Finite Sample Inference for the Maximum Score Estimand^{*}

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Abstract

We provide a finite sample inference method for the structural parameters of a semiparametric binary response model under a conditional median restriction originally studied by Manski (1975, 1985). Our inference method is valid for any sample size and irrespective of whether the structural parameters are point identified or partially identified, for example due to the lack of a continuously distributed covariate with large support. Our inference approach exploits distributional properties of observable outcomes conditional on the observed sequence of exogenous variables. Moment inequalities conditional on this size n sequence of exogenous covariates are constructed, and the test statistic is a monotone function of violations of sample moment inequalities. The critical value used for inference is provided by the appropriate quantile of a known function of n independent Rademacher random variables. We investigate power properties of the underlying test and provide simulation studies to support the theoretical findings.

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1 Introduction

In Chapter 41 of Volume 4 of the *Handbook of Econometrics* on the estimation of semiparametric models, Powell (1994) on page 2488 cites Manski (1975) as the earliest example of semiparametric analysis of limited dependent variable models. Subsequently Manski (1985) provided further analysis for the binary outcome version of the model, in which the outcome is determined by the same linear index formulation as in the binary probit model,

$$Y = 1\{X\beta + U \ge 0\},$$
(1.1)

for obervable variables $Y \in \{0, 1\}$ and X a row vector in \mathbb{R}^{K} , but where the unobservable variable U is restricted to satisfy a conditional median restriction

$$Q_{1/2}(U \mid X) = 0, (1.2)$$

rather than full independence from X and normality.¹ This semiparametric model is thus distribution-free with regard to unobservable U, and allows for the conditional distribution of U given X = x to vary with the conditioning value x, for instance accommodating general forms of heteroskedasticity. Under both a rank condition and a large support condition on a continuous regressor Manski (1985) established point identification of β as well as the large deviations convergence rate of the maximum score estimator in the model given by (1.1) and (1.2).

Several further analyses of the maximum score and similar estimators for this and closely related semiparametric binary response models have since been provided, and the literature on the *asymptotic* properties of the maximum score estimator is now vast. Kim and Pollard (1990) showed that the convergence rate of the maximum score estimator is cube-root and established its nonstandard asymptotic distribution after appropriate centering and scaling. Horowitz (1992) developed a smoothed maximum score estimator that converges faster than the $n^{-1/3}$ rate and is asymptotically normal under some additional smoothness assumptions. Additional papers that study large sample estimation and inference in the maximum score context include Manski and Thompson (1986), Delgado, Rodríguez-Poo, and Wolf (2001), Abrevaya and Huang (2005), Léger and MacGibbon (2006), Komarova (2013), Blevins (2015), Chen and Lee (2017, 2018), and Cattaneo, Jansson, and Nagasawa (2018).

In contrast to prior approaches for inference on β that employ asymptotic distributional approximations, in this paper we develop a method for conducting *finite sample* inference on β . To do this we employ a conditional moment inequality characterization of the observable implications of the binary response model in the finite sample. Moment inequality characterizations of the model's implications have been previously used by Komarova (2013), Blevins (2015), and Chen and Lee (2017), but none of these papers proposed a method for conducting finite sample inference. As was the case in the analysis provided in these papers, we do not require that β is point identified. For instance, we do not require that any component of X is continuously distributed, much less with large support.

In fact, even if β is point identified, and regardless of the support of X in the population, in any finite sample the observable support of X is discrete. Indeed, Manski (1985, page 320) defines "the maximum score estimate \hat{B}_n to be the *set* of solutions to the problem $\max_{b \in \mathcal{B}} S_n(b)$ " where \mathcal{B} is the parameter space and $S_n(\cdot)$ denotes the sample score function.² He shows that if β is point identified, then the distance between

¹As noted in Manski (1985), his analysis easily generalizes to cover the restriction that $Q_{\tau}(U \mid X) = 0$ for any $\tau \in (0, 1)$.

²Manski (1985) used B to denote the parameter space and upper case N to denote sample size, which we have changed to \mathcal{B} and lower case n to match our notation.

 \hat{B}_n and β converges almost surely to zero, implying consistency of any sequence of $\hat{\beta}_n \in \hat{B}_n$ for β . Intuitively, the set of possible maximum score point estimators shrinks to a point as $n \to \infty$. Given that our aim in this paper is to conduct finite sample inference, we must own up to the fact that even if β is point identified, there is a proper set of values b to which β is observationally equivalent on the basis of only values of X observed in the finite sample.

We thus introduce the concept of the *finite sample identified set* as the set of parameters vectors $b \in \mathcal{B}$ that satisfy the observable implications of the binary response model *conditional* on a size *n* sequence of observable covariate vectors $\mathcal{X}_n \equiv (X_1, ..., X_n)$. Our finite sample inference approach is driven by observable implications regarding $Y_1, ..., Y_n$ conditional on \mathcal{X}_n , and will be explicit in not being able to detect violations of conditional moment inequalities that condition on values of X not observed in the sample.

The approach taken here exploits the implication of the binary response model that conditional on \mathcal{X}_n , each outcome Y_i is distributed Bernoulli with parameter $p(X_i, \beta)$. In practice the Bernoulli probabilities $p(X_i, \beta)$ are unknown. Nonetheless, conditional on \mathcal{X}_n , each $p(X_i, \beta)$ is bounded from above or below by 1/2according to the sign of $X_i\beta$. Consequently, for any nonnegative-valued function $g(\cdot) : \mathcal{X}_n \to \mathbb{R}$, the finite sample distributions of $\omega_i^+(\beta, g) \equiv (2Y_i - 1)1\{X_i\beta \ge 0\}g(X_i)$ and $\omega_i^-(\beta, g) \equiv (1 - 2Y_i)1\{X_i\beta \le 0\}g(X_i)$ can be bounded from below. The test statistic $T_n(b)$ that we use to implement our test of the null hypothesis $H_0 : \beta = b$ is a supremum of sample averages of $\omega_i^+(\beta, g)$ and $\omega_i^-(\beta, g)$, where the supremum is taken over particular collections of functions $g(\cdot)$. The test statistic $T_n(b)$ is shown to be bounded above by a function $\overline{T}_n(b)$ of n independent Rademacher random variables, such that the exact distribution of $\overline{T}_n(b)$ is known. Then, under the null hypothesis, we have that

$$\mathbb{P}\left(T_n(b) > q_{1-\alpha} \mid X_1, \dots, X_n\right) \le \alpha,$$

where $q_{1-\alpha}$ is the $1-\alpha$ quantile of $\overline{T}_n(b)$. We establish that if particular functions $g(\cdot)$ are used, the moment functions which $T_n(b)$ incorporates preserve all the identifying information contained in finite sample identified set, and we establish a power result for alternatives that lie outside this set.

Among the aforementioned papers from the literature on maximum score, the most closely related is that of Chen and Lee (2017), who also cast the implications of Manski's (1985) model as conditional moment inequalties for the sake of delivering a new insight, albeit one that is entirely different from ours. Chen and Lee (2017) expand on the conditional moment inequalities used by Komarova (2013) and Blevins (2015) to develop a novel conditional moment inequality characterization of the identified set which involves conditioning on two linear indices instead of on the entire exogenous covariate vector. They apply intersection bound inference from Chernozhukov, Lee, and Rosen (2013) to this conditional moment inequality characterization to achieve asymptotically valid inference. This cleverly exploits the model's semiparametric linear index restriction in order to side step the curse of dimensionality. Although a good deal of focus is given to Manski's (1985) binary response model, their method can also be applied to other semiparametric models.

To the best of our knowledge, this paper is the first to propose a method for finite sample inference on β in Manski's (1985) semiparametric binary response model. It is also the first to introduce the concept of a finite sample identified set, explicitly defining the set of model parameters logically consistent with the modeling restrictions and only information that can be gathered from observable implications conditional on realizations of exogenous variables observed in the finite sample. There are however two precedents for employing finite sample inference in the context of two rather different partially identifying models. Manski (2007) considers the problem of predicting choice probabilities for the choices individuals would make if subjected to counterfactual variation in their choice sets. In the absence of the structure afforded by commonly

used random utility models, he shows that counterfactual choice probabilities are partially identified, and proposes a procedure for inference using results from Clopper and Pearson (1934). Chernozhukov, Hansen, and Jansson (2009) propose a method for finite sample inference in quantile regression models in which the outcome is continuously distributed. Their approach exploits a "conditionally pivotal property" to bound the finite sample distribution of a GMM criterion incorporating moment equalities, but which does not require point identification for its validity. The approach taken in this paper for finite sample inference in the context of Manski's (1985) binary response model is different from both of these.

The rest of this paper is organized as follows. Section 2, formally sets out the testing problem and the moment inequality representation of the finite sample identified set. Section 3 lays out the construction of the test statistic and corresponding critical value, and establishes the finite sample validity of the test. Section 4 provides a result concerning the power of the test. Section 5 demonstrates the performance of the approach by reporting results from Monte Carlo simulation and Section 6 concludes. The proofs are in the Appendix.

2 Testing Problem and Moment Restrictions

This section is divided into two subsections, the first of which formally presents the modeling restrictions imposed, and the testing problem at hand. The second subsection describes the observable implications of the binary response model *conditional* on a size n sequence of covariate vectors, \mathcal{X}_n , in contrast to those observable implications obtainable from knowledge of the population distribution of observable variables. Based on these observable implications, this second subsection introduces our definition of the finite sample identified set. It clarifies what violations of our model's implications the proposed test could feasibly detect, which is useful for power considerations. The developments of Subsection 2.2 are however not essential for establishing the size control of the test presented in Section 3.

2.1 Model and Test

The following assumption formalizes the restrictions of the semiparametric binary response model under study and the requirements on the sampling process.

Assumption 1. (i) Random vectors $\{(Y_i, X_i, U_i) : i = 1, ..., n\}$ reside on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where \mathcal{F} contains the Borel sets on Ω . (ii) Variables $\{(Y_i, X_i) : i = 1, ..., n\}$ are observed. (iii) There is a $\beta \in \mathbb{R}^K$ such that

$$\mathbb{P}\left(Y_i = 1\{X_i\beta + U_i \ge 0\} \mid \mathcal{X}_n\right) = 1$$

and $\mathbb{P}(U_i \ge 0 \mid \mathcal{X}_n) = 1/2$ for every i = 1, ..., n. (iv) There is a known set $\mathcal{B} \subseteq \mathbb{R}^K$ to which β belongs. (v) The events $\{\{U_i \ge 0\} : i = 1, ..., n\}$ are independent given \mathcal{X}_n .

The requirements of Assumption 1 are slightly weaker than those required by (1.1) and (1.2) in the Introduction. Parts (i), (ii) and (iv) are standard. Although it is not necessary in this paper, the parameter space \mathcal{B} can be restricted by imposing one of the usual scale normalizations from the literature, such as $|b_1| = 1$ for all $b \in \mathcal{B}$. Part (iii) imposes the binary response structure and the requirement that $\mathbb{P}(U_i \ge 0 | \mathcal{X}_n) = 1/2$ for each *i*. Binary response models typically require that U_i is continuously distributed in a neighborhood of zero, in which case this is implied by the usual conditional median restriction. Strictly speaking, we do not need to impose that each U_i is continuously distributed at zero, and hence we replace the median restriction with this requirement. Part (v) of Assumption 1 holds if (Y_i, X_i, U_i) are independent and identically distributed, but is much more general. The observations $\{(Y_i, X_i) : i = 1, ..., n\}$ can be non-independent and non-identically distributed. Throughout the text, $\mathbb{E}[\cdot]$ is used to denote population expectations taken with respect to \mathbb{P} , and $\mathbb{E}_n[\cdot] \equiv n^{-1} \sum_{i=1}^n [\cdot]$.

For a given value $b \in B$, in this paper we consider the hypothesis test

$$H_0 : \beta = b$$
$$H_1 : \beta \neq b$$

on the basis of *n* observations $\{(Y_i, X_i) : i = 1, ..., n\}$ following the restrictions of the semiparametric binary response model given by Assumption 1. These restrictions all hold in the typical semiparametric binary response models in which the maximum score estimator has been studied. As noted in the Introduction, our method does not require point identification of β , and thus we do not assume sufficient conditions for point identification. Most notably, the existence of a continuous covariate – much less one with full support on \mathbb{R} – is not required.

2.2 Observable Implications Conditional on \mathcal{X}_n

To conduct finite sample inference, we focus solely on the implications obtainable from a sequence of n draws of (Y, X) in a sample $\{(Y_i, X_i) : i = 1, ..., n\}$ and not on features of the population distribution of these variables that could be obtained on the basis of infinitely many observations. Consequently our focus is not on the identified set that could be obtained from knowledge of the population distribution of (Y, X) in an infinitely large population, but rather on the set obtainable solely from knowledge of a size n sample of observations in accord with Assumption 1. By definition, this is the set of parameter vectors $b \in \mathcal{B}$ such that, conditional on \mathcal{X}_n , the observed distribution of $Y_1, ..., Y_n$ matches that of $1\{X_i b + \tilde{U}_i \ge 0\}$ for a sequence of random variables $\tilde{U}_1, ..., \tilde{U}_n$ that satisfy the restrictions placed on the conditional distribution of $U_1, ..., U_n$ in Assumption 1. We refer to this set as the finite sample identified set and denote it as \mathcal{B}_n^* .

Definition 1. The finite sample identified set for β under Assumption 1, denoted by \mathcal{B}_n^* , is the set of $b \in \mathcal{B}$ for which there are n random variables $\{\tilde{U}_i : i = 1, ..., n\}$ such that $\mathbb{P}\left(Y_i = 1\{X_i b + \tilde{U}_i \ge 0\} \mid \mathcal{X}_n\right) = 1$ and $\mathbb{P}\left(\tilde{U}_i \ge 0 \mid \mathcal{X}_n\right) = 1/2$ for every i = 1, ..., n, and that $\{\{\tilde{U}_i \ge 0\} : i = 1, ..., n\}$ are independent given \mathcal{X}_n .

Our next task is to express \mathcal{B}_n^* with a moment inequality representation useful for inference. The following lemma sets out two observable implications that will be useful for this purpose.

Lemma 1. Let Assumption 1 hold. Then

$$X_i\beta \geq 0 \implies \mathbb{E}[2Y_i - 1 \mid \mathcal{X}_n] \geq 0, \tag{2.1}$$

$$X_i\beta \leq 0 \implies \mathbb{E}[2Y_i - 1 \mid \mathcal{X}_n] \leq 0.$$
(2.2)

From the inequalities of the lemma, it further follows that if $X_i\beta = 0$ then $\mathbb{E}[2Y_i - 1 \mid \mathcal{X}_n] = 0$, and furthermore that

$$\mathbb{E}[2Y_i - 1 \mid \mathcal{X}_n] > 0 \implies X_i\beta > 0,$$

and

$$\mathbb{E}[2Y_i - 1 \mid \mathcal{X}_n] < 0 \implies X_i \beta < 0.$$

Moreover (2.1) and (2.2) and the implications of them described above hold with β replaced by any b that is an element of the finite sample identified set \mathcal{B}_n^* , which can be proven by following precisely the same steps as in the lemma with \tilde{U}_i from Definition 1 replacing U_i .

With Lemma 1 in hand, the following theorem provides a moment inequality characterization of the finite sample identified set.

Theorem 1. The finite sample identified set for β under Assumption 1 is

$$\mathcal{B}_{n}^{*} = \left\{ b \in \mathcal{B} : \forall i = 1, \dots, n, \qquad \begin{array}{l} \mathbb{E} \left[(2Y_{i} - 1) \, 1\{X_{i}b \ge 0\} \mid \mathcal{X}_{n} \right] \ge 0, \\ \mathbb{E} \left[(1 - 2Y_{i}) \, 1\{X_{i}b \le 0\} \mid \mathcal{X}_{n} \right] \ge 0. \end{array} \right\}.$$
(2.3)

The conditional moment inequalities characterizing \mathcal{B}_n^* in (2.3) are equivalent to (2.1) or (2.2) for all i = 1, ..., n. However, using this conditional moment inequality representation to conduct inference on β is complicated by the fact that in a sample of n observations the distribution of Y_i given \mathcal{X}_n can vary across i, even if $(Y_i, X_i) : i = 1, ..., n$ are identically distributed.

Some level of aggregation of these implications across i is therefore required. One way to do this is to interact the expressions inside the conditional expectation operators in (2.3) with nonnegative-valued instrument functions and take sample averages. Specifically, consider the conditional moment inequalities

$$m_n^+(b,v^+) \equiv \mathbb{E}\left[\mathbb{E}_n\left[(2Y-1)1\{Xb \ge 0\}g^+(X,v^+)\right] \mid \mathcal{X}_n\right] \ge 0, \quad v^+ \in \mathcal{V}_n^+,$$
(2.4)

$$m_n^-(b, v^-) \equiv \mathbb{E}\left[\mathbb{E}_n\left[(1 - 2Y)1\{Xb \le 0\}g^-(X, v^-)\right] \mid \mathcal{X}_n\right] \ge 0, \quad v^- \in \mathcal{V}_n^-,$$
(2.5)

where $g^+(\cdot, v^+)$ and $g^-(\cdot, v^-)$ are collections of such instrument functions indexed by $v^+ \in \mathcal{V}_n^+$ and $v^- \in \mathcal{V}_n^-$, respectively. The inequalities (2.4) and (2.5) are valid for all $b \in \mathcal{B}_n^*$ because they are implications of the conditional moment inequalities characterizing \mathcal{B}_n^* in (2.3). Indeed, they are valid for *any* such collections of nonnegative-valued instrument functions. A potential drawback to aggregation of the conditional moment however is that (2.4) and (2.5) do not in general characterize \mathcal{B}_n^* , so that using the latter inequalities can result in a loss of identification power.

Collections $\{g^+(\cdot, v^+) : v^+ \in \mathcal{V}_n^+\}$ and $\{g^-(\cdot, v^-) : v^- \in \mathcal{V}_n^-\}$ are now defined so as ensure preservation of the full identifying power of the conditional on \mathcal{X}_n moment inequalities in (2.3). These collections differ from those used by Andrews and Shi (2013) for translating the identifying power of conditional moment inequalities to a collection of unconditional moment inequalities. In the present setting, there is no issue of converting inequalities conditional on continuous variables to unconditional ones, because the conditioning set in the inequalities characterizing \mathcal{B}_n^* is finite. Instead, the problem to be addressed is how best to aggregate these implications across observations *i* given the non-i.i.d. nature of Y_i conditional on \mathcal{X}_n . In constructing our collection of information-preserving instrument functions, we exploit two features specific to the task at hand, namely first that our focus is on finite sample inference conditional on \mathcal{X}_n and second that whether or not $\mathbb{E}\left[2Y_i - 1 \mid \mathcal{X}_n\right] \ge 0 \ (\leq 0)$ depends only on whether the linear index $X_i\beta$ is at least (at most) zero. First, consider the following two sequences of binary indicators:

$$\begin{aligned} r_n^+(b) &\equiv (1\{X_1b \ge 0\}, ..., 1\{X_nb \ge 0\}) \,, \\ r_n^-(b) &\equiv (1\{X_1b \le 0\}, ..., 1\{X_nb \le 0\}) \,. \end{aligned}$$

Note that irrespective of how $g^+(X, v^+)$ and $g^-(X, v^-)$ are defined, $m_n^+(b, v^+) = m_n^+(b', v^+)$ whenever $r_n^+(b) = r_n^+(b')$ and $m_n^-(b, v^-) = m_n^-(b', v^-)$ whenever $r_n^-(b) = r_n^-(b')$. Using the functions $r_n^+(\cdot)$ and $r_n^-(\cdot)$, we denote by V_n^+ the coimage of the function $r_n^+(\cdot)$ on \mathcal{B} , and by V_n^- that of the function $r_n^-(\cdot)$.³ The collections V_n^+ and V_n^- consist of sets whose members b all produce the same sequences of inequalities $X_i b \ge 0$ and $X_i b \le 0$, i = 1, ..., n, respectively. Since n is finite, there are only finitely many elements of each of V_n^+ and V_n^- . With regard to the inequalities (2.4) and (2.5), all members v^+ of any set in V_n^+ and all members v^- of any set in V_n^-) produce the same values of $m_n^+(b, v^+)$ and $m_n^-(b, v^-)$ for any $b \in \mathcal{B}$. Thus it will suffice to work with a single representative from each set, i.e., the full identifying power is preserved as long as \mathcal{V}_n^+ and \mathcal{V}_n^- have a representative from each element of V_n^+ and V_n^- , respectively.

With sets V_n^+ and V_n^- now defined, the following Theorem establishes a representation of the finite sample identified set \mathcal{B}_n^* given in (2.3) that takes the form of a finite collection of unconditional inequalities of the form (2.4) and (2.5).

Theorem 2. Let Assumption 1 hold and let $b \in \mathcal{B}$. If $b \in \mathcal{B}_n^*$ then

$$\forall v^+ \in \mathcal{V}_n^+ : \mathbb{E}\left[\mathbb{E}_n\left[(2Y-1)\mathbf{1}\{Xb \ge 0\}\mathbf{1}\{Xv^+ < 0\}\right] \mid \mathcal{X}_n \right] \ge 0, \tag{2.6}$$

and

$$\forall v^{-} \in \mathcal{V}_{n}^{-} : \mathbb{E} \left[\mathbb{E}_{n} \left[(1 - 2Y) \mathbf{1} \{ Xb \leq 0 \} \mathbf{1} \{ Xv^{-} > 0 \} \right] \mid \mathcal{X}_{n} \right] \geq 0.$$
(2.7)

Moreover, the converse also holds if \mathcal{V}_n^+ and \mathcal{V}_n^- have at least one element from each member of \mathcal{V}_n^+ and \mathcal{V}_n^- , respectively.

The moment inequalities (2.6) and (2.7) are conditional on \mathcal{X}_n and are thus different from those employed previously in the literature. Our representation is perhaps most closely related to that of Chen and Lee (2017) for the identified set in the underlying population. Their characterization uses inequalities that condition on the values of two linear indices in X: Xb and $X\gamma$, leading to significant dimension reduction when estimating conditional moments employed for asymptotic inference. In this paper our goal is finite sample inference, made operational by conditioning on \mathcal{X}_n . Our construction leading to (2.6) and (2.7) exploits the finite nature of \mathcal{X}_n . This is done by establishing that given \mathcal{X}_n , one can partition the parameter space \mathcal{B} into equivalence classes V_n^+ and V_n^- whose members comprise elements that all produce the same values of moment functions $m_n^+(b, v^+)$ and $m_n^-(b, v^-)$, respectively. Then, using our proposed instrument functions the moment inequalities (2.6) and (2.7) aggregate values of $2Y_i - 1$ and $1 - 2Y_i$ according to whether a pairs of indices Xb and Xv in each of the two inequalities disagree in particular directions.

³The coimage of a function f is defined as the quotient set of the equivalence relation defined by f.

3 Test Statistic and Critical Value

To perform inference, we incorporate sample analogs of the moments appearing in (2.6) and (2.7), which are

$$\hat{m}_n^+(b,v^+) \equiv \mathbb{E}_n \left[(2Y-1) 1\{Xb \ge 0, Xv^+ < 0\} \right], \quad v^+ \in \mathcal{V}_n^+, \\ \hat{m}_n^-(b,v^-) \equiv \mathbb{E}_n \left[(1-2Y) 1\{Xb \le 0, Xv^- > 0\} \right], \quad v^- \in \mathcal{V}_n^-,$$

into our test statistic $T_n(b) \equiv \max\{0, T_n^+(b), T_n^-(b)\}$, where

$$T_{n}^{+}(b) \equiv \sup_{v^{+} \in \mathcal{V}_{n}^{+}} \sqrt{n} \frac{-\hat{m}_{n}^{+}(b, v^{+})}{\max\{\epsilon, \sqrt{\mathbb{E}_{n}[1\{Xb \ge 0, Xv^{+} < 0\}] - \hat{m}_{n}^{+}(b, v^{+})^{2}}\}}$$

and

$$T_{n}^{-}(b) \equiv \sup_{v^{-} \in \mathcal{V}_{n}^{-}} \sqrt{n} \frac{-\hat{m}_{n}^{-}(b, v^{-})}{\max\{\epsilon, \sqrt{\mathbb{E}_{n}[1\{Xb \le 0, Xv^{-} > 0\}] - \hat{m}_{n}^{-}(b, v^{-})^{2}}\}}$$

Here ϵ is an arbitrarily small positive number taken to ensure a non-zero denominator.

Instead of deriving the finite sample distribution of $T_n(b)$ under H_0 , we construct a random variable $\overline{T}_n(b)$ which has a known finite sample distribution given \mathcal{X}_n such that

$$T_n(b) \le \overline{T}_n(b) \text{ under } H_0: b = \beta.$$
(3.1)

We define $\overline{Y}_1, \ldots, \overline{Y}_n$ by

$$\bar{Y}_i = 1\{U_i \ge 0\}$$

for every i = 1, ..., n. Define $\bar{T}_n(b) = \max\{0, \bar{T}_n^+(b), \bar{T}_n^-(b)\}$, where

$$\begin{split} \bar{m}_{n}^{+}\left(b,v^{+}\right) &= \mathbb{E}_{n}\left[(2\bar{Y}-1)1\{Xb \geq 0, Xv^{+} < 0\}\right], \quad v^{+} \in \mathcal{V}_{n}^{+}, \\ \bar{m}_{n}^{-}\left(b,v^{-}\right) &= \mathbb{E}_{n}\left[(1-2\bar{Y})1\{Xb \leq 0, Xv^{-} > 0\}\right], \quad v^{-} \in \mathcal{V}_{n}^{-}, \\ \bar{T}_{n}^{+}\left(b\right) &= \sup_{v^{+} \in \mathcal{V}_{n}^{+}} \sqrt{n} \frac{-\bar{m}_{n}^{+}\left(b,v^{+}\right)}{\max\{\epsilon, \sqrt{\mathbb{E}_{n}[1\{Xb \geq 0, Xv^{+} < 0\}] - \bar{m}_{n}^{+}\left(b,v^{+}\right)^{2}\}}, \\ \bar{T}_{n}^{-}\left(b\right) &= \sup_{v^{-} \in \mathcal{V}_{n}^{-}} \sqrt{n} \frac{-\bar{m}_{n}^{-}\left(b,v^{-}\right)}{\max\{\epsilon, \sqrt{\mathbb{E}_{n}[1\{Xb \leq 0, Xv^{-} > 0\}] - \bar{m}_{n}^{-}\left(b,v^{-}\right)^{2}}}. \end{split}$$

Note that the finite sample distribution of $\overline{T}_n(b)$ given \mathcal{X}_n is known, since $\overline{T}_n(b)$ is a function of $(2\overline{Y}_1 - 1, \ldots, 2\overline{Y}_n - 1)$ given \mathcal{X}_n and $(2\overline{Y}_1 - 1, \ldots, 2\overline{Y}_n - 1)$ are independent Rademacher random variables.

Thus, for a given level $\alpha \in (0, 1)$, the critical value used for our test is the conditional $1 - \alpha$ quantile of $\overline{T}_n(b)$ given \mathcal{X}_n , namely

$$q_{1-\alpha} \equiv \inf\{c \in \mathbb{R} : \mathbb{P}\left(\overline{T}_n(b) \le c \mid \mathcal{X}_n\right) \ge 1-\alpha\}.$$

It can be computed up to arbitrary accuracy by drawing a large number of simulations, each of which comprises a sequence of n independent Rademacher random variables.

The relationship between $T_n(b)$ and $\overline{T}_n(b)$ in (3.1) implies Theorem 3, establishing finite sample size control of the proposed test. As is the case with all formal mathematical results stated in the paper, the proofs of inequality (3.1) and Theorem 3 are in the Appendix. **Theorem 3.** Under the null $H_0: \beta = b$, $\mathbb{P}(T_n(b) \leq q_{1-\alpha} \mid \mathcal{X}_n) \geq 1 - \alpha$.

Theorem 3 establishes finite sample size control. While it is possible that $\mathbb{P}(T_n(\beta) \leq q_{1-\alpha} \mid \mathcal{X}_n)$ strictly exceeds $1-\alpha$, the following theorem shows that a test with a smaller critical value cannot achieve size control if the critical value is a deterministic function of \mathcal{X}_n . It should however be noted that Theorem 4 is silent with regard to critical values that are a function of both $X_1, ..., X_n$ and $Y_1, ..., Y_n$.

Theorem 4. Suppose cv is a random variable which depends on \mathcal{X}_n . Assume that $cv < q_{1-\alpha}$ given \mathcal{X}_n . There is a distribution of (U_1, \ldots, U_n) given \mathcal{X}_n under which cv does not achieve size control:

$$\mathbb{P}(T_n(\beta) \le cv \mid \mathcal{X}_n) < 1 - \alpha.$$

4 **Power Properties**

In this section, we establish a power result for the proposed test. This result imposes an additional restriction relative to Assumption 1, namely that the binary variables $Y_1, ..., Y_n$ are independently distributed conditional on \mathcal{X}_n . Then Hoeffding's inequality is used to establish a lower bound on finite sample power for certain violations of the inequalities (2.6) and (2.7) from Theorem 2. The result is given in the following Theorem.

Theorem 5. Let ρ be any number in (0,1). Assume Y_1, \ldots, Y_n are independent given \mathcal{X}_n . If there is $v^+ \in \mathcal{V}_n^+$ such that

$$\mathbb{E}\left[\mathbb{E}_{n}\left[(2Y-1)1\{Xb \ge 0, Xv^{+} < 0\}\right] \mid \mathcal{X}_{n}\right] \\
\leq -\frac{1}{\sqrt{n}}\left(q_{1-\alpha}\max\left\{\epsilon, \sqrt{\frac{\mathbb{E}_{n}\left[1\{Xb \ge 0, Xv^{+} < 0\}\right]}{1+q_{1-\alpha}^{2}/n}}\right\} + \sqrt{2\log(1/\rho)\mathbb{E}_{n}\left[1\{Xb \ge 0, Xv^{+} < 0\}\right]}\right), \quad (4.1)$$

or there is $v^- \in \mathcal{V}_n^-$ such that

$$\mathbb{E}\left[\mathbb{E}_{n}\left[(1-2Y)1\{Xb \leq 0, Xv^{-} > 0\} \right] \mid \mathcal{X}_{n} \right]$$

$$\leq -\frac{1}{\sqrt{n}} \left(q_{1-\alpha} \max\left\{ \epsilon, \sqrt{\frac{\mathbb{E}_{n}[1\{Xb \leq 0, Xv^{-} > 0\}]}{1+q_{1-\alpha}^{2}/n}} \right\} + \sqrt{2\log(1/\rho)\mathbb{E}_{n}[1\{Xb \geq 0, Xv^{-} < 0\}]} \right),$$

then the rejection probability is at least $1 - \rho$, i.e.,

$$\mathbb{P}(T_n(b) > q_{1-\alpha} \mid \mathcal{X}_n) \ge 1 - \rho.$$

5 Monte Carlo Experiments

In this section, we present some Monte Carlo results on the performance of our method as well as that of the intersection bound test in Chernozhukov, Lee, and Rosen (2013) – henceforth CLR – applied to the moment inequalities of which our test statistic $T_n(b)$ is comprised. The sample size in the Monte Carlo experiments reported here is 250 and the number of simulations is 500. The variable ϵ was set to MATLAB's eps value of approximately $2.2 \cdot 10^{-16}$. To compute the critical value, we use 500 random draws of n Rademacher random variables { $\omega_i : i = 1, \ldots, n$ } and 500 samples for the multiplier bootstrap in our application of CLR. The implementation of CLR is described further in Section 5.3.

We considered eight designs for these Monte Carlo experiments, four in which there is a continuous covariate with large support resulting in point identification of β up to scale, and four in which covariates are discretely distributed and point identification is lacking. We normalize the first component of β so that $\beta = (1, \theta)'$ for an unknown parameter θ . In all cases we considered tests with size $\alpha = 0.10$.

5.1 Designs with point identification

In this subsection we report the results of Monte Carlo experiments using the simulation designs in Horowitz (1992). $X = (X_1, X_2)$ is a bivariate normal random vector with $\mathbb{E}[X_1] = 0$, $\mathbb{E}[X_2] = 1$, $Var(X_1) = Var(X_2) = 1$ and $Cov(X_1, X_2) = 0$.

The distribution of unobservable U in each design is as follows.

- Design 1: U is distributed according to the logistic distribution with mean zero and variance one.
- Design 2: U is uniformly distributed on $\left[-\sqrt{12}/2, \sqrt{12}/2\right]$.
- Design 3: U is distributed according to the Student's t distribution normalized to have variance one.
- Design 4: $U = 0.25(1 + 2Z^2 + Z^4)$ where $Z = X_1 + X_2$ and V is distributed according to the logistic distribution with mean zero and variance one.

Figure 1 presents non-rejection frequencies for θ ranging from 0 to 3 for $\alpha = 0.10$. The non-rejection frequency is maximized around the population value of $\theta = 1$ and exceeds 0.90 in a range spanning from about 0.6 to 1.7. Our finite sample inference method performs slightly better than our implementation of CLR does here, applied to the same moment inequalities. The CLR critical value makes use of an asymptotic Gaussian approximation for studentized versions of each of the sample moments. Although our finite sample approach does slightly better, the figure suggests that the asymptotic approximation is quite accurate at this moderate sample size of n = 250. The CLR approach also incorporates a moment selection procedure (described in Section 5.3) that in a first step discards moment inequalities that are sufficiently far from binding, which can lead to power improvements relative to no moment selection. Our finite sample inference approach has no such moment selection step.⁴

Since our method and our implementation of CLR use precisely the same moment inequalities, their only difference stems from their use of different critical values. Figure 2 depicts these critical values, which in all cases were somewhere between 2.5 and 3. The figure shows that for each θ the CLR critical value is slightly higher than our proposed critical value. This comparison between the two critical values held up similarly across each of Designs 1-4. In order to economize on space the comparison is only illustrated for Design 1.

Figures 3-5 show the analog of Figure 1, depicting non-rejection probabilities across θ for each of Designs 2-4. The qualitative comparison between the two approach is similar to that for Design 1, with the finite sample valid critical value achieving size control, while also having an equal or slightly lower non-rejection frequency across all values of θ , which translates to slightly greater power.

The finite sample approach to inference seems to perform well in these experiments. Future Monte Carlo experiments will additionally compare the approach to other inference methods from the maximum score literature, while also experimenting with both larger and smaller sample sizes.

⁴Moment selection approaches from the literature on inference based on moment inequalities guarantee that binding moments are selected with probability approaching one as $n \to \infty$. This is sufficient to guarantee asymptotic validity, but not for finite sample validity.

5.2 Designs without point identification

In this subsection, we consider designs in which the covariates are discrete and point identification fails. We modify the distribution of X such that X_1 takes -1 and 1 with equal probabilities and X_2 takes 0 and 2 with equal probabilities. This distribution was used because X_1 and X_2 have the same means and variances as in the previous designs. The identified set is $\theta \geq 1/2$.

The designs implemented were as follows:

- Design 5: U is distributed according to the logistic distribution with mean zero and variance one.
- Design 6: U is uniformly distributed on $\left[-\sqrt{12}/2, \sqrt{12}/2\right]$.
- Design 7: U is distributed according to the Student's t distribution normalized to have variance one.
- Design 8: $U = 0.25(1 + 2Z^2 + Z^4)$ where $Z = X_1 + X_2$ and V is distributed according to the logistic distribution with mean zero and variance one.

The nonrejection frequencies of both our finite sample procedure and the CLR procedure in Design 5 are reported in Figure 6. The results reflect an earlier finding from Komarova (2013), where it was shown that under mild conditions with discrete covariates an analog set estimator for the identified set for β converges to the identified at an arbitrarily fast polynomial rate.⁵ In our simulations both procedures appear to internalize this super consistency property, producing non-rejection probabilities of one for elements of the identified set, and rejection probabilities of one for all elements not in the identified set. Figure 7 additionally reports the critical values attained by each procedure. The CLR intersection bound procedure produces a smaller critical value for all sample sizes in these experiments, although as Figure 6 illustrates, the difference has no effect on the outcome of the test for any of the parameter values considered.

The Monte Carlo results for Designs 6-8 bore identical results to those of Design 5 in terms of non-rejection probabilities, and a similar qualitative comparison between critical values. Due to the close similarity of the results to those for Design 5 reported in Figures 6 and 7, figures for these results are omitted.

5.3 Implementation of Chernozhukov, Lee, and Rosen (2013)

Using the test statistic $T_n(b)$ we also implement the inference method in CLR to compute an asymptotically valid critical value based on a large sample Gaussian approximation to each of the constituent moments. The implementation is similar to that of Chen and Lee (2017), but instead applied to the moment functions described here, and with the supremum applied over finite sets of v^+ and v^- indexing the instrument functions $g^+(\cdot, v^+)$ and $g^-(\cdot, v^-)$.

The critical value cv_2 is computed in two steps. Consider *n* independent standard normal random variables (η_1, \ldots, η_n) . Denote by cv_1 the $1 - 0.1/\log(n)$ quantile of

$$\max\left\{0,\sup_{v^+\in\mathcal{V}_n^+}\hat{\mu}_n^{+*}\left(b,v^+;\eta_1,\ldots,\eta_n\right),\sup_{v^-\in\mathcal{V}_n^-}\hat{\mu}_n^{-*}\left(b,v^-;\eta_1,\ldots,\eta_n\right)\right\}$$

⁵That is, for any c > 0, $n^c \rho(\hat{B}, B_I) \xrightarrow{p} 0$ as $n \to \infty$, where \hat{B} denotes the set estimator and B_I denotes the population identified set. See also Blevins (2015).

given $\{(Y_i, X_i, U_i) : i = 1, ..., n\}$, where

$$\begin{split} \hat{\mu}_{n}^{+*}\left(b,v^{+};\eta_{1},\ldots,\eta_{n}\right) &\equiv \sqrt{n}\frac{-\mathbb{E}_{n}\left[\eta((2Y-1)1\{Xb\geq0,Xv^{+}<0\}-\hat{m}_{n}^{+}\left(b,v^{+}\right)\right)\right]}{\max\{\epsilon,\sqrt{\mathbb{E}_{n}[1\{Xb\geq0,Xv^{+}<0\}]-\hat{m}_{n}^{+}\left(b,v^{+}\right)^{2}\}}}, \quad v^{+}\in\mathcal{V}_{n}^{+}, \\ \hat{\mu}_{n}^{-*}\left(b,v^{-};\eta_{1},\ldots,\eta_{n}\right) &\equiv \sqrt{n}\frac{-\mathbb{E}_{n}\left[\eta((1-2Y)1\{Xb\leq0,Xv^{-}>0\}-\hat{m}_{n}^{-}\left(b,v^{-}\right)\right)\right]}{\max\{\epsilon,\sqrt{\mathbb{E}_{n}[1\{Xb\leq0,Xv^{-}>0\}]-\hat{m}_{n}^{-}\left(b,v^{-}\right)^{2}\}}}, \quad v^{-}\in\mathcal{V}_{n}^{-}. \end{split}$$

The critical value cv_1 is used to implement the adaptive inequality selection step of CLR. Moment inequalities that are sufficiently far from binding can be safely discarded, and those that remain are collected in the sets $\hat{\mathcal{V}}_n^+$ and $\hat{\mathcal{V}}_n^-$ defined as

$$\hat{\mathcal{V}}_{n}^{+} = \left\{ v^{+} \in \mathcal{V}_{n}^{+} : \sqrt{n} \frac{-\hat{m}_{n}^{+}(b,v^{+})}{\max\{\epsilon, \sqrt{\mathbb{E}_{n}[1\{Xb \ge 0, Xv^{+} < 0\}] - \hat{m}_{n}^{+}(b,v^{+})^{2}\}}} > -2cv_{1} \right\}$$
$$\hat{\mathcal{V}}_{n}^{-} = \left\{ v^{-} \in \mathcal{V}_{n}^{-} : \sqrt{n} \frac{-\hat{m}_{n}^{-}(b,v^{-})}{\max\{\epsilon, \sqrt{\mathbb{E}_{n}[1\{Xb \le 0, Xv^{-} > 0\}] - \hat{m}_{n}^{-}(b,v^{-})^{2}\}}} > -2cv_{1} \right\}.$$

The critical value used for the test statistic $T_n(b)$ based on CLR is then cv_2 , defined as the $1-\alpha$ quantile of

$$\max\left\{0, \sup_{v^{+}\in\hat{\mathcal{V}}_{n}^{+}}\hat{\mu}_{n}^{+*}\left(b, v^{+}; \eta_{1}, \dots, \eta_{n}\right), \sup_{v^{-}\in\hat{\mathcal{V}}_{n}^{-}}\hat{\mu}_{n}^{-*}\left(b, v^{-}; \eta_{1}, \dots, \eta_{n}\right)\right\}$$

conditional on the data $\{(Y_i, X_i) : i = 1, ..., n\}$.

6 Conclusion

In this paper we have proposed an approach to conduct finite sample inference on the parameters of Manski's (1985) semiparametric binary response model, for which the maximum score estimator has been shown to be cube-root consistent with a non-normal asymptotic distribution. Our finite sample inference approach circumvents the need to accommodate the complicated asymptotic behavior of this point estimator. Since our goal was finite sample inference, we considered the problem of making inference conditional on n covariate vectors observable in a finite sample. With covariates having only finite observed values, the parameter vector β is not point-identified. We therefore employed moment inequality implications for β for the sake of constructing our test statistic for inference, as the moment inequalities are valid no matter whether β is point identified. In order to exposit what observable implications can be distilled on only the basis of exogenous variables observed in the finite sample, we defined the notion of a finite sample identified set. We showed how to make use of the full set of observable implications conditional on the size n sequences of exogenous variables in our construction of a test statistic $T_n(b)$. Finite sample valid critical values were established, and were shown to be easily computed by making use of many simulations of size n sequences of independent Rademacher variables. A finite sample power property was presented and various Monte Carlo experiments were reported.



Figure 1: Non-rejection frequencies with $1 - \alpha = 90\%$ for Design 1 with n = 250.



Figure 2: Differences between the critical values in this paper and Chernozhukov, Lee, and Rosen (2013) for Design 1 with n = 250.



Figure 3: Non-rejection frequencies with $1 - \alpha = 90\%$ for Design 2 with n = 250.



Figure 4: Non-rejection frequencies with $1 - \alpha = 90\%$ for Design 3 with n = 250.



Figure 5: Non-rejection frequencies with $1 - \alpha = 90\%$ for Design 4 with n = 250.



Figure 6: Non-rejection frequencies with $1 - \alpha = 90\%$ for Design 5 with n = 250.



Figure 7: Differences between the critical values in this paper and Chernozhukov, Lee, and Rosen (2013) for Design 5 with n = 250.

References

- ABREVAYA, J., AND J. HUANG (2005): "On the Bootstrap of the Maximum Score Estimator," *Econometrica*, 73(4), 1175–1204.
- ANDREWS, D. W. K., AND X. SHI (2013): "Inference Based on Conditional Moment Inequalities," Econometrica, 81(2), 609–666.
- BLEVINS, J. R. (2015): "Non-standard Rates of Convergence of Criterion-Function-Based Set Estimators for Binary Response Models," *The Econometrics Journal*, 18(2), 172–199.
- CATTANEO, M. D., M. JANSSON, AND K. NAGASAWA (2018): "Bootstrap-Based Inference for Cube Root Consistent Estimators," Working paper, arXiv:1704.08066.
- CHEN, L.-Y., AND S. LEE (2017): "Breaking the Curse of Dimensionality in Conditional Moment Inequalities for Discrete Choice Models," CeMMAP working paper CWP 51/17.
- (2018): "Best Subset Binary Prediction," Journal of Econometrics, 206(1), 39 56.
- CHERNOZHUKOV, V., C. HANSEN, AND M. JANSSON (2009): "Finite Sample Inference for Quantile Regression Models," *Journal of Econometrics*, 152(2), 93 103.
- CHERNOZHUKOV, V., S. LEE, AND A. M. ROSEN (2013): "Intersection Bounds: Estimation and Inference," *Econometrica*, 81(2), 667–737.
- CLOPPER, C., AND E. S. PEARSON (1934): "The Use of Confidence or Fiducial Limits Illustrated in the Case of the Binomial," *Biometrika*, 26(4), 404–413.
- DELGADO, M. A., J. M. RODRÍGUEZ-POO, AND M. WOLF (2001): "Subsampling Inference in Cube Root Asymptotics with an Application to Manski's Maximum Score Estimator," *Economics Letters*, 73(2), 241–250.
- HOEFFDING, W. (1963): "Probability Inequalities for Sums of Bounded Random Variables," Journal of the American Statistical Association, 58(301), 13–30.
- HOROWITZ, J. L. (1992): "A Smoothed Maximum Score Estimator for the Binary Response Model," *Econo*metrica, 60(3), 505–531.
- KIM, J., AND D. POLLARD (1990): "Cube Root Asymptotics," Annals of Statistics, 18(1), 191–219.
- KOMAROVA, T. (2013): "Binary Choice Models with Discrete Regressors: Identification and Misspecification," Journal of Econometrics, 177(1), 14 – 33.
- LÉGER, C., AND B. MACGIBBON (2006): "On the Bootstrap in Cube Root Asymptotics," The Canadian Journal of Statistics / La Revue Canadienne de Statistique, 34(1), 29–44.
- MANSKI, C. F. (1975): "Maximum Score Estimation of the Stochastic Utility Model of Choice," Journal of econometrics, 3(3), 205–228.
- (1985): "Semiparametric Analysis of Discrete Response: Asymptotic Properties of the Maximum Score Estimator," *Journal of econometrics*, 27(3), 313–333.
- (2007): "Partial Identification of Counterfactual Choice Probabilities," International Economic Review, 48(4), 1393–1410.
- MANSKI, C. F., AND T. S. THOMPSON (1986): "Operational Characteristics of Maximum Score Estimation," Journal of Econometrics, 32(1), 85–108.
- POWELL, J. L. (1994): "Estimation of Semiparametric Models," in *The Handbook of Econometrics*, ed. by R. F. Engle, and D. L. McFadden, vol. 4. North-Holland.

A Proofs

Proof of Lemma 1. If $X_i \beta \ge 0$, then

 $Y_i = 1\{X_i\beta + U_i \ge 0\} \ge 1\{U_i \ge 0\}$

and therefore

$$\mathbb{E}[2Y_i - 1 \mid \mathcal{X}_n] \ge 2\mathbb{P}(U_i \ge 0 \mid \mathcal{X}_n) - 1 = 0.$$

If $X_i \beta \leq 0$, then

$$Y_i = 1\{X_i\beta + U_i \ge 0\} \le 1\{U_i \ge 0\}$$

and therefore

$$\mathbb{E}[2Y_i - 1 \mid \mathcal{X}_n] \le 2\mathbb{P}(U_i \ge 0 \mid \mathcal{X}_n) - 1 = 0$$

Proof of Theorem 1. That $b \in \mathcal{B}_n^*$ implies $\mathbb{E}\left[(2Y_i - 1) \ 1\{X_i b \ge 0\} \mid \mathcal{X}_n\right] \ge 0$ and $\mathbb{E}\left[(1 - 2Y_i) \ 1\{X_i b \le 0\} \mid \mathcal{X}_n\right] \ge 0$ follows directly from Lemma 1. To demonstrate the other direction, let b be any element of \mathcal{B} such that $\mathbb{E}\left[(2Y_i - 1) \ 1\{X_i b \ge 0\} \mid \mathcal{X}_n\right] \ge 0$ and $\mathbb{E}\left[(1 - 2Y_i) \ 1\{X_i b \le 0\} \mid \mathcal{X}_n\right] \ge 0$ for every $i = 1, \ldots, n$. For any such b:

$$X_i b \ge 0 \implies \mathbb{P}(Y_i = 1 \mid \mathcal{X}_n) \ge 1/2, \tag{A.1}$$

$$X_i b \le 0 \implies \mathbb{P}(Y_i = 1 \mid \mathcal{X}_n) \le 1/2.$$
(A.2)

To show that b is in \mathcal{B}_n^* as defined in Definition 1, we now construct a sequence of random variables $\{\tilde{U}_i : i = 1, ..., n\}$ such that for all i = 1, ..., n: (i) $\mathbb{P}(Y_i = 1\{X_i b + \tilde{U}_i \ge 0\} | \mathcal{X}_n) = 1$, and (ii) $\mathbb{P}(1\{U_i \ge 0\} = 1\{\tilde{U}_i \ge 0\} | \mathcal{X}_n) = 1$. To do so, let $\kappa_i : i = 1, ..., n$ be n random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$ each with support on $(0, \infty)$ and consider each of the cases $X_i\beta < 0, X_i\beta = 0$, and $X_i\beta > 0$ in turn as follows. Case 1: $X_i\beta < 0$: Let

$$\tilde{U}_i \equiv 1\{U_i \ge -X_i\beta\} \cdot \max\{-X_ib, 0\} + 1\{0 \le U_i < -X_i\beta\} \cdot 0 + 1\{U_i < 0\} \cdot (\min\{-X_ib, 0\} - \kappa_i).$$

From this construction of \tilde{U}_i , $1{\tilde{U}_i \ge 0} = 1{U_i \ge 0}$, which verifies (ii). To verify (i), we use the following equality:

$$1\{X_i b + U_i \ge 0\} = Y_i + 1\{0 \le U_i < -X_i \beta, X_i b \ge 0\}.$$
(A.3)

If $X_i b < 0$ then (A.3) implies $1\{X_i b + \tilde{U}_i \ge 0\} = Y_i$, which verifies (i) when $X_i b < 0$. For the rest of Case 1, we assume $X_i b \ge 0$. (2.2) and (A.1) imply that $\mathbb{P}(Y_i = 1 \mid \mathcal{X}_n) = 1/2$. Therefore,

$$\mathbb{P}(0 \le U_i < -X_i\beta \mid \mathcal{X}_n) = \mathbb{P}(U_i \ge 0 \mid \mathcal{X}_n) - \mathbb{P}(U_i \ge -X_i\beta \mid \mathcal{X}_n) = \mathbb{P}(U_i \ge 0 \mid \mathcal{X}_n) - \mathbb{P}(Y_i = 1 \mid \mathcal{X}_n) = 0.$$

Thus, together with (A.3),

$$\mathbb{P}(Y_i = 1\{X_i b + \tilde{U}_i \ge 0\} \mid \mathcal{X}_n) = \mathbb{P}(1\{0 \le U_i < -X_i\beta\} = 0 \mid \mathcal{X}_n) = 1,$$

which verifies (i).

Case 2: $X_i\beta = 0$: Let

$$\tilde{U}_i \equiv 1\{U_i \ge 0\} \cdot \max\{-X_i b, 0\} + 1\{U_i < 0\} \cdot (\min\{-X_i b, 0\} - \kappa_i).$$

Then $1{\tilde{U}_i \ge 0} = 1{U_i \ge 0}$, which verifies (ii). Since $X_i\beta = 0$, we have $1{X_ib + \tilde{U}_i \ge 0} = 1{U_i \ge 0} = 1{X_i\beta + U_i \ge 0}$, which verifies (i).

Case 3: $X_i\beta > 0$: Let

$$\tilde{U}_i \equiv 1\{U_i \ge 0\} \cdot \max\{-X_i b, 0\} + 1\{-X_i \beta \le U_i < 0\} \cdot (-X_i b) + 1\{U_i < -X_i b\} \cdot (\min\{-X_i b, 0\} - \kappa_i).$$

From this construction of \tilde{U}_i ,

$$1\{X_i b + \tilde{U}_i \ge 0\} = 1\{U_i \ge -X_i \beta\} = Y_i,$$

which verifies (i). To verify (ii), we use the following equality:

$$1\{\tilde{U}_i \ge 0\} = 1\{U_i \ge 0\} + 1\{-X_i\beta \le U_i < 0, -X_ib \ge 0\}.$$
(A.4)

If $X_i b > 0$ then (A.4) implies $1{\tilde{U}_i \ge 0} = 1{U_i \ge 0}$, which verifies (ii) when $X_i b > 0$. For the rest of Case 3, we assume $X_i b \le 0$. (2.1) and (A.2) imply that $\mathbb{P}(Y_i = 1 | \mathcal{X}_n) = 1/2$. Therefore,

$$\mathbb{P}(-X_i\beta \le U_i < 0 \mid \mathcal{X}_n) = \mathbb{P}(U_i \ge -X_i\beta \mid \mathcal{X}_n) - \mathbb{P}(U_i \ge 0 \mid \mathcal{X}_n) = \mathbb{P}(Y_i = 1 \mid \mathcal{X}_n) - \mathbb{P}(U_i \ge 0 \mid \mathcal{X}_n) = 0.$$

Thus, together with (A.4),

$$\mathbb{P}(1\{\tilde{U}_i \ge 0\} = 1\{U_i \ge 0\} \mid \mathcal{X}_n) = \mathbb{P}(1\{-X_i\beta \le U_i < 0\} = 0 \mid \mathcal{X}_n) = 1,$$

which verifies (ii).

Proof of Theorem 2. That $b \in \mathcal{B}_n^*$ implies (2.6) and (2.7) is immediate. To demonstrate the other direction, we are going to show that (2.6) and (2.7) imply that

$$\mathbb{E}\left[(2Y_i - 1) \, \mathbb{1}\{X_i b \ge 0\} \mid \mathcal{X}_n\right] \ge 0 \text{ and } \mathbb{E}\left[(1 - 2Y_i) \, \mathbb{1}\{X_i b \le 0\} \mid \mathcal{X}_n\right] \ge 0 \tag{A.5}$$

for every i = 1, ..., n. Note that under Assumption 1 (iii), there is a $v^+ \in \mathcal{V}_n^+$ and a $v^- \in \mathcal{V}_n^-$ such that

 $r_n^+(v^+)=r_n^+(\beta)$ and $r_n^-(v^-)=r_n^-(\beta).$ Lemma 1 implies

$$-\mathbb{E}_{n} \left[|\mathbb{E} \left[2Y - 1 \mid \mathcal{X}_{n} \right] | 1\{Xb \ge 0, X\beta < 0\} \right] \\= \mathbb{E}_{n} \left[\mathbb{E} \left[2Y - 1 \mid \mathcal{X}_{n} \right] 1\{Xb \ge 0, X\beta < 0\} \right] \\= \mathbb{E}_{n} \left[\mathbb{E} \left[2Y - 1 \mid \mathcal{X}_{n} \right] 1\{Xb \ge 0, Xv^{+} < 0\} \right] \\= \mathbb{E} \left[\mathbb{E}_{n} \left[(2Y - 1)1\{Xb \ge 0, Xv^{+} < 0\} \right] \mid \mathcal{X}_{n} \right] \\\ge 0$$
(A.6)
$$-\mathbb{E}_{n} \left[|\mathbb{E} \left[2Y - 1 \mid \mathcal{X}_{n} \right] | 1\{Xb \le 0, X\beta > 0\} \right] \\= \mathbb{E}_{n} \left[\mathbb{E} \left[1 - 2Y \mid \mathcal{X}_{n} \right] 1\{Xb \le 0, X\beta > 0\} \right] \\= \mathbb{E}_{n} \left[\mathbb{E} \left[1 - 2Y \mid \mathcal{X}_{n} \right] 1\{Xb \le 0, Xv^{-} > 0\} \right] \\= \mathbb{E} \left[\mathbb{E}_{n} \left[(1 - 2Y) 1\{Xb \le 0, Xv^{-} > 0\} \right] \mid \mathcal{X}_{n} \right] \\\ge 0.$$
(A.7)

Since both $|\mathbb{E}[2Y-1 | \mathcal{X}_n]| 1\{Xb \ge 0, X\beta < 0\}$ and $|\mathbb{E}[2Y-1 | \mathcal{X}_n]| 1\{Xb \le 0, X\beta > 0\}$ are non-negative, we have

$$|\mathbb{E}[2Y_i - 1 \mid \mathcal{X}_n]|1\{X_i b \ge 0, X_i \beta < 0\} = |\mathbb{E}[2Y_i - 1 \mid \mathcal{X}_n]|1\{X_i b \le 0, X_i \beta > 0\} = 0$$
(A.8)

for every $= 1, \ldots, n$. For every i with $\mathbb{E}[2Y_i - 1 | \mathcal{X}_n] = 0$, Eq. (A.5) holds with equality. For every i with $\mathbb{E}[2Y_i - 1 | \mathcal{X}_n] > 0$, we have $X_i\beta > 0$ from Lemma 1, and therefore Eq. (A.8) implies $X_ib > 0$, which in turn implies Eq. (A.5). For every i with $\mathbb{E}[2Y_i - 1 | \mathcal{X}_n] < 0$, we have $X_i\beta < 0$ from Lemma 1, and therefore Eq. (A.8) implies $X_ib < 0$, which in turn implies Eq. (A.8) implies $X_ib < 0$, which in turn implies Eq. (A.5). \Box

Proof of Theorem 3. If Eq. (3.1) holds under H_0 , then

$$\mathbb{P}\left(T_n(\beta) \le q_{1-\alpha} \mid \mathcal{X}_n\right) \ge \mathbb{P}\left(\bar{T}_n(\beta) \le q_{1-\alpha} \mid \mathcal{X}_n\right) \ge 1 - \alpha.$$

For the rest of the proof, we are going to show Eq. (3.1) under H_0 . Since

$$\begin{cases} Y_i \ge \bar{Y}_i & \text{if } X_i \beta \ge 0\\ Y_i \le \bar{Y}_i & \text{if } X_i \beta \le 0, \end{cases}$$

for every $i = 1, \ldots, n$, we have

$$\hat{m}_n^+ \left(\beta, v^+ \right) \ge \bar{m}_n^+ \left(\beta, v^+ \right), \quad v^+ \in \mathcal{V}_n^+ \hat{m}_n^- \left(\beta, v^- \right) \ge \bar{m}_n^- \left(\beta, v^- \right), \quad v^- \in \mathcal{V}_n^-.$$

By the construction of $\bar{T}_n(\beta)$ and $T_n(\beta)$, it suffices to show that, for every $v^+ \in \mathcal{V}_n^+$, the function

$$f(t) \equiv \max\left\{0, \sqrt{n} \frac{-t}{\max\{\epsilon, \sqrt{\mathbb{E}_n[1\{X\beta \ge 0, Xv^+ < 0\}] - t^2}\}}\right\}$$
(A.9)

is weakly decreasing and that, for every $v^- \in \mathcal{V}_n^-$, the function

$$t \mapsto \max\left\{0, \sqrt{n} \frac{-t}{\max\{\epsilon, \sqrt{\mathbb{E}_n[1\{X\beta \ge 0, Xv^- < 0\}] - t^2}\}}\right\}$$

is weakly decreasing. For the rest of the proof, we focus on the mapping in Eq. (A.9). Consider t_1 and t_2 with $t_1 < t_2$. If $t_2 \ge 0$, we have $f(t_1) \ge 0 = f(t_2)$. So we assume $t_1 < t_2 < 0$. Since $t_2^2 < t_1^2$, we have

$$\mathbb{E}_n[1\{X\beta \ge 0, Xv^+ < 0\}] - t_1^2 < \mathbb{E}_n[1\{X\beta \ge 0, Xv^+ < 0\}] - t_2^2,$$

 \mathbf{SO}

$$0 < \max\left\{\epsilon, \sqrt{\mathbb{E}_n[1\{X\beta \ge 0, Xv^+ < 0\}] - t_1^2}\right\} \le \max\left\{\epsilon, \sqrt{\mathbb{E}_n[1\{X\beta \ge 0, Xv^+ < 0\}] - t_2^2}\right\}.$$

Since $-t_1 > -t_2 > 0$, we have

$$\frac{-t_1}{\max\{\epsilon, \sqrt{\mathbb{E}_n[1\{X\beta \ge 0, Xv^+ < 0\}] - t_1^2}\}} > \frac{-t_2}{\max\{\epsilon, \sqrt{\mathbb{E}_n[1\{X\beta \ge 0, Xv^+ < 0\}] - t_2^2}\}}.$$

Therefore, $f(t_1) > f(t_2)$.

Proof of Theorem 4. We have

$$\mathbb{P}\left(\bar{T}_n(\beta) \le cv \mid \mathcal{X}_n\right) < 1 - \alpha - \eta$$

for some η . Consider a conditional distribution of (U_1, \ldots, U_n) given \mathcal{X}_n under which U_1, \ldots, U_n are independent given \mathcal{X}_n and $U_i \mid \mathcal{X}_n \sim N(0, \sigma^2)$ for every $i = 1, \ldots, n$, where σ is defined by

$$\sigma^{-n}\phi(0)^n \prod_{i=1}^n |X_i\beta| = 1 - \eta.$$

For every i = 1, ..., n, define $D_i = 1\{Y_i = \overline{Y}_i\}$. Note that if $D_i = 1$ for every i = 1, ..., n, then $T_n(\beta) = \overline{T}_n(\beta)$. Then

$$\begin{split} \mathbb{P}\left(T_{n}(\beta) \leq cv \mid \mathcal{X}_{n}\right) &\leq \mathbb{P}\left(T_{n}(\beta) \leq cv \land D_{i} = 1 \text{ for every } i = 1, \dots, n \mid \mathcal{X}_{n}\right) \\ &+ 1 - \mathbb{P}\left(D_{i} = 1 \forall i = 1, \dots, n \mid \mathcal{X}_{n}\right) \\ &= \mathbb{P}\left(\bar{T}_{n}(\beta) \leq cv \land D_{i} = 1 \text{ for every } i = 1, \dots, n \mid \mathcal{X}_{n}\right) \\ &+ 1 - \mathbb{P}\left(D_{i} = 1 \forall i = 1, \dots, n \mid \mathcal{X}_{n}\right) \\ &= \mathbb{P}\left(\bar{T}_{n}(\beta) \leq cv \mid \mathcal{X}_{n}\right) + 1 - \mathbb{P}\left(D_{i} = 1 \forall i = 1, \dots, n \mid \mathcal{X}_{n}\right) \\ &< 1 - \alpha - \eta + 1 - \mathbb{P}\left(D_{i} = 1 \forall i = 1, \dots, n \mid \mathcal{X}_{n}\right). \end{split}$$

The statement of this theorem follows from

$$\mathbb{P}(D_i = 1 \text{ for every } i = 1, \dots, n \mid \mathcal{X}_n) = \prod_{i=1}^n \mathbb{P}(D_i = 1 \mid \mathcal{X}_n)$$

$$\leq \prod_{i=1}^n \left(\Phi\left(\frac{|X_i\beta|}{\sigma}\right) - \frac{1}{2} \right)$$

$$\leq \prod_{i=1}^n \frac{|X_i\beta|}{\sigma} \phi(0)$$

$$= \eta.$$

Proof of Theorem 5. In this proof, we focus on Eq. (4.1). Define $W = (2Y - 1)1\{Xb \ge 0, Xv^+ < 0\}$. First, we are going to show that

$$\sqrt{n}\mathbb{E}_n[W] < -q_{1-\alpha} \max\left\{\epsilon, \sqrt{\frac{\mathbb{E}_n[W^2]}{1+q_{1-\alpha}^2/n}}\right\} \implies T_n(b) > q_{1-\alpha}.$$
(A.10)

Suppose $\sqrt{n}\mathbb{E}_n[W] < -q_{1-\alpha} \max\left\{\epsilon, \sqrt{\frac{\mathbb{E}_n[W^2]}{1+q_{1-\alpha}^2/n}}\right\}$. Note that

$$\mathbb{E}_n[W] < 0 \tag{A.11}$$

and

$$n\mathbb{E}_{n}[W]^{2} > \max\left\{\epsilon^{2}q_{1-\alpha}^{2}, \mathbb{E}_{n}[W^{2}]\frac{q_{1-\alpha}^{2}}{1+q_{1-\alpha}^{2}/n}\right\}.$$

The second inequality implies

$$n\mathbb{E}_{n}[W]^{2} > \max\{\epsilon^{2}q_{1-\alpha}^{2}, \mathbb{E}_{n}[W^{2}]q_{1-\alpha}^{2} - \mathbb{E}_{n}[W]^{2}q_{1-\alpha}^{2}\}$$

Using Eq. (A.11),

$$-\sqrt{n}\mathbb{E}_n[W] > q_{1-\alpha}\max\{\epsilon, \sqrt{\mathbb{E}_n[W^2] - \mathbb{E}_n[W]^2}\}.$$

Then

$$\sqrt{n} \frac{-\mathbb{E}_n[W]}{\max\{\epsilon, \sqrt{\mathbb{E}_n[W^2] - \mathbb{E}_n[W]^2}\}} > q_{1-\alpha}$$

which implies $T_n > q_{1-\alpha}$.

Then, we are going to show $\mathbb{P}(T_n > q_{1-\alpha} \mid \mathcal{X}_n) \ge 1 - \rho$. Using Eq. (A.10), we have

$$\mathbb{P}(T_n > q_{1-\alpha} \mid \mathcal{X}_n) \ge \mathbb{P}\left(\sqrt{n}\mathbb{E}_n[W] < -q_{1-\alpha} \max\left\{\epsilon, \sqrt{\frac{\mathbb{E}_n[W^2]}{1+q_{1-\alpha}^2/n}}\right\} \mid \mathcal{X}_n\right).$$

Eq. (4.1) implies

$$\mathbb{P}(T_n > q_{1-\alpha} \mid \mathcal{X}_n) \geq \mathbb{P}\left(\mathbb{E}_n[W] < \mathbb{E}\left[\mathbb{E}_n[W] \mid \mathcal{X}_n\right] + \sqrt{\frac{2\log(1/\rho)\mathbb{E}_n[1\{Xb \ge 0, Xv^+ < 0\}]}{n}} \mid \mathcal{X}_n\right)$$

$$= 1 - \mathbb{P}\left(\mathbb{E}_n[W] \ge \mathbb{E}\left[\mathbb{E}_n[W] \mid \mathcal{X}_n\right] + \sqrt{\frac{2\log(1/\rho)\mathbb{E}_n[1\{Xb \ge 0, Xv^+ < 0\}]}{n}} \mid \mathcal{X}_n\right).$$

Since

$$-1\{Xb \ge 0, Xv^+ < 0\} \le W_i \le 1\{Xb \ge 0, Xv^+ < 0\}$$

for every i = 1, ..., n, Hoeffding (1963)'s inequality implies

$$\mathbb{P}(T_n > q_{1-\alpha} \mid \mathcal{X}_n) \ge 1 - \exp\left(-\frac{2n^2 \left(\sqrt{\frac{2\log(1/\rho)\mathbb{E}_n[1\{Xb \ge 0, Xv^+ < 0\}]}{n}}\right)^2}{4\sum_{i=1}^n 1\{X_ib \ge 0, X_iv^+ < 0\}}\right) = 1 - \rho.$$