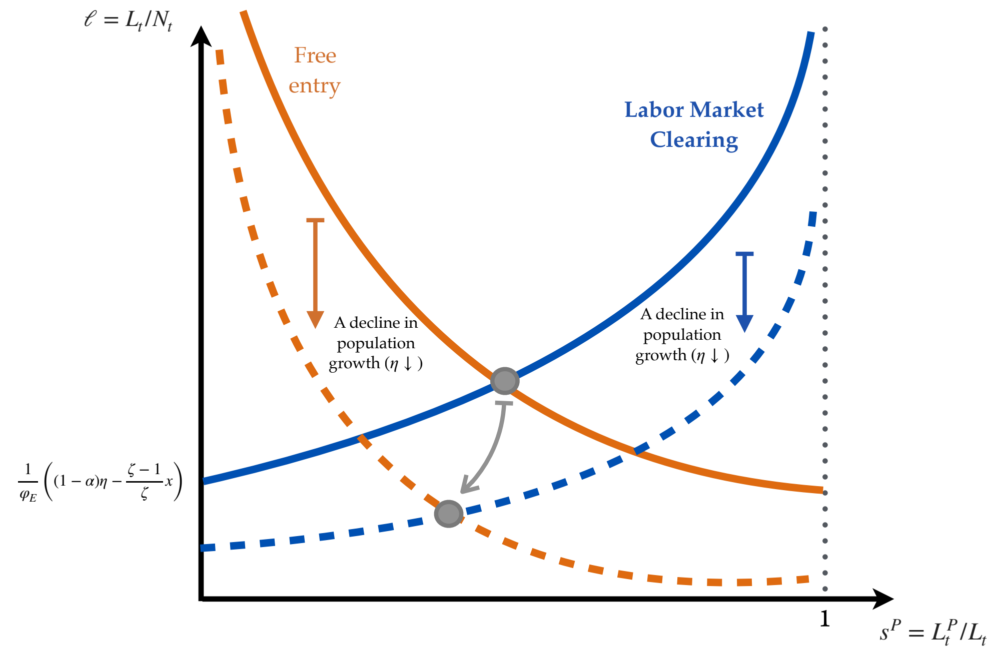
Second, the rate of population growth directly affects the rate of growth. It does so in two ways. First, population growth determines variety creation, which is in itself a form of growth. Second, and more importantly, population growth also affects creative destruction and hence the rate of efficiency growth g^Q. While the effect of population growth on variety growth is always positive, its affect on efficiency growth depends on the average efficiency of newly created products $\overline{\omega}$ and the increment of creative destruction λ.

Third, the *level* of varieties N_t relative to the population is jointly determined with the equilibrium growth rate and the the production share. Moreover, a decline in population growth reduces the labor intensity ℓ or equivalently increases the variety intensity N_t/L_t. This is seen in Figure 2, where we depict the free entry condition (shown in orange) and the labor market clearing condition (shown in blue) from Proposition 2. Because a decline in population growth shifts both schedules down, the labor intensity ℓ unambiguously declines. Hence, lower population growth increases the number of varieties per worker. Note that this a countervailing force to the growth implications highlighted in Proposition 2. Because increases in N_t/L_t are a source of welfare gains, lower population growth has positive welfare consequences through a higher level of varieties (a "static" effect) but negative consequences via a decline in the growth rate (a "dynamic" effect).

Interestingly, the effect of population growth on the long-run share of production workers s^P is theoretically ambiguous. In Figure 2, we show the case where a decline in population growth reduces the share of production workers s^P and hence increases the share of workers employed in research $1 - s^P$. This is the case that emerges in our quantitative analysis. Hence, lower population growth can simultaneously increase the resources deployed in R&D and reduce the rate of growth. This pattern is qualitatively consistent with the trends of research investment and productivity growth in the US (see De Ridder (2019) and Bloom et al. (2020)).

Proposition 2 also highlights that our model is a semi-endogenous growth model in the spirit of Jones (1995): the rate of growth is fully determined from the rate of population growth and is independent of the level of the population. This is in stark contrast to the baseline model of Klette and Kortum (2004), where the growth rate is increasing in the size of the population and hence features "strong scale effects". This difference arises because in our theory the number of varieties N_t is endogenous. Hence, as in Young (1998), a larger population increases the number of goods available (and hence the level of income) but not its growth rate. This result does not hinge on taking I to be exogenous, which we assumed for purely expositional purposes. Even if we treat I as endogenous (as we do below), it is still the case that the rate of growth is independent of level of the population. However, growth is endogenous in the sense that for example the cost of innovation affect the rate of growth.

Figure 2: Population Growth and Level of Varieties



Note: This figure shows the determination of (ℓ, s^P) along the BGP (see Proposition 2). It also depicts the consequences of a decline in population growth η.

2.4 Population Growth and Firm Dynamics

Proposition 2 is also the key ingredient to to analyze how population growth affects the process of firm-dynamics. Because firms increase the efficiency of their own products and accumulate new products as they age, firms' survival chances are a key aspect of the process of firm-dynamics. And because declining population growth reduces aggregate creative destruction τ relative to the expansion rate by incumbents, lower population growth increases firms' expected life-span and their average growth rate conditional on survival. This has direct implications for the distribution of firm size and the rate of entry which our theory allows us to characterize analytically. In particular, we show how population growth affects (i) firm survival and the distribution of firm age, (ii) the size distribution and industry concentration, (iii) the entry rate and (iv) the extent of variety creation.

Population Growth, Firm Survival and the Age Distribution. Consider first the impact of population growth on firms' chances to survive. To do so, define firms' *net* rate of product accumulation $\psi = x - \tau$, which is exactly the difference between the rate of product loss τ and the accumulation of products x. Using (11) to express τ in terms of the rate of population growth η yields

$$\psi = x - \frac{\alpha}{1-\alpha}\eta,$$

i.e. a decline in the rate of population growth increases the rate of product accumulation at the firm-level as firms' face less of a threat of creative destruction.

As we show in Section B.1.5 in the Appendix, this net accumulation rate ψ emerges as one key determinant for the process of firm dynamics. Let $S(a)$ denote the survival function, i.e. the probability that a given firm survives until age a. This survival function is fully parameterized by ψ and given by

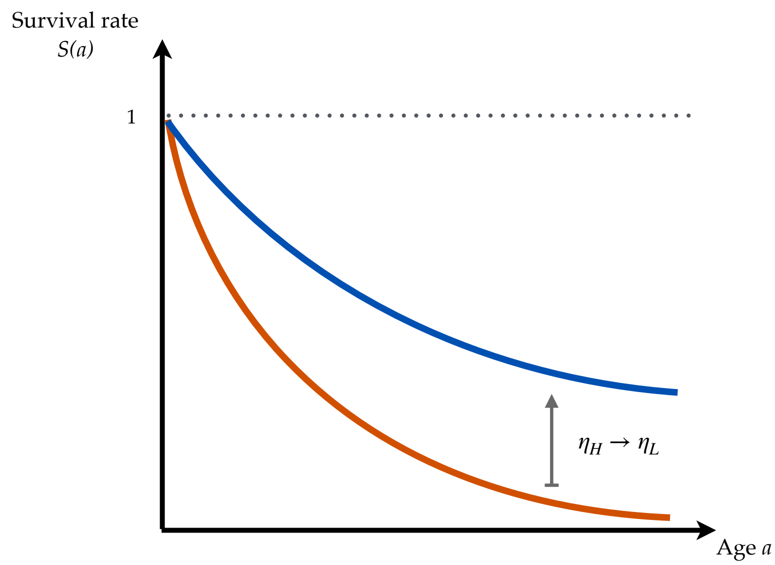
$$S(a) = \frac{\psi e^{\psi a}}{\psi - x\left(1 - e^{\psi a}\right)}. \tag{13}$$

In Figure 3 we display $S(a)$ graphically. Naturally, $S(a)$ is declining in a and satisfies $\lim_{a\to\infty} S(a) = 0$ as all firms exit eventually. More importantly, lower population growth *increases* firms' survival rates through an increase in the accumulation rate ψ. Hence, firms exit at a lower rate and become older on average. In fact, one can show that the average age of firms is given by $\mathbb{E}[\text{Age}] = \frac{1}{x}\ln\frac{\alpha\eta}{\alpha\eta-(1-\alpha)x}$, which is decreasing in η.

Population Growth, Concentration and Firm Size. Because firms on average grow as they age conditional on survival, lower population growth increases firm size and concentration by shifting the age distribution towards older firms. In addition, by increasing the net accumulation rate ψ, lower population growth also increases the whole profile of life-cycle growth, i.e. firms are becoming bigger *conditional* on age since their expansion incentives do not chance but they lose products less often. In particular, let $\overline{n}(a)$ denote the average number of products of a firm of age a. Then it can be shown that

$$\overline{n}(a) = 1 - \frac{x}{\psi}\left(1 - e^{\psi a}\right), \tag{14}$$

Figure 3: Population Growth and Firm Survival



Note: This figure shows the relationship between population growth η and firms' survival probabilities $S(a)$ (see (13)).

which we also display in Figure 4. Not only is $\overline{n}(a)$ increasing in a, but it is also declining in η.

These two forces imply that market concentration rises. To see this, consider the right tail of the product distribution. As we show in the Section B.1.5 in the Appendix, as long as $\eta > \psi > 0$, the distribution of the number of products n_f has a pareto tail ζ_n, which is given by

$$\zeta_n = \frac{\eta}{\psi} = \frac{(1-\alpha)\,\eta}{x(1-\alpha) - \alpha\eta}. \tag{15}$$

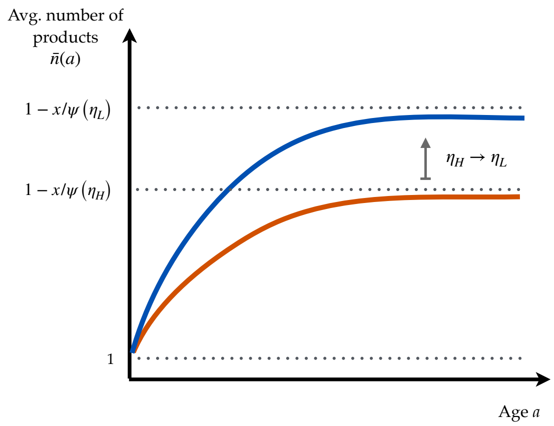
Hence, the Pareto tail of the product distribution is a closed form expression of the rate of population growth η and a decline in population growth increases concentration, i.e lowers ζ_n towards unity. Equation (15) also highlights that lower population growth affects the firm size distribution in two ways.[9] Holding firms' net expansion rate ψ constant, lower population growth increases concentration because it reduces the rate at which new firms, which are on average small by virtue of being young, enter. In addition, lower population growth endogenously increases the net accumulation rate ψ by lowering creative destruction. This further increases market concentration and lowers the

[9]Equation (15) focuses on the distribution of the number of products, i.e. the extensive margin of firm growth. Firm employment is given by

$$l_{ft} = \sum_{i=1}^{n_f} l_{it} = \ell \times \sum_{j=1}^{n_f} (q_i/Q_t)^{\sigma-1},$$

where, recall, $\ell = L_t^P/N_t$. Hence, firm employment is determined both by the number of products n_f and the efficiency of the firms' products. As we show in Section (B.1.6) in the Appendix, the efficiency distribution also has a Pareto tail. Whether the Pareto tail of the efficiency distribution or the number of products (given in (15)) dominates is a quantitative questions.

Figure 4: Population Growth and Product Expansion



Note: This figure shows the relationship between population growth η and the average number of products $\overline{n}(a)$ (see (14)).

tail of the product distribution.[10]

Note that these increases in concentration and firm size goes hand in hand with an *increase* in the aggregate variety intensity N_t/L_t. This is due to multi-product nature of our theory: while population growth reduces the number of firms per workers, it increases the number of products per worker because each existing firm offers a larger product portfolio.

Population Growth and the Entry Rate. Finally, our theory highlights the implications of population growth for the equilibrium entry rate. Letting $\mathcal{F}_t$ denote the number of firms at time t, the entry rate is given by

$$\text{Entry rate}_t = \frac{Z_t}{\mathcal{F}_t} = z \times \frac{N_t}{\mathcal{F}_t},$$

i.e. the entry rate is the product of the entry flow (per existing product) and the number of products per firm $N_t/\mathcal{F}_t$. Holding the number of products per firm constant, a lower entry flow z reduces the rate of entry. Conversely, for a given entry intensity z, an increase in $N_t/\mathcal{F}_t$ increases the entry rate. Our theory reflects these two counteracting forces. A decline in population growth lowers z, which all else equal pushes the entry rate lower. At the same time, we showed in (13) and (14) that a lower rate of population growth increases the number of products by age and shifts the firm distribution towards older firms. Hence, $N_t/\mathcal{F}_t$ increases in response to a decline in population growth. Quantitatively, we find that the first effect decisively dominates: declining population growth lowers the rate of entry in equilibrium.[11]

[10]Note that (15) can also be written as $\zeta_n = \frac{\eta}{\eta - z}$, i.e. concentration is large if the flow of new entrants z is small *relative* to population growth η. Our theory, in particular Proposition 2, implies that a decline in η will reduces both z and $\frac{\eta}{\eta - z}$.

[11]We have not found an analytic expression for $N_t/\mathcal{F}_t$. However, it is straightforward to calculate. Let $\nu(n) = \frac{\omega_t(n)}{N_t}$

2.5 Decreasing Returns in the Entry Technology

So far we assumed that entry is subject to constant returns at the aggregate level. We now discuss which of our results hinge on this assumption.

Assume that the productivity of entrant labor hired to produce new ideas for research is given by

$$\varphi_E(z_t) = \tilde{\varphi}_E z_t^{-\chi} \text{ where } \chi \geq 0. \tag{16}$$

Here, z_t is the aggregate entry rate that each entrant takes as given. For $\chi = 0$, this specification yields the constant returns to case analyzed above. For $\chi > 0$, the cost of entry rises with the aggregate entry rate. We refer to χ as the strength of congestion.

Under (16), free entry requires that

$$V_t^{Entry} = \alpha \frac{V_t^{CD}}{w_t} + (1-\alpha)\frac{V_t^{NV}}{w_t} = \frac{1}{\varphi_E(z_t)} = \frac{1}{\tilde{\varphi}_E} z_t^{\chi}. \tag{17}$$

Hence, to the extent that there is congestion, i.e. $\chi > 0$, the average value of product creation (relative to the wage) is increasing in the aggregate entry rate. Alternatively, the aggregate entry supply curve is increasing in the value of entry V_t^{Entry} with an elasticity $1/\chi$. For our baseline case of $\chi = 0$, entry is infinitely elastic.

Irrespective of the entry technology, it is still the case that the rate of product creation is equal to the rate of population growth η. This directly implies that two important results of Proposition 2 still apply: the rate of creative destruction is still given by $\tau = \frac{\alpha}{1-\alpha}\eta$ (see (11)) and both the aggregate growth rate g_y and the rate of efficiency growth g^Q are still given in (12).

In contrast, the composition of the rate of creative destruction into the entry flow z and incumbents' rate of product creation x, depends on the strength of congestion χ. Note that the policy function of incumbents (6) and the congestion-adjusted free entry condition in (17) imply that

$$\tau = \alpha(z+x) = \alpha\left(z + \left(\frac{\varphi_x}{\zeta}\right)^{\frac{1}{\zeta-1}}\left(\alpha\frac{V_t^{CD}}{w_t} + (1-\alpha)\frac{V_t^{NV}}{w_t}\right)^{\frac{1}{\zeta-1}}\right) = \alpha\left(z + \left(\frac{\varphi_x}{\zeta\tilde{\varphi}_E}\right)^{\frac{1}{\zeta-1}} z^{\frac{\chi}{\zeta-1}}\right).$$

Using $\tau = \frac{\alpha}{1-\alpha}\eta$, this implies that the product entry flow z is uniquely determined from the equation

$$\frac{\eta}{1-\alpha} = z + \left(\frac{\varphi_x}{\zeta\tilde{\varphi}_E}\right)^{\frac{1}{\zeta-1}} z^{\frac{\chi}{\zeta-1}}.$$

denote the share of firms with n products. As we show in Section B.1.5 in the Appendix, $\nu(n)$ is given by

$$\nu(n+1) = \begin{cases} \left(2\frac{\alpha\eta}{1-\alpha}\right)^{-1}\left(\nu(1)\left(\frac{\eta}{1-\alpha}+x\right)-z\right) & \text{if } n=1 \\ \left((n+1)\frac{\alpha\eta}{1-\alpha}\right)^{-1}\left(\nu(n)\,n\left(\frac{\alpha\eta}{1-\alpha}+x\right)+\nu(n)\,\eta-\nu(n-1)(n-1)\,x\right) & \text{if } n>2 \end{cases}.$$

Together with the consistency condition $\sum_{n=1}^{\infty}\nu(n)\,n = 1$, these equations fully determine $\{\nu(n)\}_n$ as a function of parameters. Then, $N_t/\mathcal{F}_t = \left(\sum_{n=1}^{\infty}\nu(n)\right)^{-1}$. For the case of $\eta = 0$, the solution is the same as in Klette and Kortum (2004).

It easy to see that z is declining in η. Given z, the rate of incumbent product creation is given by

$$x = \left(\frac{\varphi_x}{\zeta \tilde{\varphi}_E}\right)^{\frac{1}{\zeta-1}} z^{\frac{\chi}{\zeta-1}}.$$

For the case of no congestion, $\chi = 0$, the solution is exactly as in Proposition 2 and x does not depend on population growth. If $\chi > 0$, x is increasing in z and hence also declining in population growth.

Whether changes in population growth affect entrants or incumbents relatively more depends on the congestion elasticity χ relative to the convexity of the cost function ζ. In particular, it is easy to show that entrants respond more changes in population growth if the entry cost elasticity χ is smaller than the incumbent cost elasticity $\zeta - 1$. Formally,

$$\frac{\partial z/x}{\partial \eta} > 0 \text{ if and only if } \chi < \zeta - 1.$$

Hence, qualitatively, all the results derived above hold true as long as $\chi < \zeta - 1$. The case of $\chi = 0$ makes the "entry dependence" particularly salient.

2.6 Discussion of the Mechanism: Supply or Demand?

So far we have characterized some of the implications of population growth for firm dynamics. The primary mechanism is simple: the growth rate of the number of products is tied to growth rate of production labor, and so a slowdown in the the long-run growth rate of the latter implies a slowdown in the former. The fact that this slowdown is absorbed (mainly) by entrants drives the results above.

However, less clear at first glance are the economic forces driving this mechanism. Our theory is a closed economy model, where a decline in population growth lowers both the growth rate of the labor force and the growth rate of the mass of consumers. This naturally raises the question if our mechanisms operates through tighter supply of workers , or lower growth in demand for goods.

To see that our mechanism is about labor supply, let aggregate spending in the economy be given by S_t. S_t does not necessarily have to equal domestic income, but could be determined by growing demand from abroad. We can write total profits per product as

$$\pi_t(q) = (\mu - 1)\left(\frac{q}{Q_t}\right)^{\sigma-1} \frac{S_t}{N_t},$$

where S_t/N_t is average sales per product. Letting g^S denote the growth rate of spending, the value function is given by

$$V_t(q) = \frac{\pi_t(q)}{\rho + g^W + \tau - g^S + g^N + (\sigma - 1)(g - I)} + \frac{\frac{\zeta-1}{\varphi_x} x^{\zeta} w_t}{\rho + \tau}$$

Hence, the free entry condition requires that

$$\frac{1}{\varphi_E} = \frac{(\mu - 1)\left(\alpha\lambda^{\sigma-1} + (1-\alpha)\,\overline{\omega}^{\sigma-1}\right)}{\rho + g^W + \tau - g^S + g^N + (\sigma - 1)(g - I)} \frac{S_t}{N_t w_t} + \frac{\frac{\zeta-1}{\varphi_x}\left(\frac{1}{\zeta}\frac{\varphi_x}{\varphi_E}\right)^{\frac{\zeta}{\zeta-1}}}{\rho + \tau},$$

so that free entry requires that ratio of average sales per product, S_t/N_t, relative to the wage w_t has to be constant. If markups are constant, the wage payments to production workers are a proportional to total profits, i.e.

$$\frac{1}{\mu - 1} = \frac{w_t L_t^P}{N_t \int \pi_t(q)\, dF_t(q)} = \frac{w_t L_t^P}{(\mu - 1) S_t}.$$

From this we can see that changes in the rate of growth of spending S_t can change the *number* of products N_t on the balanced growth path, but not their long-run growth rate. This remains firmly tied to the growth of production labor L_t^P, which in the long-run must be equal to the growth rate of the labor force L_t. Spending growth and the entry rate are unconnected in the long-run. If for a given growth rate in spending, growth in the labor force slows down, wages must rise faster to compensate and keep the share of aggregate production going to workers constant. This makes entry more expensive, and keeps the number of products growing at the same rate as the number of workers.

3 Extensions for the Quantitative Model

So far we assumed that markups were constant and equal to the standard CES markup. We now generalize our model to our model with endogenous markups by assuming that firms compete *a la* Bertrand within product lines. Doing so allows us to study the effects of declining population growth on market power.

Given the CES structure of demand, each firm would like to charge a markup of $\frac{\sigma}{\sigma-1}$ over marginal cost. However, the presence of competing firms within their product line implies that the most efficient producer might have to resort to limit pricing. If they are unable to price at the optimal markup without inviting competition, they will set their price equal to the marginal cost of the next most efficient producer of that good, who is then indifferent between producing or not. The markup charged in product i, μ_i, is thus given by

$$\mu_i = \min\left\{\frac{\sigma}{\sigma - 1}, \frac{q_i}{q_i^C}\right\} \equiv \min\left\{\frac{\sigma}{\sigma - 1}, \Delta_i\right\}, \tag{18}$$

where q_i denotes the productivity of current producer in product i, q_i^C is the productivity of the most efficient competitor and $\Delta_i \equiv q_i/q_i^C$ is the firm's productivity advantage relative to it competitors (we also refer to this as the "gap"). Equation (18) highlights that markups are rising in the gap Δ.

The static equilibrium allocations generalize in a straight-forward way and the aggregate allocations given (2) take a similar form: aggregate output and equilibrium wages are now given by

$$Y_t = \mathcal{M}_t Q_t N_t^{\frac{1}{\sigma-1}} L_t^P \text{ and } w_t = \Lambda_t Y_t / L_t^P = \Lambda_t \mathcal{M}_t Q_t N_t^{\frac{1}{\sigma-1}},$$

where

$$\mathcal{M}_t = \frac{\left(\int \mu^{1-\sigma} \left(q/Q_t\right)^{\sigma-1} dF_t\left(q,\mu\right)\right)^{\frac{\sigma}{\sigma-1}}}{\int \mu^{-\sigma} \left(q/Q_t\right)^{\sigma-1} dF_t\left(q,\mu\right)} \quad \text{and} \quad \Lambda_t = \frac{\int \mu^{-\sigma} \left(q/Q_t\right)^{\sigma-1} dF_t\left(q,\mu\right)}{\int \mu^{1-\sigma} \left(q/Q_t\right)^{\sigma-1} dF_t\left(q,\mu\right)}. \tag{19}$$

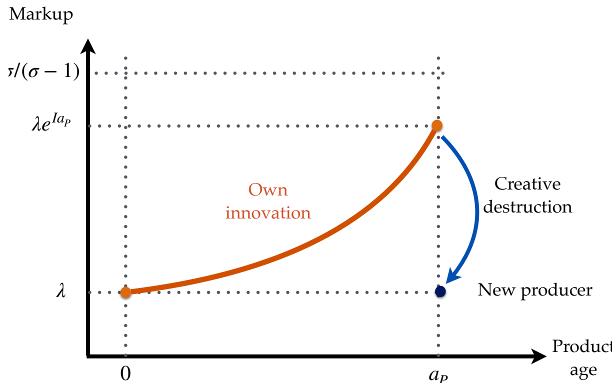
and now $F_t(q, \mu)$ denotes the joint distribution of productivity and markups. The two aggregate statistics $\mathcal{M}_t$ and Λ_t fully summarize the static macroeconomic consequences of monopoly power. Market power reduces both production efficiency (the misallocation term $\mathcal{M}_t$) and lowers factor prices relative to their social marginal product (the labor share Λ_t). In particular, a common increase in markups reduces the labor wedge Λ_t but keeps the allocation efficiency $\mathcal{M}_t$ unchanged. The latter is affected by the dispersion of markups. Because our model generates the joint distribution distribution of markups and efficiency $F_t(q, \mu)$ endogenously and this distribution is a function of the rate of population growth, a decline in the rate of population growth affects allocative efficiency via $\mathcal{M}_t$ and has distributional consequences through Λ_t.

Perhaps more surprisingly, the dynamic implications are very similar to our baseline model. While the value function is more involved, we show in Section A.1 in the Appendix that we can still derive an analytic expression. More importantly, all the results of Proposition 2 *exactly* hold in the model with Bertrand competition, i.e. the equilibrium rate of creative destruction τ, the entry rate z and the rate of incumbent expansion x are still given by (11). Hence, our findings that lower population growth increases concentration and shifts the age distribution towards older firms directly carries over to the environment with Bertrand competition.

To see why these results have important implications for the equilibrium distribution of markups, note that our model implies a crucial difference between productivity growth due to creative destruction and own-innovation. Suppose the current producer of product i has an efficiency gap of Δ_i. If this firm is replaced by another producer, the productivity gaps *reduces* to λ as the new firm is only a single step ahead of the previous producer, reducing the markup of that product. In contrast, if the existing firm successfully increases its productivity through own-innovation, the efficiency gap and hence the markup increase at rate I (as long as $\Delta_i \leq \frac{\sigma}{\sigma-1}$). Hence, own-innovation is akin to a positive drift for the evolution of markups, while creative destruction is similar to a "reset" shock, which lowers markups and keeps the accumulation of market power in check.

This process is displayed in Figure 5. When a firm adds a product to its portfolio, the initial markup is λ. Conditional on survival, markups increase at rate I. A faster rate of creative destruction lowers the expected time a given firm produces a particular product and limits the opportunities for incumbent firms to accumulate market power.

Figure 5: The Life-Cycle of Product Markups



Notes: This figure shows a stylized example of how markups evolve at the product level, When a firm takes over a product, markups increase through own-innovation. Once the product is lost to another firm, markups are reset to the baseline level of λ.

The stochastic process shown in Figure 5 gives rise a stationary distribution of quality gaps Δ and hence markups. Newly created varieties do not face any competitor and hence charge a markup of $\frac{\sigma}{\sigma-1}$. Products that have been creatively destroyed at some point in the past are subject to Bertrand competition and the markup depends on Δ. Let N_t^{NC} denote the mass of products without any competitor and N_t^C be the mass of products that are subject to competition. Consistency requires that $N_t = N_t^{NC} + N_t^C$.

In Section B.2.4 in the Appendix we prove two results. First, we show that, along a BGP, the share of product without any competitor is given by

$$N_t^{NC}/N_t = 1-\alpha,$$

i.e. it is simply given by the share of product creation that results in new varieties (rather than creative destruction).[12] Second, the distribution of quality gaps among products with a competitor is given by

$$F^C(\Delta) = 1 - \left(\frac{\lambda}{\Delta}\right)^{\frac{\tau+\eta}{I}}, \tag{20}$$

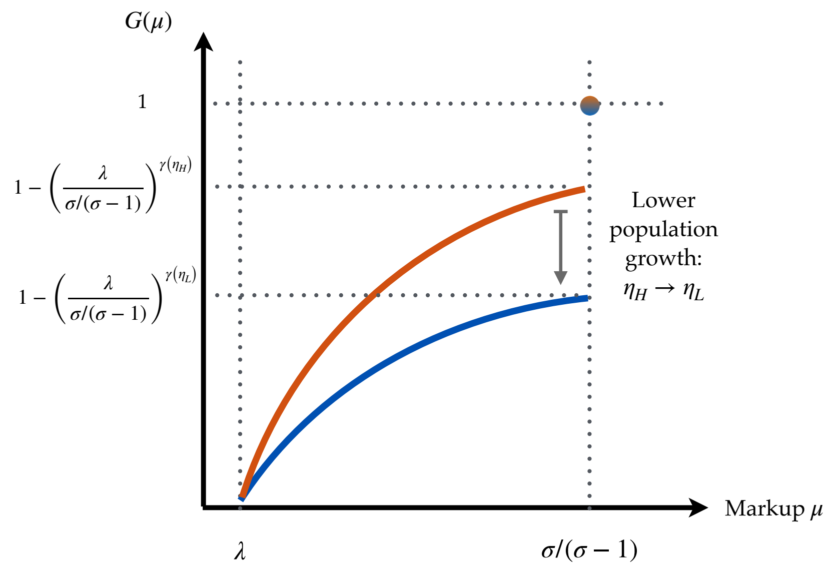
i.e. the marginal distribution of quality gaps is a Pareto distribution with tail parameter of $\frac{\tau+\eta}{I}$. As such, slower population growth increases the equilibrium distribution of efficiency gaps in a first-order stochastic dominance sense. First of all, slower population (and hence product) growth

[12] Let $N_t^{NC}(a)$ the number of products without a competitor that have been around for a years at time t. Because $(1-\alpha)(z+x)N_t e^{-\eta a}$ such products entered at time $t-a$ and receive a competitor at the rate of creative destruction τ, $N_t^{NC}(a) = (1-\alpha)(z+x)N_t e^{-(\eta+\tau)a}$. Hence,

$$N_t^{NC} = \int_{a=0}^{\infty} N_t^{NC}(a)\,da = N_t\left(\frac{1-\alpha}{\alpha}\right)\left(\frac{\tau}{\tau+\eta}\right) = (1-\alpha)N_t,$$

because $\tau = \frac{1-\alpha}{\alpha}\eta$.

Figure 6: Population Growth and Market Power



shifts the product distribution towards old products. More interestingly, because slower population growth also reduces creative destruction, this effect is amplified, i.e. the average product age is increasing even for a given cohort of firms. In addition, lower population growth also increases the dispersion in efficiency gaps.

To see intuitively, why our model gives rise to a Pareto distribution as in (20), let a_P denote the age of a product, i.e the time a given product has been produced by a given firm. Because the quality gap Δ increases at rate I as long as the firm is not replaced, $\Delta(a_P) = e^{Ia_P}$. Second, because the current producer of a given product gets replaced at rate τ and the number of products increases at rate η, the distribution of a_P is simply$P(a_P \leq a) = 1 - e^{-(\eta+\tau)a}$. Hence, $P\left(e^{Ia_P} < \Delta\right) = 1 - e^{-\left(\frac{\eta+\tau}{I}\right)\ln\Delta}$,which is (20).

To translate the distribution of gaps into the distribution of markups, recall that $\mu(\Delta) = \min\left\{\frac{\sigma}{\sigma-1}, \Delta\right\}$. Hence, for the case where markups are below the "unconstrained", monopolistically competitive markup $\frac{\sigma}{\sigma-1}$, the distribution of markups is a truncated Pareto. Among products without a competitor, the markup is given by $\frac{\sigma}{\sigma-1}$. Hence, the cross-sectional distribution of markups across products is given by

$$G(\mu) = \begin{cases} \alpha F^C(\mu) & \mu < \frac{\sigma}{\sigma-1} \\ 1 & \mu = \frac{\sigma}{\sigma-1} \end{cases}.$$

A reduction in population growth therefore increases markups along the whole distribution and shifts more mass towards the maximum CES markup. In Figure 6 we depict how the distribution of markups changes in response to a decline in population growth from η_H to η_L.

The macroeconomic consequences of misallocation are summarized by $\mathcal{M}$ and Λ, which depend on the joint distribution between quality gaps Δ and quality q. To derive this distribution, define relative efficiency $\hat{q} = \ln(q/Q_t)^{\sigma-1}$ and let $\hat{\lambda} = \ln \lambda^{\sigma-1}$. Denote $F_t^C(\Delta, \hat{q})$ as the joint distribution of quality gaps and relative efficiency for products which have a next best competitor. Similarly, denote $F_t^{NC}(\hat{q})$ as the distribution of relative efficiency for products that do not have a competitor. We show in Appendix B.2.3 that these objects evolve according to laws of motion given by

$$\begin{aligned}
\frac{\partial F_t^C(\Delta,\hat{q})}{\partial t} &= \underbrace{-\frac{\partial F_t^C(\Delta,\hat{q})}{\partial \Delta} I\Delta - (\sigma-1) g_t^Q \frac{\partial F_t^C(\Delta,\hat{q})}{\partial \hat{q}}}_{\text{drift from own innovation}} \underbrace{- (\tau+\delta+\eta) F_t\left(\Delta,\hat{q}\right)}_{\text{product loss}} \\
&+ \underbrace{\lim_{s\to\infty} \tau_t F_t^C\left(s, \hat{q}-\hat{\lambda}\right)}_{\text{creative destruction of } C \text{ products}} + \underbrace{\tau_t \frac{1-\alpha}{\alpha} F_t^{NC}(\hat{q}-\hat{\lambda})}_{\text{creative destruction of } NC \text{ products}} , \\
\frac{\partial F_t^{NC}(\hat{q})}{\partial t} &= \underbrace{\frac{\partial F_t^{NC}(\hat{q})}{\partial \hat{q}} (\sigma-1) g^Q}_{\text{drift from own innovation}} - \underbrace{(\tau_t+\delta+\eta) F_t^{NC}(\hat{q})}_{\text{product loss}} + \underbrace{\frac{(1-\alpha)}{\alpha} \tau \Gamma\left(\frac{\exp(\hat{q})}{\sigma-1}\right)}_{\text{new products}} .
\end{aligned}$$

These expressions highlight the separate roles of own innovation and creative destruction in influencing the evolution of efficiency and markups. Own innovation causes both the production efficiency and the gap to drift upwards at the deterministic rate I, while creative destruction "resets" the mass in the distribution above Δ to have a gap of λ. Note too that there is a one-way flow of products from the non-competitive mass to the competitive through creative destruction events, while the entrant distribution Γ only directly affects the non-competitive mass.

Though these distributions do not have a closed form solution on the BGP, they can easily be computed. And given $F^C(\Delta, \hat{q})$ and $F^{NC}(\Delta, \hat{q})$, the economy-wide joint distribution is given by

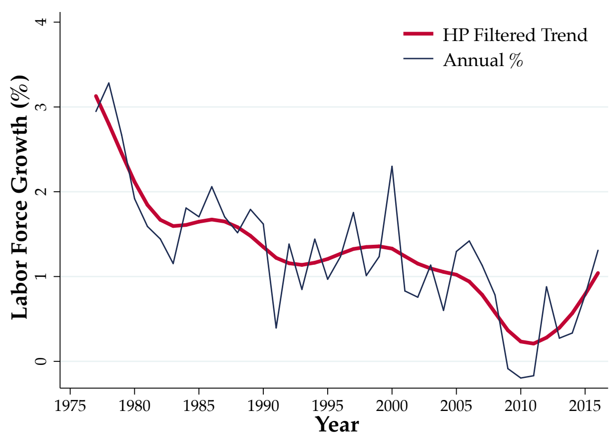
$$F(\Delta, \hat{q}) = (1-\alpha) F^C(\Delta, \hat{q}) + \alpha F^{NC}(\Delta, \hat{q}),$$

because α is exactly the steady-state fraction of products that have a competitor. Given $F(\Delta, \hat{q})$ we can then quantify the aggregate consequences of market power. Because higher markups reduce the labor share Λ and more dispersed markups reduce allocative efficiency $\mathcal{M}$, lower population growth tends to increase profits relative factor payments and has adverse effects on static allocation efficiency. Below we quantify the strength of these forces and solve for the joint distribution $F_t^C(\Delta, \hat{q})$ computationally.

4 Quantifying the Effects of Lower Population Growth

To quantify the importance of the decline in population growth we now calibrate our model to data from the US. The US experienced a sustained decline in the growth rate of the labor force, which we display in Figure 7. Our exercise to quantify the aggregate impact of this decline is conceptually simple. We parametrize the model to a balanced growth path matching key moments of the data from 1980, when labor force growth was approximately 2%. In Section 5 we then study the aggregate impact of population growth by reducing the population growth rate by 1%, the magnitude of the decline observed until 2008, and trace out the long run implications.

Figure 7: Labor Force Growth in the US: 1980 - 2010



Notes: The figure shows the growth rate of the labor force in the U.S., with the raw series in blue and a HP-filtered trend component in red. The data is sourced from the BLS, accessed through FRED.

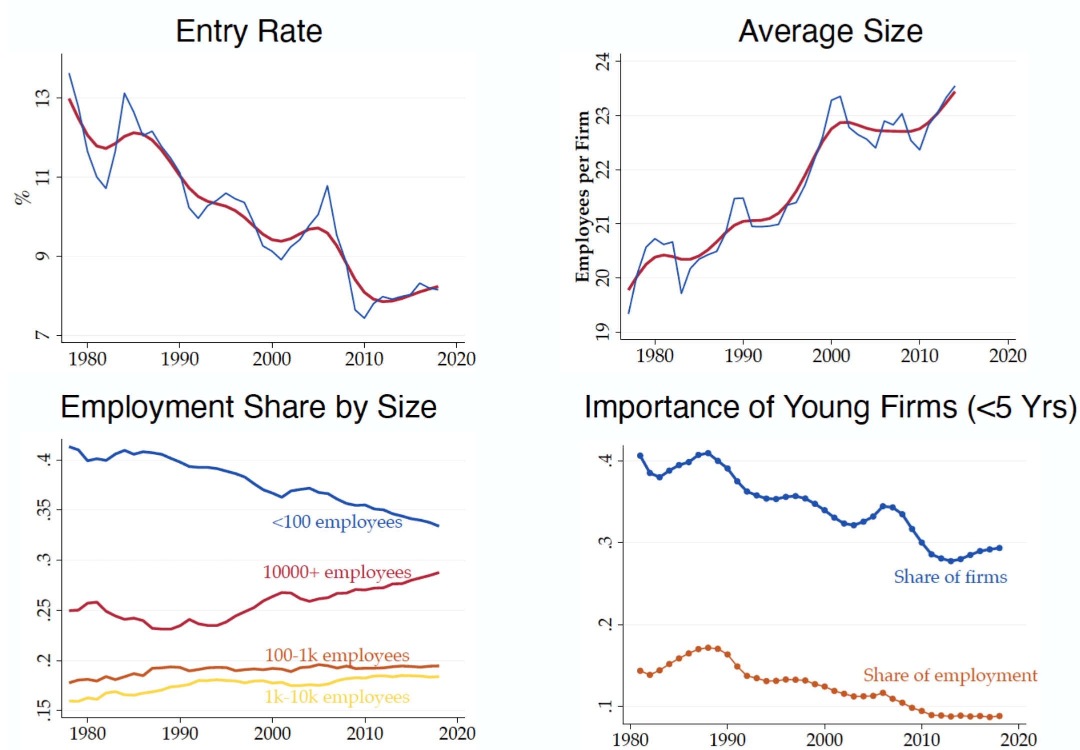
4.1 Data

Our main dataset is the the U.S. Census Longitudinal Business Database (LBD). The LBD is an administrative dataset containing information on the universe of employer establishments since 1978. It contains information on the age, industry, employment and payroll of each establishment, along with identifiers at the firm level that allows us to track the ownership of each establishment over time. We define the age of the firm in the LBD as the age of the oldest establishment that the firm owns. The birth of a new firm requires both a new firm ID in the Census and a new establishment record. We also modify the Census firm ID's to deal with some issues involving multi-establishment firms in the same way as developed in Walsh (2019).

In Figure 8 we display the aggregate evolution of key statistics of our dataset. The entry rate (shown in the upper left panel) declined markedly in the last 30 years from around 12% in the 1980s to around 8% in the mid 2000s. Note that this series of the entry rate tracks the evolution of population growth shown in Figure 7 very closely, and indeed the contemporaneous correlation is 0.74.[13] At the same time, average firm size (shown in the upper right panel) rose from 20 to 23 employees, i.e. increased

[13] Karahan et al. (2016) and Hathaway and Litan (2014) study this link directly in the geographic cross-section, showing that states with slower labor force growth, as predicted by lagged birth rates in previous decades, see lower rates of firm entry.

Figure 8: Firm Dynamics and Concentration in the US: 1980 - 2010



Notes: The figure shows the time-series of the firm entry rate, average firm size, the aggregate employment shares of firms of different size and and the share of firms and employment of young firms, i.e. firms that are younger than 5 years.

by around 15%. The two bottom panels depict two aspects of the rise in concentration. The aggregate employment share of small firms (i.e. firms with less than 100 employees) declined markedly. This decline was mainly absorbed by very large firms, i.e. firms with more than 10,000 employees. At least part of this reallocation is driven by changes in the age structure of firms with young firms experiencing a decline in economic importance: as see in the bottom-right panel both their share in aggregate employment in the number of firms declined since the mid 1980s. All of these are patterns are *qualitatively* consistent implications of a decline in population growth. Below we show that this mechanism is also *quantitatively* important to explain the patterns in Figure 8.

To measure firms' markups, we also require information on sales. We augment the LBD data with information on firm revenue from administrative data contained in the Census' Business Register, following Moreira (2015) and Haltiwanger et al. (2016). The Business Register is the master list of establishments and firms for the U.S. Census and we are able to match approximately 70% of the records to the LBD. In Section B.2 in the Appendix we describe this process in more detail and compare our matched sample to the LBD data.

4.2 Calibration

The model is parsimoniously parametrized and rests on 11 parameters:

$$\Psi = \left\{ \underbrace{\alpha, \zeta, \varphi_E, \varphi_x, I, \bar{\omega}}_{\text{Innovation \& Entry technology}}, \underbrace{\delta}_{\text{Exog. exit}}, \underbrace{\eta}_{\text{Pop. growth}}, \underbrace{\lambda}_{\text{Stepsize}} \underbrace{\rho, \sigma}_{\text{Preferences}} \right\}.$$

Three of them - the discount rate ρ, the demand elasticity σ and the convexity of the innovation cost function ζ - we set exogenously. We fix the elasticity of substitution between products σ at 4, following Garcia-Macia et al. (2016), set the discount rate ρ to 0.95 and assume a quadratic innovation cost function (i.e. $\zeta = 2$) as in Acemoglu et al. (2012).

The rate of labor force growth η is directly observed in the data and is our key parameter for the comparative statics. The remaining seven parameters are calibrated internally. We target key moments from the cross-sectional firm-size distribution in 1980 and observed life-cycle dynamics of markups and sales.[14] We are able to match these moments with arbitrary precision. Building a quantitatively accurate picture of the dynamic evolution of sales, employment and markups at the firm-level is crucial to credibly quantify the consequences of declining population growth. In Table 1 we report the parameters and the main moments we target.

While all moments are targeted simultaneously, there is nevertheless a tight mapping between particular moments and particular parameters which highlights how the different parameters are identified.

Innovation efficiency of incumbent firms: I and φ_x We identify the relative efficiency of the different sources of innovation from two dynamic moments: the life-cycle profile of sales and the life-cycle of markups. Because markup growth is driven by incumbents' own-innovation activities (see Figure 5), this moment is informative about the rate of efficiency improvement I. Sales growth is in addition also affected by the rate of incumbent product creation, which depends directly on the cost of product expansion φ_x.

As we show in detail in Section B.3.4 in the Appendix, we can derive the two life-cycle moments of sales and markups (essentially) explicitly. This is not only convenient from a quantitative standpoint but also clarifies our identification strategy. The main insight to derive these moments is to first express markups and sales of a given product as a function of the product age a_P. Average relative sales as a function of a product age a_P are then given by

$$s_P(a_P) \equiv E\left[\frac{p_i y_i}{Y}\middle| a_p\right] = E\left[\mu_i^{1-\sigma}\left(\frac{q_i}{Q_t}\right)^{\sigma-1}\middle| a_p\right] = \mu(a_p)^{1-\sigma} e^{(\sigma-1)(I-g^Q)a_p}\left(\alpha\lambda^{\sigma-1} + (1-\alpha)\bar{\omega}^{\sigma-1}\right),$$

[14] The LBD data does not contain direct information on products. Argente et al. (2019) use data from Nielsen to provide direct evidence on the process of life-cycle growth at the product-level. Akcigit et al. (2021) analyze a related model and show that their model, calibrated to employment data, replicates the product-level distribution well. Cao et al. (2017) identify products (in the theory) with plants (in the data). For an early analysis of product-level data, see Bernard et al. (2011).

Table 1: Model Parameters

	Structural Parameters Description	Value	Moments	Data	Model
η	Labor force growth in 1980	0.02	Data from BLS	2%	2%
λ	Step size on quality ladder	1.11	Aggregate poductivity growth	2%	2%
I	Rate of own innovation	0.023	Markup growth by age 10 (RevLBD)	10.2%	10.2%
φ_X	Cost of inc. product creation	0.04	Sales growth by age 10 (RevLBD)	58%	58%
φ_E	Cost of entry	0.12	Avg. firm size (BDS)	20.7	20.7
δ	Destruction rate of products	0.06	Entry Rate in 1980 (BDS)	11.6 %	11.6 %
α	Share of creative destruction	0.59	Markup of entrants	-	18 %
$\bar{\omega}$	Relative efficiency of new products	0.09	Pareto tail of LBD employment distribution in 1980	1.1	1.1
ζ	Curvature of innovation cost	2	Set exogenously		
σ	Demand elasticity	4	Set exogenously		
ρ	Discount rate	0.05	Set exogenously		

Note: This table reports the calibrated parameters for the full model. Data for the firm lifecycle comes from the U.S. Census Longitudinal Database, augmented with revenues from tax-information using the Census' Business Register. Data for average firm size and the firm entry rate in 1980 are taken from the public use Business Dynamics Statistics.

where $\mu\left(a_P\right) = \min\left\{\frac{\sigma}{\sigma-1}, \Delta\left(a_P\right)\right\} = \min\left\{\frac{\sigma}{\sigma-1}, e^{Ia_P}\right\}$ and the remaining terms are the average relative quality . Because own-quality q increases at rate I while average quality Q increases at the g^Q, $e^{(\sigma-1)\left(I-g^Q\right)a_p}$ is the relative drift of these random variables. The last term $(\alpha\lambda^{\sigma-1} + (1-\alpha)\bar{\omega}^{\sigma-1})$ reflects that the initial average quality when the firm adds the product to its portfolio.

With this expression for relative product sales $s\left(a_P\right)$ in hand, we can calculate the life-cycle of sales and markups at the firm-level. In particular, average sales and markups as a function of firm age a_f are given by[15]

$$\begin{aligned} s_f\left(a_f\right) &= E\left[\left.\sum_{n=1}^{N_f} s_P\left(a_P\right)\right| a_f\right] \\ \mu_f\left(a_f\right) &= E\left[\left.\left(\sum_{i=1}^{N_f} \mu\left(a_p\right)^{-1} \frac{s_P\left(a_P\right)}{\sum_{i=1}^{N_f} s_P\left(a_P\right)}\right)^{-1}\right| a_f\right], \end{aligned}$$

where the expectations are taken with respect to the conditional distribution of N_f and a_P, conditional on a_f. Note that the conditional distribution of product age will in general depend on the age of the

[15]Note that the firm-level markup μ_f can also be expressed as a cost-weighted average of product-level markups μ_i, i.e. $\mu_f = \sum_{i=1}^{N_f} \mu_i \frac{wl_i}{\sum_{i=1}^{N_f} wl_i}$.

firm a_f, and will the conditional distribution of the number of products N. As we show in Section B.3.4 in the Appendix, we can calculate these conditional distributions of product age a_P and the number of products N_f given firm age a_f essentially explicitly. We can therefore calculate $s_f\left(a_f\right)$ and $\mu_f\left(a_f\right)$ without having to simulate the model.

As we have shown in Proposition 2, if incumbent firms face low costs of expansion, i.e. φ_x is high, incumbent innovation x is high *relative* to creative destruction τ. This implies that older firms have on average more but younger products. On net, the first extensive margin effect dominates making the sales life-cycle an increasing function of φ_x. Markups, in contrast, are directly affected by the rate of own innovation I. The higher I, the steeper the markup-age relationship. In particular, we show in Section B.3.4 in the Appendix that neither the distribution of product age a_P, nor the distribution of the number of products N is a function of I. Hence, $\mu_f\left(a_f\right)$ is only a function of I via $\mu\left(a_P\right)$.

Empirically, we measure markups at the firm level by the inverse labor share. In other words, we measure the markup of firm f as

$$\mu_f = \frac{py_f}{wl_f}, \tag{21}$$

where py_f is the total revenue of the firm, and wl_f is the total wage bill. We calculate the total wage bill by aggregating establishment payroll. Our theory implies that this average markup is given by $\mu_f = \sum_i \mu_i \frac{l_i}{\sum_i^{N_f} l_i}$, i.e. firms' markups are an average of the product-level markup μ_i weighted by the employment (or cost) shares (see also Edmond et al. (2018)).

While this allows us in principle to measure markups for the population of U.S. firms, we only use firms' markup *growth* to calibrate our model. More specifically, letting $\mu_{f,t}$ be the mark-up of firm f at time t, we run a regression of the form

$$\ln \mu_{f,t} = \sum_{a=0}^{20} \gamma_a^{\mu} \mathbb{I}_{Age_{ft}=a} + \theta_f + \theta_t + \epsilon_{f,t}, \tag{22}$$

where $\mathbb{I}_{Age_{ft}=a}$ is an indicator for whether the firm is of age a and θ_f and θ_t are firm and time fixed effects respectively. Hence, γ_a^{μ} provides a non-parametric estimate of the rate of markup growth. We calibrate our model to the growth rate at the 10-year horizon, γ_{10}^{μ}. By focusing on (22) we control for a firm fixed effect when measuring properties of firms' markups and hence to not have to take a stand on firms' output elasticities as long as they are constant with age.[16] We follow the same approach when we estimate the life-cycle of sales, i.e. we also estimate (22) using log sales as the dependent variable and target γ_{10}^{py} in our quantitative model. In the LBD, firms increase their average markup by roughly ten percentage points and grow in size by about 60%.

[16] If, for example, firms within sectors had different production functions with different output elasticities, neither the level nor the dispersion of markups as measured from (21) could be distinguished from such differences in technology (see De Loecker and Warzynski (2012) and Peters (2018)). Also, by targeting markup growth, we avoid estimating output elasticities for labor, which is not feasible with the data we have as it does not contain data on capital of material inputs. Doing so would also complicate the mapping from model to data, since in our model labor is the only factor of production.

Entry Costs and product loss: φ_E and δ We choose φ_E and δ to jointly match the entry rate and average firm size. The free condition determines market size L_t^P/N_t as a function of entry efficiency φ_E,. This in turn is a key component of average firm employment. We thus choose φ_E to match an average firm employment of 20.76 in 1980 from the BDS. The higher the entry efficiency, the lower market size and the smaller the average size of firms.

The exogenous rate of product loss δ directly influences the exit and hence - in a BGP - the entry rate of firms. While there is no closed form expression for the entry rate as a function of δ, we can solve for it computationally from expressions derived in Appendix B.1.5. We target the entry rate in 1980 of 11.6%.

The creative destruction step size and the efficiency of new products: λ and $\overline{\omega}$ The parameters λ and $\overline{\omega}$ determine the relative quality of creatively destroyed products and newly generated varieties. We infer these parameters from the aggregate rate of growth and the tail of the firm size distribution. That λ and $\overline{\omega}$ directly affect the growth rate is apparent from Proposition 2.

To see that they are also important determinants of the size distribution, recall that there are two ways for firms to be very large in our theory: through having many products, or having a particularly good product that employs many workers in its production. Hence, the firm size distribution is determined by the marginal distribution of product quality. Denote this distribution by $H(\hat{q})$. We show in Appendix B.1.6 that this distribution solves a differential equation given by

$$\frac{dH\left(\hat{q}\right)}{dq}(\sigma-1)(g^Q-I)=(\delta+\eta+\tau)H\left(\hat{q}\right)-\tau H(\hat{q}-(\sigma-1)log(\lambda))-\frac{1-\alpha}{\alpha}\tau\Gamma\left(exp\left(\frac{\hat{q}}{\sigma-1}\right)\right)$$

As long as the entrant efficiency distribution Γ has a thin tail, the solution to this differential equation has a Pareto tail, which we denote by κ. This tail parameter is implicitly defined by

$$\kappa\left(\left(\frac{1-\alpha}{\alpha}\right)\tau\left(\bar{\omega}-1\right)+\tau\left(\lambda^{\sigma-1}-1\right)\right)=-(\delta+\eta+\tau)+\tau e^{\kappa(\sigma-1)\ln\lambda}, \tag{23}$$

and hence depends on λ and $\bar{\omega}$. As $\lambda\to 1$, κ approaches

$$\kappa=-\frac{\delta+\eta}{\left(\frac{1-\alpha}{\alpha}\right)\tau\left(\bar{\omega}-1\right)}=\frac{1}{1-\bar{\omega}},$$

i.e. if entrants are relative unproductive, i.e. $\bar{\omega}$ is small, the product efficiency distribution has thick tails.

For our calibration we chose λ and $\bar{\omega}$ to target a rate of productivity growth of 2% and a tail parameter of the firm size distribution of 1.1, i.e. close to Zipf law. Quantitively, we find in our calibration that the condition in Section 2.3 for a fat tail in the product distribution does not hold so that the employment tail is indeed governed by κ given in (23).

New varieties vs. creative destruction: α

As shown in Figure 6, the share of new products in innovation, $1-\alpha$, plays an important role for the level of markups in the economy. Since a new entrant either begins life with a single new product (in which case the markup is $\frac{\sigma}{\sigma-1}$), or steals a single product with a productivity gap of unity (and hence a markup of λ), the average markup of entering firms is given by

$$E\left[\mu_f(0)\right] = \alpha\lambda + (1-\alpha)\frac{\sigma}{\sigma-1}.$$

Given λ and σ we can directly infer α from this moment. We target an economy-wide profit share of 25%. This implies that entrants begin charge an average markup of 19%.

4.3 Estimates and Non-Targeted Moments

As seen in Table 1, our model is able to match the targeted moments perfectly. To match the fact that markups grow by around ten percentage points at the ten year horizon, our model implies a rate of own-innovation of around 2.3%. In terms of creative destruction we estimate, we estimate a productivity increase of 11%. This is required to match an annual growth rate of 2%. The initial quality of new products is estimated to be low - they are about 10% as productive as the average product in the economy. This relative low value is required to match the thickness of the tail of the employment distribution.

In addition to the targeted moments, our model, despite it parsimonious parametrization, also matches a variety of additional non-targeted moments. Consider first the sales and markup life cycle. In Figure 9 we show the model's performance against our targets of sales and markups growth by age by plotting the estimated coefficients γ_a^{μ} and γ_a^{py} from specification (22) estimated in the model and in the data. As highlighted in Table 1, we calibrate our model to match γ_{10}^{py} and γ_{10}^{μ}, i.e. average firm size and the average markup of firms of age 10 relative to entrants.

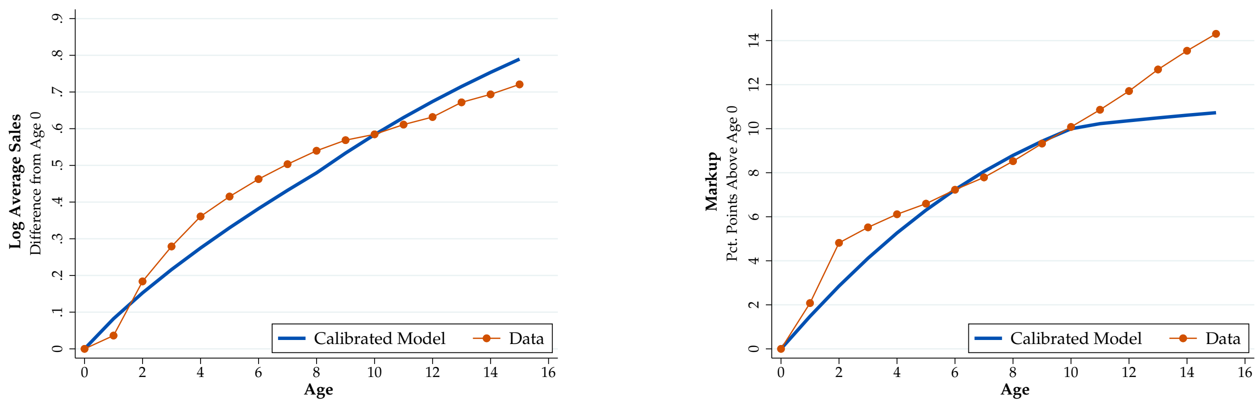
In Figure 9 we display the lifecycle of sales (left panel) and markups (right panel) both in the data and in the model. Even though the model is only calibrated to match the growth from birth to age 10, Figure 9 shows that the whole age profile of sales and markups is quite close in the model to what is observed in the data.

For the case of sales, the model replicates the slight concavity of log sales well. In the model, this shape reflects survivorship bias; small firms either grow or are destroyed, while large firms can have products stolen and shrink without exiting. As such, average growth conditional on survival is declining with age for young firms before, eventually, becoming log-linear for large old firms, matching Gibrat's law. Quantitatively, firms in the US grow their sales by about 60 log points during their first 10 years.

The fit for markups in Panel (b) is also relatively good, even though in the data markups appear somewhat more linear with age than emerge from the model. Empirically, markups are increasing almost linearly by 1% each year. In the model, the rate of markup growth is much more concave, reflecting the fact that markups are bounded from above by $\frac{\sigma}{\sigma-1}$.

Figure 9: Lifecycle Growth in Firm Sales and Markups

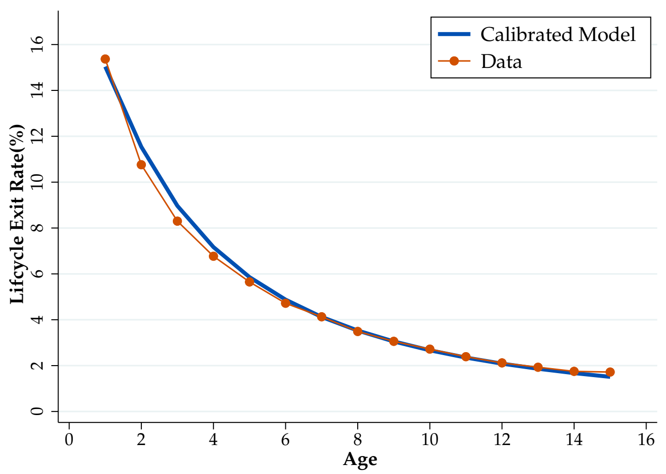
(a) Sales Growth By Age (b) Markup Growth By Age



Note: Panel (a) in this Figure compares the lifecycle of firm sales in the model to the estimated lifecycle in the data. The data lifecycle plots the age coefficients from estimating equation (22) in the LBD. $N = \{35,300,000\}$, where this number has been rounded to accord with Census Bureau disclosure rules. The lifecycle of sales in the calibrated model is computed by simulating a panel of 10^6 firms , and averaging sales within age groups. Panel (b) does the same for relative markups.

Our model also makes precise predictions for the exit rate by age, which should be declining. These declining exit rates by age reflect the fact that older firms have more product lines. Since the risk of product line destruction is independent across products, owning more products makes it progressively less likely that they will all be destroyed within a particular year. In Figure 10 we compare the model's predictions for exit rates by age to the data. To construct exit rates by age, we estimate a non-parametric Kaplan-Meier survival function by age for firms in the LBD. We select the cohort of firms born between 1980 and 1990, and follow them until 2015. We then take the exit rates to be the increments of the estimated survival functions. Each estimate is essentially the fraction of the sample that exits at age a (though the estimator accounts for the truncation from ceasing to observe firms after 2015). Figure 10 shows that our model is remarkably successful to replicate theses exit rates, despite the fact that we do not target them in the estimation.

Figure 10: Firm Exit Rates: Model and Data

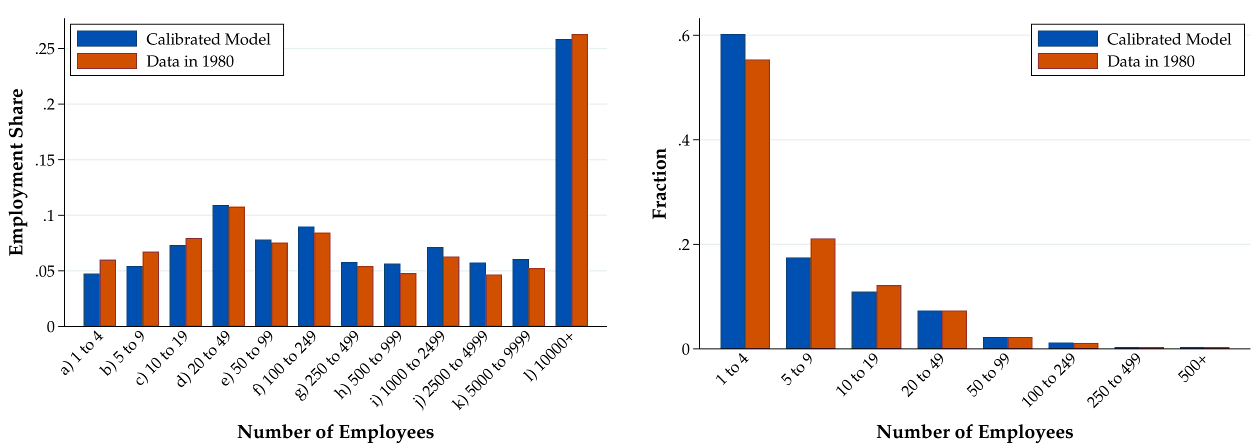


Note: This figure presents a comparison of lifecycle exit rates between model and data. The exit rates in the data are taken from the increments in a Kaplan-Meier survival function estimated on all firms in the LBD born between 1980 and 1990. The model exit rates come from simulating a panel of 10^6 firms and calculating the fraction of the panel that exit at yearly frequencies. Age of a on the horizontal axis indicates that the firm exited between age $a-1$ and age a. $N = 70,000,000$, where this count has been rounded to accord with U.S. Census disclosure rules.

Figure 11: Size Distribution in Model and Data

(a) Employment Size Distribution

(b) Firm Counts in Size Bins



Notes: Panel (a) of this figure plots the employment shares by firm size in the calibrated model (blue bars) and the data (orange bars). Panel (b) shows the shares of the firm counts in model and data. The data is from the BDS release of 1980.

We can also compare our model's predictions for the size distribution with the data. While we have explicitly targeted average size and the Pareto tail, our model also matches the full non-parametric firm size and employment distribution very well.[17] In Figure 11 we plot the distribution of employ-

[17]Calculating the full employment size distribution in the model requires solving for the joint distribution of quality

Table 2: Sources of Growth

	Entrants	Incumbents	Total
New Varieties	0.1	0.5	0.7
Creative Destruction	0.3	1.2	1.5
New Product Efficiency	-0.5	-2.1	-2.7
Own Innovation	0	2.5	2.5
Total	-0.1	2.1	2

Note: The table reports a decomposition of the growth rate. The columns distinguish between entrants and incumbents. The rows decompose the growth rate into the four components: pure variety gains $\left(\frac{1}{\sigma-1}g_N\right)$, creative destruction $\left(\frac{\lambda^{\sigma-1}-1}{\sigma-1}\tau\right)$, own-innovation (I) and the efficiency of new products $\left(\frac{\overline{\omega}-1}{\sigma-1}g_N\right)$.

ment (left panel) and the number of firms (right panel) for both the model and the data in 1980.[18]

Finally, we can decompose the aggregate growth rate into its different components. This decomposition is contained in Table 2. Two interesting patterns emerge. First of all, new varieties impact product efficiency growth negatively. The reason is that we estimate the quality of new products to be substantially smaller than average quality, i.e. $\overline{\omega} < 1$. Second, because entrants - by construction - do not engage in own-innovation, their direct contribution to growth is small. In fact, in our calibration it is slightly negative. Hence, most growth is accounted for by incumbent firms and is due to a combination of own-innovation and creative destruction, with own-innovation accounting for most of it. This, of course, does not mean that the economy would be better off without entering firms, because entering firms turn into incumbents who engage in own-innovation on their products.

5 The Aggregate Impact of a Decline in Population Growth

While the patterns on the changes in concentration in the US shown in Figure 8 are qualitatively consistent with our theory, we now examine the implications implications of a 1% slow down in labor force growth quantitatively by comparing BGP's and holding all other parameters constant. We focus both on the positive and normative aspects of our theory. On the positive side we focus on changes in the process of firm-dynamics, in particular the entry rate, average firm size, measures of concentration, the distribution of markups and firms' lifecycle growth. On the normative side we quantify the effect of the observed population growth decline on the economy-wide growth g^y and the static increase in the variety intensity N_t/L_t.

and efficiency $F^C(\Delta, \hat{q})$ for an individual product, which we plot in Figure B-3. Then, since the gap Δ is a deterministic function of age, this distribution is then integrated over the product age by firm age distribution for a single product. Multiple products are incorporated via a recursive convolution. Finally we integrate over the firm age distribution. More detail on this procedure is provided in Appendix B.4.1.

[18] For replicability we chose size bins that are also available in the publicly available data from the BDS.

Figure 12: Population Growth and Firm Dynamics

Note: The figure shows the prediction for the BGP in the calibrated model for a decline in the population growth rate η from 1.5% to 0.5%. All other calibrated parameters are held constant. The figure shows the entry rate (top left), average firm size (top right), the share of young firms (bottom left) and the employment share of young firms (bottom right). We define young firms as firms with less than 5 years old.

5.1 Positive Implications: Population Growth and Firm Dynamics

In Figure 12 we display the effect of a 1% decline in population growth on the entry rate, average firm size and the aggregate importance of young firms, both in terms of the share of firms and their share of employment. We always display the actual data from Figure 8, our calibrated model (in dark blue) and the long-run counterfactual in orange.

In the top row we show that our model can explain the entirety of the decline in the entry rate between 1980 and 2010. In terms of firm size, our model predicts too strong an increase in firm size, at least if we interpret the data in 2010 as the new BGP. Note that we used both the entry rate and average size in 1980 as a calibration target and hence match these numbers by construction.

In the bottom row we show on snapshot of the joint distribution of age and size by focusing on the importance of young firms that we define as being younger then 5 years old. We did not use any information in our calibration, hence both the "old" BGP, that was calibrated to data from 1980, and the counterfactual BGP due to changes in population growth are non-targeted. The left panel shows that our model matches both the level and change of the share of young firms very well. The right panel shows that our model overestimates the employment share of young firms. Given that we match the share of young firms, our model implies that young firms are slightly larger than observed in the data. However, our model does a decent job in explaining the change in the employment share of young firm due to a decline in population growth.

Population Growth and Markups. In Figure 13 we report the aggregate implications of the slowdown of population growth on markups. We show the model's prediction for the distribution of gaps, which translates into markups below truncation at the maximum markup of $\sigma/(\sigma-1)$. As implied by our theoretical results, the decline in population growth increases the average markup. Moreover, there is more mass on the maximum markup of $\sigma/(\sigma-1)$, which in our calibration is 33%. This reflects the fact that more products see enough innovation without being destroyed to reach the maximum markup incumbents would like to charge on the product.

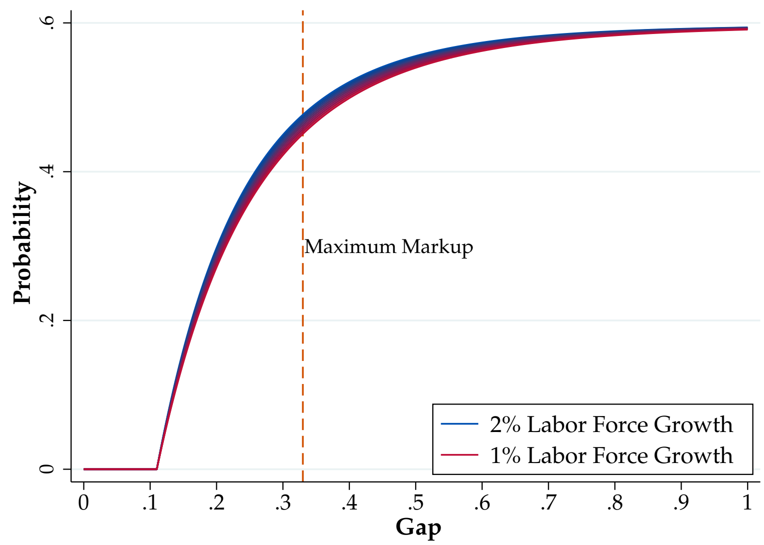
The average product markup increases by almost 1%. In terms of the firm lifecycle, this is an almost entirely across firm phenomenon, as firms on average become older. Within firms, products tend to become older for a given firm as the labor force slows and there is less creative destruction, since products are destroyed less frequently. On its own, this would tend to raise average markups. However, firms tend to accumulate more products, and as we show in Figure B-4, the average product age of firms with more products is lower. Quantitatively, these two forces almost exactly offset one another, and all the action in occurs in the shift in the firm age distribution.

5.2 Normative Implications: Population Growth and Aggregate Productivity

Finally, we turn to the implications for aggregate productivity, which we summarize in Table 3. The first three rows report the aggregate growth rate and its composition between variety gains and efficiency growth. A decline in population growth reduces the long-run equilibrium growth rate from 2% to 1.8%. Furthermore, this decline stems almost entirely from falling variety growth. In fact, efficiency growth rises slightly in response to the decline in population growth. The reason is that we estimate the quality of new varieties $\overline{\omega}$ to be relatively low. The declining rate of variety creation therefore impacts average efficiency growth positively.

In the remaining two rows we report the equilibrium allocation of labor and the long-run variety intensity, which is constant along a BGP. First note that the share of production workers declines in response to population growth. Hence, the share of researchers increases. Falling population growth therefore causes growth to decline and the share of resources devoted to R&D to increase. Qualitatively, this is in line with the US experience, which experienced falling productivity growth and rising research employment. Second, note that falling population growth increases the level of productivity by increasing the variety intensity of the economy.

Figure 13: Model Counterfactual for Markups



Note: This Figure shows the markup distribution over competitive products as the population growth rate falls from 2% to 1%. The total mass of products who have direct competitors is 0.59 and equal to α.

Table 3: Population Growth and Economic Growth

	Growth			Production labor	Variety intensity
	g^y	Variety (N_t)	Efficiency (Q_t)	L_t^P/L_t	N_t/L_t
Calibrated Baseline	0.02	0.007	0.013	0.91	1
Declining pop. growth	0.018	0.003	0.014	0.9	1.102

Note: The table reports the aggregate growth rate (g^y), the growth stemming from variety gains $\left(\frac{1}{\sigma-1}g_N\right)$ and efficiency growth $\left(g^Q\right)$, the share of workers employed in researcher $\left(L^P/L\right)$ and the variety intensity (N_t/L_t). The first row refers to the calibrated model. The second row to the counterfactual where we reduce population growth by 1%. We normalize N_t/L_t to 1 in the calibrated model.

6 Conclusion

Most countries have seen declining rates of fertility and slowdowns in population growth in recent decades. We have proposed a general framework for understanding the interaction of population growth, market concentration and economic welfare. Quantitatively, this framework predicts significant changes in response to slowing population growth, including rising markups and slowdowns in creative destruction. Future work should consider the policy implications of these changes. Misallocation rises as population growth falls, and the urgency of anti-competitive policy may become increasingly acute in a world of slow growth.

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Appendix A: Theoretical Results

A Model Extensions

A.1 Endogenous own-innovation

Suppose now that firms can chose the rate of own-innovation I subject to some costs. In particular, assume that the cost function (in terms of labor) of achieving a drift I of a particular product is given by

$$c\left(I;q/Q\right)=\left(\frac{q}{Q_t}\right)^{\sigma-1}\frac{1}{\varphi_I}I^{\zeta}.$$

Hence, the cost of innovation are convex in I (for simplicity we assume the same convexity as for firms' product creation technology). Additionally, the cost of innovation depend on firms' relative efficiency q/Q_t. Allowing for this cost-shifter is required to make the model consistent with balanced growth (see e.g. Atkeson and Burstein (2010)) and Gibrat's law for large firms.

Most results of the baseline model generalize in a straightforward way. In particular, Proposition 2 is exactly the same in this more general framework, except I in the expression for the growth rate is no longer a parameter but a choice variable. The characterization of the value function contained in Proposition 1 is also strikingly similar. The value function is still additive across products and the value of a given product with efficiency q is given by

$$V_t\left(q\right)=\frac{\pi_t\left(q\right)}{\rho+\tau+\left(g^Q-\frac{\zeta-1}{\zeta}I\right)\left(\sigma-1\right)}+\frac{\frac{\zeta-1}{\varphi_x}x^{\zeta}w_t}{\rho+\tau}. \tag{A-1}$$

Hence, the only difference to the baseline model is the term $\frac{\zeta-1}{\zeta}$ in front of I in the discount rate. Given $V_t^I\left(q\right)$ the optimal rate of own-innovation is therefore defined by

$$\max_I\left\{I\frac{\partial V_t\left(q\right)}{\partial q}q-\left(\frac{q}{Q_t}\right)^{\sigma-1}\frac{1}{\varphi_I}I^{\zeta}w_t\right\}. \tag{A-2}$$

Using (A-1), the optimal innovation rate associated with (A-2) is given by

$$I=\left(\frac{\left(\sigma-1\right)\left(\mu-1\right)\ell}{\rho+\tau+\left(g^Q-\frac{\zeta-1}{\zeta}I\right)\left(\sigma-1\right)}\frac{\varphi_I}{\zeta}\right)^{\frac{1}{\zeta-1}}, \tag{A-3}$$

where again $\ell=L_t^P/N_t$. Hence, the optimally chosen drift is indeed independent of the efficiency q and constant in a BGP. Importantly, because ℓ, τ and g_Q depend on the rate of population growth η, I also changes when population growth declines.

To see how I depends on the rate of population growth η, note that the free entry condition now implies

$$\frac{1}{\varphi_E} = \frac{(\mu-1)\left(\alpha\lambda^{\sigma-1} + (1-\alpha)\,\overline{\omega}^{\sigma-1}\right)\ell}{\rho+\tau+\left(g^Q - \frac{\zeta-1}{\zeta}I\right)(\sigma-1)} + \frac{\frac{\zeta-1}{\varphi_x}\left(\frac{1}{\zeta}\frac{\varphi_x}{\varphi_E}\right)^{\frac{\sigma}{\zeta-1}}}{\rho+\tau}.$$

Hence,

$$I = \varsigma\left(1 - \frac{\left(\frac{\zeta-1}{\zeta}\right)\left(\frac{1}{\zeta}\frac{\varphi_x}{\varphi_E}\right)^{\frac{1}{\zeta-1}}}{\rho+\tau}\right)^{\frac{1}{\zeta-1}}, \tag{A-4}$$

where ς is a collection of structural parameters.[1] Importantly, this expresses the optimal rate of own-innovation I directly as a function of parameters and a single endogenous variable - the rate of creative destruction. In particular, I only depends on the rate of population growth through τ. And because I is increasing the rate of creative destruction, a decline in population growth *reduces* firms' own innovation incentives.

The fact that I is increasing in the rate of creative destruction might at first seem surprising. After all, a higher rate of creative destruction reduces the expected life-span, which should reduce firms' incentives to invest in productivity improvements. To see that this intuition is correct, consider the (A-3): holding market size ℓ and the rate of efficiency growth g_Q constant, an increase in τ indeed *lowers I*. However, once one realizes that all these objects are linked through the free entry condition, the general equilibrium effect of a higher rate of creative destruction becomes positive. Economically: free entry requires the average production value *plus* the innovation value to be equal to the entry costs. A higher rate of creative destruction *lowers* the innovation value. Hence, for the free entry condition to be satisfied, the production value has to *increase.* This increase is achieved through an increase in market size ℓ. And as the returns to own-innovation scale with the production value but not the innovation value, the returns to own-innovation are higher in an environment with higher creative destruction. Conversely, lower population growth lowers own-innovation. This endogenous response of incumbents' own-innovation efforts amplifies the negative consequences of population growth.

In Appendix 3 we provide the full solution when own-innovation is endogenous *and* markups are determined by Bertrand competition. In this case, the optimal rate of own-innovation I varies across firms. In particular, we show that it is given by a function $I(\Delta)$, which is declining and satisfies $I(\Delta) = I$ for $\Delta \geq \frac{\sigma}{\sigma-1}$. Hence, similar to model of neck-to-neck competition (see e.g. Aghion et al. (2001), Acemoglu and Akcigit (2012) or Celik et al. (2020)), firms have a higher incentive to innovate if competition limit prices are binding because own-innovation increases both the quantity sold *and* the markups they can charge. Once the efficiency gap exceeds the CES markup and the limit price

[1] In particular, $\varsigma = \left(\frac{\sigma-1}{\alpha\lambda^{\sigma-1}+(1-\alpha)\omega^{\sigma-1}}\frac{\varphi_I}{\zeta}\right)^{\frac{1}{\zeta-1}}\frac{1}{\varphi_E}$.

becomes binding, the second effect disappears and the optimal rate of own-innovation is constant as in the constant-markup version of Section A.1.[2]

Second, in contrast the baseline model, the equilibrium depends on the joint distribution between efficiency q and efficiency gaps Δ (see 19). In Section A.1 in the Appendix we provide a characterization of this joint distribution. The exact form of the marginal distribution of gaps $v(\Delta)$ depends on the initial efficiency gap of newly created varieties. If this initial efficiency gap is equal to λ (as it is for the creatively destroyed products), the stationary distribution of efficiency gaps is the Pareto distribution given in (20). If the initial gap is drawn from a general distribution $H(\Delta)$, the stationary distribution of efficiency gaps is determined from a differential equation that we provide in the Appendix below.

A.2 Endogenous choice of production creation direction α

In the baseline model in we assume that innovation was undirected, i.e. the share of product innovation resulting in creative destruction (rather than new varieties) was constant and equal to α. In this section we show that we can extend our theory to a setting where the direction of innovation is a choice variable of the firm.

A.2.1 Incumbent Innovation and the Value Function

Suppose that the firm can chose the flow of new varieties x_N and creative destruction x_{CD}. The value function is then given by

$$r_t V_t(q) - \dot{V}_t(q) = \pi_t(q) + I\frac{\partial V_t(q)}{\partial q}q - \tau_t V_t(q) + \Xi_t$$

where

$$\Xi_t \equiv \max_{x_N}\left\{x_N V_t^N - \frac{1}{\varphi_N}x_N^{\zeta}w_t\right\} + \max_{x_{CD}}\left\{x_{CD}V_t^{CD} - \frac{1}{\varphi_{CD}}x_{CD}^{\zeta}w_t\right\}, \tag{A-5}$$

where φ_{CD} and φ_N parametrize the efficiency of creative destruction and new variety creation and V_t^N and V_t^N denote the value of creative destruction and new variety creation respectively. Along the BGP, the solution of $V_t(q)$ is given by

$$V_t(q) = \frac{(\mu-1)}{\rho+(g_N-\eta)+(g_Q-I)(\sigma-1)+\tau}\left(\frac{q}{Q_t}\right)^{\sigma-1}\frac{L_t^P}{N_t}w_t + \frac{\Xi_t}{r+\tau-g_{\Xi_t}}.$$

[2] The aggregate rate of efficiency growth in Proposition 2 therefore depends on the average rate of own-innovation and is given by

$$g_Q = \frac{1}{\sigma-1}\left(\frac{\alpha\left(\lambda^{\sigma-1}-1\right)}{1-\alpha}+\overline{\omega}^{\sigma-1}-1\right)\eta + I^* \text{ where } I^* = \int_{\Delta=\lambda}^{\infty} I(\Delta)\,dG(\Delta),$$

where $G(\Delta)$ is the distribution of efficiency gaps (which is stationary along a BGP).

Optimal Innovation and the Value of Innovation The optimal innovation rates associated with (A-5) are given by

$$x_N = \left(\frac{\varphi_N}{\zeta}\frac{V_t^N}{w_t}\right)^{\frac{1}{\zeta-1}} \text{ and } x_{CD} = \left(\frac{\varphi_{CD}}{\zeta}\frac{V_t^{CD}}{w_t}\right)^{\frac{1}{\zeta-1}}. \tag{A-6}$$

Note that this implies that the endogenous share of product creation directed to creative destruction is given by

$$\tilde{\alpha} = \frac{\left(\varphi_{CD}\frac{V_t^{CD}}{w_t}\right)^{\frac{1}{\zeta-1}}}{\left(\varphi_N\frac{V_t^N}{w_t}\right)^{\frac{1}{\zeta-1}} + \left(\varphi_{CD}\frac{V_t^{CD}}{w_t}\right)^{\frac{1}{\zeta-1}}},$$

i.e. the relative "bias" of innovation depends on the relative valuations. This also implies that

$$\Xi_t = \left(\frac{\zeta-1}{\varphi_{NV}}x_{NV}^{\zeta} + \frac{\zeta-1}{\varphi_{CD}}x_{CD}^{\zeta}\right)w_t, \tag{A-7}$$

where x_{NV} and x_{CD} are constant (see below). Hence, the value of product creation grows at rate w_t, i.e.

$$g_{\Xi_t} = g_w = r - \rho.$$

Similarly, along the BGP we have $g_N = \eta$. Hence,

$$V_t(q) = \frac{(\mu-1)}{\rho + (g_Q - I)(\sigma-1) + \tau}\left(\frac{q}{Q_t}\right)^{\sigma-1}\frac{L_t^P}{N_t}w_t + \frac{\Xi_t}{\rho+\tau},$$

where Ξ_t is given in (A-7).

V_t^{CD} and V_t^N The value of creative destruction is given by

$$\begin{aligned} V_t^{CD} = \int V(\lambda q)\,dF_t(q) &= \frac{(\mu-1)\lambda^{\sigma-1}}{\rho + (g_Q - I)(\sigma-1) + \tau}\frac{L_t^P}{N_t}w_t + \frac{\Xi_t}{\rho+\tau} \\ &= \left(\frac{(\mu-1)\lambda^{\sigma-1}}{\rho + (g_Q - I)(\sigma-1) + \tau}\frac{L_t^P}{N_t} + \frac{\frac{\zeta-1}{\varphi_{NV}}x_{NV}^{\zeta} + \frac{\zeta-1}{\varphi_{CD}}x_{CD}^{\zeta}}{\rho+\tau}\right)w_t. \end{aligned}$$

Hence, upon substituting for x_{NV} and x_{CD},

$$\frac{V^{CD}}{w_t} = \frac{(\mu-1)\lambda^{\sigma-1}}{\rho + (g_Q - I)(\sigma-1) + \tau}\frac{L_t^P}{N_t} + \frac{1}{\rho+\tau}\frac{\zeta-1}{\zeta^{\frac{\zeta}{\zeta-1}}}\left(\left(\varphi_{NV}^{1/\zeta}\frac{V_t^{NV}}{w_t}\right)^{\frac{\zeta}{\zeta-1}} + \left(\varphi_{CD}^{1/\zeta}\frac{V_t^{CD}}{w_t}\right)^{\frac{\zeta}{\zeta-1}}\right) \tag{A-8}$$

Similarly, the value of new variety creation is given by

$$\frac{V^{NV}}{w_t} = \frac{V_t(\omega Q_t)}{w_t} = \frac{(\mu-1)\,\omega^{\sigma-1}}{\rho+(g_Q-I)(\sigma-1)+\tau}\frac{L_t^P}{N_t} + \frac{\Xi_t/w_t}{\rho+\tau} \tag{A-9}$$

$$= \frac{(\mu-1)\,\omega^{\sigma-1}}{\rho+(g_Q-I)(\sigma-1)+\tau}\frac{L_t^P}{N_t} + \frac{1}{\rho+\tau}\frac{\zeta-1}{\zeta^{\frac{\zeta}{\zeta-1}}}\left(\left(\varphi_{NV}^{1/\zeta}\frac{V_t^{NV}}{w_t}\right)^{\frac{\zeta}{\zeta-1}} + \left(\varphi_{CD}^{1/\zeta}\frac{V_t^{CD}}{w_t}\right)^{\frac{\zeta}{\zeta-1}}\right) \tag{A-10}$$

Note that

$$\frac{V_t^{CD}}{w_t} - \frac{V_t^{NV}}{w_t} = \frac{(\mu-1)\left(\lambda^{\sigma-1}-\omega^{\sigma-1}\right)}{\rho+(g_Q-I)(\sigma-1)+\tau}\frac{L_t^P}{N_t}.$$

Hence, generically we have that $\frac{V_t^{CD}}{w_t}$ and $\frac{V_t^{NV}}{w_t}$ are different. In the empirically plausible case where $\lambda^{\sigma-1} > 1 > \omega^{\sigma-1}$, we have that

$$\frac{V_t^{CD}}{w_t} > \frac{V_t^{NV}}{w_t}.$$

A.2.2 Entry

Suppose entrants can either come up with new varieties or creatively destroy existing varieties. Let the entry flow be z_{NV} and z_{CD} respectively. Suppose the entry technology is linear in labor with productivities

$$\varphi_E\varphi_{NV}z_{NV}^{-\chi} \text{ and } \varphi_E\varphi_{CD}z_{CD}^{-\chi}.$$

Free entry requires

$$\frac{V_t^j}{w_t} \le \frac{1}{\varphi_E\varphi_j}z_j^{\chi} \text{ with equality if } z_j > 0.$$

Hence, in an equilibrium with entry,

$$\frac{V_t^{NV}}{w_t} = \frac{z_{NV}^{\chi}}{\varphi_E\varphi_{NV}} \text{ and } \frac{V_t^{CD}}{w_t} = \frac{z_{CD}^{\chi}}{\varphi_E\varphi_{CD}}. \tag{A-11}$$

Note that if $\chi = 0$ and if there was entry in both "sectors", we would need that $\frac{1}{\varphi_E} = \varphi^{NV}\frac{V^{NV}}{w_t} = \varphi^{CD}\frac{V^{CD}}{w_t}$. Note that this will (generically) never be the case.

A.2.3 BGP equilibrium

From (A-8) and (A-9) we have

$$\frac{V^{NV}}{w_t} = \frac{(\mu-1)\,\omega^{\sigma-1}}{\rho+(g_Q-I)(\sigma-1)+\tau}\ell^P + \frac{1}{\rho+\tau}\frac{\zeta-1}{\zeta^{\frac{\zeta}{\zeta-1}}}\left(\left(\varphi_{NV}^{1/\zeta}\frac{V_t^{NV}}{w_t}\right)^{\frac{\zeta}{\zeta-1}} + \left(\varphi_{CD}^{1/\zeta}\frac{V_t^{CD}}{w_t}\right)^{\frac{\zeta}{\zeta-1}}\right)$$

$$\frac{V^{CD}}{w_t} = \frac{(\mu-1)\,\lambda^{\sigma-1}}{\rho+(g_Q-I)(\sigma-1)+\tau}\ell^P + \frac{1}{\rho+\tau}\frac{\zeta-1}{\zeta^{\frac{\zeta}{\zeta-1}}}\left(\left(\varphi_{NV}^{1/\zeta}\frac{V_t^{NV}}{w_t}\right)^{\frac{\zeta}{\zeta-1}} + \left(\varphi_{CD}^{1/\zeta}\frac{V_t^{CD}}{w_t}\right)^{\frac{\zeta}{\zeta-1}}\right)$$

Substituting the free entry condition in (A-11)

$$\frac{z_{NV}^{\chi}}{\varphi_E\varphi_{NV}} = \frac{(\mu-1)\,\omega^{\sigma-1}}{\rho+(g_Q-I)(\sigma-1)+\tau}\ell^P + \frac{1}{\rho+\tau}\frac{\zeta-1}{\zeta^{\frac{\zeta}{\zeta-1}}}\left(\frac{1}{\varphi_{NV}}\left(\frac{z_{NV}^{\chi}}{\varphi_E}\right)^{\frac{\zeta}{\zeta-1}} + \frac{1}{\varphi_{CD}}\left(\frac{z_{CD}^{\chi}}{\varphi_E}\right)^{\frac{\zeta}{\zeta-1}}\right)$$

$$\frac{z_{CD}^{\chi}}{\varphi_E\varphi_{CD}} = \frac{(\mu-1)\,\lambda^{\sigma-1}}{\rho+(g_Q-I)(\sigma-1)+\tau}\ell^P + \frac{1}{\rho+\tau}\frac{\zeta-1}{\zeta^{\frac{\zeta}{\zeta-1}}}\left(\frac{1}{\varphi_{NV}}\left(\frac{z_{NV}^{\chi}}{\varphi_E}\right)^{\frac{\zeta}{\zeta-1}} + \frac{1}{\varphi_{CD}}\left(\frac{z_{CD}^{\chi}}{\varphi_E}\right)^{\frac{\zeta}{\zeta-1}}\right)$$

These are two equations in 4 unknowns: $\left(z_{CD}, z_{NV}, (g_Q-I)(\sigma-1)+\tau, \ell^P\right)$. Hence, we need two additional equations:

1. From $g_N = \eta$ we have

$$\eta = g_N = z_{NV} + x_{NV} = z_{NV} + \left(\frac{\varphi_N}{\zeta}\frac{V_t^N}{w_t}\right)^{\frac{1}{\zeta-1}} = z_{NV} + \left(\frac{1}{\zeta}\frac{1}{\varphi_E}\right)^{\frac{1}{\zeta-1}} z_{NV}^{\frac{\chi}{\zeta-1}}$$

   This equation uniquely determines z_{NV} as a function of η, i.e. $z_{NV}(\eta)$.

2. The rate of efficiency growth g_Q is given by

$$g_Q = \frac{\lambda^{\sigma-1}-1}{\sigma-1}\tau + \frac{\omega^{\sigma-1}-1}{\sigma-1}g_N + I$$

   Hence,

$$\begin{aligned}(g_Q-I)(\sigma-1)+\tau &= \left(\lambda^{\sigma-1}-1\right)\tau + \left(\omega^{\sigma-1}-1\right)g_N + \tau \\ &= \lambda^{\sigma-1}\tau + \left(\omega^{\sigma-1}-1\right)\eta\end{aligned}$$

   Now note that

$$\tau = x_{CD} + z_{CD} = \left(\frac{\varphi_{CD}}{\zeta}\frac{V_t^{CD}}{w_t}\right)^{\frac{1}{\zeta-1}} + z_{CD} = \left(\frac{1}{\zeta}\frac{1}{\varphi_E}\right)^{\frac{1}{\zeta-1}} z_{CD}^{\frac{\chi}{\zeta-1}} + z_{CD}$$

   so that

$$\tau = \tau(z_{CD}).$$

These 4 equations uniquely determine $\left(z_{CD}, z_{NV}, (g_Q-I)(\sigma-1)+\tau, \ell^P\right)$.

A.3 Imperfect substitution between research and production labor

In our baseline analysis we assumed that labor was the only factor of production and was supplied perfectly elastically between research (for new products and innovation) and for production. In this section we generalize our analysis and allow for an upward sloping sectoral supply curve.

More specifically, suppose that individuals can work either as production workers or as researchers. Letting w_t denote the wage for production workers and v_t denote the wage for researchers, we model the share of people working as researchers, s_t^R as

$$s_t^R\left(\frac{v_t}{w_t}\right)=\frac{hv_t^\theta}{w_t^\theta+hv_t^\theta}=\frac{h\left(\frac{v_t}{w_t}\right)^\theta}{1+h\left(\frac{v_t}{w_t}\right)^\theta}. \tag{A-12}$$

Similarly, total labor input in the two sectors is given by

$$\begin{aligned} L_t^P &= L_t\left(s_t^P\right)^{\frac{\theta-1}{\theta}}=L_t\left(\frac{w_t}{\left(w_t^\theta+hv_t^\theta\right)^{\frac{1}{\theta}}}\right)^{\theta-1} \\ L_t^R &= L_t h^{\frac{1}{\theta}}\left(s_t^R\right)^{\frac{\theta-1}{\theta}}=L_t\left(\frac{hv_t}{\left(w_t^\theta+hv_t^\theta\right)^{\frac{1}{\theta}}}\right)^{\theta-1}. \end{aligned} \tag{A-13}$$

Hence, θ parametrizes the labor supply elasticity and h is a parameter governing the level of human capital in research.[3] If $\theta\to\infty$ and $h=1$, labor supply is perfectly elastic as in our benchmark model and factor prices across activities are equal, i.e. $w_t=v_t$.

It turns out most of our results are unaffected by this change. First of all, note that along a BGP where the share of researchers is constant, the relative research wage v_t/w_t is constant and the number of production workers grows at the rate of population growth η. Stationarity therefore still requires that the number of varieties N_t grows at rate η. Hence, the rate of creative destruction τ, the entry rate z and the rate of incumbent innovation x are still given by the same expressions as in the text.

The only aspect of the model that changes with this addition is the effect of changes in population growth on the *number* of varieties available N_t. The number of varieties is determined from the market clearing equation for researchers (9)

$$L_t h^{\frac{1}{\theta}}\left(s^R\left(v^{BGP}\right)\right)^{\frac{\theta-1}{\theta}}=N_t\left(\frac{1}{\varphi_e}z+\frac{1}{\varphi_x}x^\zeta\right),$$

[3] The labor supply function in (A-12) can be micro-founded as the outcome of a discrete choice occupation choice model where individuals idiosyncratic preference shocks are Frechet distributed. In particular, suppose each individual draws a level of human capital in the research and production sector $\epsilon=\left(\epsilon^P,\epsilon^R\right)$. Suppose that ϵ^j are distributed according to a Frechet distribution with shape θ and location parameter h^j, i.e.

$$P\left[\epsilon^R\le x\right]=F_j\left(x\right)=e^{-h^j x^{-\theta}}.$$

Without less of generality we can normalize $h^P=1$ and let $h^R=h$ denote the human capital in research.

where $v^{BGP} = v_t/w_t$ is the relative research wage along the BGP. Using the results of Proposition 2 yields

$$\frac{N_t}{L_t} = \frac{1}{\frac{1}{\varphi_e}\left(\frac{\eta+\delta}{1-\alpha} - \left(\frac{\zeta-1}{\zeta}\right)x\right)} h^{\frac{1}{\theta}} \left(s^R\left(v^{BGP}\right)\right)^{\frac{\theta-1}{\theta}}. \tag{A-14}$$

Equation (A-14) shows that holding the share of researchers s^R fixed, a decline in population growth increases the variety intensity N_t/L_t or alternatively reduces the number of workers per variety. Intuitively, a reduction in population growth reduces the amount of entry z but keeps incumbent innovation *per product* x constant. This frees up productive resources in the research sector. Thus, holding the share of researchers fixed, the number of products N_t has to increase for the market for researchers to clear. Similarly, an increase in the human capital of research h increases the variety intensity N_t/L_t holding s^R constant.

Note also that, using (A-13), (A-14) implies that

$$\frac{L_t^P}{N_t} = \frac{1}{\varphi_e}\left(\frac{\eta+\delta}{1-\alpha} - \left(\frac{\zeta-1}{\zeta}\right)x\right)\frac{1}{h}\left(\frac{v_t}{w_t}\right)^{-(\theta-1)}. \tag{A-15}$$

Hence, the number of production workers (as measured in human capital units) per variety is decreasing in the relative wage of researchers v_t relative to the production worker wage w_t.

To determine the equilibrium number of varieties, the free entry condition can be written as

$$\frac{1}{\varphi_E} = \left\{\frac{\alpha u(\lambda)\lambda^{\sigma-1} + (1-\alpha)\int_\Delta u(\Delta)\,dG(\Delta)\int_\omega \omega^{\sigma-1}d\Gamma(\omega)}{g(\sigma-1)+\rho+\tau+\delta-\eta}\right\}\frac{1}{\mathcal{M}^{\sigma-1}\Lambda^\sigma}\frac{L_t^P}{N_t}\frac{w_t}{v_t} + \frac{\frac{\zeta-1}{\varphi_x}\left(\frac{1}{\zeta}\frac{\varphi_x}{\varphi_e}\right)^{\frac{\zeta}{\zeta-1}}}{\rho+\tau+\delta}. \tag{A-16}$$

Note that g, $\tau$and $u(\Delta)$ can be solved as a function of parameters. Hence, holding $\mathcal{M}^{\sigma-1}\Lambda^\sigma$ constant, the free entry condition (A-16) implies a positive relationship between the relative cost of research $\frac{v_t}{w_t}$ and equilibrium market size $\frac{L_t^P}{N_t}$: because a higher market size increases the profits per product, the cost of research has to go up to ensure that the free entry condition is satisfied.

Hence, holding $\mathcal{M}^{\sigma-1}\Lambda^\sigma$ constant, (A-15) and (A-16) are two equations in two unknowns and together determine the relative wage for researchers $\frac{v_t}{w_t}$ and the equilibrium market size L_t^P/N_t. In the left panel of Figure A-1 we depict the determination of the equilibrium for two cases of the research supply elasticity θ. If $\theta \approx 1$, the supply of researchers is inelastic and the equilibrium market size L_t^P/N_t is fully determined from parameters. If θ is large, the supply of researchers is elastic and a given change in relative factor prices v_t/w_t induces a large response in the number of varieties provided.

In the right panel of Figure A-1 we depict the consequences of a decline in population growth. Equation (A-15) implies that this unambiguously decreases equilibrium market size. The consequences on the free entry condition are more involved and the comparative static in principle ambiguous. In Figure A-1 we focus on the case where market size also declines. As seen from A-1, a decline in

Figure A-1: Research Supply and Variety Creation in Equilibrium

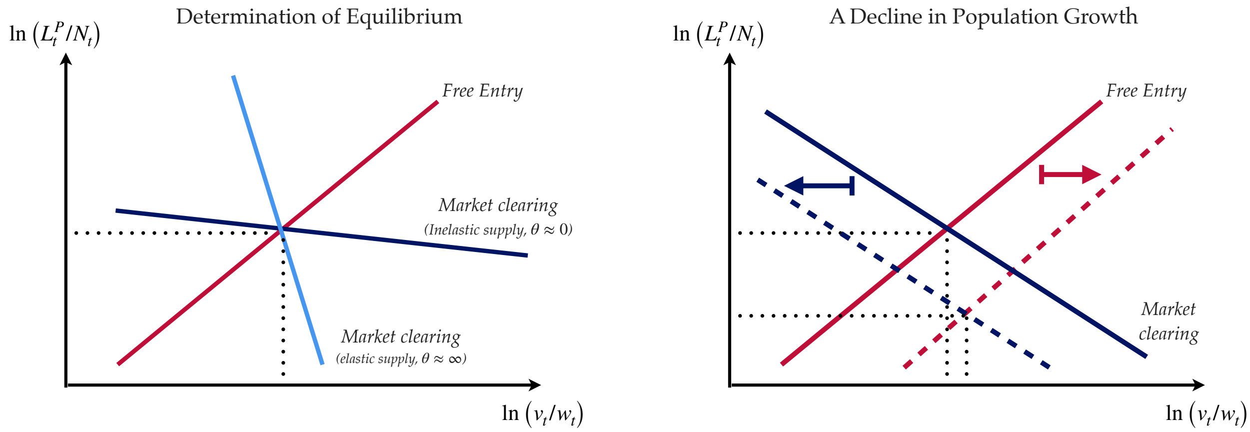

Figure A-2: Research Supply and Variety Creation

HERE: FIGURE where we plot the change in N/L when we reduce pop growth by 1 percent on y axis and θ on x axis. For $\theta \to \infty$ we should get our initial calibration back.

population growth will trigger variety creation and reduce the number of production workers per product. The effect on relative wages is ambiguous. Also note that the response of variety creation is particularly strong if the sectoral labor supply elasticity is large. In particular, lower population growth can increases the variety intensity N_t/L_t. This "level effect" of declining population growth increases welfare and is particularly strong, if the supply of researchers is very responsive to changes in factor prices. Whether declining in population growth increases or reduces welfare is therefore a horse-race between the static variety-creation effect and the dynamic growth effect.

B Theoretical Results

B.1 Characterization of the Baseline Model

This section contains the derivation of all results for the baseline model characterized in Section 2.

B.1.1 Static Equilibrium

Consider first the static equilibrium allocations, in particular (2). Letting μ_i denote the markup in product i, the equilibrium wage is given by

$$w_t = \left(\int_0^{N_t} \mu_i^{1-\sigma} q_i^{\sigma-1} di \right)^{\frac{1}{\sigma-1}} = N_t^{\frac{1}{\sigma-1}} \left(\int \mu^{1-\sigma} q^{\sigma-1} dF_t(q,\mu) \right)^{\frac{1}{\sigma-1}}. \tag{A-17}$$

Similarly, aggregate output Y_t is given by

$$Y_t = N_t^{\frac{1}{\sigma-1}} \frac{\left(\int \mu^{1-\sigma} q^{\sigma-1} dF_t(q,\mu)\right)^{\frac{\sigma}{\sigma-1}}}{\int \mu^{-\sigma} q^{\sigma-1} dF_t(q,\mu)} L_t^P. \tag{A-18}$$

Defining $Q_t = \left(\int q^{\sigma-1} dF_t(q)\right)^{\frac{1}{\sigma-1}} = \left(E\left[q^{\sigma-1}\right]\right)^{\frac{1}{\sigma-1}}$ we can write (A-18) as

$$Y_t = N_t^{\frac{1}{\sigma-1}} Q_t \mathcal{M}_t L_t^P \qquad \text{where} \qquad \mathcal{M}_t = \frac{\left(\int \mu^{1-\sigma} (q/Q_t)^{\sigma-1} dF_t(q,\mu)\right)^{\frac{\sigma}{\sigma-1}}}{\int \mu^{-\sigma} (q/Q_t)^{\sigma-1} dF_t(q,\mu)}.$$

Similarly,

$$w_t L_P = \Lambda_t Y_t \qquad \text{where} \qquad \Lambda_t = \frac{\int \mu^{-\sigma} (q/Q_t)^{\sigma-1} dF_t(q,\mu)}{\int \mu^{1-\sigma} (q/Q_t)^{\sigma-1} dF_t(q,\mu)}. \tag{A-19}$$

For the case of $\mu_i = \mu$, $\mathcal{M}_t$ and Λ_t reduce to $\mathcal{M}_t = 1$ and $\Lambda_t = 1/\mu$ as required in (2).

Product-level sales and profits are given by

$$py_i = \mu_i^{1-\sigma} \left(\frac{q_i}{Q_t}\right)^{\sigma-1} \left(\frac{1}{\mathcal{M}_t \Lambda_t}\right)^{\sigma-1} \frac{Y_t}{N_t} \tag{A-20}$$

$$\pi_i = \left(1 - \frac{1}{\mu_i}\right) \times \mu_i^{1-\sigma} \left(\frac{q_i}{Q_t}\right)^{\sigma-1} \left(\frac{1}{\mathcal{M}_t \Lambda_t}\right)^{\sigma-1} \frac{Y_t}{N_t}. \tag{A-21}$$

If markups are constant, (A-20) reduces to

$$py_i = \left(\frac{q_i}{Q_t}\right)^{\sigma-1} \frac{Y_t}{N_t} \quad \text{and} \quad \pi_i = \left(\frac{\mu-1}{\mu}\right)\left(\frac{q_i}{Q_t}\right)^{\sigma-1} \frac{Y_t}{N_t}.$$

B.1.2 Proof of Proposition 1

We first derive the value function stated in Proposition 1. Upon rewriting the innovation value $\Xi_t([q_i])$ as

$$\Xi_t([q_i]) = n \times \max_x \left\{ x\left(\alpha \int V_t([q_i], \lambda q) dF_t(q) + (1-\alpha) \int V_t([q_i], \omega Q_t) d\Gamma(\omega) - V_t([q_i])\right) - \frac{1}{\varphi_x} x^{\zeta} w_t \right\},$$

it is immediate that the value function is additive, i.e. $V_t([q_i]) = \sum_{i=1}^n V_t(q_i)$. The HJB equation associated with $V_t(q_i)$ is given by

$$rV_t(q) - \dot{V}_t(q) = \pi_t(q) + I \frac{\partial V_t(q)}{\partial q} q - \tau V_t(q) + \Xi_t, \tag{A-22}$$

where $\Xi_t = \max_x \left\{ x\left(\alpha V_t^{CD} + (1-\alpha) V_t^{NV}\right) - \frac{1}{\varphi_x} x^{\zeta} w_t \right\}$ with $V_t^{CD} = \int V_t(\lambda q) dF_t(q)$ and $V_t^{NV} = \int V_t(\omega Q_t) d\Gamma(\omega)$.

Suppose the value function takes the following forms

$$V_t(q) = q^{\sigma-1}U_t + M_t,$$

where M_t and U_t grow at some constant rates g_M and g_U respectively. Then

$$I\frac{\partial V_t(q)}{\partial q}q = I(\sigma-1)q^{\sigma-1}U_t.$$

Using that

$$\pi_t(q) = (\mu-1)\mu^{-\sigma}q^{\sigma-1}\frac{Y_t}{w_t^{\sigma-1}} = (\mu-1)\left(\frac{q}{Q_t}\right)^{\sigma-1}\frac{L_t^P}{N_t}w_t$$

(A-22) can be written as

$$(r_t+\tau-g_U)q^{\sigma-1}U_t + (r+\tau-g_M)M_t = \left((\mu-1)\left(\frac{1}{Q_t}\right)^{\sigma-1}\frac{L_t^P}{N_t}w_t + I(\sigma-1)U_t\right)q^{\sigma-1} + \Xi_t.$$

It is easy to show that this implies that

$$\begin{aligned} U_t &= \frac{(\mu-1)\left(\frac{1}{Q_t}\right)^{\sigma-1}\frac{L_t^P}{N_t}w_t}{\rho+\tau+(\sigma-1)(g_Q-I)} \\ M_t &= \frac{\Xi_t}{\rho+\tau}, \end{aligned}$$

as $\Xi_t \propto w_t$. To see this note that

$$\Xi_t = \max_x\left\{x\left(\alpha V_t^{CD} + (1-\alpha)V_t^{NV}\right) - \frac{1}{\varphi_x}x^{\zeta}w_t\right\} = \frac{\zeta-1}{\varphi_x}x^{\zeta}w_t,$$

where

$$x = \left(\frac{\varphi_x}{\zeta}\right)^{\frac{1}{\zeta-1}}\left(\alpha\frac{V_t^{CD}}{w_t} + (1-\alpha)\frac{V_t^{NV}}{w_t}\right)^{\frac{1}{\zeta-1}}. \tag{A-23}$$

The value function is therefore given by

$$\begin{aligned} V_t(q) &= \frac{(\mu-1)\left(\frac{q}{Q_t}\right)^{\sigma-1}\frac{L_t^P}{N_t}w_t}{\rho+\tau+(\sigma-1)(g_Q-I)} + \frac{\frac{\zeta-1}{\varphi_x}x^{\zeta}w_t}{\rho+\tau} \\ &= \frac{\pi_t(q)}{\rho+\tau+(\sigma-1)(g_Q-I)} + \frac{\frac{\zeta-1}{\varphi_x}x^{\zeta}w_t}{\rho+\tau}. \end{aligned}$$

Note also that

$$V_t^{CD} = \int V_t(\lambda q)\,dF_t(q) = V_t(\lambda Q_t) \text{ and } V_t^{NV} = \int V_t(\omega Q_t)\,d\Gamma(\omega) = V_t(\overline{\omega}Q_t).$$

This concludes the proof of Proposition 1.

B.1.3 Characterization of Equilibrium and BGP

In this section we characterize the full equilibrium of our economy. We maintain the assumption that the free entry condition is binding along the equilibrium path. The equilibrium is characterized by the following conditions:

1. The evolution of aggregate productivity is given by (see Section B.1.4 below)

$$\frac{\dot{Q}_t}{Q_t} = \frac{\lambda^{\sigma-1}-1}{\sigma-1}\tau_t + \frac{\overline{\omega}^{\sigma-1}-1}{\sigma-1}g_t^N + I$$

2. The rate of creative destruction and is linked to the growth rate of N_t according to

$$\tau = \frac{\alpha}{1-\alpha}g_t^N, \tag{A-24}$$

where $g_t^N = (1-\alpha)(z_t + x)$. Note that x is constant because of the binding free entry condition.

3. Labor market clearing requires $L_t = L_{Pt} + L_{Rt}$, where

$$L_{Rt} = N_t\left(\frac{1}{\varphi_E}z_t + \frac{1}{\varphi_x}x^{\zeta}\right) = N_t\frac{1}{\varphi_E}\left(z_t + \frac{1}{\zeta}x\right)$$

Hence,

$$\frac{L_t}{N_t} = \frac{L_{Pt}}{N_t} + \frac{1}{\varphi_E}\left(z_t + \frac{1}{\zeta}x\right) \tag{A-25}$$

4. The Euler equation is given by

$$r_t = \rho + g_c \tag{A-26}$$

where g_c is the growth rate of per capita consumption. Wages and output are given by $Y_t = N_t^{\frac{1}{\sigma-1}}Q_tL_t^P$ and $w_t = \frac{1}{\mu}Y_t/L_t^P$. Note that market clearing requires $C_t = Y_t$. Hence, the growth rate of per capita consumption is given by

$$g_c = g_Y - \eta = g_w + g_{L^P} - \eta, \tag{A-27}$$

where $g_w = \frac{1}{\sigma-1}g_N + g_Q$. The Euler equation in (A-26) there implies that the real interest rate is given by

$$r_t = \rho + g_w + g_{L^P} - \eta.$$

5. The free entry condition requires that

$$\frac{1}{\varphi_E} = \frac{V_t^{Entry}}{w_t} = \overline{q}\frac{U_t}{w_t} + \frac{M_t}{w_t} \tag{A-28}$$

where

$$M_t = \frac{\frac{\zeta-1}{\varphi_x} x^{\zeta} w_t}{\rho + g_w + g_{L^P} - \eta + \tau_t - g_M}$$

$$U_t = \frac{(\mu-1)\frac{L_t^P}{N_t} w_t}{\rho + g_w + g_{L^P} - \eta + \tau_t - g_U - I(\sigma-1)}$$

and

$$\bar{q} = \left(\alpha\lambda^{\sigma-1} + (1-\alpha)\,\overline{\omega}^{\sigma-1}\right)$$

and g_U and g_M are the growth rates of M_t and U_t. Now define $u_t \equiv \frac{U_t}{wt}$ and $m_t = \frac{M_t}{w_t}$. Then we can write the free entry condition as

$$\frac{1}{\varphi_E} = \bar{q} u_t + m_t$$

where

$$m_t = \frac{\frac{\zeta-1}{\varphi_x} x^{\zeta}}{\rho + g_{L^P} - \eta + \frac{\alpha}{1-\alpha} g_N - g_m} \tag{A-29}$$

$$u_t = \frac{(\mu-1)\frac{L_t^P}{N_t}}{\rho + g_{L^P} - \eta + \frac{\alpha}{1-\alpha} g_N - g_u - I(\sigma-1)} \tag{A-30}$$

Now define

$$s_t^P \equiv \frac{L_t^P}{L_t} \quad \text{and} \quad \ell_t = \frac{L_t}{N_t}.$$

Hence, the free entry condition is given by

$$\frac{1}{\varphi_E} = \bar{q}\frac{(\mu-1)\,\ell_t s_t^P}{\rho + g_{s^P} + \frac{\alpha}{1-\alpha} g_N - g_u - I(\sigma-1)} + \frac{\frac{\zeta-1}{\varphi_x} x^{\zeta}}{\rho + g_{s^P} + \frac{\alpha}{1-\alpha} g_N - g_m}.$$

Hence, the equilibrium is characterized by a path $\left\{s_t^P, \ell_t\right\}_t$ that satisfies the the free entry condition and labor market clearing

$$\frac{1}{\varphi_E} = \bar{q}\frac{(\mu-1)\,\ell_t s_t^P}{\rho + g_{s^P} + \frac{\alpha}{1-\alpha} g_N - g_u - I(\sigma-1)} + \frac{\frac{\zeta-1}{\varphi_x} x^{\zeta}}{\rho + g_{s^P} + \frac{\alpha}{1-\alpha} g_N - g_m}. \tag{A-31}$$

$$\ell_t\left(1 - s_t^P\right) = \frac{1}{\varphi_E}\left(\frac{g_N}{1-\alpha} - \frac{\zeta-1}{\zeta} x\right), \tag{A-32}$$

where g_u and g_m are growth rates of u_t and m_t given in (A-29) and (A-30).

Balanced Growth Path

Along a BGP, income per capita grows at a constant rate. (A-27) implies that

$$g_c = g_w + g_{L^P} - \eta = \frac{1}{\sigma - 1}\left(\frac{\left(\lambda^{\sigma-1} - 1\right)\alpha}{1-\alpha} + \overline{\omega}^{\sigma-1}\right) g_t^N + I + g_{s^P}.$$

Along the BGP it also has to be the case that $s^P = L_t^P / L_t$ is constant. Hence, g^N is constant along a BGP. (A-32) therefore implies that ℓ_t has to be constant, i.e.

$$\ell_t = \eta - g_N = 0.$$

Hence, along the BGP the mass of products N_t grows at the same rate as the population. With s^P and ℓ constant, (A-29) and (A-30) imply that $g_u = g_m = 0$ along the BGP. Hence, $\left(\ell, s^P\right)$ are given by

$$\begin{aligned} \frac{1}{\varphi_E} &= \bar{q}\frac{(\mu-1)\,\ell_t s_t^P}{\rho + \frac{\alpha}{1-\alpha}\eta - I(\sigma-1)} + \frac{\frac{\zeta-1}{\varphi_x}x^\zeta}{\rho + \frac{\alpha}{1-\alpha}\eta} \\ \ell_t\left(1 - s_t^P\right) &= \frac{1}{\varphi_E}\left(\frac{\eta}{1-\alpha} - \frac{\zeta-1}{\zeta}x\right). \end{aligned}$$

These equations have a unique solution for $\ell > 0$ and $s^P \in (0,1)$.

B.1.4 Proof of Proposition 2

To derive 11, note first that

$$\frac{\dot{N}_t}{N_t} = \eta = (1-\alpha)(x+z) = \frac{1-\alpha}{\alpha}\tau.$$

Hence, $\tau = \frac{\alpha}{1-\alpha}\eta$. Now note that the free entry condition in (7) and the optimality condition for x in (6) implies

$$x = \left(\frac{\varphi_x}{\zeta}\right)^{\frac{1}{\zeta-1}}\left(\alpha\frac{V_t^{CD}}{w_t} + (1-\alpha)\frac{V_t^{NV}}{w_t}\right)^{\frac{1}{\zeta-1}} = \left(\frac{1}{\zeta}\frac{\varphi_x}{\varphi_E}\right)^{\frac{1}{\zeta-1}}.$$

Hence, $z = \frac{\tau}{\alpha} - x = \frac{1}{1-\alpha}\eta - x$.

To derive the equilibrium growth rate g^y in (12), note that along a BGP $w_t \propto y_t = \frac{Y_t}{L_t} \propto N_t^{\frac{1}{\sigma-1}} Q_t$,so that

$$g^y = \frac{1}{\sigma-1}\frac{d\ln N_t}{dt} + \frac{d\ln Q_t}{dt} = \frac{1}{\sigma-1}\eta + g^Q.$$

Note that $g^Q = \frac{1}{\sigma-1} d \ln E_t q^{\sigma-1}$. Within a small interval ι, we have

$$\begin{aligned}
\mathbb{E}_t q^{\sigma-1} &= \int_q q^{\sigma-1} dF_t(q) \\
&= \underbrace{\left(1-e^{-\eta\iota}\right)\int_q q^{\sigma-1} dF_{t-\iota}(q) \int \omega^{\sigma-1} d\Gamma(\omega)}_{\text{New varieties}} \\
&\quad \underbrace{+\left(1-e^{-\tau\iota}\right)\int_q (\lambda q)^{\sigma-1} dF_{t-\iota}(q)}_{\text{Creative destruction}} \\
&\quad \underbrace{+\left(1-\left(1-e^{-\eta\iota}\right)-\left(1-e^{-\tau\iota}\right)\right)\int_q \left(e^{I\iota} q\right)^{\sigma-1} dF_{t-\iota}(q)}_{\text{Own innovation}}.
\end{aligned}$$

Taking the limit of $\iota \to 0$ and rearranging terms yields

$$d \ln E_t q^{\sigma-1} = \eta\left(\overline{\omega}^{\sigma-1}-1\right)+\tau\left(\lambda^{\sigma-1}-1\right)+(\sigma-1) I.$$

Hence, $g_Q = \eta\left(\frac{\overline{\omega}^{\sigma-1}-1}{\sigma-1}\right)+\tau\left(\frac{\lambda^{\sigma-1}-1}{\sigma-1}\right)+I$. Substituting $\tau = \frac{\alpha}{1-\alpha}\eta$ yields (12).

B.1.5 Population Growth and Firm Dynamics (Section 2.4)

In this section we derive the relationship between population growth η and the different moments of the process of firm dynamics. In particular, we derive

1. the survival function $S(a)$ in (13),
2. the average number of products by age $\overline{n}(a)$ in (14),
3. the pareto tail of the product distribution ζ_n in (15).

Firm survival $S(a)$ and the average number of products $\overline{n}(a)$ Let $p_n(a)$ be the probability that a firm has n products at age a. This evolves according to

$$\dot{p}_n(a) = (n-1) x p_{n-1}(a) + (n+1)(\tau+\delta) p_{n+1}(a) - n(x+\tau+\delta) p_n(a).$$

Because exit is an absorbing state, $\dot{p}_0(a) = (\tau+\delta) p_1(a)$. The solution to this set of differential equations is (see Klette and Kortum (2004))

$$\begin{aligned}
p_0(a) &= \frac{\tau+\delta}{x}\gamma(a) && \text{(A-33)} \\
p_1(a) &= (1-p_0(a))(1-\gamma(a)) \\
p_n(a) &= p_{n-1}(a)\gamma(a) && \text{(A-34)}
\end{aligned}$$

where

$$\gamma(a) = \frac{x\left(1 - e^{-(\tau+\delta-x)a}\right)}{\tau + \delta - x \times e^{-(\tau+\delta-x)a}}. \tag{A-35}$$

Given that $\frac{1-\alpha}{\alpha}\tau = \delta + \eta$, the net rate of accumulation ψ is given by

$$\begin{aligned}\psi &\equiv x - \tau - \delta = x - \tau - \frac{1-\alpha}{\alpha}\tau + \eta = x - \frac{1}{\alpha}\tau - \eta \\ &= \eta - z \\ &= x - \frac{\alpha}{1-\alpha}\eta\end{aligned} \tag{A-36}$$

Note also that $\psi = x - \frac{\alpha\eta+\delta}{1-\alpha}$,i.e. ψ is decreasing in η.

To make the firm-size distribution stationary, we need that $\eta > x - \tau - \delta$. Using equation (A-36), this implies that $z > 0$, i.e. stationary requires the entry flow to be positive. From this solution for $p_n(a)$ we can calculate both the survival rate and the cross-sectional age distribution.

The survival function $S(a)$**.** Let $S(a)$ denote share of firms that survive until age a. Then

$$S(a) \quad = \quad 1 - p_0(a) = \frac{\psi e^{\psi a}}{\psi - x(1 - e^{\psi a})}, \tag{A-37}$$

which is equation (13) in the main text.

The expected number of products by age $\overline{n}(a)$**.** To derive $\overline{n}(a)$ in (14), let $\overline{p}_n(a)$ denote the share of firms of age a with n production *conditional on survival. Then,*

$$\overline{p}_n(a) = \frac{p_n(a)}{1 - p_0(a)} \text{ for } n \geq 1.$$

Using $p_n(a)$ in (A-33)-(A-34), this implies that

$$\overline{p}_n(a) = \gamma(a)^{n-1}(1 - \gamma(a)). \tag{A-38}$$

Then,

$$\overline{n}(a) = E\left[N \middle| A_f = a\right] \quad = \quad \sum_{n=1}^{\infty} n\overline{p}_n(a) = (1 - \gamma(a)) \sum_{n=1}^{\infty} n\gamma(a)^{n-1} = \frac{1}{1 - \gamma(a)}. \tag{A-39}$$

Using (A-35), this implies

$$\overline{n}(a) = 1 - \frac{x}{\psi}\left(1 - e^{\psi a}\right),$$

which is the expression in (14).

The pareto tail of the product distribution ζ_n. To derive the tail of the product distribution, let $\omega_t(n)$ be the mass of firms with n products at time t. Consider $n \geq 2$. Then

$$\dot{\omega}_t(n) = \underbrace{\omega_t(n-1)(n-1)x}_{\text{From } n-1 \text{ to} n \text{ products}} + \underbrace{\omega_t(n+1)(n+1)(\tau+\delta)}_{\text{From } n+1 \text{ to} n \text{ products}} - \underbrace{\omega_t(n)n(\tau+x+\delta)}_{\text{From } n \text{ to } n-1 \text{ or } n+1 \text{ products}} .$$

For $n = 1$ we have

$$\dot{\omega}_t(1) = Z_t + \omega_t(2)2(\tau+\delta) - \omega_t(1)(\tau+x+\delta).$$

Along the BGP the mass of firms grows at rate η. Intuitively: the distribution of firms across products is stationary and the number of products N_t is increasing at rate η. Hence, the mass of firms is increasing at rate η. Hence, along the BGP we have

$$\dot{\omega}_t(n) = \eta\omega_t(n).$$

Denote $\nu(n) = \frac{\omega_t(n)}{N_t}$ and $z = \frac{Z_t}{N_t}$. Along the BGP, $\{\nu(n)\}_{n=1}^{\infty}$ is determined by

$$\nu(2) = \frac{\nu(1)(\tau+x+\delta+\eta) - z}{2(\tau+\delta)} \tag{A-40}$$

and

$$\nu(n+1) = \frac{\nu(n)n(\tau+x+\delta) + \nu(n)\eta - \nu(n-1)(n-1)x}{(n+1)(\tau+\delta)} \quad \text{for } n \geq 2 \tag{A-41}$$

Given $\nu(1)$, these equations fully determine $[\nu(n)]_{n\geq 2}$ as a function of (x, z, τ). We can then pin down $\nu(1)$ from the consistency condition that

$$\sum_{n=1}^{\infty}\nu(n)n = \sum_{n=1}^{\infty}\frac{\omega_t(n)}{N_t}n = \frac{\sum_{n=1}^{\infty}\omega_t(n)n}{N_t} = 1. \tag{A-42}$$

Hence, equations (A-40), (A-41) and (A-42) fully determine the firm-size distribution $[\nu(n)]_{n\geq 1}$. In particular, the average number of products per firm are given by $\overline{n} = \frac{1}{\sum_{n=1}^{\infty}\nu_t(n)}$.

Importantly, the distribution described by (A-40), (A-41) and (A-42) has a pareto tail as long as

$$\eta > x - \tau - \delta > 0.$$

In particular, applying Proposition 3 in Luttmer (2011), the tail index of the product distribution is given by[4]

$$\zeta_n = \frac{\eta}{x-\tau-\delta}.$$

Using that $\tau = \frac{\alpha}{1-\alpha}(\eta+\delta)$ we get that

$$\zeta_n = \frac{(1-\alpha)\eta}{x(1-\alpha)-\delta-\alpha\eta} = \frac{\eta}{\eta - z},$$

where the second equality uses that $z = \frac{\eta+\delta}{1-\alpha} - x$. Also

$$\frac{\partial \zeta_n}{\partial \eta} = (1-\alpha)\frac{x(1-\alpha)-\delta}{(x(1-\alpha)-\delta-\alpha\eta)^2} > 0.$$

Note that the requirement that $x-\tau-\delta > 0$ ensures that $x(1-\alpha)-\delta > 0$.[5] Hence, a decline in population growth reduces the pareto towards unity and causes concentration.

B.1.6 Marginal Efficiency Distribution

In this section we derive the marginal distribution of efficiency q. In particular we derive (23), which we use to calibrate $\overline{\omega}$.

Define $\hat{q}_t$ as the relative productivity of a product

$$\hat{q}_t \equiv \ln(q_t/Q_t)^{\sigma-1}. \quad \text{(A-43)}$$

The drift of $\hat{q}_t$ (conditional on survival) is given by

$$\frac{\partial \hat{q}_t}{\partial t} = (\sigma-1)I - (\sigma-1)d\ln Q_t = -\left(\frac{\alpha\left(\lambda^{\sigma-1}-1\right)}{1-\alpha} + \overline{\omega}^{\sigma-1} - 1\right)(\eta+\delta), \quad \text{(A-44)}$$

where the second equality uses (12).

Let $F_t(\hat{q})$ denote the share of products at time t with $\hat{q}_i \leq \hat{q}$. This cdf evolves according to the

[4]To map the formulation of Luttmer (2011) to our model, note that he expresses the law of motion for the number of products as

$$DM_1 = \lambda 2M_2 + \nu N - (\mu+\lambda)M_1$$

and

$$DM_n = \mu(n-1)M_{n-1} + \lambda(n+1)M_{n+1} - (\mu+\lambda)nM_n.$$

This is the same law of motion as ours once we chose $\nu = z$, $\mu = x$ and $\lambda = \tau+\delta$. He shows that the pareto tail is given by $\frac{\eta}{\mu-\lambda}$ or (using our notation) $\frac{\eta}{x-\tau-\delta}$.

[5]Using that $\tau = \frac{\alpha}{1-\alpha}(\eta+\delta)$, it follows that

$$x-\tau-\delta = \frac{1}{1-\alpha}(x(1-\alpha)-\alpha\eta-\delta).$$

Hence, $x-\tau-\delta > 0$ implies that $x(1-\alpha)-\delta > \alpha\eta > 0$.

differential equation

$$\frac{\partial F_t(\hat{q})}{\partial t} = \underbrace{-\frac{\partial F_t(\hat{q})}{\partial \hat{q}}\frac{\partial \hat{q}_t}{\partial t}}_{\text{Drift of }\hat{q}} + \underbrace{\tau\left(F_t(\hat{q}-\hat{\lambda}) - F_t(\hat{q})\right)}_{\text{Creative destruction}} \underbrace{- (\delta+\eta)\left(F_t(\hat{q}) - \Gamma\left(\exp\left(\frac{\hat{q}}{\sigma-1}\right)\right)\right)}_{\text{Product loss vs new product creation}},$$

where $\hat{\lambda} = \ln \lambda^{\sigma-1}$. In the steady state, $\frac{\partial F_t(\hat{q})}{\partial t} = 0$ so that

$$\frac{dF(\hat{q})}{dq}\frac{\partial \hat{q}_t}{\partial t} = \tau\left(F_t(\hat{q}-\hat{\lambda}) - F_t(\hat{q})\right) - (\delta+\eta)\left(F_t(\hat{q}) - \Gamma\left(\exp\left(\frac{\hat{q}}{\sigma-1}\right)\right)\right). \tag{A-45}$$

Guess that F is exponential in the tail with index κ, that is

$$\underset{\hat{q}\to\infty}{lim}\, e^{\kappa\hat{q}}(1-F(\hat{q})) = a$$

for some a and κ. If we assume that Γ has a thin tail[6] then as $\hat{q}\to\infty$, (A-45) implies that

$$\lim_{\hat{q}\to\infty}\left(ae^{-\kappa\hat{q}}\kappa\frac{\partial \hat{q}_t}{\partial t}\right) = \lim_{\hat{q}\to\infty}\left[(\delta+\eta+\tau) - \tau e^{\kappa\hat{\lambda}}\right]ae^{-\kappa\hat{q}} - (\delta+\eta).$$

Hence, the tail coefficient κ solves the equation

$$-\kappa\frac{\partial \hat{q}_t}{\partial t} = -(\delta+\eta+\tau) + \tau e^{\kappa\hat{\lambda}}.$$

Substituting for (A-44) and noting that $\tau = \frac{\alpha}{1-\alpha}(\eta+\delta)$ yields

$$\kappa\left(\left(\lambda^{\sigma-1}-1\right)\tau + \frac{1-\alpha}{\alpha}\left(\overline{\omega}^{\sigma-1}-1\right)\tau\right) = -(\delta+\eta+\tau) + \tau e^{\kappa\hat{\lambda}}.$$

This is equation (23) in the main text.

For the special case where creative destruction does not lead to any productivity advancements, i.e. $\lambda = 1$ and $\hat{\lambda} = \ln\lambda = 0$, the tail coefficient is given by

$$\kappa = \frac{-(\delta+\eta)}{\frac{1-\alpha}{\alpha}\left(\overline{\omega}^{\sigma-1}-1\right)\tau} = \frac{1}{\left(1-\overline{\omega}^{\sigma-1}\right)}\frac{\delta+\eta}{\frac{1-\alpha}{\alpha}\tau} = \frac{1}{1-\overline{\omega}^{\sigma-1}}.$$

B.2 Characterization of the Model with Bertrand Competition (Section 3)

In this section we derive the results for the model with Bertrand competition described in Section 3

[6] Formally, assume that for any κ, we have $\underset{\hat{q}\to\infty}{lim}\, e^{\kappa\hat{q}}(1-\Gamma\left(\exp\left(\frac{\hat{q}}{\sigma-1}\right)\right)) = 0$.

B.2.1 The Value Function

The only difference relative to the baseline case characterized in Section B.1.2 is that the static profit function is given by (A-20), i.e.

$$\pi\left(q_i, \Delta_i\right) = \left(1 - \frac{1}{\mu\left(\Delta_i\right)}\right) \mu\left(\Delta_i\right)^{1-\sigma} \left(\frac{q_i}{Q_t}\right)^{\sigma-1} \frac{1}{\left(\mathcal{M}_t \Lambda_t\right)^{\sigma-1}} \frac{Y_t}{N_t}. \tag{A-46}$$

Along the BGP, $\left(\mathcal{M}_t \Lambda_t\right)^{\sigma-1}$ is constant. Hence, the value function (of a single product) is given by

$$V_t\left(q, \Delta\right) = \frac{\pi_t\left(q, \Delta\right)}{\rho + \tau + (\sigma - 1)\left(g_Q - I\right)} + \frac{\frac{\zeta-1}{\varphi_x} x^{\zeta} w_t}{\rho + \tau},$$

where all other terms are defined as in Section B.1.2. The respective values of creative destruction and new variety creation are given by

$$V_t^{CD} = \int V_t\left(\lambda q, \lambda\right) dF_t\left(q\right) = V_t\left(\lambda Q_t, \lambda\right) \text{ and } V_t^{NV} = \int V_t\left(\omega Q_t, \frac{\sigma}{\sigma - 1}\right) d\Gamma\left(\omega\right) = V_t\left(\overline{\omega} Q_t, \frac{\sigma}{\sigma - 1}\right).$$

Note that the quality gap for a creative destruction event is equal to λ. For notational simplicity we denote the quality gap for the creation of a new variety by $\frac{\sigma}{\sigma-1}$ to indicate that new varieties are able to charge the monopolistic markup. ,

B.2.2 Proposition 2 in the model with Bertrand Competition

To see that Proposition 2 still applies in the model with Bertrand competition, note first that creative destruction and the rate of variety creation are still given by

$$\begin{aligned} \tau &= \alpha\left(z + x\right) \\ g_N &= \left(1 - \alpha\right)\left(z + x\right). \end{aligned}$$

Moreover, the optimality condition for incumbent expansion x is still given by (A-23) and the free entry condition still holds. Hence,

$$x = \left(\frac{\varphi_x}{\zeta}\right)^{\frac{1}{\zeta-1}} \left(\alpha \frac{V_t^{CD}}{w_t} + \left(1 - \alpha\right) \frac{V_t^{NV}}{w_t}\right)^{\frac{1}{\zeta-1}} = \left(\frac{1}{\zeta} \frac{\varphi_x}{\varphi_E}\right)^{\frac{1}{\zeta-1}}.$$

These three equations together with BGP condition $g_N = \eta$ are sufficient to derive Proposition 2.

B.2.3 The Joint Distribution of Gaps and Productivity

In the model with Bertrand competition in Section 3, the joint distribution of quality q and quality gaps Δ emerges as a key endogenous object. This distribution is characterized from the two differential equations $\frac{\partial F_t^C(\Delta,\hat{q})}{\partial t}$ and $\frac{\partial F_t^{NC}(\Delta,\hat{q})}{\partial t}$ given in the main text. In this section we derive these expressions. Let $\bar{F}_t^C(\Delta,\hat{q})$ denote the mass of products with a gap less than Δ and relative productivity less than $\hat{q}$, for products with a direct competitor. Let N_t^C the mass of such products. Similarly, let $F_t^{NC}(\hat{q})$ denote the mass of the products who have no direct competitor at time t with relative productivity less than $\hat{q}$ and N_t^{NC} their mass. Recall that $\hat{q}_t = \ln(q_t/Q_t)^{\sigma-1}$ (see (A-43)) and that $\hat{q}_t$ has a drift of $g_{\hat{q}}$ given in (A-44).

The evolution of the non competitor mass $\bar{F}_t^{NC}(\hat{q})$ satisfies

$$\bar{F}_t^{NC}(\hat{q}) = \underbrace{\bar{F}_{t-\iota}^{NC}\left(\hat{q} - g_{\hat{q}}\iota\right)\left(1-(\tau+\delta)\,\iota\right)}_{\text{existing mass that survives and improves/falls}} + \underbrace{\left(\frac{1-\alpha}{\alpha}\right)\tau\iota N_t^{NC}\Gamma\left(\frac{\exp(\hat{q})}{\sigma-1}\right)}_{\text{new products that enter}}.$$

As ι becomes small this leads to the differential equation

$$\frac{\partial \bar{F}_t^{NC}(\hat{q})}{\partial t} = -g_{\hat{q}}\frac{\partial \bar{F}_t^{NC}(\hat{q})}{\partial \hat{q}} - (\tau+\delta)\,\bar{F}_{t-\iota}^{NC}(\hat{q}) + \left(\frac{1-\alpha}{\alpha}\right)\tau N_t^{NC}\Gamma\left(\frac{\exp(\hat{q})}{\sigma-1}\right). \tag{A-47}$$

Defining the distribution $F_t^{NC} \equiv \bar{F}_t^{NC}/N_t^{NC}$, A-47 implies

$$\frac{\partial F_t^{NC}(\hat{q})}{\partial t} = -g_{\hat{q}}\frac{\partial F_t^{NC}(\hat{q})}{\partial \hat{q}} - (\tau+\delta+\eta)\,F_{t-\iota}^{NC}(\hat{q}) + \left(\frac{1-\alpha}{\alpha}\right)\tau\Gamma\left(\frac{\exp(\hat{q})}{\sigma-1}\right).$$

This is the equation reported in Section 3.

For the mass of products with a competitor, $\bar{F}_t^C(\Delta,\hat{q})$, we not only need to keep track of the relative quality $\hat{q}$ but also of the quality gap Δ. This mass evolves according to

$$\begin{aligned}\bar{F}_t^C(\Delta,\hat{q}) = {} & \underbrace{\bar{F}_{t-\iota}^C\left(\Delta e^{-I\iota}, \hat{q} - g_{\hat{q}}\iota\right)\left(1-(\tau_t+\delta)\,\iota\right)}_{\text{existing mass that survives and improves/falls}} \\ & + \underbrace{\lim_{s\to\infty}\iota\tau_t\bar{F}_{t-\iota}^C(s,\hat{q}-\hat{\lambda})}_{\text{Creative destruction of } C \text{ products}} + \underbrace{\tau\iota\bar{F}_{t-\iota}^{NC}(\hat{q}-\hat{\lambda})}_{\text{Creative destruction of } NC \text{ products}},\end{aligned}$$

where again we defined $\hat{\lambda} = \ln\lambda^{\sigma-1}$ as in Section B.1.6 above. Using again $\bar{F}_t^C(\Delta,\hat{q}) = N_t F_t^C(\Delta,\hat{q})$, the joint distribution of quality gaps Δ and relative quality $\hat{q}$ solves the differential equation

$$\begin{aligned}\frac{\partial \bar{F}_t^C(\Delta,\hat{q})}{\partial t} = {} & -\frac{\partial \bar{F}_t^C(\Delta,\hat{q})}{\partial \Delta}I\Delta + g_{\hat{q}}\frac{\partial \bar{F}_t^C(\Delta,\hat{q})}{\partial \hat{q}} - \bar{F}_t^C\left(\Delta,\hat{q}\right)(\tau+\delta) \\ & + \lim_{s\to\infty}\tau\bar{F}_t^C(s,\hat{q}-\hat{\lambda}) + \tau\bar{F}_t^{NC}(\hat{q}-\hat{\lambda}).\end{aligned}$$

Defining $\bar{F}_t^C(\Delta,\hat{q}) = N_t^C F_t^C(\Delta,\hat{q})$, we get

$$\frac{\partial F_t^C(\Delta,\hat{q})}{\partial t} = -\Delta I \frac{\partial F_t^C\left(\Delta,\hat{q}\right)}{\partial \Delta} + g_{\hat{q}} \frac{\partial F_t^C\left(\Delta,\hat{q}\right)}{\partial \hat{q}} - (\tau+\delta+\eta)\, F_t^C(\Delta,\hat{q}) + \lim_{s\to\infty} \tau F_t^C(s,\hat{q}-\hat{\lambda}) + \tau \frac{N_t^{NC}}{N_t^C} F_t^{NC}(\hat{q}-\hat{\lambda}).$$

Note that the latter term depends on the relative share of products without a competitor N_t^{NC}/N_t^C.

To derive N_t^{NC}/N_t^C, note that N_t^{NC} evolves according to

$$\dot{N}_t^{NC} = N_t \tau \left(\frac{1-\alpha}{\alpha}\right) - N_t^{NC}\left(\delta+\tau\right).$$

The steady state share of NC products is therefore given by

$$\frac{N_t^{NC}}{N_t} = \frac{\tau\left(\frac{1-\alpha}{\alpha}\right)}{\eta+\delta+\tau} = 1-\alpha, \tag{A-48}$$

i.e. the steady-state share of NC products is simply given by its share in the process of product creation. Hence,

$$\frac{\partial F_t^C(\Delta,\hat{q})}{\partial t} = -\Delta I \frac{\partial F_t^C\left(\Delta,\hat{q}\right)}{\partial \Delta} + g_{\hat{q}} \frac{\partial F_t^C\left(\Delta,\hat{q}\right)}{\partial \hat{q}} - (\tau+\delta+\eta)\, F_t^C(\Delta,\hat{q}) + \lim_{s\to\infty} \tau F_t^C(s,\hat{q}-\hat{\lambda}) + \tau \frac{1-\alpha}{\alpha} F_t^{NC}(\hat{q}-\hat{\lambda}).$$

B.2.4 Marginal gap distribution

We now derive the distribution of efficiency gaps given in (20). Let $F_t^C(\Delta)$ denote the cdf of quality gaps among products with a competitor. Let, as before, denote the number of competitor and non-competitor products as N_t^C and N_t^{NC}. The distribution $F_t^C(\Delta)$ the solves the differential equation

$$\frac{\partial F_t^C(\Delta)}{\partial t} + F_t^C(\Delta)\frac{1}{N_t^C}\frac{\partial N_t^C}{\partial t} = \underbrace{-I\Delta \frac{\partial F_t^C(\Delta)}{\partial \Delta}}_{\text{Upward drift of own-innovation}} \underbrace{-\delta F_t^C(\Delta)}_{\text{Exit}} + \underbrace{(1-F_t^C(\Delta))\tau}_{\text{Inflow through CD}} + \underbrace{\frac{N_t^{NC}}{N_t^C}\tau}_{\text{Inflow from } NC\text{products}}.$$

Along a BGP, this distribution is stationary (i.e. $\frac{\partial F_t^C(\Delta)}{\partial t} = 0$), the number of competitive products grows at rate η and $N_t^{NC}/N_t^C = \frac{1-\alpha}{\alpha}$ (see (A-48)). Hence,

$$\begin{aligned} I\Delta \frac{\partial F^C(\Delta)}{\partial \Delta} &= -(\delta+\eta)\, F^C(\Delta) + (1-F^C(\Delta))\tau + \frac{1-\alpha}{\alpha}\tau \\ &= -(\delta+\eta+\tau)\, F^C(\Delta) + \frac{1}{\alpha}\tau. \end{aligned}$$

Together with the initial condition $F^C(\lambda) = 0$ and the fact that $\frac{1-\alpha}{\alpha}\tau = \eta + \delta$, it is easy to verify that the solution to this differential equation is

$$F^C(\Delta) = 1 - \left(\frac{\lambda}{\Delta}\right)^{\frac{\delta+\eta+\tau}{I}}.$$

B.3 Computing the sales and markups lifecycle

In this section we derive the details of our characterization of the firms' lifecycle of markup and sales that we use to calibrate the model (see Section 4.2). In particular, we show that relative sales by age is given by

$$s_P(a_P) \equiv E\left[\frac{p_i y_i}{Y}\middle| a_p\right] = \mu(a_p)^{1-\sigma} e^{(\sigma-1)(I-g^Q)a_p}\left(\alpha\lambda^{\sigma-1} + (1-\alpha)\varpi^{\sigma-1}\right). \tag{A-49}$$

Moreover we derive the distribution of product age a_P as a function of firm age a_f and the number of products N. Given this distribution we can then easily evaluate $s_f(a_f)$ and $\mu_f(a_f)$ computationally.

B.3.1 Derivation of $s_P(a_P)$ in (A-49)

Consider a BGP where $\mathcal{M}_t$ and Λ_t are constant. (A-20) then implies that sales of product i relative to average sales are

$$s_P(a_P) \equiv E\left[\frac{p_i y_i}{Y_t/N_t}\middle| a_p\right] = E\left[\mu_i^{1-\sigma}\left(\frac{q_i}{Q_t}\right)^{\sigma-1}\middle| a_p\right]\left(\frac{1}{\mathcal{M}_t\Lambda_t}\right)^{\sigma-1}.$$

Note first that markups are a deterministic function of Δ and Δ is a deterministic function of the age of the product. In particular,

$$\mu_i = \mu(a_P) = \min\left\{\lambda e^{Ia_P}, \frac{\sigma-1}{\sigma}\right\}.$$

Similarly, Q_t is given by $Q_t = e^{g_Q a_p} Q_{t-a_p}$.

Now consider the distribution of q_i conditional on a_P. This distribution is given by

$$P(q_i \le q|a_P) = P(q_i \le q|a_P, CD)\alpha + P(q_i \le q|a_P, NV)(1-\alpha),$$

where $P(q_i \le q|a_P, CD)$ and $P(q_i \le q|a_P, NV)$ denotes the conditional probability, conditional on the firm having acquired product i through creative destruction or new variety creation respectively. Then

$$P(q_i \le q|a_P, CD) = F_{t-a_P}\left(\frac{1}{\lambda} q e^{-Ia_P}\right),$$

where $F_{t-a_P}(q)$ denotes the productivity distribution at time $t - a_P$. Similarly,

$$P(q_i \le q|a_P, NV) = \Gamma\left(q e^{-Ia_P}\frac{1}{Q_{t-a_P}}\right).$$

Hence.

$$\begin{aligned} E\left[q_i^{\sigma-1}\middle| a_P\right] &= \alpha\int q^{\sigma-1}dF_{t-a_P}\left(\frac{1}{\lambda}qe^{-Ia_P}\right)+(1-\alpha)\int q^{\sigma-1}d\Gamma\left(qe^{-Ia_P}\frac{1}{Q_{t-a_P}}\right) \\ &= e^{(\sigma-1)Ia_P}Q_{t-a_P}^{\sigma-1}\left(\alpha\lambda^{\sigma-1}+(1-\alpha)\,\overline{\omega}^{\sigma-1}\right), \end{aligned}$$

so that

$$s_P\left(a_P\right)=\mu\left(a_P\right)^{1-\sigma}e^{(\sigma-1)\left(I-g_Q\right)a_P}\left(\alpha\lambda^{\sigma-1}+(1-\alpha)\,\overline{\omega}^{\sigma-1}\right)\left(\frac{1}{\mathcal{M}_t\Lambda_t}\right)^{\sigma-1},$$

which is the expression in (A-49).

B.3.2 Life-Cycle Dynamics

Relative sales and markups at the product level as a function of the state variables Δ and q are given by

$$\begin{aligned} \mu_i &= \mu\left(\Delta_i\right)=\min\left\{\frac{\sigma}{\sigma-1},\Delta_i\right\} \\ \frac{s_i}{Y_t/N_t} &= s_P\left(\Delta_i,q_i\right)=\left(\frac{1}{\mathcal{M}_t\Lambda_t}\right)^{\sigma-1}\mu\left(\Delta_i\right)^{1-\sigma}\left(\frac{q_i}{Q_t}\right)^{\sigma-1}. \end{aligned}$$

Relative sales and average markups of firm f level as a function of the random vector $[\Delta_i,q_i]_{i=1}^{N_f}$ are then given by

$$\frac{s_{ft}}{Y_t/N_t}=\sum_{n=1}^{N_f}s_P\left(\Delta_i,q_i\right)\quad\text{and}\quad\mu_f=\frac{1}{N_f}\sum_{i=1}^{N_f}\mu\left(\Delta_n\right).$$

Expected relative sales as a function of firm age a_f are given by

$$\begin{aligned} E\left[\frac{s_{ft}}{Y_t/N_t}\middle| a_f\right] &= E\left[E\left[\frac{s_{ft}}{Y_t/N_t}\middle| a_f,a_P,N_f\right]\middle| a_f\right] \\ &= E\left[\sum_{n=1}^{N_f}E\left[s_P\left(\Delta_i,q_i\right)\middle| a_f,a_P,N_f\right]\middle| a_f\right] \\ &= E\left[\sum_{n=1}^{N_f}s_P\left(a_P\right)\middle| a_f\right], \end{aligned}$$

where $s_P\left(a_P\right)$ is given in (A-49). The last equality exploits the fact that conditional on product age a_P, product level sales are independent of firm age a_f and the number of products N_f. Letting $f_{a_P|A_f,N}\left(a_P|a,n\right)$ denote the conditional distribution of product age a_P conditional on firm age a_f and the number of products n and $\overline{p}_n\left(a_f\right)$ the probability a firm of age a_f having n products (conditional on survival). Then

$$E\left[\left.\frac{s_{ft}}{Y_t/N_t}\right| a_f\right] = \sum_{n=1}^{\infty} n \left(\int_{a_P} s_P\left(a_P\right) f_{a_P|A_f,N}\left(a_P|a_f,n\right) da_P\right) \overline{p}_n\left(a_f\right).$$

Using the expression for $\overline{p}_n\left(a_f\right)$ in (A-38) yields

$$E\left[\left.\frac{s_{ft}}{Y_t/N_t}\right| a_f\right] = \left(1-\gamma\left(a_f\right)\right) \sum_{n=1}^{\infty} n \left(\int_{a_P} s_P\left(a_P\right) f_{a_P|A_f,N}\left(a_P|a_f,n\right) da_P\right) \gamma\left(a_f\right)^{n-1},$$

where $\gamma\left(a\right)$ is given in (A-35). Using the same logic, the average markup as a function of firm age a_f is given by

$$E\left[\mu_f|\, a_f\right] = \sum_{n=1}^{\infty} \left(\int_{a_P} \mu\left(a_P\right) f_{a_P|A_f,N}\left(a_P|a_f,n\right) da_P\right) \left(1-\gamma\left(a_f\right)\right) \gamma\left(a_f\right)^{n-1}.$$

Given the density $f_{a_P|A_f,N}\left(a_P|a_f,n\right)$, these expressions can be directly evaluated. We now show how to compute this density.

B.3.3 Calculating the conditional density $f_{a_P|A_f,N}\left(a_P|a_f,n\right)$

We now derive the conditional density of product age a_P, $f_{a_P|A_f,N}\left(a_P|a_f,n\right)$.

B.3.4 Lifecycle product age distribution and expected markups by firm age

We want to derive the expected age of the products in a firm's portfolio as it ages. To do so, consider the mass of firms with n products at age A. We are going to derive the law of motion for the *total number of years* the products that this mass of firms owns have been alive (think of products accumulating years for every instant they have been alive). Call this object $\Psi_A(n)$, where

$$\Psi_A(n) = \underbrace{\Lambda_A(n)n}_{\text{Total number of products by firms of age } A} \underbrace{\mathbb{E}_A[a|n]}_{\text{Average age of products of firms of age } A \text{ and } n \text{ products}}.$$

The pool of total years $\Psi_A(n)$ is equal to the number of firms of age A with n products, denoted $\Lambda_A(n)$, times the number of products they own n, times the average age of all those products $\mathbb{E}_A[a|n]$.

We are going to consider how this object evolves through a discrete time approximation. For a small time interval ι,

$$\mathbb{E}_A[a|n]\Lambda_A(n)n = \underbrace{(\mathbb{E}_{A-\iota}[a|n] + \iota)\Lambda_{A-\iota}(n)n(1-(\tau+\delta+x)n\iota)}_{\text{drift from existing mass}}$$
$$+ \underbrace{\iota x(n-1)\Lambda_{A-\iota}(n-1)\Big((n-1)\mathbb{E}_{A-\iota}[a|n-1])\Big)}_{\text{flow in from n-1 firms}}$$
$$+ \underbrace{\iota(\tau+\delta)(n+1)\Lambda_{A-\iota}(n+1)\Big(n\mathbb{E}_{A-\iota}[a|n+1]\Big)}_{\text{flow in from n+1 firms}}$$

The first term in this expression is the drift in total years from an increment of time ι, multiplied by the fraction of firms who don't drop or gain a product in this increment. Intuitively, these products age with a unit drift. The second term is the flow of total years into the pool $\Psi_A(n)$ from the mass of firms with $n-1$ products who are each gaining a product. Importantly, while they bring n products each into the year pool, only $n-1$ have a positive age, and their average age is $\mathbb{E}_{A-\iota}[a|n-1]$. Lastly, the third term is the flow from the mass of firms with $n+1$ products who are losing a product. They bring n products with average age $\mathbb{E}_A[a|n+1]$ with them.

Rewrite this as

$$\frac{\Psi_A(n) - \Psi_{A-\iota}(n)}{\iota} = \Lambda_A(n)n - (\tau+\delta+x)n\mathbb{E}_{A-\iota}[a|n]\Lambda_A(n)n$$
$$+ x(n-1)\Lambda_A(n-1)\Big((n-1)\mathbb{E}_{A-\iota}[a|n-1])\Big)$$
$$+ (\tau+\delta)(n+1)\Lambda_A(n+1)\Big(n\mathbb{E}_{A-\iota}[a|n+1]\Big)$$

so that

$$\frac{\Psi_A(n)}{dA} = \Lambda_A(n)n - (\tau+\delta+x)n\mathbb{E}_A[a|n]\Lambda_A(n)n$$
$$+ x(n-1)\Lambda_A(n-1)\Big((n-1)\mathbb{E}_A[a|n-1])\Big)$$
$$+ (\tau+\delta)(n+1)\Lambda_A(n+1)\Big(n\mathbb{E}_A[a|n+1]\Big) \tag{A-50}$$

This gives us a set of equations for the evolution of $\Psi_A(n)$ for all $n > 1$ that can be solved computationally given initial conditions. We also need one for $n = 1$, which comes from

$$\frac{d\mathbb{E}_A[a|1]\Lambda_A(1)1}{dA} = \Lambda_A(1) - (\tau+\delta+x)\mathbb{E}_{A-\iota}[a|1]\Lambda_A(1) + (\tau+\delta)(2)\Lambda_A(2)\Big(\mathbb{E}_A[a|2]\Big)$$

The initial condition is that

$$\mathbb{E}_0[a|n]\Lambda_0(n)n = \Psi_0(n) = 0$$

for all n. The equations we solve computationally are

$$\frac{\Psi_A(n)}{dA} = \Lambda_A(n)n - (\tau+\delta+x)n\Psi_A(n) + x(n-1)\Psi_A(n-1) + (\tau+\delta)n\Psi_A(n+1) \tag{A-51}$$

Lastly, to recover $\mathbb{E}_A[a|n]$ after computing $\Psi_A(n)$, note that

$$\Lambda_A(n) = F_0 p_A(n)$$

where F_0 is the initial number of firms, and $p_A(n)$ as above is the probability that a firm of age A will have n products , for which we have closed form expressions. Then

$$\mathbb{E}_A[a|n] = \frac{\Psi_A(n)}{\Lambda_A(n)n}$$

Finally, to compute the expected age of products for surviving firms of age A, we have

$$E_A[a] = \sum_{n=1}^{\infty} \mathbb{E}_A[a|n]\frac{p_A(n)}{1-p_A(0)}$$

We use this object in computing markups and sales by firm age, since product markup is a deterministic function of product age.